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Market Share Dynamics in a Duopoly Model with Word-of-Mouth Communication*

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Abstract

We analyze dynamic price competition in a homogeneous goods duopoly, where consumers exchange information via word-of-mouth communication. A fraction of consumers, who do not learn any new information, remain locked-in at their previous supplier in each period. We analyze Markov perfect equilibria in which firms use mixed pricing strategies. Market share dynamics are driven by the endogenous price dispersion. Depending on the parameters, we obtain different 'classes' of dynamics. When firms are impatient, there is a tendency towards equal market shares. When firms are patient, there are extended intervals of market dominance, interrupted by sudden changes in the leadership position.

Keywords: dynamic duopoly; homogeneous goods; price competition; consumer lock-in; mixed pricing; Markov perfect equilibrium
JEL classification: C73, D83, L11

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1 Introduction

Consider a market in which consumers can learn about the available products and their prices in two ways: via own experiences from previous purchases and by asking fellow consumers about products or suppliers. Such gathering of information via word-of-mouth communication is often a costless byproduct of social interaction, but is unlikely to reveal all decision-relevant information to all consumers in any market. The lack of information may then lead to stickiness in the demand. In this paper we study the dynamics of prices and market shares in a homogeneous-goods duopoly with sticky demand stemming from imperfect word-of-mouth communication. While firms act as perfectly rational forward-looking profit maximizers, consumers behave in a simple fashion. Whenever they learn about the prices charged by both suppliers, they purchase the good from the firm that charges the lower price in that period. If a consumer does not discover the price charged by the alternative supplier via word-of-mouth, then she remains locked-in and returns to the supplier visited in the previous period.[1] As a result of the sticky demand, the firm's strategic decisions become dependent on its customer base.

A central question in the analysis of dynamic oligopoly games is whether market shares tend to equalize over time, or whether a firm may be able to build up and subsequently defend a dominant market position persistently. Two basic effects determine whether persistent dominance is likely to occur. On the one hand, having a large customer base implies more monopoly power over locked-in consumers, which firms tend to exploit by charging higher prices (*anti-competitive effect*). This leads to a tendency towards equal market shares. On the other hand, charging a low price today increases the future customer base. If this incentive is sufficiently strong, then a firm that already has a dominant position in the market, may price very aggressively whenever it is threatened, in order to defend its dominant position (*pro-competitive effect*). This, in turn, leads to a tendency towards extreme market shares.

The main contribution of this paper is that it provides a single framework that can explain surprisingly rich dynamics, including a tendency towards equal market shares, as well as persistence of dominance and changes in the role of the dominant firm. The prevalence of word-of-mouth communication plays a key role in the determination of firms' incentives to build up and subsequently to defend a dominant market position. We thus identify different 'types' of dynamics, and relate them to the basic parameters of the model, in particular the discount rate, and parameters that determine the effectiveness of word-of-mouth communication. Our model helps to explain why a firm that has dominated a market for a long period of time, may lose this position again, in which case the competitor takes over the dominant position.

In our model, dynamics never 'die out', even though we do not assume any exogenous

[1] Such behavior may arise, for instance, due to imperfect recall.

source of uncertainty. More specifically, due to the assumption of homogeneous goods, firms adopt mixed pricing strategies in the Markov perfect equilibria we identify. The endogenous price dispersion determines the probability of each firm to gain or lose markets shares, depending on the current state represented by the customer base. Similarly as in Varian (1980), firms adopt randomized pricing strategies in order to be unpredictable to the competitor. As Varian (1980) argues, randomized pricing can be interpreted as limited-time sales that do not exhibit any systematic patterns. Such behavior can be observed for grocery stores or supermarkets, that frequently offer few selected products at discounted prices. Empirical evidence for price dispersion is provided, for instance, by Lach (2002), who uses a dataset involving homogeneous products sold by different sellers. Lach (2002, p. 444) also argues that the identified price dispersion is consistent with randomized prices (or sales) as studied by Varian (1980).

In this paper, we extend the idea of sales to dynamic pricing games, and demonstrate how mixed-strategy equilibria can generate plausible dynamics. In particular, we identify a tendency towards skewed market share splits when future profits are important. This tendency becomes more pronounced when many consumers rely on word-of-mouth communication. As a distinctive consequence of word-of mouth communication, a firm with a smaller customer base can attract less additional consumers when charging a lower price in the market, because there are less consumers who can share this information. As a result, market shares are more sticky near the extremes of the market share space than in the center, where information spreads more efficiently. A firm that has reached a dominant position in the market can then easily *defend* this position against the smaller competitor whenever it is at stake (pro-competitive effect). This is indeed the case when future profits are sufficiently important and market shares are skewed but not extreme. The firm with a larger customer base then starts to price aggressively and tends to gain market shares. On the other hand, when market shares are closer to one of the extremes, the opposite (anti-competitive) effect dominates: the firm with the larger customer base now prefers to exploit the locked-in consumers in its customer base by charging higher prices. The combination of these two effects often induces a zig-zag pattern near one of the extremes of the market share space, with one firm dominating the market for many consecutive periods. This persistence of dominance, however, can be interrupted by sudden changes in the leadership position. In contrast, when few consumers rely on word-of-mouth and most consumers are fully informed, market shares are very volatile and the role of the leader changes frequently. In this case, the size of the discount factor has little impact upon the dynamics.

From the technical point of view, we offer a new treatment of Markov perfect equilibria in mixed strategies. Via a discretization of the state space, we are able to approximate the evolution of market shares, allowing us to derive analytical results. For a certain

range of parameter values where word-of-mouth communication plays a major role in consumers' information acquisition, we show that a particularly simple market share grid represents market share dynamics sufficiently well.

Related Literature

The model introduced in this paper builds on a strand of literature that was initiated by Salop and Stiglitz (1977) and Varian's (1980) 'Model of Sales'. Similarly as in Varian (1980), we also assume that some consumers learn only one supplier's price, while others are fully informed and purchase from the supplier that currently offers the lower price. Whereas Varian's model is static, in our model an intertemporal link (inertia) in demand arises because consumers always learn their previous supplier's current price (e.g., because they remember this supplier's location), but do not always discover the other firm's offer. Varian (1980), in contrast, considers only markets that are *ex-ante* symmetric. From the technical point of view, our paper is related to later contributions, for example, Baye, Kovenock, and de Vries (1992 and 1996), Baye and Morgan (2004), and others that offer a more rigorous treatment of mixed-strategy equilibria and give additional explanations for their occurrence. Our main contribution to that literature is to extend the concept of mixed strategies to a dynamic game where players randomize over prices in each period and for any given state (reflecting the size of firms' customer bases).

Furthermore, our paper contributes to a sizable strand of literature that seeks to reveal conditions under which market shares in duopolies tend to become more or less skewed over time.[2] For instance, Budd, Harris, and Vickers (1993) and Cabral (2011) introduce an external source of uncertainty and show that a rise in the discount factor for future profits can induce a tendency towards more skewed market splits (relative to the myopic case where future profits are fully discounted away). While Budd et al. (1993) show that asymmetric market splits can emerge also due to self-reinforcing cost effects under strategic interaction, in Cabral (2011) and Mitchell and Skrzypacz (2006), a tendency towards skewed market splits results from network effects. Related results are also presented by Cabral and Riordan (1994) in a model with learning by doing. Athey and Schmutzler (2001) offer a more general dynamic oligopoly framework that can encompass patent races, learning by doing as well as network externalities as special cases. The dynamics are guided by investments and the authors derive predictions about the market evolution by comparing investment incentives of leading and lagging firms.

In contrast to most papers from this literature, in our model market share dynamics are generated solely due to the firms' usage of randomized pricing strategies.[3] The en-

[2]See also Sutton (2007) for an empirical investigation on the persistence of leadership.

[3]Budd et al. (1993) point out that in their model, an equilibrium typically fails to exist when the variance of the random drift is zero. The reason for this is that the authors focus on Markov perfect equilibria in pure strategies, and neglect the possibility of mixed strategies. In this sense, our analysis is

dogenous price dispersion in our model can help to explain rather complicated dynamics. For instance, both the volatility and the skewedness of market shares are endogenous. Depending on the parameter values, persistence of leadership as well as frequent changes in the leadership position can be obtained. The wealth of dynamics our model can generate is remarkable, given our parsimonious basic setup (e.g., homogeneous goods).

A dynamic duopoly model with randomized pricing strategies is presented also by Chen and Rosenthal (1996). Similarly as in our model, a firm that currently offers the higher price loses market shares. However, the authors assume that a loss in market shares is always associated with the same number of consumers switching the supplier. The resulting uniform "step size" in the market share space is an assumption that seems poorly justified. Although our approach is far more general, it can accommodate also this special case.[4]

There are several features of our model shared also with other strands of literature. For instance, price dispersion is often obtained also in models with search (e.g., Stahl, 1989; Janssen and Moraga-Gonzalez, 2004).[5] Alternatively, price dispersion can also be obtained by introducing bounded rationality. Baye and Morgan (2004), for instance, demonstrate that firms adopt mixed pricing strategies already under a small degree of bounded rationality among sellers in a pricing game with homogeneous goods.[6] The feature that firms are perfectly rational, but consumers use only simple decision rules is rather common in the industrial organization literature. Consumers' behavior is then often interpreted as a result of bounded rationality. Ellison (2006) argues that consumers who are active in many markets may pay little attention to the characteristics of a specific market and may, thus, behave in a boundedly rational way, whereas firms are more likely to exhibit rational decision making. In Spiegler (2006), this behavior is not derived from an explicit intertemporal optimization problem. Spiegler assumes that the agents sample all suppliers in the market once. We consider a similar process where consumers collect information via word-of-mouth, but the sampling stops irrespective of whether useful information was obtained or not.

Relatedly, the effects of word-of-mouth communication or learning from popularity have been analyzed in different strands of literature. For instance, Juang (2001), Banerjee and Fudenberg (2004), Ellison and Fudenberg (1995) analyze the effects of word-of-mouth learning in non-market environments where agents choose between alternatives with stochastic payoffs. Different sampling rules are applied, that do not result from an

complementary to their analysis.

[4]Chen and Rosenthal's (1996) model predicts that market shares are close to the center most of the time when firms have identical discount factors. This finding is related to the simplifying assumption of a uniform step size in the market share space.

[5]See also Fishman and Rob (1995) for a model with search where consumers purchase the good repeatedly.

[6]The authors also show that similar patterns can be observed in laboratory experiments and in the internet.

optimization problem and may, therefore, reflect bounded rationality. Rob and Fishman (2005), and Vettas (1997), for example, analyze the role of word-of-mouth communication among consumers in the context of firms' optimization problems. For more classical approaches to incorporate demand inertia and popularity effects into oligopoly models, see also Selten (1965) and Bass (1969).[7]

Finally, a link also exists between our model and the literature on switching costs (for an overview, see Klemperer, 1995). While in our model, some consumers are locked-in at their previous supplier due to the lack of decision-relevant information about the alternative supplier (e.g., its location or this firm's current offer), models with switching costs assume that consumers can always change their supplier but, e.g., due to learning costs of how to use the other product, incur costs when they switch. Our model is closely related to Beggs and Klemperer (1992), who assume that due to prohibitively high switching costs, consumers who have made a purchase before, are effectively locked-in at their previous supplier. New consumers who enter the market are not yet attached to any of the firms. Therefore, firms may compete vigorously for those consumers, in order to build up a larger customer base that is valuable in future periods.

The remainder of this paper is organized as follows. In Section 2, the model is introduced and equilibrium conditions are derived. In Section 3, we provide a micro-foundation for consumers' behavior based on word-of-mouth communication. Section 4 derives general results under a discretization of the state space. In Section 5 we provide a detailed analysis using a market share grid with five positions. Section 6 uses numerical simulations that serve as a robustness check. Section 7 concludes. All proofs are relegated to the Appendix.

2 The model

Consider a market for a homogeneous good with two firms ($i \in 1, 2$). Time is discrete, the time horizon is infinite. The good is non-durable and lasts only for one period. The firms choose simultaneously prices $p_{i,t}$ in every period $t = 1, 2, ...$ Both firms' marginal costs are constant and normalized to zero. On the demand side, there is a continuum of consumers with mass 1 who make repeated purchases of the good. Each consumer purchases either zero or one unit of the good in each period, and all consumers have the same reservation price, normalized to 1. Prices above 1 are eliminated from the strategy space without loss of generality. Hence, the market demand in each period equals 1.

We consider the situation where market shares in period t depend on prices in period t, and on market shares in period $t-1$. Such "sticky demand" or "inertia" may arise

[7] Hehenkamp (2002) presents an evolutionary approach where sellers decide via experimentation and imitation, and consumers search.

for various reasons. Within our motivation it reflects consumers' imperfect information acquisition; see Section 3 for more details.[8] Firm 1's demand (equal to its market share) in period $t-1$ represents its *customer base* in period t; we denote it $n_t \equiv D_{1,t-1}$. The size of firm 2's customer base in period t is then $D_{2,t-1} = 1 - n_t$.

Let $n_1 \in (0,1)$ be the initial state such that both firms have positive customer bases. We assume that in each period firm 1's market share increases to the value $h(n_t)$ if it charges a lower price in period t, and drops to the value $l(n_t)$ if it charges a higher price. If both firms charge identical prices, their market shares remain constant. Hence, market share dynamics are guided by the following transition function:

$$n_{t+1} = D(p_{1,t}, p_{2,t}, n_t), \qquad \text{where} \qquad D(p_1,p_2,n) = \begin{cases} h(n) & \text{if} \quad p_1 < p_2, \\ n & \text{if} \quad p_1 = p_2, \\ l(n) & \text{if} \quad p_1 > p_2. \end{cases} \tag{1}$$

We assume that the functions $h, l : [0,1] \to [0,1]$ are continuous and strictly increasing, and that

$$0 < l(n) < n < h(n) < 1 \qquad \text{for all } n \in (0,1), \tag{2}$$

$$h(0) > 0 \qquad \text{and} \qquad l(1) < 1. \tag{3}$$

Hence, a firm that charges a lower price in the current period gains market shares, but it does not serve the entire market.[9]

Moreover, we assume that the above process is symmetric for both firms. Thus, the functions $h(\cdot)$ and $l(\cdot)$ are required to satisfy the following consistency condition:

$$l(n) + h(1-n) = 1, \qquad \text{for all } n \in [0,1]. \tag{4}$$

It follows from this condition that the aggregate demand equals 1 in each period, for any given market split (captured by the state variable n_t). In Section 3 we present an information transmission technology that gives rise to the functions $h(\cdot)$ and $l(\cdot)$. Observe that it is sufficient to specify the function $l(\cdot)$: Function $h(\cdot)$ then follows from condition (4).

Note that in the above specification, the change in the demand (or market share) depends, besides the customer base, only on whether the firm charges a higher or a lower price than its competitor; the size of the price difference is irrelevant. In this respect, our model represents only a minor departure from the Bertrand model, which would be

[8] Other possible reasons include consumer search, switching costs, and/or network effects.

[9] Given the assumption $n_1 \in (0,1)$, condition (2) assures that market shares never reach the boundaries of the market share space ($n = 0$ or $n = 1$).

obtained by setting $h(n) = 1$ and $l(n) = 0$. We will see later that our simple setup generates surprisingly rich dynamics.

2.1 Profit maximization and equilibrium

In the following, we characterize the firms' profit maximization problems for a general specification of the functions $h(\cdot)$ and $l(\cdot)$. We derive equilibrium conditions that can be used to solve the model.

Given the prices $p_{1,t}$ and $p_{2,t}$, firm 1's profit in period t is $p_{1,t}D(p_{1,t}, p_{2,t}, n_t)$. Let us also denote $\pi^E_{1,t}(n_t, p_{1,t}) = p_{1,t}E_{p_{2,t}}[D(p_{1,t}, p_{2,t}, n_t)]$ firm 1's expected profit, if firm 2 uses a (potentially) mixed strategy, where the expectation is taken over firm 2's price. As we show later, in equilibrium the firms indeed use only mixed strategies. Firm 2's profit is obtained in an analogous way (we replace n by $1 - n$, and swap the indices of the firms). Let $\delta \in [0, 1)$ be the discount factor for future profits. In period t, firm i maximizes the present discounted value of future expected profits over an infinite horizon:

$$V_i(H_t) \equiv \max_{\{p_{i,t}\}} \sum_{\tau=t}^{\infty} \delta^{\tau-t} \pi^E_{i,\tau}(n_\tau, p_{i,\tau} \mid H_\tau), \tag{5}$$

where H_t denotes the history up to period t. The history H_t contains all prices chosen up to period $t - 1$, as well as the initial state n_1.[10] The expectation in (5) is taken over firm j's prices ($j \neq i$), conditional on the history. Note that there is no external source of uncertainty in the model. As we show below, in equilibrium, uncertainty arises only due to the firms' usage of mixed pricing strategies.

If firms can condition their choice in period t on the entire history H_t, the set of equilibria may potentially be overwhelming. To narrow the set of equilibria, the Markov perfection equilibrium (MPE) refinement is used. The payoff-relevant *state variable* in period t is the customer base n_t. Hence, we are looking for equilibria, where the firms condition their price only on the current state n_t. This is a natural requirement, as the profit in each period depends only on the prices in this period, and market shares in the previous period.

We start by considering all Markov perfect equilibria and derive some of their properties (Propositions 1 and 2). Later, we will restrict our attention only to the symmetric equilibria.

The solution of firm i's optimization problem satisfies the following Bellman-equation:

$$V_i(n) = \max_p \{\pi^E_i(n, p) + \delta E[V_i(n') \mid n, p]\}, \tag{6}$$

[10]Note also that given a specification of the functions $h(\cdot)$ and $l(\cdot)$, the history of the market shares up to period t is fully described by the initial size of firm 1's customer base (n_1), and a trinary sequence that indicates whether firm 1 gains, loses, or retains the market share in each period.

where n and n' stand for customer base n_t and current demand n_{t+1}, respectively.[11]

As a first simple observation, note that $V_i(n)$ is non-negative. This follows from the fact that the firm can always set price $p_{i,t} = 0$ in every period, guaranteeing itself zero profit. Second, observe that the joint profit $V_1(n) + V_2(1-n)$ is bounded from above by $1 + \delta + \delta^2 + \cdots = 1/(1-\delta)$, the discounted monopoly profit with full market, as the joint profit in each period is at most 1. Further properties of Markov perfect equilibria are summarized in the following propositions.[12]

Proposition 1. *If δ is sufficiently small, there is no Markov perfect equilibrium (MPE), where both firms use pure strategies in some state $n \in [0,1]$.*

Proposition 2. *Consider a MPE. If δ is sufficiently small, then for any $n \in (0,1)$ the following statements hold:*

(i) *Both firms use mixed strategies and the price distribution functions $F_1(p \mid n)$ and $F_2(p \mid 1-n)$ have the same support of prices.*[13]

(ii) *The support of the price distribution is an interval of the form $[\underline{p}(n), 1]$.*

(iii) *At most one firm attaches a positive probability mass to some price. If so, the mass point is located at $p = 1$ (the monopoly price).*

We require δ to be sufficiently small for most of the above results (see the proof of Proposition 2 for details). In this case, the future profits are relatively unimportant and undercutting becomes beneficial due to an increase in the current profit. The difficulty that arises for δ large, is that an increase in the size of a firm's customer base may lead to *reduced* profits in the future when the intensity of future price competition increases. Hence, the value function $V(n)$ may not be generally monotone. If δ is large, a firm may be reluctant to undercut the competitor's price if this price can be predicted with certainty. However, the logic of the mixed pricing strategies requires that, within the support of $F_i(\cdot)$, a firm would always undercut the competitor's price if it were known.

From now on, let us consider only *symmetric* Markov perfect equilibria. We thus omit the index i identifying the firm. In the rest of this section we also assume that δ is sufficiently small (as required in Propositions 1 and 2).

[11] We replace the full history H_t in the argument of V by the payoff relevant state $n = n_t$.

[12] The propositions provide necessary conditions for Markov perfect equilibria. The existence of the equilibrium can be potentially proved using similar methods as in Dasgupta and Maskin (1986a, 1986b). We prove the existence of an equilibrium for a special case, a discrete approximation with 5 positions, in Section 5.

[13] Let the cumulative distribution function be defined as $F(p \mid n) \equiv \Pr(P < p \mid n)$. Under this convention, $1 - F_i(\bar{p}(n) \mid n)$ is the probability mass at the maximal price $\bar{p}(n)$.

The expected profit of a firm with market share n (let it be firm 1) in the current period can be written as

$$\pi^E(n,p) = p\left[l(n)F(p \mid 1-n) + h(n)(1-F(p \mid 1-n))\right] \tag{7}$$

where $l(n)$ and $h(n)$ is firm 1's demand if it ends up having a higher and a lower price, and $F(p \mid 1-n)$ and $1-F(p \mid 1-n)$ are the probabilities of these events, respectively. Firm 1's expected value in the next period is

$$E[V(n') \mid n,p] = V(l(n))F(p \mid 1-n) + V(h(n))(1-F(p \mid 1-n)) \tag{8}$$

where $V\left(l(n)\right)$ and $V(h(n))$ is the firm's value after losing and gaining market shares, respectively.

In a mixed strategy equilibrium, the maximum in (6) is attained over a range of prices, and the support of $F(p \mid n)$ contains only prices within this set. Among all prices p outside the support of $F(p \mid n)$, there is no profitable deviation, i.e., $\pi^E(n,p)+\delta E[V(n') \mid n,p] \leq V(n)$. Using (7) and (8) in (6), we obtain the following expression for the value function:

$$V(n) = \left[pl(n) + \delta V(l(n))\right]F(p \mid 1-n) + \left[ph(n) + \delta V(h(n))\right](1-F(p \mid 1-n)). \tag{9}$$

for all prices p within firm 1's support. The max-operator has been omitted because, for prices within the support of $F(\cdot)$, the right-hand side must be independent of p. Equation (9) can be solved for $F(p \mid 1-n)$:

$$F(p \mid 1-n) = \frac{\delta V(h(n)) - V(n) + ph(n)}{\delta[V(h(n)) - V(l(n))] + p[h(n) - l(n)]}, \tag{10}$$

for all $n \in (0,1)$. Note that the cumulative distribution function $F(\cdot)$ is a ratio of two linear functions of the price p. If the function $V(\cdot)$ is known, (10) can be used to evaluate $F(\cdot)$ for all n.

In the following we derive equilibrium conditions that can be used to determine $V(\cdot)$. It follows from Proposition 2 that $F(\underline{p}(n) \mid n) = F(\underline{p}(n) \mid 1-n) = 0$ for all $n \in (0,1)$. Using (10), after eliminating $\underline{p}(n)$, we obtain the first equilibrium condition:

$$h(n)[V(1-n) - \delta V(h(1-n))] = h(1-n)[V(n) - \delta V(h(n))]. \tag{11}$$

We obtain further conditions by considering the maximal price $p = 1$, as at most one of the firms chooses the monopoly price with positive probability (see Proposition 2).

Hence, it must hold that

$$\begin{array}{llllll} \text{either} & \text{(A)} & F(1 \mid n) = 1 & \text{and} & F(1 \mid 1-n) \leq 1, \\ \text{or} & \text{(B)} & F(1 \mid n) < 1 & \text{and} & F(1 \mid 1-n) = 1. \end{array}$$

We refer to the former as *case (A)* and to the latter as *case (B)*.

Suppose that for a given value of the state n, case (A) applies.[14] Condition $F(1 \mid n) = 1$ can then be used to derive an equilibrium condition for the given value of n. Using (10), we obtain:

$$V(1-n) - \delta V(l(1-n)) = l(1-n). \tag{12}$$

If case (B) applies (for a given n), then it similarly follows from (10) that:

$$V(n) - \delta V(l(n)) = l(n). \tag{13}$$

Together with an (until now) unknown rule that states whether case (A) or case (B) applies for every given state n, conditions (11), (12), and (13) implicitly define the value function $V(\cdot)$. Note, that this is a *continuum* of equations because these conditions must be fulfilled for all $n \in (0, 1)$. Unfortunately when $\delta > 0$, it is not possible to solve this system in general. In Section 4 we analyze a discrete version of the model which can indeed be solved. Furthermore, we provide an explicit solution for a discrete model with 5 positions (states) in Section 5. As it turns out, this simple model serves as a good approximation of the dynamics for certain transition functions, when the state is continuous.

2.2 Benchmark: Myopic case

As a benchmark, we first analyze the myopic case ($\delta = 0$). This corresponds to a repetition of a static game, as the future profits are fully discounted. As in the full dynamic game, the dynamics are governed by each firm's probability of gaining or losing market shares, depending on the current state n. Results for the myopic case can be obtained analytically.

First of all, note that Propositions 1 and 2 also hold when $\delta = 0$. In order to derive the price distribution, let us first consider the price $p = \underline{p}(n)$. Then $F(\underline{p}(n) \mid 1-n) = 0$

[14] Until now, it is not clear under what conditions case (A) or case (B) applies. Intuitively, we would expect that case (A) applies (firm 2 chooses the monopoly price with positive probability) whenever firm 1's customer base n is not greater than $\frac{1}{2}$, because it should, then, have a stronger incentive to compete for new customers, while firm 2 plays less aggressively and charges the monopoly price with positive probability. As shown below, this intuition is generally correct for sufficiently small values of the discount factor δ.

and it follows from (9) that $V(n) = \underline{p}(n)h(n)$. Substituting this into (10), we obtain that

$$F(p \mid 1-n) = \frac{h(n)}{h(n)-l(n)}\Big(1 - \frac{\underline{p}(n)}{p}\Big) \tag{14}$$

for $p \in [\underline{p}(n), 1]$. Now recall that given n, both firms' price distributions have the same support (Proposition 2), i.e., $\underline{p}(1-n) = \underline{p}(n)$. Comparing the cumulative distribution functions of firms with market shares n and $1-n$, we obtain the following proposition.

Proposition 3. *In the myopic case ($\delta = 0$), firm 2's price distribution function $F(p \mid 1-n)$ first-order stochastically dominates firm 1's price distribution function $F(p \mid n)$ if and only if $n < \frac{1}{2}$.*

The proposition reveals that in the myopic case, it is always the firm with the *smaller* customer base that prices more aggressively than its larger rival.[15] This, in turn, implies that there is a tendency towards even splits of the market. In Section 6, we illustrate this result using numerical simulations.

As the last step we specify the lower bound of the price distribution $\underline{p}(n)$. It follows from Proposition 3 that for the monopoly price $p = 1$ and for $n < \frac{1}{2}$, we obtain $F(1 \mid 1-n) < F(1 \mid n) = 1$, which means that case (A) applies. Clearly, for $n > \frac{1}{2}$, case (B) applies. Substituting this into (14), we obtain

$$\underline{p}(n) = \begin{cases} l(1-n)/h(1-n), & \text{if } n \leq \frac{1}{2}, \\ l(n)/h(n), & \text{if } n > \frac{1}{2}. \end{cases}$$

This, together with (14), gives a complete description of the firms' equilibrium strategies.

3 Information transmission

In this section we provide a micro-foundation for consumers' behavior that gives rise to the functions $h(\cdot)$ and $l(\cdot)$. In particular, we introduce two specifications of the underlying technology of information transmission, based on two ways how consumers may exchange information via word-of-mouth. The first specification of the information technology yields a linear-quadratic functional form, while the other one yields an exponential form. Under both specifications, consumers remember the relevant information about the supplier they visited in the previous period. In addition, at the beginning of the period, they may learn the other supplier's price via communication with other consumers.

Under the first specification, consider a consumer from firm 1 with customer base n. We assume that the consumer learns the other firm's price with an exogenous probability

[15] Proposition 3 can be seen as an extension of Varian's (1980) model, for an asymmetric distribution of uninformed consumers.

$1-\phi$, where $\phi \in (0,1)$. In this case, the consumer becomes *informed* and buys from the firm with a lower price (under equal prices, she visits the same firm as in the previous period).[16] With a complementary probability ϕ, the consumer does not learn the price automatically and tries to learn it from a fellow consumer. In particular, with a probability μ she can meet a fellow and ask her about her price. Should the fellow be from the other firm's customer base, which occurs with probability $1-n$, the consumer learns the other firm's price and again chooses the firm with a lower price. For simplicity, the communication occurs only in one direction, i.e., the fellow does not learn anything from the agent who met him. Should the consumer meet a fellow from the same firm, which occurs with probability n, she does not learn anything and visits the same supplier as in the previous period.[17] In this specification we assume that each agent meets at most one fellow.[18] This then yields the following functional form for the function $l(\cdot)$:

$$l(n) = \phi n(1-\mu+\mu n). \tag{15}$$

Recall that $l(n)$ is firm 1's demand in the current period when it loses market shares (as compared to its previous market share n). It is given by the mass of consumers who visited firm 1 in the previous period and remain ignorant about firm 2's offer in the current period, because they do not learn the price of firm 2 directly (which occurs with probability ϕ), and either do not meet a fellow (with probability $1-\mu$), or meet a fellow who also purchased from firm 1 (with probability μn).[19] For example, if 20% of consumers become informed automatically in each period, while 80% of consumers rely on word-of-mouth communication and all of them meet one fellow, then $\phi = 0.8$, $\mu = 1$, so $l(n) = 0.8n^2$ and $h(n) = 1 - 0.8(1-n)^2$.

The second specification differs from the first one only in the number of fellows a consumer can meet and ask. Each consumer can meet several fellows and ask each of them about the supplier they visited in the previous period. If the consumer meets at least one fellow who visited the alternative supplier, the consumer learns the other firm's price. Otherwise, she remains locked-in at her own previous supplier. Here we assume that the number of fellows a consumer meets is drawn from a Poisson distribution with

[16] An alternative interpretation is that a fraction $1-\phi$ of consumers (randomly chosen from the population) exits, and an equal mass of new consumers enters the market. Upon entry, the new consumers become informed about the offers of both suppliers (e.g., via active search).

[17] Due to imperfect recall, information from older periods is not available. Alternatively, one may assume that consumers are replaced in each period, and new consumers arriving in the market can ask old consumers about the location or other characteristics of their supplier.

[18] This is a simplified version of a communication process introduced by Ellison and Fudenberg (1995). In a non-market environment, these authors assume that each agent can ask a sample of N other agents about the payoff received from choosing one (out of two) alternatives. Here we focus on the case where the sample size is restricted to 1. This is sufficient to capture the popularity weighting property characterizing word-of-mouth communication. In the second specification, we assume that the sample size follows a Poisson process.

[19] Note, that the corresponding specification of the function $h(\cdot)$ follows immediately from (4).

an arrival rate of λ. In that case, firm 1's demand in the current period, given that it loses customers, is given by:

$$l(n) = \phi n e^{-\lambda(1-n)}. \tag{16}$$

As before, firm 1's demand $h(n)$ when it gains customers, follows from (4).

Results are qualitatively similar under the two technologies of information transmission. Therefore, we focus on the simpler linear-quadratic specification (15). Moreover, we will be mostly interested in settings where market share dynamics are mainly driven by word-of-mouth communication. This means that ϕ as well as μ are relatively large.[20] This is the case when the share of informed consumers is rather small and meeting a fellow is likely.

4 Discretizing the state space

Departing from the myopic case, the computation of equilibrium becomes much more complex. The derivation of a closed-form solution for a dynamic pricing game with non-trivial market share dynamics is often not feasible. Various authors have developed different approaches in order to characterize the outcome of a dynamic game in light of this caveat. Maskin and Tirole (1988) discretize the *action space* (by introducing a price grid) in order to derive tractable results. Budd et al. (1993) conduct an asymptotic expansion (similar to a Taylor expansion) around a discount factor of zero in order to obtain analytical results.

As shown in Section 2.1, the conditions that can be used to determine Markov perfect equilibria for the game introduced in this paper, are linear in the value function $V(\cdot)$. Therefore, a useful starting point for the derivation of analytical results for this model is to discretize the *state space*. In that case, the value function can be found by solving a finite system of linear equations. For the discretization, we consider finite market share grids with N positions: $G_N \equiv \{a_1, a_1, \ldots, a_N\}$, where $a_1 = 0 < a_2 < \cdots < a_{N-1} < a_N = 1$. A raise (a loss) in market shares is represented by a move to the right (left) along the grid. Symmetry in demand then requires that the grid is symmetric, i.e., that $a_k + a_{N+1-k} = 1$ holds for all $k = 1, ..., N$. In order to solve the equilibrium conditions (11), (12), and (13) for a market share grid, we consider modified functions $l(\cdot)$ and $h(\cdot)$ that are defined *only* on the set G_N and take values from G_N.[21] Hence, also the state variable n takes on values only on the grid: $n \in G_N$.

[20] In the second specification we would assume that ϕ as well as λ are large enough.

[21] To spare notation, we use the *same* symbols l and h for the original functions defined in (15) and (4) that reflect our assumptions about the information transmission process, and modified functions that approximate this process when using a market share grid. Note that these functions satisfy weaker assumptions than in the continuous case. In particular, we require them only to be non-decreasing and also abandon the assumption that $l(n) > 0$ and $h(n) < 1$ for all $n \in (0, 1)$.

For a given set of parameter values specifying the word-of-mouth process (ϕ and μ), it is in general possible to design a specific market share grid with a small number of positions, and corresponding state transition functions, that describe the evolution of market shares quite accurately for any sequence of events.[22] In order to illustrate this, consider the following two examples: (i) $\phi = 0.1$ and $\mu = 1$, which reflects a situation where most consumers are fully informed, and the remaining consumers communicate via word-of-mouth (each of them meets one other consumer), and (ii) $\phi = 0.8$ and $\mu = 1$, which corresponds to a situation where few consumers automatically learn the other firm's price, and all others communicate via word-of-mouth.

Market share dynamics under example (i) can be described quite accurately by a grid with four positions: $\{0, 0.1, 0.9, 1\}$. To see this, note that for these parameter values, we obtain (using (15)): $l(1) = 0.1$, and $l(l(1)) = 0.001$. Starting from any position on this grid, a move 'upwards' or 'downwards' in the market share space yields a market split close to one of the positions on the grid (e.g., $l(h(0.9)) = 0.0998 \approx 0.1$). The state transition functions for this grid are: $l(1) = l(0.9) = 0.1$, $l(0.1) = l(0) = 0$, and $h(.)$ defined according to (4). While the use of a finer grid can further increase the accuracy of the approximation, the behavior of market shares remains virtually identical (see Section 6). It can be shown more generally that for sufficiently small values of ϕ, grids with four positions can be used to describe market share dynamics sufficiently well.[23] We discuss some results for this grid in Section 5.1.

Under example (ii), a different market share grid with different transition functions can be used to approximate the transition between states, but once more, a grid with a small number of positions is sufficient to describe the dynamics quite accurately. For the parameter values $\phi = 0.8$ and $\mu = 1$, we obtain (using (15)): $l(1) = 0.8$, $l(l(1)) = 0.512$, $l(l(l(1))) = 0.20971$, and $l(l(l(l(1)))) = 0.0351$. This suggests that a grid with five positions: $\{0, 0.2, 0.5, 0.8, 1\}$ can be used to describe any sequence of events quite accurately, using the transition function: $l(1) = 0.8$, $l(0.8) = 0.5$, $l(0.5) = 0.2$, $l(0.2) = l(0) = 0$, and $h(.)$ defined according to (4). For instance, if firm 1 starts with a customer base of size 0.8 in period 1, loses market shares once, and in the following period gains market shares, its resulting customer base size in period 3 is $h(l(0.8)) = 0.8094 \approx 0.8$. For other parameter values (that are sufficiently close to $\phi = 0.8$ and $\mu = 1$), positions a_2 and a_4 of this grid can be modified to minimize the errors of approximation.

Grids with five positions, as the one in example (ii), turn out to be suitable to approximate the dynamics of state transitions in situations where word-of-mouth plays a

[22] Grids with small numbers of positions have the advantage to produce analytically simple results (see Section 5).

[23] If ϕ and μ are varied, positions a_2 and a_3 of the grid should be modified to maintain a good approximation of state transitions. For sufficiently small values of ϕ, a good approximation is obtained using $a_2 = \phi$ and $a_3 = 1 - \phi$. Related numerical simulations, that give rise to such a grid endogenously are illustrated in Section 6.

major role in consumers' information acquisition. In Section 5, we generalize the above grid with five positions, by introducing a new parameter that allows to vary the location of the second and the fourth position. This allows us to capture higher or lower amounts of word-of-mouth communication among consumers in a simple way. This is also the simplest grid where both the pro- and the anti-competitive effect can occur simultaneously. We will be particularly interested in the interplay of these two effects in the second (and fourth) position, where the firm with the larger customer base faces a trade-off between defending its dominant market position and abusing it.

In the remainder of this section, we focus on an important class of transition functions which preserve the property from example (ii) that a gain or loss of market shares by a firm is always represented by a move to the direct neighboring position on the grid. Formally, this means that $l(a_k) = a_{k-1}$ for $k \geq 2$ and $h(a_k) = a_{k+1}$ for $k \leq N-1$ (as well as $l(0) = 0$ and $h(1) = 1$). Expressed in terms of the state transition functions $h(\cdot)$ and $l(\cdot)$, this property implies that

$$l(h(n)) = n \tag{17}$$

holds for any position $n < 1$ on the grid.[24] For such grids (with an arbitrary finite number of positions), we are able to extend our earlier results of Propositions 1 and 2 to discount factors arbitrarily close to 1.

Proposition 4. *Consider a finite market share grid that fulfills* (17) *for all values of* $n < 1$ *on the grid. Then the results of Proposition 1 also hold for all values of the discount factor* $\delta \in (0,1)$ *and in every position* n*. The results of Proposition 2 hold for all values of the discount factor* $\delta \in (0,1)$ *and in every position* n *where* $l(n) > 0$ *and* $h(n) < 1$*.*

Due to Proposition 4, the use of such a grid will not only enable us to derive a simple closed-form solution (see below). It also allows us to drop the restriction to "sufficiently small δ" in Propositions 1 and 2. This restriction was used in the proof of Proposition 1 to rule out the possibility that a firm may be reluctant to undercut the competitor's price (if it were known), as this may lead to a loss of market shares in the next period, due to tougher price competition. The problem is that for an arbitrary specification of the functions $h(\cdot)$ and $l(\cdot)$, we cannot compare the values $V(n)$ and $V(l(h(n)))$, which is the value after gaining and subsequently loosing market share, without knowing the shape of the value function $V(\cdot)$. However, given a market share grid that fulfills (17), such knowledge is not necessary since $l(h(n)) = n$, and thus $V(l(h(n))) = V(n)$ for all $n < 1$.

[24]Note that due to symmetry of the grid, the consistency condition (4) is then automatically satisfied. It is also important to note that the assumption (2) does not hold here at points $n = a_2$ and $n = a_{N-1}$, as $l(a_2) = 0$ and $h(a_{N-1}) = 1$. In this case it may well happen that some firm reaches position $a_N = 1$ and serves the entire market.

Note that the last claim of Proposition 4 does not refer to all positions of the grid. It requires that $l(n) > 0$ and $h(n) < 1$, which excludes positions a_1, a_2, a_{N-1}, and a_N. In these positions, some deviations to higher prices may not deliver higher profits even if the competitor's price were known, because the deviating firm faces zero demand. Due to this complication, not all arguments used in the proof of Proposition 2 can be used. In fact, as we show in the Appendix, a broader class of equilibrium strategies than allowed by Proposition 2 can occur in the second position of the grid, $n = a_2$. We characterize these strategies in Lemma 3 (in the Appendix). In particular, we show that if case (A) fails to apply (in equilibrium) in position $n = a_2$, then instead of case (B) the larger firm conducts *limit pricing* and undercuts the rival with probability 1. The supports of the firms' price distribution functions then differ. We refer to this as *case (C)*; see Lemma 3 in the Appendix, as well as the comments preceding the lemma for more details.

As noted earlier, the introduction of a market share grid transforms the *continuum* of equilibrium conditions (11), (12), resp. (13) into a finite set of linear equations. The difficulty that remains is that a rule to determine when case (A) and case (B) (or case (C)) apply, that is, whether (12) or (13) must be used for a given position n on the market share grid, is *a priori* not known. Having chosen some combination of these cases, the corresponding system of linear equations can be solved. Then it remains to verify whether the value function fulfills $F(1 \mid n) \leq 1$ (using (10)) for *all* values of n on the market share grid. However, as the number of combinations grows exponentially in the size of the grid, verifying all combinations can be viable only for small grids.[25]

Note, however, that case (A) *always* applies in position $n = 0$. Otherwise, (13) would imply that $V(0) = 0$.[26] Furthermore, the equilibrium conditions for case (A) and case (B) (conditions (12) and (13)) coincide in position $n = 1/2$. Therefore, the only difficulty that remains is to determine which case applies in positions $0 < n < 1/2$. In case of a grid with 5 positions as the one introduced above, this means that the only case distinction that remains relates to position a_2 (see the following section).

[25]The most obvious guess is that case (A) applies if $n \leq \frac{1}{2}$, and case (B) otherwise. If this does not yield an equilibrium, we proceed with specifications where there is a single location on the grid within the interval $[0, \frac{1}{2})$ where the case switches from (A) to (B).

[26]This cannot hold in equilibrium because $V(0) = 0$ is only possible when firm 1 does not gain market share by setting any positive price. Thus, the probability that firm 2 chooses a positive price is equal to zero. However, this cannot occur in equilibrium in position $n = 0$, as firm 2, that already has the full demand, would clearly benefit from charging a positive price.

5 Market share grid with 5 positions

Let us generalize the simple market share grid with five positions that was introduced above, by considering the following grid:

$$G_5 \equiv \big\{0, s, 1/2, 1-s, 1\big\}, \tag{18}$$

where $s \in (0, \frac{1}{2})$ is a parameter. For this grid, we specify the function $l(\cdot)$ as follows:

$$l(0) = l(s) = 0, \qquad l(1/2) = s, \qquad l(1-s) = 1/2, \qquad l(1) = 1-s, \tag{19}$$

while the function $h(\cdot)$ is defined according to the consistency condition (4). Clearly, these transition functions satisfy condition (17) used in Proposition 4.

By varying the new parameter s, we can adjust the grid to reflect different choices of the original parameters μ and ϕ. Not all combinations can be adequately reflected. However, for large values of ϕ (around 0.8) and large values of μ, this grid can be used to approximate the true market share dynamics sufficiently well. This is also the range of parameter values that we are especially interested in, because high values of ϕ and μ imply a large amount of word-of-mouth communication among consumers.

Figure 1 illustrates the accuracy of our approximation for $\phi = 0.8$ and $\mu = 1$, using the grid G_5 with $s = 0.2$. The curve shows the true function $l(n)$. The dots represent the discrete approximation of $l(\cdot)$ as given by (19); the vertical coordinate of each dot is given by the horizontal location of the next step on the grid to the left (because a loss of market shares is reflected by a motion to the left along the grid). The figure shows that for the given parameter values, the function $l(n)$ is indeed reflected quite accurately by setting $s = 0.2$.[27]

Small values of s can be used to describe markets where most consumers communicate via word-of-mouth, corresponding to large values of ϕ and μ. Intuitively, in such a case, a firm with a small customer base gains only a moderate amount of market shares when it charges the lower price in the market, because few consumers discover this firm's offer via word-of-mouth. Hence, market shares tend to be more volatile near the center of the market share space than near the extremes. On the other hand, larger values of s (closer to $\frac{1}{2}$) can be interpreted as market shares that are more volatile near the extremes than near the center of the market share space. This corresponds more closely to a situation where few (or no) consumers communicate via word-of-mouth.[28]

[27] Alternatively, we can choose s to minimize the sum of squared errors implied by imposing the grid. For example, for $\phi = 0.8$ and $\mu = 1$, the sum of squared errors $LS = \sum_{k=2}^{N} [a_{k-1} - l(a_k)]^2$ is minimized by $s = 0.2013$, for which $LS < 0.0012$. Note, that it is impossible to match the true function $l(\cdot)$ *precisely* with any finite grid, as $l(n) > 0$ holds for any $n > 0$, while $l(a_2) = a_1 = 0$ holds for the grid.

[28] In this range of parameters, however, the grid G_5 yields a less accurate representation of our micro-foundation. Therefore, the interpretation of the results derived using this grid for larger values of s must

We can now derive a closed-form solution for the value function given the market share grid G_5 introduced above. For this we only need to determine which case applies in position a_2. Let us first assume that case (A) applies in position a_2. We will see below that this is the case when the discount factor remains below a critical value. Applying condition (12) to positions $n = 0$, $n = s$, and $n = \frac{1}{2}$, and condition (11) for $n = 1 - s$ and $n = 1$, then, yields

$$V(1) - \delta V(1-s) = 1 - s, \quad V(1-s) - \delta V(1/2) = 1/2, \quad V(1/2) - \delta V(s) = s, \quad (20)$$

$$V(s) - \delta V(1/2) = V(1-s)/2 - \delta V(1)/2, \qquad V(0) - \delta V(s) = s(1-\delta)V(1). \quad (21)$$

Equations (20)–(21) form a system of five linear equations with five unknowns that can be easily solved. The complete solution can be found in the proof of Proposition 5 in the Appendix.

Proposition 5. *Consider the grid G_5. For every $s \in (0, \frac{1}{2})$ there exists a critical value $\delta_{crit} > 0$ such that: A symmetric MPE in which case (A) applies in position a_2 of the grid exists, if and only if $\delta < \delta_{crit}$. In that case the equilibrium is unique. The critical value δ_{crit} is increasing in s and fulfills $\delta_{crit} \to \sqrt{2} - 1$ for $s \to 0$ and $\delta_{crit} = 1$ for $s = \frac{1}{3}$.*

The proposition implies that an equilibrium where the firm with the larger customer base charges the monopoly price with a positive probability always exists for sufficiently small values of δ (conforming to our results for the myopic case from Section 2.2). When δ increases, the critical value δ_{crit} (which is a function of s) marks the turning point towards an MPE where the firm with the *larger* customer base conducts limit pricing in positions s and $1 - s$, as captured by case (C), and hence, gains market shares with probability 1. We provide a detailed discussion of case (C) in the Appendix.

Proposition 6. *Consider the grid G_5 with $s \in (0, \frac{1}{2})$. An equilibrium that involves case (C) in position $n = s$ exists if and only if $s \leq \frac{1}{3}$ and $\delta \geq \delta_{crit}$.*

Compared to the case $\delta < \delta_{crit}$, where the equilibrium is unique (Proposition 5), there are potentially multiple equilibria when $\delta > \delta_{crit}$. In all of them, case (C) applies in position s. However, it turns out that there is an equilibrium that yields the highest values in all states (see the proof of Proposition 6). This equilibrium is characterized by the same condition that one obtains also when applying case (B) to position s. Under the assumption that firms coordinate on the strategies that deliver the highest profits to both of them, the derivation of an equilibrium condition is, thus, not more complicated than for the other positions on the grid.

In the following, we analyze the resulting dynamics of market shares. Market share dynamics in an MPE that uses mixed strategies are governed by the probability that

be seen in light of this caveat.

firm 1 charges the higher (the lower) price in the current period. If a firm with a small customer base prices aggressively (chooses a lower price in the current period with a high probability), then the resulting dynamics exhibit a tendency towards the *center* of the market share space. Conversely, if the firm with a larger customer base prices aggressively when market shares are skewed but not extreme (positions s and $1-s$ on the grid), then a tendency towards the *extremes* of the market share space emerges. Note also that in a symmetric equilibrium, in position $n = \frac{1}{2}$ (center of the market share space), the probability of gaining or losing market shares is always equal to $\frac{1}{2}$ for both firms.

Proposition 7. *Given the grid G_5, for δ sufficiently small, $F(p \mid 1-n)$ first-order stochastically dominates $F(p \mid n)$ in positions $n = 0$ and $n = s$ of the grid.*

The proposition implies that for small values of δ, the firm with the *smaller* customer base (firm 1 for $n < \frac{1}{2}$) is more likely to gain market shares, and a tendency towards the center of the market share space results. Intuitively, for small δ, the firm with a larger customer base has a stronger incentive to exploit the locked-in consumers in its customer base than a firm with a smaller customer base, while the smaller firm competes more vigorously for new customers. The stochastic dominance result of Proposition 7 also confirms earlier findings obtained in the myopic case (Proposition 3).

Figure 2 illustrates these findings. The figure contains the state n on the horizontal and prices on the vertical axis. The vertical lines show the support of firm 1's price distribution, and the dots indicate the expected price. The lower dashed line is the lower boundary of the price distribution functions, $\underline{p}(n)$, and the upper dashed line is the monopoly price, which is the upper boundary of the support. The horizontal arrows indicate the probability of moving upwards/downwards in the market share space for the corresponding state. For instance, when the state is $n = s$, firm 1 gains market shares with a probability of 0.75, and when $n = 0$, this probability is 0.9. The figure shows that market shares have a strong tendency to move towards the center of the market share space whenever the market split is an uneven one.[29]

For larger values of δ, an additional effect strongly influences the dynamics. Since future profits are valuable, the firm with the larger customer base may start to defend its dominant position in the market by pricing very aggressively whenever it is threatened. Hence, for larger values of δ, we might expect tougher price competition around the center of the market share space than at the extremes. If this holds, then a tendency towards skewed market splits emerges. If price competition is very intense in the center of the market share space, the firm with a smaller customer base may "shy away" from this region, which allows the larger firm to maintain its dominant position in the market.

[29]The value function for G_5, given $s = 0.2$ and $\delta = 0$, is: $V(0) = 0.16$, $V(s) = 0.25$, $V(\frac{1}{2}) = 0.2$, $V(1-s) = 0.5$, $V(1) = 0.8$. Hence, already in the myopic case, the value function is non-monotonic. For numerical simulations of market share dynamics in the myopic case, see Section 6.

This is highlighted by the following proposition.

Proposition 8. *Consider the grid G_5 and let $s \leq \frac{1}{3}$. If $\delta \to \delta_{crit}^-$, then $F(p \mid n)$ first-order stochastically dominates $F(p \mid 1-n)$ in position $n = s$, and the probability that the firm with the larger customer base (firm 2) gains market share converges to 1.*

If follows from the proposition that, for larger values of δ, a tendency towards the extremes of the market share space emerges.[30] If δ is sufficiently large ($\delta \geq \delta_{crit}$), market shares never cross the center of the market share space in equilibrium, and a firm that has reached a dominant position in the market, maintains this position forever. Proposition 8 conforms to our intuition indicated earlier. Small values of s correspond to situations where most consumers communicate via word-of-mouth. It is intuitive that in such markets, a high market share is particularly valuable, since firms with a small customer base can attract few additional customers when gaining market shares. The market shares, thus, tend to become more skewed when s is small and δ is raised.

Figure 3 illustrates these findings for a discount factor of $\delta = 0.66$, which is, given $s = 0.2$, just below $\delta_{crit} \approx 0.6653$.[31] We observe that — whereas prices are quite dispersed at the center of the market share space ($n = \frac{1}{2}$) — the mean is almost equal to the minimal price in position $1 - n$. Recall, that in this position of the market share grid, firm 1 conducts limit pricing, because its dominant market position is at stake. As a result, this firm gains market share with a probability of (almost) 1. Market shares, thus, fluctuate around one of the extremes of the market share space, as illustrated also by our numerical simulations conducted in Section 6. Note also that the lowest price $\underline{p}(n)$ is below zero when $n = \frac{1}{2}$. Price competition is, thus, rather intense at the center of the market share space, where each firm competes to obtain the dominant market position. This effect becomes more pronounced when the discount factor is raised further.[32]

In markets where few consumers communicate via word-of-mouth, we would expect this effect to be less pronounced. A larger value of s can be interpreted as a smaller fraction of consumers communicating via word-of-mouth (smaller μ and ϕ), which makes it less valuable for a firm with a large customer base to defend its dominant market position. It is, thus, important to note that there are fundamental differences in the dynamics when comparing situations where s is small ($s < \frac{1}{3}$) to situations where s is large ($s > \frac{1}{3}$). Namely, when $s < \frac{1}{3}$ and δ is sufficiently large, market shares never cross

[30] We show in the Appendix that the results of Proposition 8 are preserved when $\delta > \delta_{crit}$; see also Proposition 6.

[31] The value function for these parameters is: $V(0) = 0.15$, $V(s) = 0.10$, $V(\frac{1}{2}) = 0.27$, $V(1-s) = 0.68$, $V(1) = 1.25$.

[32] In fact, it can be shown that as $s \to 0$, price competition becomes so intense in this position that both firms charge a negative limit price when $\delta > \delta_{crit}$, which eliminates all payoffs when $n_1 = \frac{1}{2}$ is the initial state. This is because only a dominant firm can earn a positive profit under these conditions, so firms compete in a Bertrand-fashion for this payoff.

the center, and a firm that has obtained a dominant position in the market, maintains this position indefinitely. Conversely, when $s > \frac{1}{3}$, then even for discount factors arbitrarily close to 1, markets shares continue to cross the center, and price competition is not particularly intense in position $n = \frac{1}{2}$ (as compared to the other positions).

These differences in the dynamics can be explained intuitively by highlighting pro- and anti-competitive effects in this dynamic model, compared to a Bertrand model without locked-in consumers. An anti-competitive effect occurs when a firm has an incentive to exploit the locked-in consumers in its customer base by charging high prices. A counteracting pro-competitive effect arises when the discount factor becomes sufficiently large, and firms start to "invest" in the size of their customer base by charging very low prices (even below zero). Depending on the state of competition (captured by n), either the pro- or the anti-competitive effect can dominate.[33]

If $s < \frac{1}{3}$, the anti-competitive effect dominates in position $n = 0$, as the larger firm 2 tries to exploit the monopoly power over its locked-in consumers. This leads to positive expected prices in equilibrium. In position s (and $1 - s$), the pro-competitive effect dominates if δ is sufficiently large, because the dominant firm vigorously defends its position, and even charges a negative limit price if $\delta \geq \delta_{crit}$. Moreover, the pro-competitive effect also dominates in position $n = \frac{1}{2}$, as each firm is trying to become dominant in the next period. Hence, most of the probability mass lies at prices below the competitive price (zero) when the discount factor is sufficiently large.

When $s > \frac{1}{3}$ and δ is sufficiently large, price competition is also more intense for even market splits than for skewed ones. However, which effect dominates depends on the exact size of s. If s is large (close to $\frac{1}{2}$), then the anti-competitive effect dominates in all positions of the grid (with the highest expected prices in positions $n = 0$ and $n = 1$). However, as s becomes smaller (closer to $\frac{1}{3}$), price competition intensifies in all positions of the grid (for fixed δ). If s is close to $\frac{1}{3}$, then the anti-competitive effect still dominates in position $n = 0$, whereas the pro-competitive effect now dominates in position s where the expected price of the firm with the larger customer base becomes negative.

5.1 Other market share grids

We have shown in the previous sections that the grid G_5 is especially suitable to analyze market share dynamics when many consumers communicate via word-of-mouth. For this grid, we were able to derive analytical results in a simple way, and could extend our basic findings from Propositions 1 and 2 to discount factors arbitrarily close to 1. Depending

[33]Related effects can also be found in dynamic models with switching costs. Similarly as in our model, firms can be tempted to exploit their monopoly power over locked-in consumers. However, also the firms' desire to attract new customers may be higher when consumers are (partially) locked-in due to switching costs. Beggs and Klemperer (1992) show that, overall, the first effect tends to dominate, so switching costs make markets less competitive.

on the parameters ϕ and μ that characterize the process of consumers' information acquisition, it is generally possible to construct a specific grid with a small number of positions N, designed to represent this set of parameter values particularly well. The advantage of this approach is analytical tractability.[34]

In this section, we briefly discuss the usage of two alternative market share grids (with a small number of positions). First, we add two additional positions (near the extremes) to the grid G_5, which converts this into a new grid G_7, with new state transition functions. The advantage of this is that the G_7 allows to approximate the state transitions even more accurately than the G_5. The disadvantage is that (as we will show below) the state transition functions no longer satisfy condition (17), which implies that Proposition 4 does not apply.[35] We find that the results for this grid remain qualitatively very similar to the results for G_5. We again obtain a critical value of the discount factor so that case (A) applies in all positions (with $n \leq \frac{1}{2}$) when δ lies below the critical value. For δ large, case (B) applies in position a_3. When δ increases, market shares become more skewed, and cross the center of the market share space less frequently. In particular, when δ is sufficiently large, the firm with the larger customer base prices very aggressively in position $n = a_3$, and market shares tend to fluctuate near one of the extremes of the market share space. Hence, extended intervals of dominance are obtained, interrupted by sudden changes in the identity of the leading firm.

Second, we will discuss a market share grid with only four positions (G_4). This grid has been introduced earlier (see Section 4). It is particularly suitable to approximate market share dynamics when most (but not all) consumers are fully informed. More specifically, let $G_4 \equiv \{0, s, 1-s, 1\}$, where $s = \phi$ assures a good approximation when ϕ is small. The state transition functions for this grid are: $l(1) = l(1-s) = s$, $l(s) = 0$, and $h(.)$ again defined by (4).[36] For this grid, we can easily compute the value function, using the assumption that in all positions of the grid, the firm with the larger customer base charges the monopoly price with positive probability (hence, case (A) applies in positions $n = 0$ and $n = s$). We find that when s is small, the evolution of market shares is almost independent of the discount factor δ. Furthermore, the outcome converges to the Bertrand equilibrium when the fraction of fully informed consumers converges to 1 (i.e., $s \to 0$). Intuitively, when most consumers are fully informed, then firms have little incentive to invest into a large customer base. Even for large values of the discount factor δ, dynamics reflect mainly firms' *current* profit maximization, and price competition is intense in all positions of the market share grid. Market shares exhibit a zig-zag pattern between the extremes of the market share space, and their frequency of crossing the center

[34]The alternative approach is to use a finer grid with a large number of positions. This allows to approximate the state-transition process more accurately, but analytical results are hard to obtain. This is the approach we follow in Section 6, where we perform a numerical analysis using simulations.

[35]As an example, we consider $G_7 \equiv \{0, 0.035, 0.2, 0.5, 0.8, 0.965, 1\}$, and assume $l(1) = l(a_6) = a_5$.

[36]Note, that also the G_4 violates condition (17).

declines only slightly when δ becomes large (see Section 6 for numerical simulations).

6 Numerical simulations

In this section, we simulate market share dynamics numerically in order to provide a robustness check for our earlier results. For this purpose we again use a discretization of the state space, where we abandon the assumption (17). In particular, we use a finer grid with a larger number of positions (we use $N = 400$ positions for our simulations), and allow for bigger steps along the grid when a firm gains or looses market share.[37] Note, however, that results are still obtained by deriving a *closed-form solution* for the value function (for all positions of the grid) using the equilibrium conditions (11), (12), and (13). Conceptually, the only difference is that for finer grids, the closed-form solution for $V(.)$ becomes so complicated that a solution is only derived for a given value of the discount factor δ, and given state transition functions $l(.)$ and $h(.)$ defined on the grid, reflecting the parameter values μ and ϕ.

In order to disentangle the driving forces behind the dynamics in the model, we first analyze the dynamics in the myopic case (analyzed in Section 2.2), and compare our results to the dynamics obtained for larger values of the discount factor δ. Furthermore, the results in the myopic case are contrasted against the backdrop of a stochastic process that assigns an equal probability to each firm and in each period of gaining or losing market shares, *independently* of the current state n.

Figure 4 shows a simulation for this state-independent stochastic process assigning equal probabilities of gaining or losing market shares in each period, using (15) with the parameter values $\phi = 0.8$, $\mu = 1$ considered also in previous sections. The left panel shows the evolution of market shares for 100 rounds. The right panel shows the invariant distribution for the same parameter values, using a simulation with 5 million rounds.[38] Figure 4 shows that information processes characterized by a large amount of word-of-mouth communication, have an inherent tendency to generate skewed market splits. The invariant distribution indicates that there are natural 'hotspots' in the market share space, where most of the probability mass is found near them. In particular, in Figure 4 these hotspots are around the values $\{0, 0.2, 0.5, 0.8, 1\}$, which correspond to the positions of the grid G_5 when $s = 0.2$.

[37] This can be described using functions $L, H : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\}$ that specify by how many positions firm 1's market share "jumps" to the left/right along the grid, when firm 1 looses/gains market share, i.e., $l(a_k) = a_{L(k)}$ and $h(a_k) = a_{H(k)}$. In the simulations we further assume equal distances between the positions of the grid: $a_k = (k-1)/(N-1)$. Note that if N is large, this is sufficient to represent market share dynamics accurately for *any* set of parameter values. For a given set of parameter values, the transition functions $L(.)$ and $H(.)$ are specified by choosing the position on the grid nearest to the *true* value of $l(a_k)$, i.e., $L(k) = \arg\min_{k'} |a_{k'} - l(a_k)|$.

[38] Note, that the irregularities (peaks) in the invariant distributions are *not* due to numerical imprecision. They result from the specification of the process of information transmission.

Figure 5 uses the same parameter values, but adds strategic interaction. Future profits are fully discounted away (myopic case). By Propositions 3 and 7, we expect market shares to be (on average) *less* skewed than under the state-independent stochastic process. This reflects the tendency of the firm with the smaller customer base to price more aggressively than its larger rival in order to gain market shares. Figure 5 confirms this prediction.

Figure 6 simulates the full dynamic game for the same parameter values and $\delta = 0.8$. According to Proposition 8, we expect market shares to be skewed most of the time. The invariant distribution in the right panel confirms this prediction. Whenever market shares are skewed but not extreme (n near 0.2 or 0.8), the firm with the *large* customer base starts to price very aggressively in order to defend the dominant position in the market. When market shares are closer to the extremes (n near 0 or 1), the dominant firm prices less aggressively and charges the monopoly price with positive probability in order to exploit the locked-in consumers in its customer base. This explains the zig-zag pattern of market shares in the left panel. Confirming our theoretical predictions (see Section 5), Figure 6 illustrates that when many consumers rely on word-of-mouth, and the discount factor is sufficiently large, then extended intervals of dominance are observed, interrupted by sudden changes in the identity of the leading firm.

Summing up, when word-of-mouth communication among consumers is the dominant source of information, a "natural" tendency towards skewed market splits emerges, that can be observed in the absence of strategic interaction. Intuitively, when most consumers rely on word-of-mouth, then few people find out about the offer of a firm that has served a small fraction of the market in the previous period. This implies that market shares are more volatile in the center than near the extremes of the market share space. In the presence of strategic interaction, for small values of the discount factor δ, the smaller firm's attempts to gain market shares counteract the tendency towards skewed market splits, so market shares tend to be more evenly distributed. When future profits are important, the tendency towards skewed market splits is reinforced and becomes more visible again. For sufficiently large values of δ, a firm that has reached a dominant position in the market, tends to maintain this position for many consecutive periods.

For comparison, we also simulate two additional situations: when there is no word-of-mouth communication, and when most consumers are informed. Figure 7 shows a simulation of the full dynamic game for the the former situation under parameter values $\phi = 0.8$, $\mu = 0$, and $\delta = 0.99$. Then 20 percent of the consumers are informed, whereas the remaining 80 percent of consumers remain locked-in at their previous supplier. Figure 7 illustrates that market shares are more centered than in a market characterized by word-of-mouth communication. Furthermore, the size of the discount factor δ does not have a strong impact on the dynamics. For smaller values of δ, market shares are somewhat

less skewed, but the effect is not as pronounced (not shown here).

Figure 8 shows a simulation of the full dynamic game for the latter situation (mentioned above) under parameter values $\phi = 0.1$, $\mu = 1$, and $\delta = 0.99$ (which is the example (i) in Section 4). In this case, 90 percent of consumers are informed and all remaining consumers communicate (with one other consumer) via word-of-mouth. Figure 8 illustrates that market shares are mostly concentrated in four extreme "hot spots" with frequent jumps from one to the other extreme. Contrary to the case where many consumers communicate via word-of-mouth, the size of the discount factor has very little impact upon equilibrium dynamics. The reason is that — since market shares are very volatile — firms have little incentive to "invest" in the size of their customer base. Hence, dynamics for δ close to 1 resemble those in the myopic case.

7 Conclusion

This paper presents a model of the dynamics of a duopoly, in which dynamics are generated by the firms' usage of mixed pricing strategies. We demonstrate that with a small set of assumptions, surprisingly rich dynamics are obtained. Depending on the parameter values, our model can generate dynamics where market shares tend to equalize over time, as well as dynamics where dominance persists over many consecutive periods. Mixed pricing strategies are a feature well-known, for instance, from models with consumer search. They characterize situations in which a firm is willing to undercut the competitor's price if this price can be predicted with certainty. This often applies in markets with homogeneous products. We apply this concept to a dynamic pricing game.

In dynamic duopoly models, the state often tends to evolve into a direction where the joint payoff increases.[39] In our model, the joint (expected) payoff in the myopic case is higher when market shares are skewed, because the firm with a larger customer base then tends to price less aggressively. One may, therefore, suspect that market shares would become more skewed when future profits are more important. Our results confirm this prediction. Depending on the amount of word-of-mouth communication as well as on the discount factor, different "classes" of dynamics are obtained in our model. When most consumers are informed, market shares are very volatile, and firms have little incentive to invest in the size of their customer base. The discount factor, then, does not strongly affect the dynamics. When word-of-mouth plays a major role in consumers' information acquisition, market shares tend to become less volatile. If the discount factor is sufficiently large, they rarely cross the center of the market share space, and extended periods of dominance, interrupted by sudden changes in the leadership position, are obtained.

The intuition for our results can be illustrated by highlighting pro- and anti-competitive

[39] See Budd et al. (1993), Cabral and Riordan (1994), and Athey and Schmutzler (2001).

effects due to partial consumer lock-in and word-of-mouth. The anti-competitive effect arises if a firm tries to exploit the locked-in consumers in its customer base, which occurs near the extremes of the market share space. When the discount factor is sufficiently large, then the counter-acting pro-competitive effect arises. If market shares are skewed but not extreme, then the larger firm tries to defend its dominant position. This effect can be so strong that this firm charges a negative limit price in order to gain market shares.

From the policy perspective, the anti- and pro-competitive effects can shed some light on the behavior of dominant firms. Besides their incentive to exploit the locked-in consumers, dominance can also benefit the consumers when a dominant firm defends its position by pricing aggressively. While in the model, we relate this behavior to the position in the market share space (or grid), it can also be interpreted in the following way: a dominant firm tends to adopt an aggressive pricing strategy when it loses a significant market share after being undercut by the rival, threatening its dominant position. In order to verify this condition, standard methods, like estimation of the elasticity of the firm-specific demand, can be used.

Finally, let us briefly revisit some of our assumptions. If our assumption of homogeneous goods is relaxed, a larger firm's desire to undercut the competitor's price in order to defend its dominant market position may be reduced, if product differentiation gives firms some monopoly power irrespective of past market shares. Therefore, product differentiation may weaken some of our effects. Similarly, the size of the price differential could, in addition to just the ranking of prices, also matter for consumers' choices. This feature seems relevant for some markets, and may generate dynamics that are less volatile than in our model. Another interesting starting point for future research would be the introduction of forward-looking consumers (e.g. Cabral, 2011). In our model, consumers who become informed about both prices always choose the supplier with the lower price. This behavior may not always be rational, if this firm is expected to charge higher prices in future periods.

A Appendix: Proofs

A.1 Proofs for Section 2

Before starting with the proof of Proposition 1, we state and prove the following lemma. It claims that the gain $h(n) - n$ from having a lower price is not arbitrarily small for the firm with the smaller customer base. This sets a lower bound on the relative gain from undercutting the rival and thus suggests a profitable deviation in potential equilibrium candidates in pure strategies.

Lemma 1. *There exists $\delta_0 > 0$ such that $h(n) - n > \delta/(1-\delta)$ for every $n \in (0, \frac{1}{2}]$ and every $\delta \in (0, \delta_0)$,*

Proof of Lemma 1. As $h(n) - n$ is continuous on the compact interval $[0, \frac{1}{2}]$, it has a minimum, let us denote it α. Clearly $\alpha > 0$. As $\delta/(1-\delta)$ is increasing in δ, the statement follows by choosing δ_0 such that $\delta_0/(1-\delta_0) < \alpha$. □

Proof of Proposition 1. We prove the claim by contradiction. Suppose there is an equilibrium where both firms use pure strategies p_1 and p_2 in a state $n \in (0,1)$. We consider several cases. If $p_1 = p_2 \equiv p$ and $p > 0$, either firm would benefit from marginally undercutting the common price p. This leads to a discontinuous rise in demand that, for sufficiently low δ, more than compensates for (potential) future losses resulting from the increase in the size of the firm's customer base.[40] More precisely, a necessary condition for price $p > 0$ to be best response of firm i (with market share n) is that $p'h(n) \leq pn/(1-\delta) = V_i(n)$ for all $p' < p$. This implies that $n > 0$ and that $h(n) \leq n/(1-\delta)$. From this we obtain the following necessary condition:

$$h(n) - n \leq \frac{n\delta}{1-\delta} \leq \frac{\delta}{1-\delta} \tag{22}$$

Now assuming without loss of generality that $n \leq \frac{1}{2}$, we obtain a contradiction to Lemma 1 when δ is small enough.

The case $p_1 = p_2 \equiv p$ with $p \leq 0$ cannot arise in equilibrium either. Otherwise, state n would remain constant, so firms choose $p_1 = p_2 \equiv p$ in *all* periods and total discounted profits are non-positive (a deviation to the monopoly price 1 yields a positive profit to a firm with a positive customer base size, and there is at least one such firm). If $p_1 \neq p_2$ and $p_1, p_2 < 1$, the high-price firm would benefit from deviating to the monopoly price because current demand and, thus, the state next period are not affected. Finally, there can be no pure strategy equilibrium where $p_i = 1$ and $p_j < 1$ (where $j \neq i$), as the

[40] Note that an increase in the size of a firm's customer base can lead to a *reduction* in profit, unless the value function is monotonically increasing. However, this cannot be imposed here.

low-price firm would benefit from deviating to a higher price, which again only affects the current period profit. □

Before proceeding with the Proof of Proposition 2, we derive some useful properties of mixed strategy equilibria. In the text below we clarify when the indifference condition known from mixed strategies equilibria in finite games holds. After that we state and prove a lemma with useful properties of the firms' price distribution supports that will be used in the Proof of Proposition 2.

Consider an MPE and a state n. Let S_1 and S_2 be the supports of $F_1(p \mid n)$ and $F_2(p \mid 1-n)$, respectively. Firm i's value from choosing the price p can be written as (let $j \neq i$)[41]

$$\begin{aligned}[pl(n) + \delta V_i(l(n))] \Pr(p_j < p \mid 1-n) + [ph(n) + \delta V_i(h(n))] \Pr(p_j > p \mid 1-n) \\ +[pn + \delta V_i(n)] \Pr(p_j = p \mid 1-n). \end{aligned} \tag{23}$$

In the case when p is a mass-point (atom) of firm i's price distribution, the indifference condition clearly holds and (23) is thus equal to $V_i(n)$. The second case where the indifference condition holds is when the price $p \in S_i$ is not a mass-point of firm j's price distribution. In that case the last term in (23) is equal to zero and firm i's value is continuous in its price. The indifference condition then becomes

$$V_i(n) = [pl(n) + \delta V_i(l(n))] F_j(p \mid 1-n) + [ph(n) + \delta V_i(h(n))] (1 - F_j(p \mid 1-n)). \tag{24}$$

For all other prices the value (23) does not exceed $V_i(n)$. Note that (23) may be strictly smaller than $V_i(n)$ even for prices from the support S_i when p is a mass-point of firm j's but not firm i's price distribution.[42]

Now, let us fix some price p and consider prices p' that are not mass-points of firm j's price distribution and are sufficiently close to p (such clearly exist). Choosing price p' does not deliver a higher value than $V_i(n)$, thus,

$$V_i(n) \geq [p'l(n) + \delta V_i(l(n))] \Pr(p_j < p' \mid 1-n) + [p'h(n) + \delta V_i(h(n))] \Pr(p_j > p' \mid 1-n). \tag{25}$$

[41] We elaborate more on this dynamic equation in the main text just below Proposition 2.

[42] Having a continuous strategy space, the indifference condition for firm i in mixed strategy equilibrium needs to hold over every set on which firm i puts a positive measure. Point-wise it is required to hold (i) in points that are mass-points of firm i's price distribution, and (ii) in points in S_i that are not mass-points of firm j's price distribution (as firm i's expected value is continuous at such points). Similar arguments can be found, for example, in Baye, Kovenock, and de Vries (1996) in the context of all-pay auctions.

Considering prices $p' < p$ and taking the limit $p' \to p^-$ we have[43]

$$V_i(n) \geq [pl(n)+\delta V_i(l(n))]\Pr(p_j < p \mid 1-n)+[ph(n)+\delta V_i(h(n))]\Pr(p_j \geq p \mid 1-n). \quad (26)$$

Similarly, for prices $p' > p$ and in the limit $p' \to p^+$ we obtain

$$V_i(n) \geq [pl(n)+\delta V_i(l(n))]\Pr(p_j \leq p \mid 1-n)+[ph(n)+\delta V_i(h(n))]\Pr(p_j > p \mid 1-n). \quad (27)$$

With these preliminaries, we can now state the following lemma.

Lemma 2. *Let $\delta \in [0,1)$ and consider an MPE and a state $n \in (0,1)$. Let $i,j \in \{1,2\}$, $j \neq i$. Then, for any price $p \in S_i$, $p < 1$ the following statements hold:*

(i) *$p \in S_j$.*

(ii) *If $(p,p') \cap S_j = \emptyset$ for some $p' \in (p,1)$, then p is a mass-point of firm j's price distribution.*

(iii) *If $[p,p'] \subseteq S_i$ for some $p' \in (p,1)$, then p is not a mass-point of firm j's price distribution.*

Proof of Lemma 2. (i) Suppose, by contradiction, that $p \in S_i$ but $p \notin S_j$. Then, by choosing price $p \in S_i$, firm i gets a value that satisfies (24). Since the complement of S_j is open, we can find $p' > p$ so that $(p,p'] \cap S_j = \emptyset$. In that case $F_j(p' \mid 1-n) = F_j(p \mid 1-n)$. As $l(n) > 0$, by choosing price p' instead of p in the current period, firm i obtains a higher current profit, but the probabilities as well as future values remain unchanged.[44] Thus, it cannot be an equilibrium. The fact that both firms use mixed strategies follows from identical supports.

(ii) If p is not a mass-point of firm j's (the firm with market share $1-n$) price distribution, then firm i by choosing price p gets a value as in (24). Choosing price p' increases the price, but changes neither firm i's probabilities of having a lower/higher price, nor its future value functions. This contradicts p being a best response (i.e., having $p \in S_i$).

(iii) Suppose to the contrary that p is such a mass-point. Note that now the indifference condition for firm i may not hold at the price p (see footnote 42). For any price $p'' \in [p,p']$ that is not a mass-point of firm j's price distribution, the indifference condition holds. Taking the limit $p'' \to p^+$ we obtain that (27) holds with equality. Now we compare

[43] This follows from the fact that $\Pr(p_j > p' \mid 1-n)$ converges to $\Pr(p_j \geq p \mid 1-n)$.

[44] Here we consider only a deviation (to price p') in the current period, leaving the future strategy unchanged.

this value to (26). After subtracting identical terms and dividing by $\Pr(p_j = p \mid 1-n)$, which is positive as p is a mass-point, we obtain

$$pl(n) + \delta V_i(l(n)) \geq ph(n) + \delta V_i(h(n)). \tag{28}$$

Finally, consider some price $p'' > p$ outside of the limit (again such that is not a mass-point of firm j's price distribution). We show that (28) implies that there is a profitable deviation to such price p''. Let V'' denote the value from choosing price p'' and let $X = \Pr(p_j \leq p'' \mid 1-n)$ and $Y = \Pr(p < p_j \leq p'' \mid 1-n)$. Using (27) that (as noted above) now holds with equality, and substituting these equalities, we obtain

$$\begin{aligned}
&V'' - V_i(n) \\
&= [p''l(n) + \delta V_i(l(n))] \Pr(p_j \leq p'' \mid 1-n) + [p''h(n) + \delta V_i(h(n))] \Pr(p_j > p'' \mid 1-n) \\
&\quad - [pl(n) + \delta V_i(l(n))] \Pr(p_j \leq p \mid 1-n) - [ph(n) + \delta V_i(h(n))] \Pr(p_j > p \mid 1-n) \\
&= [p''l(n) + \delta V_i(l(n))] X + [p''h(n) + \delta V_i(h(n))] (1-X) \\
&\quad - [pl(n) + \delta V_i(l(n))] (X-Y) - [ph(n) + \delta V_i(h(n))] (1-X+Y) \\
&= (p''-p) [l(n)X + h(n)(1-X)] \\
&\quad + \big[[pl(n) + \delta V_i(l(n))] - [ph(n) + \delta V_i(h(n))]\big] Y.
\end{aligned}$$

Now, the first term is clearly positive as $p'' > p$, whereas the second term is non-negative, as follows from (28). Thus, $V'' > V_i(n)$, which is a contradiction. □

Proof of Proposition 2. (i) It follows directly from Lemma 2, part (i) that $\tilde{S} = S_1 \setminus \{1\} = S_2 \setminus \{1\}$. However, it can still be the case that the supports S_1 and S_2 differ in the monopoly price $p = 1$. In part (ii) we will also show that this case does not occur. More precisely, we will show that $\tilde{S}$, and thus also S_1 and S_2, contains prices arbitrarily close to 1. Because S_1 and S_2 are closed, then also $1 \in S_1$ and $1 \in S_2$, completing the proof of part (i).

(ii) We show that if $p \in \tilde{S}$ for some $p < 1$, then also $[p, 1) \subseteq \tilde{S}$. The statement in the proposition then follows directly. Suppose to the contrary that the above claim is not true. So, there exist $p < p'' < 1$ such that $p \in \tilde{S}$, but $p'' \notin \tilde{S}$. We can without loss of generality choose p such that $\tilde{S} \cap (p, p'')$ is empty.[45]

Now we proceed in three steps. In Step 1 we show that both firms' price distributions have a mass-point at price p. In Step 2 we derive a necessary condition that $pn/(1-\delta)$ is an upper bound for the value $V_i(n)$. In Step 3 we show that this upper bound cannot be satisfied for δ sufficiently small (when $n \leq \frac{1}{2}$).

[45]Such p is equal to the maximum of the closed set $\tilde{S} \cap (-\infty, p'']$.

Step 1. This follows directly from Lemma 2, part (ii).

Step 2. Being a mass-point, price p satisfies firm i's indifference condition

$$V_i(n) = [pl(n) + \delta V_i(l(n))] \Pr(p_j < p \mid 1-n) + [ph(n) + \delta V_i(h(n))] \Pr(p_j > p \mid 1-n) + [pn + \delta V_i(n)] \Pr(p_j = p \mid 1-n). \quad (29)$$

Comparing this to the inequality (26), subtracting identical terms and dividing by $\Pr(p_j = p \mid 1-n) > 0$, we obtain

$$pn + \delta V_i(n) \geq ph(n) + \delta V_i(h(n)). \quad (30)$$

By the same argument, comparing (29) and (27), we obtain

$$pn + \delta V_i(n) \geq pl(n) + \delta V_i(l(n)). \quad (31)$$

Substituting inequalities (30) and (31) into (29) implies $pn + \delta V_i(n) \geq V_i(n)$, which yields the necessary condition

$$\frac{pn}{1-\delta} \geq V_i(n). \quad (32)$$

This completes the second step. Note that this argument holds for any $\delta \in [0, 1)$.

Step 3. It follows from (32) that the price p is non-negative. By rearranging (32) we obtain $pn/(1-\delta) \geq pn + \delta V_i(n)$. This, together with (30) implies that $pn/(1-\delta) \geq ph(n) + \delta V_i(h(n)) \geq ph(n)$. Thus, $h(n) \leq n/(1-\delta)$ and $h(n) - n \leq \delta/(1-\delta)$, which is exactly condition (22). Now, as all arguments above are symmetric, we can without loss of generality assume that firm j has a higher market share, i.e., $n \leq \frac{1}{2}$. Similarly as in the proof of Proposition 1, we obtain a contradiction to Lemma 1 if δ is sufficiently small.

(iii) Having established the statement (ii), it follows from Lemma 2, part (iii) that there are no mass-points at prices $p < 1$. It only remains to show that at most one firm can have a mass-point at price $p = 1$ when δ is small enough. Suppose to the contrary that both firms have a mass-point at $p = 1$ and without loss of generality let firm j be the larger firm with customer base size $1 - n$ (so $n \leq \frac{1}{2}$). As the indifference condition needs to hold, we again obtain the necessary condition (30) where we set $p = 1$. Thus, $h(n) - n \leq \delta[V_i(n) - V_i(h(n))] \leq \delta/(1-\delta)$. Now, for $n \leq \frac{1}{2}$, we obtain a contradiction to Lemma 1 if δ is sufficiently small. □

Proof of Proposition 3. We show that $n < \frac{1}{2}$ if and only if $F(p \mid 1-n) < F(p \mid n)$ for all $p > \underline{p}(n)$. This, in turn yields the required stochastic dominance result.

Rewriting the inequality $F(p \mid 1-n) < F(p \mid n)$, we obtain

$$\frac{h(n)}{h(n)-l(n)}\Big(1-\frac{\underline{p}(n)}{p}\Big) < \frac{h(1-n)}{h(1-n)-l(1-n)}\Big(1-\frac{\underline{p}(n)}{p}\Big).$$

Because $h(1-n)-l(1-n) = h(n)-l(n) > 0$, the above inequality is, for $p > \underline{p}(n)$, equivalent to $h(n) < h(1-n)$, or $n < \frac{1}{2}$ (as $h(\cdot)$ is increasing). □

A.2 Proofs for Section 4

Proof of Proposition 4. Clearly the arguments in the proofs of Propositions 1 and 2 that do not use δ small do apply here as well. We, therefore, only focus on the most relevant parts of the arguments where δ small is necessary.

First we show that there is no pure strategy equilibrium with identical (positive) prices (Proposition 1). Suppose to the contrary that prices $p_1 = p_2 = p > 0$ constitute an equilibrium for some n in period t. We derive a necessary condition for such strategies to occur in equilibrium. For this we consider the following two (one-shot) deviations: undercutting to a price p' close to p in position n, and setting the monopoly price in position $h(n)$.

Undercutting (by firm i with market share n) to price $p' < p$ should not be profitable. In the limit $p' \to p^-$ the necessary condition becomes

$$pn + \delta V(n) \geq ph(n) + \delta V(h(n)), \tag{33}$$

where the left-hand side is equal to $V(n)$.

Now consider state $h(n)$ and let us denote $q = \Pr(p_j = 1 \mid 1-h(n))$ the probability that the rival chooses the monopoly price. Choosing the monopoly price $p_i = 1$ with probability 1 should also not be a profitable deviation in state $h(n)$, thus

$$V(h(n)) \geq [h(n) + \delta V(h(n))]\, q + [n + \delta V(n)]\,(1-q). \tag{34}$$

Expressing $V(h(n))$ we obtain

$$V(h(n)) \geq \frac{q}{1-\delta q}[h(n)-n] + \frac{1}{1-\delta q}[n + \delta(1-q)V(n)], \tag{35}$$

which after substituting into (33) and rearranging yields

$$\Big(p + \frac{\delta q}{1-\delta q}\Big)[n-h(n)] \geq \frac{\delta}{1-\delta q}[n-(1-\delta)V(n)]. \tag{36}$$

The left-hand side is clearly negative,[46] whereas the right-hand side is non-negative as $V(n) = pn/(1-\delta) \leq n/(1-\delta)$. This is a contradiction.

Second, let us review Step 3 from Proposition 2, part (ii). The argument is basically identical to the one above, it just needs to be refined. The inequality (33) is now not straightforward, but follows from (30). The inequalities (34)–(36) are then obtained in the same way. In order to argue that the right-hand side of (36) is non-negative, we use (32), which yields the same contradiction as above.

Finally, we show that at most one firm can have a mass-point at the monopoly price $p = 1$ as in Proposition 2, part (iii). Assuming that both firms have a mass-point at the monopoly price $p = 1$, by the same procedure as above, we obtain the inequality (36) where we set $p = 1$. Moreover, the inequality also holds for firm $j \neq i$ which has a market share $1 - n$. Thus, we have

$$n - h(n) \geq \delta[n - (1-\delta)V_i(n)],$$
$$(1-n) - h(1-n) \geq \delta[(1-n) - (1-\delta)V_j(1-n)]$$

Summing these inequalities, we obtain

$$1 - h(n) - h(1-n) \geq \delta[1 - (1-\delta)(V_i(n) + V_j(1-n))].$$

Now, the left-hand side is clearly negative, while the right-hand side is non-negative, because the joint profit is bounded from above by $1/(1-\delta)$. This is a contradiction. □

A.3 Discussion of case (C)

In what follows we provide a characterization of equilibrium for those positions where $l(n) = 0$ or $h(n) = 1$, which are excluded from Proposition 4. Due to symmetry, it is sufficient to consider $l(n) = 0$, i.e., $n = a_1 = 0$ or $n = a_2$. Compared to Proposition 2, Lemma 3 allows for a broader class of equilibrium strategies in positions a_1 and a_2, where the supports may differ. In this case the firm with the *larger* customer base conducts limit pricing by setting the price equal to the lower bound of the rival's support.[47] Note that, although the lemma refers to positions a_1 and a_2, case (C) is relevant only for position a_2.[48]

[46] We can assume $n \leq \frac{1}{2}$ without loss of generality.

[47] We show in the following that if case (A) does not apply in position $n = a_2$ of the grid, then case (C) applies and firm 2 (the firm with the larger customer base) conducts limit pricing. Note, however, that case (B) may nevertheless apply in other positions n (with $a_2 < n < 1/2$) if the grid has more than five positions.

[48] See the main text, directly before Section 5. The argument remains valid if case (B) is replaced by case (C) in position a_1. Namely, under limit pricing in position a_1, firm 1 would earn a payoff of zero. To sustain such an equilibrium, firm 2 (the firm with a large customer base) would have to charge a

Lemma 3. *Let $\delta \in (0,1)$ and consider a MPE and n such that $l(n) = 0$. Let firm 1 have market share n. Then the results of Proposition 2 hold with case (A) being applied, or the supports satisfy*

$$(C) \quad S_2 = \{\bar{p}_2\} \subset S_1 \subseteq [\bar{p}_2, 1].$$

Proof. First we show that the supports S_1 and S_2 coincide for prices $p \leq \bar{p}_2$, i.e., that $S_1 \cap (-\infty, \bar{p}_2] = S_2$. The proof is analogous to the proof of Proposition 2, part (i). The argument however fails for prices $p > \bar{p}_2$. This failure stems from the failure of Lemma 2. In particular, in the proof of statement (i) we argue that "As $l(n) > 0$, by choosing price p' instead of p in the current period, firm i obtains a higher current profit, but the probabilities as well as future values remain unchanged." However, the profit is strictly higher only when $l(n) > 0$ or $\Pr(p_2 > p \mid 1 - n) > 0$. Thus, in a special case when

$$l(n) = 0 \qquad \text{and} \qquad p > \bar{p}_2 \tag{37}$$

it is indeed possible that $p \in S_1$ but $p \notin S_2$. Otherwise the original argument holds. Recall that $l(n) = 0$ for some $n > 0$ does not occur in our original specification with continuous values of n. However, it does occur once we consider a discrete grid.

Now we show that $S_1 \cap (-\infty, \bar{p}_2] = S_2$ is a connected set (i.e., an interval or a singleton). The proof is the same as in part (ii) of Lemma 2. However, by the same argument as above, it holds only for prices $p < \bar{p}_2$. For prices $p > \bar{p}_2$ the same difficulties as described above arise.

Next, we argue that if case (A) does not apply in position n (such that $l(n) = 0$), then[49]

$$V(n) = \delta V(0). \tag{38}$$

On the one hand, if $S_1 = S_2$, we end up in case (B), where the smaller firm sets a monopoly price with a positive probability. On the other hand, if the supports S_1 and S_2 are not identical, then $\bar{p}_1 > \bar{p}_2$. In both cases, there is a price (or range of prices) in the support S_1 that is chosen with positive probability and yields a certain loss in the market share for firm 1. Charging such price p yields the value $p \cdot l(n) + \delta V(l(n)) = \delta V(0)$. Moreover, because such price (or range of prices) belongs to S_1, it satisfies the indifference condition, which now becomes (38).[50]

As the last step, we show that if $V(a_2) = \delta V(0)$, then S_2 is a singleton (in position $n = a_2$). Assume to the contrary that S_2 is an interval. Consider a price $p \in S_2$ such that $p < \bar{p}_2$ and p is not a mass-point. Clearly there are infinitely many such prices and

non-positive limit price, which would lead to a non-positive value for this firm.

[49] Observe that for $l(n) = 0$, condition (13) reduces to (38).

[50] Note that in the case $n = 0$ this condition implies $V(0) = 0$.

all of them satisfy the indifference condition

$$V(a_2) = [p \cdot 0 + \delta V(0)] \Pr(p_2 < p \mid 1 - a_2) + [p \cdot a_3 + \delta V(a_3)] \Pr(p_2 > p \mid 1 - a_2).$$

Now we use (38) and substitute $\delta V(0) = V(a_2)$. After collecting identical terms and dividing by $\Pr(p_2 > p \mid 1 - a_2) > 0$ we obtain $V(a_2) = p \cdot a_3 + \delta V(a_3)$. This, however, cannot hold for infinitely many prices p, which is a contradiction. □

A.4 Proofs for Section 5

Proof of Proposition 5. The solution of the system (20) and (21) is given by:

$$V(0) = \frac{4s(1-s) + (1 - 2s + 4s^2)\delta - 2(1 - 5s^2)\delta^2 - (1 - 4s + 10s^2)\delta^3}{2(1 - \delta^2)(2 - \delta^2)}, \tag{39}$$

$$V(s) = \frac{1 - 2(1 - 4s)\delta - \delta^2 - 2s\delta^3}{2(1 - \delta^2)(2 - \delta^2)}, \tag{40}$$

$$V(1/2) = \frac{4s + \delta - 2(1 - s)\delta^2 - \delta^3}{2(1 - \delta^2)(2 - \delta^2)}, \tag{41}$$

$$V(1 - s) = \frac{1 + 2s\delta - \delta^2 - (1 - s)\delta^3}{(1 - \delta^2)(2 - \delta^2)}, \tag{42}$$

$$V(1) = \frac{2(1 - s) + \delta - (3 - 5s)\delta^2 - \delta^3}{(1 - \delta^2)(2 - \delta^2)}. \tag{43}$$

The values in (39)–(43) can be used in (10) to derive the firms' randomization strategies in the MPE at each position n on the grid. For the sake of brevity we present only the values of the distribution functions for price $p = 1$:

$$F(1 \mid 1) = \frac{2s[2s + \delta + (2 - 5s)\delta^2]}{4s + (1 + 4s^2)\delta - 2(1 - 2s)\delta^2 - (1 - 4s + 10s^2)\delta^3}, \tag{44}$$

$$F(1 \mid 1 - s) = \frac{(1 + \delta)[1 + 2(1 - 2s)\delta]}{2 + 2(1 + 2s^2)\delta - (1 - 2s)\delta^2 - (1 - 4s + 10s^2)\delta^3}, \tag{45}$$

$$F(1 \mid 0) = F(1 \mid s) = F(1 \mid 1/2) = 1. \tag{46}$$

The last set of equalities holds by construction.

We now verify when the necessary conditions $F(1 \mid 1) \leq 1$ and $F(1 \mid 1 - s) \leq 1$ hold. In particular, we show that both these inequalities hold if and only if $s \geq \frac{1}{3}$ or $\delta \leq \delta_{crit}$ (where δ_{crit} depends on s). First, observe that the inequalities hold for $\delta = 0$, and thus by continuity also for δ small. Second, it can be easily established that the denominator of (45), which is identical to the denominator of $1 - F(1 \mid 1 - s)$, is positive. Third, the numerator of $1 - F(1 \mid 1 - s)$, which is

$$N = 1 - (1 - 4s - 4s^2)\delta - 3(1 - 2s)\delta^2 - (1 - 4s + 10s^2)\delta^3, \tag{47}$$

has a unique root $\delta_{crit} \in (0, \infty)$ for every $s \in (0, \frac{1}{2})$. Then $N \geq 0$ is equivalent to $\delta \leq \delta_{crit}$. Moreover, δ_{crit} is increasing in s for $s \in (0, \frac{1}{2})$. A direct computation reveals that $\delta_{crit} = \sqrt{2} - 1$ for $s = 0$, and $\delta_{crit} = 1$ for $s = \frac{1}{3}$. Thus, $\delta_{crit} > 1$ when $s > \frac{1}{3}$. Finally, it can be verified that $N \geq 0$ implies that also $F(1 \mid 1) \leq 1$.[51]

We now show that the above solution indeed establishes a Markov perfect equilibrium. As the value functions are bounded, it is sufficient to verify that there is no profitable one-shot deviation to a price outside of the support. First, consider a deviation to a price $p < \underline{p}(n)$. This is straightforward, as for prices $p \leq \underline{p}(n)$, firm 1's value equals $ph(n) + \delta V(h(n))$. Note that this is also the value for $p = \underline{p}(n)$ as there are no mass-points lower than the monopoly price. Since this is increasing in p, there is clearly no benefit from charging a price below $\underline{p}(n)$. Second, consider a deviation (by firm 1) to the price $p = 1$ if the rival (firm 2) has a mass-point at this price.[52] Such a deviation is *not* profitable, if and only if

$$\begin{aligned} V(n) &= [l(n) + \delta V(l(n))]F(1 \mid 1-n) + [h(n) + \delta V(h(n))]\,(1 - F(1 \mid 1-n)) \\ &\geq [l(n) + \delta V(l(n))]F(1 \mid 1-n) + [n + \delta V(n)]\,(1 - F(1 \mid 1-n)), \end{aligned}$$

where the expression in the first line is firm 1's value in the limit $p \to 1^-$, and the expression in the second line is firm 1's value when choosing the price $p = 1$. After subtracting identical terms and dividing by $1 - F(1 \mid 1-n) > 0$, we obtain an equivalent formulation

$$h(n) + \delta V(h(n)) \geq n + \delta V(n). \tag{48}$$

It remains to verify that condition (48) holds for $n = 0$ and $n = s$. For $n = 0$, this condition becomes $4s + (1 + 4s^2)\delta - 2(1 - 2s)\delta^2 - (1 - 4s + 10s^2)\delta^3 \geq 0$, whereas, for $n = s$ it is $2(1 - 2s) + \delta + (1 - 2s)\delta^2 \geq 0$. It can be verified (for example, using a simple plot as in footnote 51) that both of these conditions are satisfied, when $N \geq 0$. □

Proof of Proposition 6. It follows from Lemma 3 that an equilibrium involves either case (A) or case (C) in position $n = s$. Now consider an equilibrium of the latter form. The value functions then satisfy the following system of equations:

$$V(1) - \delta V(1 - s) = 1 - s, \quad V(s) - \delta V(0) = 0, \quad V(1/2) - \delta V(s) = s, \tag{49}$$

$$V(0) - \delta V(s) = s(1 - \delta)V(1). \tag{50}$$

[51] A formal proof is algebraically demanding. However, both inequalities can be rewritten as quadratic inequalities in s. Solving for s as a function of δ, we can plot the solutions and the corresponding regions. The plot indeed confirms that $N \geq 0$ implies that $F(1 \mid 1) \leq 1$ for $\delta \in [0, 1]$ and $s \in [0, \frac{1}{2}]$.

[52] The deviation for the other firm is not profitable due to continuity.

The first and the third equation in (49) are identical to the equations in (20), whereas the second equation is identical to (38). Equation (50) is identical to the second equation in (21). Note that the first equation from (21) does not apply now, as firms 2's price distribution has a mass point at $\underline{p}$. We show below (equation (59)) that it holds with an inequality.

Thus, we have only four equations with five unknowns. This system then has infinitely many solutions of the form

$$V(0) = \omega, \tag{51}$$

$$V(s) = \delta\omega, \tag{52}$$

$$V(1/2) = s + \delta^2\omega, \tag{53}$$

$$V(1-s) = \frac{(1+\delta)}{\delta s}\,\omega - \frac{1-s}{\delta}, \tag{54}$$

$$V(1) = \frac{(1+\delta)}{s}\,\omega, \tag{55}$$

parametrized by $\omega \in \mathbb{R}$. Non-negativity of the value functions requires that $\omega \geq 0$. The corresponding firm 2's price $\bar{p}_2$ (in position $n = s$) can be computed from firm 2's value function:

$$V(1-s) = \bar{p}_2 \cdot 1 + \delta V(1), \tag{56}$$

which now gives

$$\bar{p}_2 = \frac{(1-\delta)(1+\delta)^2}{\delta s}\,\omega - \frac{1-s}{\delta}. \tag{57}$$

As the next step we derive some necessary conditions for the above solution to establish an equilibrium. Undercutting the lowest price $\underline{p} = \bar{p}_2$ should not be strictly profitable for firm 1 (in position $n = s$). Thus, in the limit $p \to \bar{p}_2^-$ we obtain

$$\bar{p}_2 \cdot 1/2 + \delta V(1/2) \leq V(s). \tag{58}$$

Substituting for $\bar{p}_2$ from (56) then gives the first necessary condition

$$V(1-s)/2 - \delta V(1)/2 \leq V(s) - \delta V(1/2), \tag{59}$$

which under the solution (51)–(55) yields an upper bound for ω:

$$\omega \leq \frac{s(1-s-2s\delta^2)}{(1-\delta^2)(1+\delta-2s\delta^2)}. \tag{60}$$

Moreover setting a price $p > \bar{p}_2$ (that is not a mass point of firm 1's distribution) should not be profitable for firm 2. Let us denote $q = \Pr(p_1 = 1 \mid s)$. In the limit $p \to 1^-$

we obtain

$$V(1-s) \geq [1/2 + \delta V(1/2)] \cdot (1-q) + [1 + \delta V(1)] \cdot q.$$

After using the inequality $1 + \delta V(1) \geq \bar{p}_2 + \delta V(1) = V(1-s)$, rearranging, and dividing by $1 - q > 0$, we obtain the second necessary condition

$$V(1-s) \geq 1/2 + \delta V(1/2), \tag{61}$$

which under the solution (51)–(55) yields a lower bound for ω:

$$\omega \geq \frac{s[2(1-s) + \delta + 2s\delta^2]}{2 + 2\delta - 2s\delta^4}. \tag{62}$$

It follows from the inequalities (60) and (62) that

$$\frac{s[2(1-s) + \delta + 2s\delta^2]}{2 + 2\delta - 2s\delta^4} \leq \frac{s(1-s-2s\delta^2)}{(1-\delta^2)(1+\delta-2s\delta^2)},$$

which is equivalent to

$$\frac{\delta s N}{2(1-\delta^2)(1+\delta-2s\delta^2)(1+\delta-s\delta^4)} \leq 0,$$

where N is given by (47). As the denominator is clearly positive, we obtain a necessary condition $N \leq 0$, which holds if and only if $s \leq \frac{1}{3}$ and $\delta \geq \delta_{crit}$.

Finally, we show existence of the equilibrium when $N < 0$. In particular, we show that for ω which satisfies (60) with equality (which is actually the condition for case (B)), the solution (51)–(55) indeed establishes an equilibrium. In that case, the solution becomes

$$V(0) = \frac{s(1-s-2s\delta^2)}{(1-\delta^2)(1+\delta-2s\delta^2)}, \tag{63}$$

$$V(s) = \frac{s\delta(1-s-2s\delta^2)}{(1-\delta^2)(1+\delta-2s\delta^2)}, \tag{64}$$

$$V(1/2) = \frac{s(1+\delta-3s\delta^2-\delta^3)}{(1-\delta^2)(1+\delta-2s\delta^2)}, \tag{65}$$

$$V(1-s) = \frac{\delta[1-s-2s^2+(1-3s)\delta-2s(1-s)\delta^2]}{(1-\delta)(1+\delta-2s\delta^2)}, \tag{66}$$

$$V(1) = \frac{1-s-2s\delta^2}{(1-\delta)(1+\delta-2s\delta^2)}. \tag{67}$$

Then in position $n = s$ we have

$$\bar{p}_2 = -\frac{2s\delta(s+\delta)}{1+\delta-2s\delta^2}.$$

Observe that all value functions in (51)–(55) are increasing in w. Therefore, this equilibrium yields the highest value in all states.

By the same argument as in the proof of Proposition 5, it is sufficient to consider one-shot deviations. Moreover, we can again use the arguments from the proof of Proposition 5. Thus, it remains to consider deviations to the following prices: (i) the monopoly price by firm 1 in position $n = 0$; (ii) $p < \bar{p}_2$ by firm 1 in position $n = s$; (iii) $p = \bar{p}_2$ by firm 1 in position $n = s$; (iv) $p > \bar{p}_2$ by firm 2 in position $n = s$. In order to rule out (i), we verify condition (48). This, after substituting the solution (63)–(67), becomes equivalent to $1 + (1 + s)\delta + (1 - 2s)\delta^2 \geq 0$, which clearly holds. Deviation (ii) is not profitable, if $\bar{p}_2 s + \delta V(s) \leq V(s)$. After substitution, this becomes equivalent to $1 - s + 2s^2 + 2s(1 + s)\delta \geq 0$, which clearly holds. Deviations (iii) and (iv) are not profitable due to the conditions derived above. Deviation (iii) corresponds (for $p \to \bar{p}_2^-$) to the inequality (60), which now holds with equality. Similarly, deviation (iv) corresponds to (62), which is satisfied as $N < 0$. □

Proof of Proposition 7. When $\delta = 0$, the firms' randomization strategies are (using (39)–(43) in (10)) given by:

$$F(p \mid 1/2) = \frac{1 - s - s/p}{1 - 2s}, \tag{68}$$

$$F(p \mid 1 - s) = 1 - \frac{1}{2p}, \qquad F(p \mid s) = 2 - \frac{1}{p} = 2F(p \mid 1 - s), \tag{69}$$

$$F(p \mid 1) = 1 - \frac{1 - s}{p}, \qquad F(p \mid 0) = \frac{1 - (1 - s)/p}{s} = \frac{1}{s} F(p \mid 1), \tag{70}$$

First-order stochastic dominance of $F(p \mid 1 - n)$ over $F(p \mid n)$ requires that $F(p \mid n) \geq F(p \mid 1 - n)$ for all $p \in [\underline{p}(n), 1]$. For $n = 0$ and $n = s$ we have $F(p \mid 0) = F(p \mid 1)/s > F(p \mid 1)$ (with $s < \frac{1}{2}$) and $F(p \mid s) = 2F(p \mid 1 - s) > F(p \mid 1 - s)$, respectively. The claim for δ small follows by the continuity of (39)–(43) in δ. □

Proof of Proposition 8. Let us rewrite (10) for position $n = 1 - s$ and use $h(s) = \frac{1}{2}$ and $l(s) = 0$ to obtain:

$$1 - F(p \mid 1 - s) = 1 - \frac{\delta V(h(s)) - V(s) + h(s)p}{\delta V(h(s)) - \delta V(l(s)) + [h(s) - l(s)]p} = \frac{V(s) - \delta V(0)}{\delta V(\frac{1}{2}) - \delta V(0) + p/2}. \tag{71}$$

Now recall from the proof of Proposition 5 that δ_{crit} was obtained as a root of (47), which is equivalent to $F(1 \mid 1 - s) = 1$. Moreover, $F(1 \mid 1 - s) \to 1$ when $\delta \to \delta_{crit}^-$. Then it follows from (71) that $V(s) - \delta V(0) \to 0$. Thus, also $F(p \mid 1 - s) \to 1$ when $\delta \to \delta_{crit}^-$, for all p in the support of $F(\cdot \mid n)$. This in turn implies that, in the limit, firm 2 (the

firm with the larger customer base) chooses the lower bound of the support, $\underline{p}(n)$, with probability 1, while firm 1 continues to randomize over the interval $[\underline{p}(n), 1]$. Note that here we actually obtain the case (C) where the supports are different (see Lemma 3 as well as the text preceding the lemma for more details). □

B Appendix: Figures

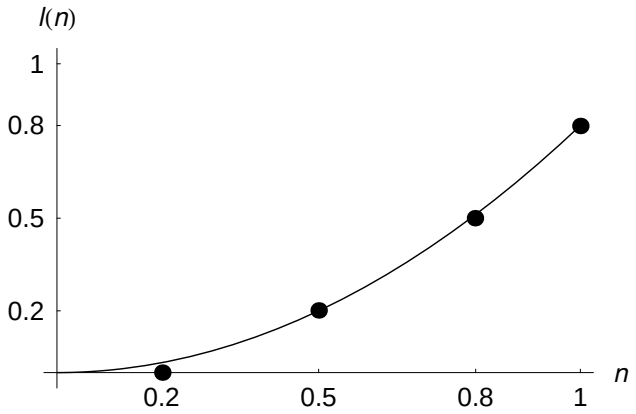

Figure 1: Approximation by a grid with 5 positions, for $\phi = 0.8$, $\mu = 1$, $s = 0.2$

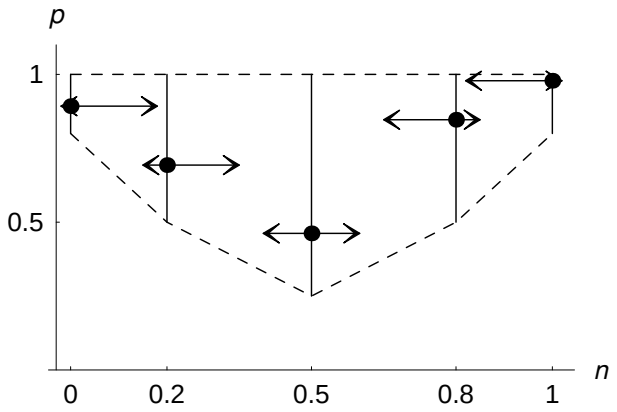

Figure 2: Prices and transition probabilities for the G_5, for $s = 0.2$ and $\delta = 0$

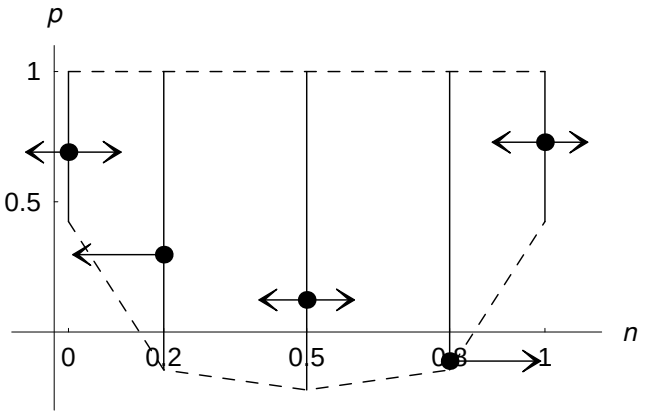

Figure 3: Prices and transition probabilities for the G_5, for $s = 0.2$ and $\delta = 0.66$

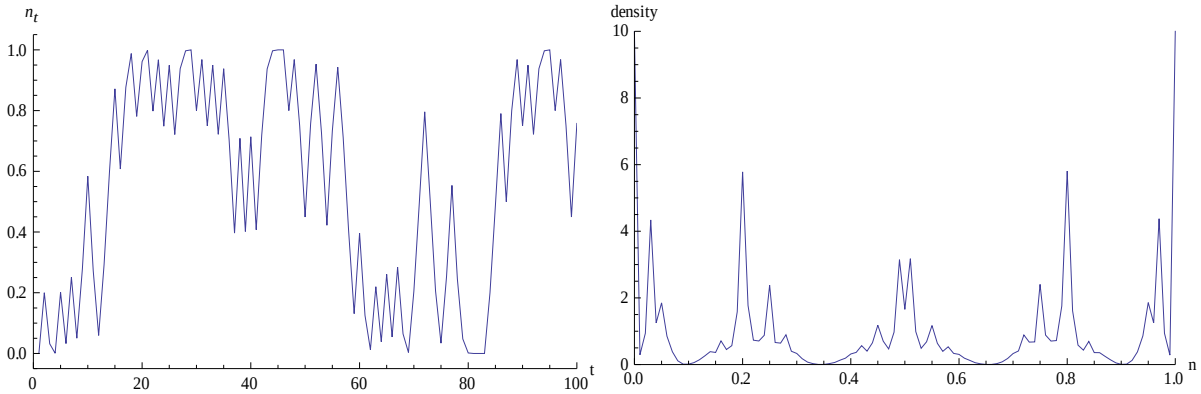

Figure 4: State-independent stochastic process, simulation for $\phi = 0.8$, $\mu = 1$

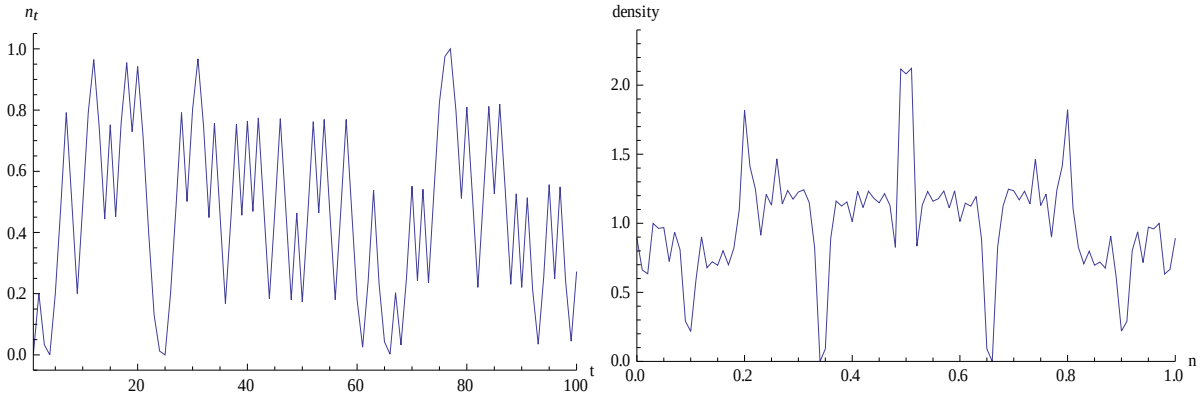

Figure 5: Myopic case ($\delta = 0$), simulation for $\phi = 0.8$, $\mu = 1$

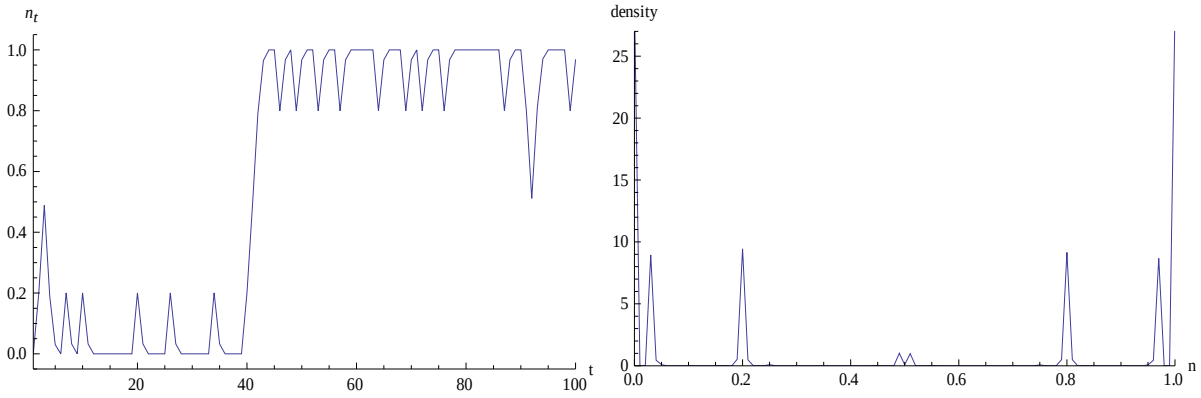

Figure 6: Full dynamic game, simulation for $\phi = 0.8$, $\mu = 1$, and $\delta = 0.8$

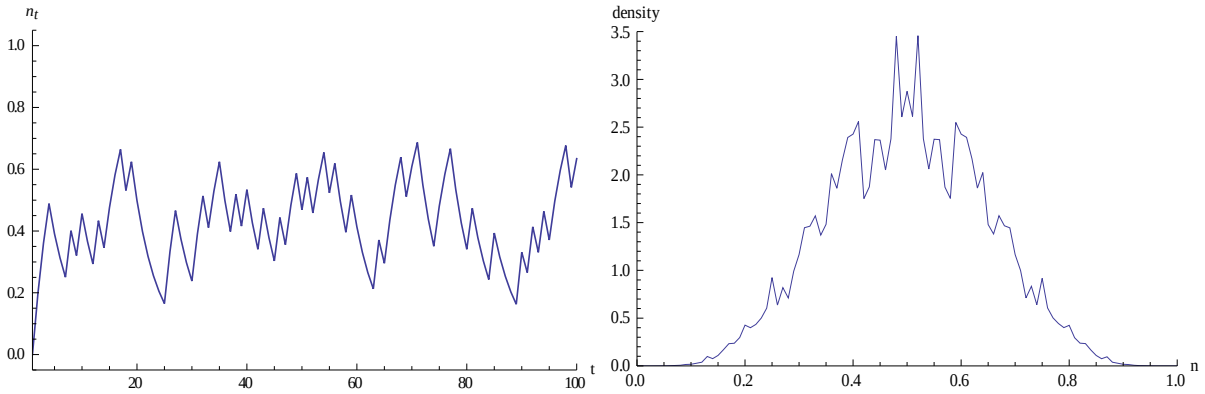

Figure 7: Full dynamic game, simulation for $\phi = 0.8$, $\mu = 0$, and $\delta = 0.99$

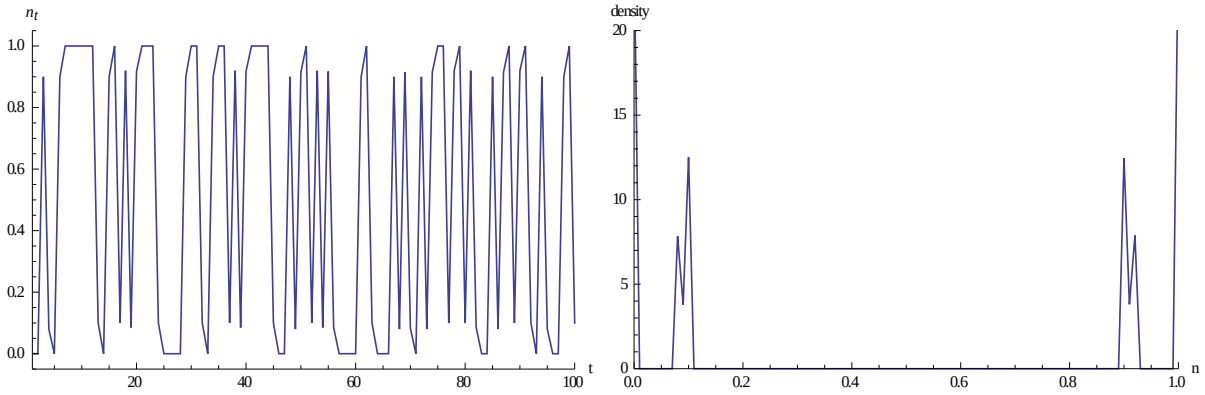

Figure 8: Full dynamic game, simulation for $\phi = 0.1$, $\mu = 1$, and $\delta = 0.99$

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