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Intertemporal Effort Provision in Sequential Tournaments

Arnd Heinrich Klein and Armin Schmutzler¹

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Abstract: This paper analyzes intertemporal effort provision in two-stage tournaments. A principal with a fixed budget for prizes faces two risk-neutral agents. He observes noisy signals of effort in both periods. His goal is to maximize either total efforts (perfect substitutes) or the product of first- and second-period efforts (imperfect substitutes). He decides (i) how to weigh performance in the two periods when awarding the second period prize, (ii) how to spread prize money across the two periods, and (iii) whether to reveal performance after the first period. Under very general conditions, the principal puts positive weight on both periods in period two. Furthermore, he sets no first-period prize provided the observations in period one are too noisy. The information revelation policy depends on the third derivative of the effort cost function.

JEL: D02, D44

Keywords: sequential tournaments, repeated contests, all-pay auctions, effort incentives

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1 Introduction

This paper analyzes incentives for intertemporal effort provision in organizations. It is common practice to use tournament-like incentive systems in such contexts rather than providing agents with contracts that are based explicitly on their own performance. A well-known argument for why tournaments are often used is that they are less prone to manipulation by the principal than standard contractual incentives: When performance is not verifiable, a principal may claim that performance was low to save on performance pay. With relative performance measures such as tournaments, this incentive is reduced, because the total payments to the agents are independent of performance.

We contribute to a substantial literature that has analyzed the effects of different designs of such contests on effort provision. We allow for two possible objectives of the principal. First, in line with the existing literature, we consider the case that a principal regards efforts in different periods and by different agents as perfect substitutes and is therefore interested in the maximization of total effort. Second, contrary to most of the existing literature, we also consider the case that the principal regards efforts in different periods as imperfect substitutes, so that he wants them to be sufficiently balanced across periods.² This is plausible when per-period profits are a concave function of efforts in the same period. We use a particularly simple way to capture such a situation: We assume that the principal maximizes the product of the period efforts. This can be rationalized if period profits are logarithmic functions of efforts.

Specifically, we consider a two-period setting with two risk-neutral agents with identical and known abilities. In each period, the principal can conduct a tournament with a fixed prize; we allow for the degenerate case that one of the prizes is zero. The principal chooses a three-dimensional incentive system. First, he decides how to spread a given total prize budget over the two periods. Second, he chooses to which extent first-period performance plays a role in the evaluation of agents in the second period. In particular, he decides whether to give more weight to first- or second-period performance. Third, the principal may or may not reveal information about performance in

²Aoyagi (2010) also allows for more general objective functions.

the first period to the agents. After observing the policy, the agents choose effort levels in each period. The principal observes the performance of each agent, which is an imperfect measure of his effort. In period 1, he awards the prize (if any) to the agent with the higher performance. In the *revelation* regime, he communicates the performance in the first period to the agents. In the *no revelation* regime, he neither communicates performance, nor who the winner was. In period 2, the agents choose efforts again. The principal then observes the second-period performance levels. He allocates the prize to the agent for whom a weighted sum of first-period and second-period performance is highest. We allow for negative and positive weights. To sum up, the principal commits to an *incentive system* consisting of the prize distribution, the information revelation regime and the weight of first-period performance in the second-period tournament. Given this incentive system, the agents choose their effort strategies.

We first characterize the equilibrium behavior of the agents for a given incentive system, then we discuss the optimal choices of the principal.

In the revelation regime, every first-period performance vector results in a second-period strategic situation that is equivalent to an ordinary asymmetric all-pay auction with noise. The asymmetry comes from potential differences in measured first-period performance. The larger the asymmetries resulting from the first period, the lower the sequentially rational second-period efforts. Moreover, both agents choose the same effort levels in the second period. First-period efforts are identical as well. Second-period asymmetry therefore results exclusively from error realizations. Potentially, first-period efforts have two effects on expected second-period payoffs. First, there is a direct effect: If, for instance, there is a positive weight of first-period performance on the probability of winning the second-period prize, then for any given second-period effort vector an increase in first-period effort increases the expected second-period payoff. Second, there is an indirect (strategic) effect: Any effort increase in period 1 affects the second-period behavior of the opponent and thus own payoffs. In the symmetric weakly perfect Bayesian equilibrium, the indirect effect vanishes. As a result, the characterization of first-period efforts becomes simple, resulting from equating marginal effort costs with the marginal benefits that consist only of the increased chances of winning

the first-period prize and the direct effect on the chances of winning the second-period prize.

In the no-revelation regime, the two players have to commit to both effort levels. As a result, second-period efforts are chosen based on the *expected* first-period performance rather than the actual performance. Strategic effects of own first-period choices on the second period choices of others are therefore absent. As in the revelation regime, the marginal benefit of higher first-period efforts therefore consists only of the direct effects. It thus turns out that first-period efforts are the same as in the revelation regime.

The preceding discussion has several implications for the optimal choice of the incentive system. First, for any given choice of the remaining instruments, the information revelation regime only affects the expected second-period effort, but not the first-period effort. The sign of the difference between the expected effort levels depends on third derivatives of the effort cost function. With quadratic effort costs, there is no difference between the two cases. When the third derivative of the effort cost function is positive, then expected efforts are lower with revelation than without; conversely for negative third derivative. For both types of objectives, this implies that the principal favors the no revelation policy for a positive third derivative, while he favors the full revelation policy for a negative third derivative. These results generalize previous results of Aoyagi (2010) who considers only the case that there is no first-period prize and the principal gives equal weight to efforts in both periods in the second period prize. By considering the optimal choices of relative prize sizes and the weights of the two periods in the second period prize, we next show under which circumstances Aoyagi's (2010) simplification is justified in the sense that it arises endogenously.

We show that, for both types of objectives and general cost functions as well as observation error distributions, the optimal weight of first-period performance in the second-period tournament is strictly positive. This result holds for both revelation regimes. The analysis is particularly simple for quadratic cost functions and normally distributed observation errors. The optimal choice of the first-period weight for perfect substitutes is then completely determined by the relative precision of the first-period and second-period observations: It is the ratio of the variances of the second-period obser-

vation error differences and first-period-period observation error differences. An interesting implication of this result is that it holds independently of the distribution of first- and second-period prizes. For imperfect substitutes, the optimal weight is the ratio of the standard deviations of second-period observation error differences and first-period-period observation error differences, given that the principal assigns no first-period prize. Moreover, because of the previous result that expected efforts are independent of the revelation regime in this simple case, the same weights should be applied in both revelation regimes.

Finally, we consider the optimal prize distribution. For both revelation regimes and for perfect and imperfect substitutes, the optimal first-period prize will be zero if the distribution of the first-period observation error difference is not too concentrated near zero, that is, sufficiently noisy. For quadratic cost functions and normally distributed observation errors, the optimal first-period prize is zero for perfect substitutes, given that the first-period weight is chosen optimally, while it is always strictly positive for imperfect substitutes.

The paper is organized as follows. In Section 2, we introduce the model. In Section 3, we analyze the behavior of agents for given policies. Section 4 characterizes optimal policy. Section 5 introduces a simple example. In Section 6, we discuss our contribution in the light of existing literature. Section 7 concludes.

2 The Model

We consider a class of two-stage tournaments. Given a fixed budget W for effort provision, a principal chooses an *incentive system* \mathcal{I} , which is a tuple $(\eta, W_1, \rho) \in (-\infty, \infty) \times [0, W] \times \{0, 1\}$ to be explained below. For given \mathcal{I} , agents $i \in \{1, 2\}$ choose effort levels $e_{it} \geq 0$ ($t \in \{1, 2\}$).³ The properties of the effort cost function $K_{it}(e_{it})$ are summarized as follows:

Assumption 1: *The effort cost function K_{it} is symmetric and time-*

³In the following, the use of i and/or j as an index always implies $i, j \in \{1, 2\}$ and $i \neq j$.

invariant, satisfies $K_{it}(0) = 0$ and $K'_{it}(0) = 0$, and is increasing and strictly convex. $K'''_{it}(e_{it}) \geq 0$ or $K'''_{it}(e_{it}) \leq 0$ must hold globally.

By Assumption 1, we can write $K \equiv K_{it}$. The agents are risk-neutral and their utility is additively separable in (expected) period-specific income and costs. At the end of each period t , the principal observes *performance*, which is an imperfect measure $s_{it} = e_{it} + \varepsilon_{it}$ of effort, where the error term ε_{it} is independently distributed across agents and periods. In each period, the error distribution is identical for both agents. However, we allow for different error distributions in both periods.

Based on the first-period performance, the principal awards the first-period prize W_1 to agent i if and only if $s_{i1} > s_{j1}$.⁴ The second-period prize $W_2 = W - W_1$ is awarded to agent i if and only if

$$s_{i2} + \eta s_{i1} > s_{j2} + \eta s_{j1}.$$

η is the first-period weight chosen by the principal and can be interpreted as the influence of the past on the chance of winning in the second period.

In the *revelation regime* ($\rho = 1$), the principal communicates his performance assessment of both players to the agents before they choose their second-period efforts. In the *no-revelation regime* ($\rho = 0$), the principal does not communicate the performance assessment; he does not even communicate who won the first-period prize.

The following notation is helpful to describe the solution of the game.

Definition 1 The **error difference** of player i in period t ($t = 1, 2$) is $\Delta\varepsilon_{it} = \varepsilon_{it} - \varepsilon_{jt}$, his **relative first-period performance** is $\Delta s_{i1} = s_{i1} - s_{j1} = \Delta e_{i1} + \Delta\varepsilon_{it}$, where $\Delta e_{it} = e_{it} - e_{jt}$.

Clearly, $\Delta e_{it} = -\Delta e_{jt}$, $\Delta\varepsilon_{it} = -\Delta\varepsilon_{jt}$, $\Delta s_{i1} = -\Delta s_{j1}$.

We make the following assumption on the error distributions:

Assumption 2 $\Delta\varepsilon_{it}$ is distributed as $F_t(s)$ with a symmetric and continuously differentiable density $f_t(s)$, for which $f_t(s) > 0 \forall s \in \mathbb{R}$ and that is

⁴In case of a tie, the principal assigns the prize to each agents with probability one half.

single-peaked at $s = 0$.⁵

The assumption implies that $f_t(s) = f_t(-s)$, $f'_t(s) = -f'_t(-s)$ and that $E(\Delta\varepsilon_{it}) = 0$.

We assume that the efforts of different agents within periods are perfect substitutes for the principal. However, we allow first- and second period efforts to be either perfect or imperfect substitutes. For *perfect substitutes*, the principal chooses the incentive system so as to maximize expected total efforts. For *imperfect substitutes*, he maximizes the expected product of first and second-period efforts. To motivate these two different objective functions, one should think of the principal as maximizing total profits. If per-period profits are linear functions of total efforts in the same period, the perfect substitutes assumption is appropriate; the imperfect substitutes case corresponds to a logarithmic relation between efforts and profits in each period, which is a simple example of a concave relation.⁶

3 Behavior of the agents

We first analyze the equilibrium behavior of agents for given policy regime. A simple auxiliary results will often be used. The proof is straightforward and will be omitted.

Lemma 1 (i) $P(s_{i1} > s_{j1}) = F_1(e_{i1} - e_{j1})$
(ii) $P(s_{i2} + \eta s_{i1} > s_{j2} + \eta s_{j1}) = F_2(\eta \Delta s_{i1} + e_{i2} - e_{j2})$

In this section, we do not yet make the principal's budget constraint $W_2 = W - W_1$ explicit and simply write W_2 for the second-period prize.

3.1 Full revelation

We first analyze the behavior of the agents in the revelation regime in period 2, then in period 1. In period 2, a player's information set consists of all

⁵It is straightforward to show that it is sufficient to make this assumption for one player, as this immediately implies the statement for the other player.

⁶Maximizing $\ln(e_{11} + e_{21}) + \ln(e_{12} + e_{22})$ is equivalent to maximizing $(e_{11} + e_{21})(e_{12} + e_{22})$.

combinations of period 1 efforts and error differences that are consistent with the own first-period effort e_{i1} and the observed relative performance Δs_{i1} . We use the solution concept of Weak Perfect Bayesian Equilibrium (WPBE) to deal with this imperfect information. The task is simplified by the fact that period 1 enters player i 's payoffs only via Δs_{i1} and e_{i1} , so that the unobservable aspects of previous play are irrelevant for the players' choices. A strategy σ_i consists of a first-period choice e_{i1} and a function E_{i2} mapping information sets $(\Delta s_{i1}, e_{i1})$ to actions e_{i2} . Given a strategy $\sigma_j = (e_{j1}, E_{j2})$, a player who played e_{i1} , observes Δs_{i1} and assumes that player j plays σ_j , will assign probability one to the event that $\Delta \varepsilon_{i1} = \Delta s_{i1} - \Delta e_{i1}$. In the following, we will always assume that beliefs are formed in this way, without specifying them explicitly.

3.1.1 Second-period efforts

Using Lemma 1(ii), the expected second-period payoff of agent i , conditional on Δs_{i1} and Δe_{i2} , is⁷

$$U_{i2}(e_{i2}, e_{j2}, \Delta s_{i1}; \eta, W_2) = F_2(\eta \Delta s_{i1} + \Delta e_{i2}) W_2 - K(e_{i2}). \quad (1)$$

Thus, the first period influences the second-period payoff via the first-period relative performance Δs_{i1} . The corresponding first-order condition is

$$f_2(\eta \Delta s_{i1} + \Delta e_{i2}) W_2 = K'(e_{i2}). \quad (2)$$

The following inequality will function as a global second-order condition:

$$f_2'(\eta \Delta s_{i1} + \Delta e_{i2}) W_2 \leq K''(e_{i2}) \quad \forall \Delta s_{i1} \in \mathbb{R}, e_{i2}, e_{j2} \in \mathbb{R}^+. \quad (3)$$

By Assumptions 1 and 2, (3) always holds if $\eta \Delta s_{i1} + \Delta e_{i2} > 0$, that is, agent i is ahead of agent j in terms of the sum of period 1 performance and period 2 effort differential. (3) requires K to be sufficiently convex.⁸

⁷Since the case of a tie has zero measure, we ignore it in the formulation of the payoff function.

⁸If f_2' is bounded above, as for standard distributions, (3) holds globally if K'' has a sufficiently high lower bound.

Though the game does not have any proper subgames because information sets in period 2 are not singletons, payoffs in period 2 are independent of the concrete node within an information set. We use this in the following definition.

Definition 2 *The second-period effort game induced by Δs_{i1} is the game with players $i = 1, 2$ strategy spaces $X_i = \mathbb{R}^+$ and payoffs given by (1) for $(e_{i2}, e_{j2}) \in X_i \times X_j$.*

We obtain the following result:

Proposition 1 *Suppose (3) holds. For any Δs_{i1} , the second-period effort game induced by Δs_{i1} has a unique NE $(e_{12}^*(\Delta s_{i1}; \eta, W_2, 1), e_{22}^*(\Delta s_{i1}; \eta, W_2, 1))$. This equilibrium is symmetric, with*

$$e_{i2}^*(\Delta s_{i1}; \eta, W_2, 1) = (K')^{-1} [f_2(\eta \Delta s_{i1}) W_2] > 0 \quad (4)$$

Proposition 1 has some simple but important comparative statics implications.

Corollary 1 *Suppose $\rho = 1$, $\eta \neq 0$ and $W_2 > 0$. Then e_{i2}^* is decreasing in $|\Delta s_{i1}|$ and η , and increasing in W_2 .*

The result on $|\Delta s_{i1}|$ implies that, if a laggard (an agent with $\Delta s_{i1} < 0$) increases own effort or his competitor decreases efforts marginal in period 1, the competitor increases his effort in period 2, whereas the converse statement holds for a leader (an agent with $\Delta s_{i1} > 0$). The other two results identify the straightforward effects of policy changes. In particular, increasing the first-period weight η comes at the cost of reducing second-period efforts.

3.1.2 First-period efforts

The expected second-period utility, conditional on first-period performance and second-period efforts, is given by (1). The expected payoff, conditional on first-period effort choices can thus be written as

$$U_{i2}^*(e_{i1}, e_{j1}; \eta, W_2) = E_{\Delta s_{i1}} U_{i2}(e_{i2}^*(\Delta s_{i1}; \eta, W_2), e_{j2}^*(-\Delta s_{i1}; \eta, W_2), \Delta s_{i1}; \eta, W_2).$$

With the characterization (4) of efforts in period 2, it is possible to characterize the expected marginal effect of first-period payoff on second-period payoff, which consists of two components. First, suppose $\eta > 0$.⁹ For any given second-period effort vector, the expected second-period payoff is increasing in first-period effort. This is the direct effect of first-period effort. Second, there is an indirect (strategic) effect: Any effort increase in period 1 affects the second-period behavior of the opponent and thus own payoffs. For any choice of efforts and any realization of first-period error differences where $\Delta s_{i1} > 0$, a higher Δs_{i1} would have led to lower e_{j2} . Hence, if player i had anticipated Δs_{i1} correctly, he would have had a strategic incentive to increase e_{i1} . Conversely, for $\Delta s_{i1} < 0$ there would have been a strategic incentive to reduce e_{i1} . However, even if player i has certain beliefs that player j chooses e_{j1} , he only knows the distribution of Δs_{i1} . In particular, he knows that $\Delta s_{i1} > 0$ is more likely if $\Delta e_{i1} > 0$. As a result, if player i has chosen $e_{i1} > e_{j1}$, a marginal increase in first-period effort makes the second-period contest less competitive on expectation and thereby induces less effort of the opponent (Corollary 1). Conversely, starting from a situation where player i has chosen $e_{i1} < e_{j1}$, marginally higher effort reduces the expected asymmetry, which tends to increase second-period effort of the opponent. The effects just described cancel out when first-period efforts are identical, so that the strategic effect is zero. Using (1), the expected marginal effect of first-period effort on second-period payoff simplifies to

$$\left. \frac{\partial U_{i2}^*}{\partial e_{i1}} \right|_{e_{i1}=e_{j1}} = 2\eta W_2 \int_0^\infty f_2(\eta s) f_1(s) ds \quad (5)$$

Using Lemma 1(i), agent i 's optimization problem in period 1 is

$$\max_{e_{i1} \geq 0} F_1(e_{i1} - e_{j1}) W_1 + U_{i2}^*(e_{i1}, e_{j1}; \eta, W_2) - K(e_{i1}).$$

The corresponding first-order conditions is

$$f_1(\Delta e_{i1}) W_1 + \frac{\partial U_{i2}^*}{\partial e_{i1}} = K'(e_{i1}). \quad (6)$$

⁹For $\eta < 0$, the argument is reversed.

The second-order condition is

$$\begin{aligned}
K''(e_{i1}) \geq & f'_1(\Delta e_{i1}) W_1 \\
& + \eta W_2 \int_0^\infty f_2(\eta s) [f'_1(s + \Delta e_{i1}) - f'_1(s - \Delta e_{i1})] ds \\
& + \eta W_2^2 \int_0^\infty \frac{f_2(\eta s) f'_2(\eta s) [f'_1(s + \Delta e_{i1}) + f'_1(s - \Delta e_{i1})]}{K''((K')^{-1}(f_2(\eta s) W_2))} ds \\
& \forall e_{i1}, e_{j1} \in \mathbb{R}^+
\end{aligned} \tag{7}$$

The left-hand side of (7) is increasing in K'' , while the right-hand side is decreasing in K'' . For given policy parameters and distributions, it therefore holds as long as K is sufficiently convex.¹⁰

The following condition is needed to exclude corner solutions in period 1:

$$f_1(0) W_1 + 2\eta W_2 \int_0^\infty f_2(\eta s) f_1(s) ds \geq 0. \tag{8}$$

(8) is only binding for negative η . For these, it essentially guarantees that the punishment by weighting first-period performance negatively in the second period is not prohibitively high. We will actually show in the later analysis that the principal will never choose negative values for η .

We are now ready to characterize the WPBE:

Proposition 2 *Suppose (3), (7) and (8) hold. For $\rho = 1$ (full revelation), the game has a unique symmetric WPBE. Second-period efforts are given as in Proposition 1. First-period efforts satisfy*

$$e_1^*(\eta, W_1, W_2, 1) = (K')^{-1} \left[f_1(0) W_1 + 2\eta W_2 \int_0^\infty f_2(\eta s) f_1(s) ds \right] > 0 \tag{9}$$

Plugging the equilibrium efforts (9) in period 1 into the equilibrium efforts (4) of the effort game and taking the expectation over $\Delta \varepsilon_{i1}$, we obtain:

¹⁰As for the second-period effort games, the first-order condition can be guaranteed to hold if K'' has a sufficiently high lower bound, whenever the slopes of f_1 and f_2 are bounded.

Corollary 2 *The expected efforts in period 2 under the full revelation regime satisfy*

$$E(e_2^*(\eta, W_2, 1)) = 2 \int_0^\infty (K')^{-1} [f_2(\eta s) W_2] f_1(s) ds \quad (10)$$

3.2 No revelation

In the no-revelation regime, the maximization problem boils down to a simultaneous choice of first- and second-period efforts. The optimization problem of agent i is thus

$$\begin{aligned} \max_{e_{i1} \geq 0, e_{i2} \geq 0} \quad & F_1(\Delta e_{i1}) W_1 + \\ & W_2 \int_{-\infty}^\infty F_2(\eta(\Delta e_{i1} + s) + \Delta e_{i2}) f_1(s) ds - K(e_{i1}) - K(e_{i2}). \end{aligned} \quad (11)$$

In this term, the integral is the probability of winning in the second period, conditional on first-period effort choice and second-period effort choice.¹¹ This leads to a simple characterization of the Nash equilibrium (NE).

Proposition 3 *Suppose that (8) and the SOC (34), (35), and (36) hold. For $\rho = 0$ (no revelation), the game has a unique symmetric NE. The equilibrium efforts satisfy*

$$\begin{aligned} e_1^*(\eta, W_1, W_2, 0) &= (K')^{-1} \left[f_1(0) W_1 + 2\eta W_2 \int_0^\infty f_2(\eta s) f_1(s) ds \right] \\ &> 0 \end{aligned} \quad (12)$$

$$e_2^*(\eta, W_2, 0) = (K')^{-1} \left[2W_2 \int_0^\infty f_2(\eta s) f_1(s) ds \right] > 0 \quad (13)$$

Both effort levels reflect standard cost-benefit calculations. Here, the marginal benefit of first-period efforts contains the increased winning probability in period 2. In the appendix, we show that in equilibrium (34), (35), and (36) are automatically implied by the convexity of the cost function.

¹¹This follows from Lemma 1(ii).

3.3 Comparing full and no revelation efforts

First-period efforts are equal under both revelation regimes, reflecting the fact that the positive marginal effect of first-period effort on expected second period payoffs is identical. Intuitively, a marginal increase of e_{i1} has beneficial second-period effects if it suffices to tip the balance in period 2 in favor of player i . The probability that this happens is independent on whether information on Δs_{i1} is revealed to players before they choose second-period efforts, which is captured by $2 \int_0^\infty f_2(\eta s) f_1(s)$ in both cases.¹² From now on, we simply write $e_1^*(\eta, W_1, W_2)$ for first-period equilibrium efforts. Using Jensen's inequality, we can compare the (expected) second-period efforts.

Proposition 4 (i) $\forall \eta, \forall W_2 > 0, e_2^*(\eta, W_2, 0) \geq E(e_2^*(\eta, W_2, 1))$ if $K''' \geq 0$.
(ii) $\forall \eta, \forall W_2 > 0, e_2^*(\eta, W_2, 0) \leq E(e_2^*(\eta, W_2, 1))$ if $K''' \leq 0$.

This result implies that second-period efforts are equal in expected value for quadratic costs under both revelation regimes. Intuitively, the role of K''' is an immediate consequence of the fact that second-period efforts are the inverse of the expectation of marginal costs for $\rho = 0$ and the expectation of the inverse of marginal costs for $\rho = 1$. Thus, concavity (convexity) of the inverse marginal costs is decisive for which regime yields higher efforts on expectation.

4 Optimal policy

We now characterize the optimal policy of the principal. To this end, we fix the total budget as W , so that $W_2 = W - W_1$ must hold. Since the effort profile is always symmetric, we can write the principal's objective in terms of the efforts of only one agent. Taking into account that first-period efforts are non-stochastic, the principal's objective functions for perfect and imperfect

¹²In this argument, it is important to start from the respective equilibria, with equal efforts in both periods.

substitutes, respectively, are:

$$V^P(\eta, W_1, \rho) \equiv e_1^*(\eta, W_1, W - W_1) + E(e_2^*(\eta, W - W_1, \rho)) \quad (14)$$

$$V^I(\eta, W_1, \rho) \equiv e_1^*(\eta, W_1, W - W_1) \cdot E(e_2^*(\eta, W - W_1, \rho)). \quad (15)$$

4.1 Choosing the revelation regime

Since first-period efforts are non-stochastic and equal under both revelation regimes, the choice of the revelation policy hinges on whether it induces higher (expected) second-period efforts, no matter whether efforts are perfect or imperfect substitutes. Proposition 4 applies to all values of η and W_1 and, in particular, to those that maximize $e_2^*(\eta, W - W_1, 0)$ or $E(e_2^*(\eta, W - W_1, 1))$. Thus, even if the principal has chosen the optimal parameters for a given revelation policy, switching to the other revelation policy is beneficial if the corresponding condition on K''' holds. Hence, we have proven:

Corollary 3 *The optimal revelation policy is the same for perfect and imperfect substitutes. Maximizing expected payoff implies setting $\rho = 0$ if $K''' \geq 0$ and $\rho = 1$ if $K''' \leq 0$. For $K''' = 0$, expected payoffs are independent of the revelation regime.*

4.2 Optimal policy choice for given revelation regime

For information revelation regimes $\rho = 0$ and $\rho = 1$ with perfect substitutes, we denote the optimal choice of η conditional on W_1 as $\eta^P(W_1, \rho)$ and the optimal choice of W_1 conditional on η as $W_1^P(\eta, \rho)$. For imperfect substitutes, we write $\eta^I(W_1, \rho)$ and $W_1^I(\eta, \rho)$.

4.2.1 Full revelation

We obtain the following results on the performance weight η .

Proposition 5 (i) $\eta^P(W_1, 1) > 0$ and $\eta^I(W_1, 1) > 0$ for all $W_1 < W$.
(ii) Suppose $K''' = 0$ and $W_1 = 0$. Then $\eta^P(W_1, 1) > 1$ if and only if

$$\int_0^\infty f_2(s) f_1(s) ds > 2 \left| \int_0^\infty s f_2'(s) f_1(s) ds \right| \quad (16)$$

Result (i), states that, no matter how high the share of period 1 of the overall prize sum is, there should still be a positive weight of past performance in the second-period contest. This result holds because, for $\eta = 0$, the marginal effect of η on first-period effort is bounded away from zero, while its marginal effect on second-period effort is zero. To understand the second-period effect of η , it is important to note why increasing η has an adverse effect on second-period efforts for $\eta > 0$. As η increases, any first-period outcome is effectively translated into a more asymmetric second-period contest, with weaker competition; the size of the effect is driven by the slope of the density at $\eta \Delta s_{i1} = \eta \Delta \varepsilon_{i1}$. As this slope tends to zero with η , so does the adverse effect of higher weight of the past. Result (ii) addresses the specific setting where there is only a second-period prize and asks which period should get higher weight in the contest. The trade-off is clear: A lower period one weight intensifies second-period competition, but reduces first-period incentives. (16) describes how this trade-off should be resolved. As before, the left-hand side captures increases in first-period incentives from a higher weight η , which are particularly high when close outcomes are likely. The right-hand side captures the reduction in second-period incentives arising from making the second-period contest less competitive.

Next, we ask under which circumstances there should only be one prize. The result will rely on the assumption that $K''' \leq 0$. This is not a serious restriction: Proposition 4 states that $K''' \leq 0$ is the case in which full revelation is optimal.

Proposition 6 Suppose $K''' \leq 0$.

(i) $W_1^P(\eta, 1) = 0$ ($W_1^P(\eta, 1) = W$) for all $\eta > 0$ if

$$f_1(0) < (>) \frac{1}{W} K' \left[(K')^{-1} \left(2\eta W \int_0^\infty f_2(\eta s) f_1(s) ds \right) + 2 \int_0^\infty (K')^{-1} (f_2(\eta s) W) f_1(s) ds \right]. \quad (17)$$

(ii) $W_1^I(\eta, 1) = 0$ for all $\eta > 0$ if

$$f_1(0) < 2\eta \int_0^\infty f_2(\eta s) \cdot f_1(s) ds. \quad (18)$$

Otherwise $W_1^I(\eta, 1) \in (0, W)$.

For perfect substitutes (i), it is never optimal to have two prizes. To understand the intuition for when the money is allocated to period 1 or 2, respectively, note that $W f_1(0)$ is the marginal benefit for the agent from exerting first-period effort when there is no second-period prize; hence, $f_1(0)$ is the increase in this marginal benefit that is induced by a marginal increase in the first-period prize. Thus, when the left-hand side of (17) is small and hence a close first-period outcome is unlikely in the symmetric equilibrium, higher first-period prizes do not improve first-period effort incentives much. The right-hand side of (17) captures increases in incentives from higher W_2 . Intuitively, a small increase in second-period effort is valuable for the agent when there is a high probability that the overall performance of the agents after two periods is similar, that is, when it is likely that one player has a lead of approximately s from period 1 and the other player has a lead of ηs in period 2. The right-hand side of (17) tends to be large when such close outcomes are likely.¹³

¹³More precisely, suppose $W = W_2$. Moreover, for simplicity, consider the benchmark case that $K''' = 0$, so that the right-hand side of (17) simplifies to

$$\frac{1}{W} K' \left[(K')^{-1} \left(2\eta W \int_0^\infty f_2(\eta s) f_1(s) ds \right) + (K')^{-1} \left(2W \int_0^\infty f_2(\eta s) f_1(s) ds \right) \right]$$

When there is no first-period prize, $2\eta \int_0^\infty f_2(\eta s) f_1(s) ds$ is the effect of a higher prize on the marginal payoff of the principal from higher first-period effort and $\int_0^\infty f_2(\eta s) f_1(s) ds$ is the effect of a higher prize on the expected marginal payoff from increasing second-period efforts. Thus increases in these marginal payoffs lead to an increase in the right-hand side.

For imperfect substitutes (ii), interior solutions are possible, reflecting the importance of an even intertemporal effort distribution. The conditions under which the prize budget is nevertheless concentrated on period 2 are similar to those for perfect substitutes: (18) requires that first-period error differences near zero are not too common, whereas error differences from period 1 and period 2 often cancel out approximately.

4.2.2 No revelation

In the revelation case, it is also true that first-period weights should be positive.

Proposition 7 $\eta^P(W_1, 0) > 0$ and $\eta^I(W_1, 0) > 0$ for all $W_1 < W$.

The intuition for the result is as in the case of full revelation. We now ask how the prize money is optimally spread across periods. Again, we restrict the third derivative of the cost function in such a way that the revelation regime is optimal by Proposition 4.

Proposition 8 Suppose $K''' \geq 0$.

(i) $W_1^P(\eta, 0) = 0$ if

$$\begin{aligned} & \frac{f_1(0) - 2\eta \int_0^\infty f_2(\eta s) f_1(s) ds}{K'' [(K')^{-1} (2\eta W \int_0^\infty f_2(\eta s) f_1(s) ds)]} \\ & - \frac{2 \int_0^\infty f_2(\eta s) f_1(s) ds}{K'' [(K')^{-1} (2W \int_0^\infty f_2(\eta s) f_1(s) ds)]} \leq 0. \end{aligned} \quad (19)$$

(ii) $W_1^P(\eta, 0) = W$ if

$$\frac{f_1(0) - 2\eta \int_0^\infty f_2(\eta s) f_1(s) ds}{K'' [(K')^{-1} (f_1(0) W)]} - \frac{2 \int_0^\infty f_2(\eta s) f_1(s) ds}{K''(0)} \geq 0. \quad (20)$$

(iii) If neither (19) nor (20) holds, $W_1^P \in [0, W]$.

Again, the reason for $W_1 > 0$ is analogous to Proposition 5. In particular, a sufficient condition for $W_1^P(\eta, 0) = 0$ is $f_1(0) - 2\eta \int_0^\infty f_2(\eta s) f_1(s) ds < 0$. Arguing as in the discussion of Proposition 5, the condition is fulfilled if the

density does not have too much weight near 0, and it is likely that a lead from period 1 is approximately cancelled out after period 2..

5 Example

We introduce a simple normal-quadratic example which allows us to obtain explicit solutions for the equilibrium efforts and the optimal policy in terms of primitives.

Example E1: *The cost function is $K(e_{it}) = \frac{k}{2}(e_{it})^2$. The error difference $\Delta\varepsilon_{it}$ is normally distributed with variance σ_t^2 . Furthermore,*

$$k \geq \frac{W}{\sigma_2^2 \sqrt{2\pi \exp(1)}} \quad (21)$$

$$k \geq \frac{W}{\sigma_1^2 \sqrt{2\pi \exp(1)}} + \frac{W^2}{\sigma_1^2 \sigma_2^2 (2\pi)^{\frac{3}{2}} \sqrt{\exp(1)}} \frac{1}{k} \quad (22)$$

Example E1 satisfies assumptions 1 and 2 directly, and satisfies the second-order conditions (3) and (7) (see the proof of Corollary 4).

Corollary 4 *In E1, a WPBE exists. The first-period equilibrium efforts for $\rho = 0$ and $\rho = 1$ are*

$$e_1^*(\eta, W_1, W_2) = \frac{W_1}{k\sigma_1\sqrt{2\pi}} + \frac{\eta W_2}{k\sqrt{2\pi}\sqrt{(\sigma_2)^2 + (\sigma_1)^2\eta^2}}. \quad (23)$$

The second period efforts for $\rho = 0$ and the expected second-period efforts for $\rho = 1$ are

$$e_2^*(\eta, W_2, 0) = E(e_2^*(\eta, W_2, 1)) = \frac{W_2}{k\sqrt{2\pi}\sqrt{(\sigma_2)^2 + (\sigma_1)^2\eta^2}}. \quad (24)$$

Comparative statics for second-period efforts are straightforward. Lower effort costs, higher second-period prize, lower first-period weight and higher second-period precision work in favor of high second-period efforts. Analogous results hold for period one. More interestingly, first-period efforts also

increase if the second-period precision increases, given $\eta > 0$: The parameter change makes first-period effort more worthwhile, because the positive effect on winning the second-period prize increases. The effect of prizes on first-period efforts is slightly more subtle. If the budget constraint ($W_1 + W_2$) is not taken into account, both prizes have positive effects on first-period efforts. However, the positive effect of an increase in the first-period prize is always stronger than the negative effect of an identical decrease in the second-period prize. As a result, the following result holds.

Corollary 5 *When the total budget is fixed at W , e_1^* is increasing in W_1 .*

Corollary 4 immediately leads to a simple characterization of the optimal first-period weight and the optimal first-period prize for the principal who regards efforts as perfect substitutes or as imperfect substitutes.

Corollary 6 *For the normal-quadratic case,*

- (i) $\eta^P(W_1, 0) = \eta^P(W_1, 1) = \frac{\sigma_2^2}{\sigma_1^2} \forall W_1 > 0$, $W^P\left(\frac{\sigma_2^2}{\sigma_1^2}, 0\right) = W^P\left(\frac{\sigma_2^2}{\sigma_1^2}, 1\right) = 0$.
- (ii) $\eta^I(0, 0) = \eta^I(0, 1) = \frac{\sigma_2}{\sigma_1}$, $W^I(\eta, 0) = W^I(\eta, 1) > 0 \forall \eta \geq 0$.

The corollary sharpens the general results of Section 4. It shows that there is only one prize, with the weight of period 1 determined entirely by the relative precision of the two performance measures.

6 Discussion and Related Literature

Previous literature has addressed some of the issues we deal with in this paper.

For instance, several papers have asked whether a principal should reveal interim information or not. Most closely related is Aoyagi (2010) whose framework is similar to ours. In particular, there is only one prize. Both periods always receive equal weights, so both handicapping and reverse handicapping are excluded by assumption. With these restrictions, he also obtains the result that third derivatives determine whether information should be revealed. Our contribution to the information revelation discussion can be seen as complementary to the one of Aoyagi: We show that the assumptions

of Aoyagi can be endogenized under certain conditions. However, we also provide clear conditions (in particular, for the normal quadratic case) under which a principal would optimally give different weights to the two periods. However, Aoyagi is also more general in some respects: He allows for partial information revelation where the agents obtain incomplete performance signals.¹⁴

Another strand of literature has dealt with the issue of whether handicapping or reverse handicapping of first-period winners should take place in settings with one prize, that is, whether the first-period weight should be lower or higher than the second-period weight. These papers usually focus on a setting where information is revealed. For instance, Meyer (1992) considers a setting similar to ours, but with risk-averse agents. She shows that reverse handicapping, that is, favoring the first-period winner, is optimal.

Gershkov and Perry (2009) address both the information revelation decision and the handicapping decision. However, they consider only the information who is ahead of the other one after one period.

7 Conclusion

This paper analyzes intertemporal effort provision in two-stage tournaments. A principal with a fixed budget for prizes faces two risk-neutral agents. He observes noisy signals of effort in both periods. He aims at maximizing either total efforts (perfect substitutes) or the product of first- and second-period efforts (imperfect substitutes). He decides (i) how to weigh performance in the two periods when awarding the second period prize, (ii) how to spread prize money across the two periods, and (iii) whether to reveal performance after the first period. Under very general conditions, the principal puts positive weight on both periods in period two. Furthermore, he sets no first-period prize provided the observations in period one are too noisy. The information revelation policy depends on the third derivative of the effort cost function.

¹⁴Ederer (2010) introduces incomplete information about ability.

References

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- [3] Gershkov, A. and Perry, M. (2009): Tournaments with midterm reviews. *Games and Economic Behavior*, 66, 162–190.
- [4] Meyer, M. (1992): Biased Contests and Moral Hazard: Implications for Career Profiles. *Annals of Economics and Statistics*, 165–187.

8 Appendix

8.1 Proof of Proposition 1

By assumption 1, $K(e_{i2})$ increases beyond bounds as e_{i2} goes to infinity. As the expected prize is bounded by W_2 , the set of undominated effort levels is contained in the compact interval $[0, K^{-1}(W_2)]$. As expected net payoffs are continuous in e_{i2} and concave by (3), a Nash equilibrium of the second-period effort game exists.

The first-order condition for agent i in period 2 is (2). Since the density is symmetric and

$$-(\eta\Delta s_{11} + \Delta e_{12}) = \eta\Delta s_{21} + e_{22} - e_{12},$$

the left-hand side of (2) is equal for both agents and hence $e_{i2}^*(\Delta s_{i1}; \eta, W_2, 1) = e_{j2}^*(\Delta s_{i1}; \eta, W_2, 1)$ for any solution. Thus, (2) becomes

$$f_2(\eta\Delta s_{i1}) W_2 = K'(e_{i2}).$$

The convexity of the cost function guarantees that K' is strictly increasing and thus invertible. As (3) is globally satisfied, (4) thus defines an equilibrium. The equilibrium must involve positive efforts, because $f_2 > 0$ and $K'(0) = 0$.

8.2 Proof of Corollary 1

The inverse function theorem yields

$$\left[(K')^{-1}\right]'(f_2(\eta\Delta s_{i1}) W_2) = \frac{1}{K''((K')^{-1}(f_2(\eta\Delta s_{i1}) W_2))},$$

so that (4) implies

$$\frac{\partial e_{i2}^*}{\partial \Delta s_{i1}} = \frac{\eta f_2'(\eta \Delta s_{i1}) W_2}{K''((K')^{-1}(f_2(\eta \Delta s_{i1}) W_2))}, \quad (25)$$

$$\frac{\partial e_{i2}^*}{\partial \eta} = \frac{\Delta s_{i1} f_2'(\eta \Delta s_{i1}) W_2}{K''((K')^{-1}(f_2(\eta \Delta s_{i1}) W_2))}, \quad (26)$$

$$\frac{\partial e_{i2}^*}{\partial W_2} = \frac{f_2(\eta \Delta s_{i1})}{K''((K')^{-1}(f_2(\eta \Delta s_{i1}) W_2))}. \quad (27)$$

From assumption 2, if $\Delta s_{i1} < (>)0 \wedge \eta \neq 0 \wedge W_2 > 0$, then $\eta f_2'(\eta \Delta s_{i1}) > (<)0$ and thus $\frac{\partial e_{i2}^*}{\partial \Delta s_{i1}} > (<)0$. This implies that e_{i2}^* is decreasing in $|\Delta s_{i1}|$. As $\Delta s_{i1} = \Delta e_{i1} + \Delta \epsilon_{i1}$ we obtain the results for e_{i1} and e_{j1} . Similar arguments show that $\frac{\partial e_{i2}^*}{\partial \eta} < 0$. Since $f_2 > 0$, we have $\frac{\partial e_{i2}^*}{\partial W_2} > 0$.

8.3 Proof of Proposition 2

We first derive an expression for $\frac{\partial U_{i2}^*}{\partial e_{i1}}$ for symmetric first-period efforts.

Lemma 2

$$\left. \frac{\partial U_{i2}^*}{\partial e_{i1}} \right|_{e_{i1}=e_{j1}} = 2\eta W_2 \int_0^\infty f_2(\eta s) f_1(s) ds \quad (28)$$

$$\begin{aligned} \frac{\partial^2 U_{i2}^*}{(\partial e_{i1})^2} &= \eta W_2 \int_0^\infty f_2(\eta s) [f_1'(s + \Delta e_{i1}) - f_1'(s - \Delta e_{i1})] ds \\ &+ \eta (W_2)^2 \int_0^\infty \frac{f_2(\eta s) f_2'(\eta s) [f_1'(s + \Delta e_{i1}) + f_1'(s - \Delta e_{i1})]}{K''((K')^{-1}(f_2(\eta s) W_2))} ds. \end{aligned} \quad (29)$$

Proof. For the payoff of agent i for a particular Δs_{i1} in period 2 we write

$$U_{i2}^s(\Delta s_{i1}; \eta, W_2) = U_{i2}(e_{i2}^*(\Delta s_{i1}; \eta, W_2), e_{j2}^*(-\Delta s_{i1}; \eta, W_2), \Delta s_{i1}; \eta, W_2).$$

Applying the envelope theorem, we obtain

$$\frac{\partial U_{i2}^s}{\partial \Delta s_{i1}} = \frac{\partial U_{i2}}{\partial e_{j2}} \frac{\partial e_{j2}^*(-\Delta s_{i1}; \eta, W_2)}{\partial \Delta s_{i1}} + \frac{\partial U_{i2}}{\partial \Delta s_{i1}}.$$

Using the symmetry of the density and (25),

$$\frac{\partial e_{j2}^* (-\Delta s_{i1}; \eta, W_2)}{\partial \Delta s_{i1}} = \frac{\partial e_{j2}^* (\Delta s_{i1}; \eta, W_2)}{\partial \Delta s_{i1}} = \frac{\eta f_2' (\eta \Delta s_{i1}) W_2}{K'' ((K')^{-1} (f_2 (\eta \Delta s_{i1}) W_2))}.$$

(1) implies

$$\begin{aligned} \frac{\partial U_{i2}}{\partial e_{j2}} &= -f_2 (\eta \Delta s_{i1} + \Delta e_{i2}) W_2, \\ \frac{\partial U_{i2}}{\partial \Delta s_{i1}} &= \eta f_2 (\eta \Delta s_{i1} + \Delta e_{i2}) W_2, \end{aligned}$$

Using $\Delta e_{i2}^* = 0$, we obtain

$$\frac{\partial U_{i2}^s}{\partial \Delta s_{i1}} = -\frac{\eta f_2 (\eta \Delta s_{i1}) f_2' (\eta \Delta s_{i1}) (W_2)^2}{K'' ((K')^{-1} (f_2 (\eta \Delta s_{i1}) W_2))} + \eta f_2 (\eta \Delta s_{i1}) W_2.$$

Let

$$U_{i2}^{e, \Delta \varepsilon_{i1}} (e_{i1}, e_{j1}; \eta, W_2) = U_{i2}^s (\Delta e_{i1} + \Delta \varepsilon_{i1}; \eta, W_2).$$

Taking into account that $\Delta s_{i1} = \Delta e_{i1} + \Delta \varepsilon_{i1}$, we obtain

$$\begin{aligned} \frac{\partial U_{i2}^{e, \Delta \varepsilon_{i1}}}{\partial e_{i1}} &= -\frac{\eta f_2 (\eta e_{i1} - e_{j1} + \Delta \varepsilon_{i1}) f_2' (\eta (\Delta e_{i1} + \Delta \varepsilon_{i1})) (W_2)^2}{K'' ((K')^{-1} (f_2 (\eta (\Delta e_{i1} + \Delta \varepsilon_{i1})) W_2))} \\ &\quad + \eta f_2 (\eta (\Delta e_{i1} + \Delta \varepsilon_{i1})) W_2. \end{aligned}$$

As $U_{i2}^* (e_{i1}, e_{j1}; \eta, W_2) = E \left(U_{i2}^{e, \Delta \varepsilon_{i1}} (\Delta s_{i1}; \eta, W_2) \right)$, integrating over $\Delta \varepsilon_{i1}$ and rearranging yields

$$\begin{aligned} \frac{\partial U_{i2}^*}{\partial e_{i1}} &= \int_{-\infty}^{\infty} \left[-\frac{\eta f_2 (\eta (\Delta e_{i1} + s)) f_2' (\eta (\Delta e_{i1} + s)) (W_2)^2}{K'' ((K')^{-1} (f_2 (\eta (\Delta e_{i1} + s)) W_2))} \right. \\ &\quad \left. + \eta f_2 (\eta (\Delta e_{i1} + s)) W_2 \right] f_1 (s) ds \\ &= \eta W_2 \int_{-\infty}^{\infty} f_2 (\eta (\Delta e_{i1} + s)) f_1 (s) ds \\ &\quad - \eta (W_2)^2 \int_{-\infty}^{\infty} \frac{f_2 (\eta (\Delta e_{i1} + s)) f_2' (\eta (\Delta e_{i1} + s))}{K'' ((K')^{-1} (f_2 (\eta (\Delta e_{i1} + s)) W_2))} f_1 (s) ds. \end{aligned}$$

Let

$$\begin{aligned} A &: = \int_{-\infty}^{\infty} f_2(\eta(\Delta e_{i1} + s)) f_1(s) ds, \\ B &: = \int_{-\infty}^{\infty} \frac{f_2(\eta(\Delta e_{i1} + s)) f_2'(\eta(\Delta e_{i1} + s))}{K''((K')^{-1}(f_2(\eta(\Delta e_{i1} + s)) W_2))} f_1(s) ds. \end{aligned}$$

With this notation, $\frac{\partial U_{i2}^*}{\partial e_{i1}} = \eta W_2 A - \eta (W_2)^2 B$. Using the substitution $t = \Delta e_{i1} + s$ in A , we obtain

$$\begin{aligned} A &= \int_{-\infty}^{\infty} f_2(\eta t) f_1(t - \Delta e_{i1}) dt \\ &= \int_{-\infty}^0 f_2(\eta t) f_1(t - \Delta e_{i1}) dt + \int_0^{\infty} f_2(\eta t) f_1(t - \Delta e_{i1}) dt. \end{aligned}$$

For $u = -t$, the symmetry of the density implies $f_2(\eta t) = f_2(\eta u)$ and $f_1(t - \Delta e_{i1}) = f_1(u + \Delta e_{i1})$. Hence,

$$\int_{-\infty}^0 f_2(\eta t) f_1(t - \Delta e_{i1}) dt = \int_0^{\infty} f_2(\eta u) f_1(u + \Delta e_{i1}) du.$$

Thus,

$$\begin{aligned} A &= \int_0^{\infty} f_2(\eta u) f_1(u + \Delta e_{i1}) du + \int_0^{\infty} f_2(\eta t) f_1(t - \Delta e_{i1}) dt \\ &= \int_0^{\infty} f_2(\eta t) [f_1(t + \Delta e_{i1}) + f_1(t - \Delta e_{i1})] dt. \end{aligned}$$

Similarly, using the substitution $t = \Delta e_{i1} + s$ in B , we obtain

$$\begin{aligned} B &= \int_{-\infty}^{\infty} \frac{f_2(\eta t) f_2'(\eta t) f_1(t - \Delta e_{i1})}{K''((K')^{-1}(f_2(\eta t) W_2))} dt \\ &= \int_{-\infty}^0 \frac{f_2(\eta t) f_2'(\eta t) f_1(t - \Delta e_{i1})}{K''((K')^{-1}(f_2(\eta t) W_2))} dt + \int_0^{\infty} \frac{f_2(\eta t) f_2'(\eta t) f_1(t - \Delta e_{i1})}{K''((K')^{-1}(f_2(\eta t) W_2))} dt. \end{aligned}$$

For $u = -t$, the symmetry of the density guarantees that $f_2(\eta t) = f_2(\eta u)$, $f'_2(\eta t) = -f'_2(\eta u)$ and $f_1(t - \Delta e_{i1}) = f_1(u + \Delta e_{i1})$, and we obtain

$$\int_{-\infty}^0 \frac{f_2(\eta t) f'_2(\eta t) f_1(t - \Delta e_{i1})}{K''((K')^{-1}(f_2(\eta t) W_2))} dt = \int_0^{\infty} \frac{f_2(\eta u) (-f'_2(\eta u)) f_1(u + \Delta e_{i1})}{K''((K')^{-1}(f_2(\eta u) W_2))} du.$$

Thus,

$$\begin{aligned} B &= \int_0^{\infty} \frac{-f_2(\eta u) f'_2(\eta u) f_1(u + \Delta e_{i1})}{K''((K')^{-1}(f_2(\eta u) W_2))} du \\ &\quad + \int_0^{\infty} \frac{f_2(\eta t) f'_2(\eta t) f_1(t - \Delta e_{i1})}{K''((K')^{-1}(f_2(\eta t) W_2))} dt \\ &= \int_0^{\infty} \frac{f_2(\eta t) f'_2(\eta t) [-f_1(t + \Delta e_{i1}) + f_1(t - \Delta e_{i1})]}{K''((K')^{-1}(f_2(\eta t) W_2))} dt. \end{aligned}$$

With the transformed expressions for A and B , and substituting back $s = t$, we obtain

$$\frac{\partial U_{i2}^*}{\partial e_{i1}} = \eta W_2 \int_0^{\infty} f_2(\eta s) [f_1(s + \Delta e_{i1}) + f_1(s - \Delta e_{i1})] ds \quad (30)$$

$$\begin{aligned} &+ \eta (W_2)^2 \int_0^{\infty} \frac{f_2(\eta s) f'_2(\eta s)}{K''((K')^{-1}(f_2(\eta s) W_2))} \cdot \\ &[f_1(s + \Delta e_{i1}) - f_1(s - \Delta e_{i1})] ds. \end{aligned} \quad (31)$$

With $\Delta e_{i1} = 0$, we obtain (28). (29) follows directly from (30). ■

Using Lemma 2, we now prove Proposition 2. The second-order condition is

$$f'_1(\Delta e_{i1}) W_1 + \frac{\partial^2 U_{i2}^*(e_{i1}, e_{j1}; \eta, W_2)}{(\partial e_{i1})^2} \leq K''(e_{i1}) \forall e_{i1}, e_{j1} \in \mathbb{R}^+$$

Using (29), we obtain (7).

Note that by assumption 1, $K(e_{i1})$ increases beyond bounds as e_{i1} goes to infinity. As the expected prize is bounded by $W_1 + W_2$, the set of undominated effort levels is contained in the compact interval $[0, K^{-1}(W_1 + W_2)]$. As expected net payoffs are continuous in e_{i1} and concave by (7), an equilibrium of the WPBE thus exists.

For $\Delta e_{i1} = 0$, we can apply (28) in (6) and obtain

$$f_1(0) W_1 + 2\eta W_2 \int_0^\infty f_2(\eta s) f_1(s) ds = K'(e_1).$$

Because K' is invertible, we obtain (9).

8.4 Proof of Corollary 2

Symmetry of the equilibrium implies $\Delta s_{i1} = \Delta \varepsilon_{i1}$. Hence, $e_{i2}^*(\Delta s_{i1}; \eta, W_2, 1) = (K')^{-1}(f_2(\eta \Delta \varepsilon_{i1}) W_2)$ from (4). Taking the expectation over $\Delta \varepsilon_{i1}$, we obtain

$$E(e_{i2}^*(\eta, W_2, 1)) = \int_{-\infty}^\infty (K')^{-1}(f_2(\eta s) W_2) f_1(s) ds.$$

From the symmetry of the density, we get (10).

8.5 Proof of Proposition 3

The first-order conditions are

$$f_1(\Delta e_{i1}) W_1 \tag{32}$$

$$+ \eta W_2 \int_{-\infty}^\infty f_2(\eta(\Delta e_{i1} + s) + \Delta e_{i2}) f_1(s) ds = K'(e_{i1});$$

$$W_2 \int_{-\infty}^\infty f_2(\eta(\Delta e_{i1} + s) + \Delta e_{i2}) f_1(s) ds = K'(e_{i2}). \tag{33}$$

Second-order conditions are

$$\begin{aligned}
& f_1'(\Delta e_{i1}) W_1 + \eta^2 W_2 \int_{-\infty}^{\infty} f_2'(\eta(\Delta e_{i1} + s) + \Delta e_{i2}) f_1(s) ds \quad (34) \\
\leq & K''(e_{i1}) \forall e_{i1}, e_{i2}, e_{j1}, e_{j2} \in \mathbb{R}^+ \\
& W_2 \int_{-\infty}^{\infty} f_2'(\eta(\Delta e_{i1} + s) + \Delta e_{i2}) f_1(s) ds \quad (35) \\
\leq & K''(e_{i2}) \forall e_{i1}, e_{i2}, e_{j1}, e_{j2} \in \mathbb{R}^+ \\
& K''(e_{i1}) W_2 \int_{-\infty}^{\infty} f_2'(\eta(\Delta e_{i1} + s) + \Delta e_{i2}) f_1(s) ds \\
& + K''(e_{i2}) \cdot \quad (36) \\
& \left[f_1'(\Delta e_{i1}) W_1 + \eta^2 W_2 \int_{-\infty}^{\infty} f_2'(\eta(\Delta e_{i1} + s) + \Delta e_{i2}) f_1(s) ds \right] \\
& - f_1'(\Delta e_{i1}) W_1 W_2 \int_{-\infty}^{\infty} f_2'(\eta(\Delta e_{i1} + s) + \Delta e_{i2}) f_1(s) ds \\
\leq & K''(e_{i1}) K''(e_{i2}) \forall e_{i1}, e_{i2}, e_{j1}, e_{j2} \in \mathbb{R}^+
\end{aligned}$$

We now show that they hold for given policy parameters and distributions as long as K is sufficiently convex. In case of (34) and (35), this is obvious, as their right-hand sides are increasing in $K''(\cdot)$. To see that it also holds for (36), let

$$\begin{aligned}
A & \equiv W_2 \int_{-\infty}^{\infty} f_2'(\eta(\Delta e_{i1} + s) + \Delta e_{i2}) f_1(s) ds \\
B & \equiv f_1'(\Delta e_{i1}) W + \eta^2 W_2 \int_{-\infty}^{\infty} f_2'(\eta(\Delta e_{i1} + s) + \Delta e_{i2}) f_1(s) ds \\
C & \equiv -f_1'(\Delta e_{i1}) W_1 W_2 \int_{-\infty}^{\infty} f_2'(\eta(\Delta e_{i1} + s) + \Delta e_{i2}) f_1(s) ds
\end{aligned}$$

With this notation, (36) can be written as

$$K''(e_{i1}) \cdot A + K''(e_{i2}) \cdot B + C \leq K''(e_{i1}) K''(e_{i2}) \quad (37)$$

Now suppose that (36) does not hold for some \tilde{K} . Let

$$D \equiv \tilde{K}''(e_{i1}) \cdot A + \tilde{K}''(e_{i2}) \cdot B + C - \tilde{K}''(e_{i1}) \tilde{K}''(e_{i2}) \quad (38)$$

and $\widehat{K}(e) = \widetilde{K}(e) + \frac{k}{2}e^2$, so that

$$\widehat{K}''(e) = \widetilde{K}''(e_{i2}) + k. \quad (39)$$

\widehat{K} is a cost function which is made more convex than \widetilde{K} by adding a quadratic term. (39) allows to write (37) for \widehat{K} as

$$\begin{aligned} & \widetilde{K}''(e_{i1}) \cdot A + \widetilde{K}''(e_{i2}) \cdot B + C + k(A + B) \leq \\ & \widetilde{K}''(e_{i1}) \widetilde{K}''(e_{i2}) + k\widetilde{K}''(e_{i1}) + k\widetilde{K}''(e_{i2}) + k^2 \end{aligned}$$

Using (38), we obtain

$$\widetilde{K}''(e_{i1}) \widetilde{K}''(e_{i2}) + D + k(A + B) \leq \widetilde{K}''(e_{i1}) \widetilde{K}''(e_{i2}) + k\widetilde{K}''(e_{i1}) + k\widetilde{K}''(e_{i2}) + k^2$$

or, cancelling out and rearranging,

$$D \leq k \left(\widetilde{K}''(e_{i1}) + \widetilde{K}''(e_{i1}) - A - B \right) + k^2 \quad (40)$$

For all A and B , the right-hand side of (40) can be made arbitrarily high by increasing k , so that (40) and thus (36) eventually hold. Consequently, it is always possible to fulfill (36) by making K sufficiently convex.

We next show that in equilibrium the SOC are automatically implied by the convexity of cost functions. Using the symmetry of the density and

taking into account that $f_1'(0) = 0$, we can write (34), (35) and (36) as

$$\eta^2 W_2 \int_{-\infty}^{\infty} f_2'(\eta s) f_1(s) ds \leq \quad (41)$$

$$K''(e_1) \forall e_{i1}, e_{i2}, e_{j1}, e_{j2} \in \mathbb{R}^+ \\ W_2 \int_{-\infty}^{\infty} f_2'(\eta s) f_1(s) ds \leq \quad (42)$$

$$K''(e_2) \forall e_{i1}, e_{i2}, e_{j1}, e_{j2} \in \mathbb{R}^+ \\ \frac{1}{K''(e_2)} W_2 \int_{-\infty}^{\infty} f_2'(\eta s) f_1(s) ds \\ + \frac{1}{K''(e_1)} \eta^2 W_2 \int_{-\infty}^{\infty} f_2'(\eta s) f_1(s) ds \leq \quad (43) \\ 1 \forall e_{i1}, e_{i2}, e_{j1}, e_{j2} \in \mathbb{R}^+$$

Because $f_1(s) = f_1(-s)$ and $f_2'(\eta s) = -f_2'(\eta s)$, we have $\int_{-\infty}^{\infty} f_2'(\eta s) \cdot f_1(s) ds = 0$, so that the left-hand sides of (41), (42) and (43) are all 0 and the inequalities hold automatically.

We now show existence of a NE. Note that by assumption 1, $K(e_{i1})$ and $K(e_{i2})$ increase beyond bounds as e_{i1} and e_{i2} go to infinity. As the expected prize is bounded by $W_1 + W_2$, the set of undominated effort levels is contained in the compact set $[0, K^{-1}(W_1 + W_2)]^2$. As expected net payoffs are continuous in e_{i1} and e_{i2} and concave by (34), (35) and (36), a NE thus exists.

Finally, for the symmetric case $\Delta e_{i1} = \Delta e_{i2} = 0$, (32) and (33) yield

$$f_1(0) W_1 + 2\eta W_2 \int_0^{\infty} f_2(\eta s) f_1(s) ds = K'(e_1); \\ 2W_2 \int_0^{\infty} f_2(\eta s) f_1(s) ds = K'(e_2).$$

Inverting K' yields (12) and (13).

8.6 Proof of Proposition 4

From (13) and (10),

$$\begin{aligned} & e_2^*(\eta, W - W_1, 0) - E(e_2^*(\eta, W - W_1, 1)) = \\ & (K')^{-1} \left(2W_2 \int_0^\infty f_2(\eta s) f_1(s) ds \right) - 2 \int_0^\infty K'^{-1}(f_2(\eta s) W_2) f_1(s) ds \end{aligned}$$

Because of the symmetry of the density, the right-hand side can be written as

$$(K')^{-1} \left(W_2 \int_{-\infty}^\infty f_2(\eta s) f_1(s) ds \right) - \int_{-\infty}^\infty K'^{-1}(f_2(\eta s) W_2) f_1(s) ds$$

Substituting $g(s) \equiv W_2 f_2(\eta s)$ we obtain

$$\begin{aligned} & e_2^*(\eta, W - W_1, 0) - E(e_2^*(\eta, W - W_1, 1)) = \\ & K'^{-1} \left(\int_{-\infty}^\infty g(s) f_1(s) ds \right) - \int_{-\infty}^\infty K'^{-1}(g(s)) f_1(s) ds. \end{aligned}$$

According to Jensen's inequality, this difference is weakly negative (weakly positive) if K'^{-1} is convex (concave), which is the case if and only if K' is concave (convex), which, in turn, is the case if and only if $K''' \leq 0$ ($K''' \geq 0$).

8.7 Proof of Proposition 5

(i) From (9) and (10),

$$\begin{aligned} V^P(\eta, W_1, 1) &= (K')^{-1} \left(f_1(0) W_1 + 2\eta (W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds \right) \\ &\quad + 2 \int_0^\infty (K')^{-1}(f_2(\eta s) (W - W_1)) f_1(s) ds. \end{aligned}$$

We first show that $\eta < 0$ never maximizes $V^P(\eta, W_1, 1)$. Suppose $\eta_1 > 0$ and $\eta_2 = -\eta_1$. Since $f_2(\eta_2 s) = f_2(\eta_1 s)$, we have

$$\begin{aligned} & V^P(\eta_1, W_1, 1) - V^P(\eta_2, W_1, 1) = \\ & (K')^{-1} \left(f_1(0) W_1 + 2\eta_1 (W - W_1) \int_0^\infty f_2(\eta_1 s) f_1(s) ds \right) \\ & - (K')^{-1} \left(f_1(0) W_1 - 2\eta_1 (W - W_1) \int_0^\infty f_2(\eta_1 s) f_1(s) ds \right) \end{aligned} \quad (44)$$

As $(K')^{-1}$ is strictly increasing from $K'' > 0$, the second term in (44) is strictly smaller than the first term. Thus, for every $\eta < 0$, $-\eta > 0$ yields strictly higher expected payoffs, so that all $\eta < 0$ are strictly dominated. To see that the optimal $\eta > 0$, note that

$$\begin{aligned} & \frac{\partial V^P(\eta, W_1, 1)}{\partial \eta} = \\ & \frac{2(W - W_1) \left[\int_0^\infty f_2(\eta s) f_1(s) ds + \eta \int_0^\infty s f_2'(\eta s) f_1(s) ds \right]}{K'' \left[(K')^{-1} (f_1(0) W_1 + 2\eta (W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds) \right]} \\ & + 2 \int_0^\infty \frac{s f_2'(\eta s) (W - W_1)}{K'' \left[(K')^{-1} (f_2(\eta s) (W - W_1)) \right]} f_1(s) ds \end{aligned} \quad (45)$$

Hence,

$$\begin{aligned} \left. \frac{\partial V^P(\eta, W_1, 1)}{\partial \eta} \right|_{\eta=0} &= \frac{2(W - W_1) f_2(0) \int_0^\infty f_1(s) ds}{K'' \left[(K')^{-1} f_1(0) W_1 \right]} \\ &+ 2 \int_0^\infty \frac{s f_2'(0) (W - W_1) f_1(s)}{K'' \left[(K')^{-1} (f_2(0) (W - W_1)) \right]} ds \\ &= \frac{(W - W_1) f_2(0)}{K'' \left[(K')^{-1} f_1(0) W_1 \right]}, \end{aligned}$$

where the second equality follows from $f_2'(0) = 0$ and $\int_0^\infty f_1(s) ds = \frac{1}{2}$. As $K'' > 0$, $\left. \frac{\partial V^S(\eta, W_1, 1)}{\partial \eta} \right|_{\eta=0} > 0$ provided $W_1 < W$. Hence, $\eta > 0$ is always optimal for $W_1 < W$.

From (9) and (10),

$$V^I(\eta, W_1, 1) = K'^{-1} \left(f_1(0) W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds \right) \cdot 2 \int_0^\infty K'^{-1}(f_2(\eta s)(W - W_1)) f_1(s) ds$$

We first show that $\eta < 0$ never maximizes $V^I(\eta, W_1, 1)$. Suppose $\eta_1 > 0$ and $\eta_2 = -\eta_1$. Since $f_2(\eta_2 s) = f_2(\eta_1 s)$, we have

$$\begin{aligned} V^I(\eta_1, W_1, 1) - V^I(\eta_2, W_1, 1) = & \quad (46) \\ & \left[K'^{-1} \left(f_1(0) W_1 + 2\eta_1(W - W_1) \int_0^\infty f_2(\eta_1 s) \cdot f_1(s) ds \right) \right. \\ & \left. - K'^{-1} \left(f_1(0) W_1 - 2\eta_1(W - W_1) \int_0^\infty f_2(\eta_1 s) \cdot f_1(s) ds \right) \right] \cdot \\ & 2 \int_0^\infty K'^{-1}(f_2(\eta_1 s)(W - W_1)) f_1(s) ds \end{aligned}$$

As $(K')^{-1}$ is strictly increasing from $K'' > 0$, the second term in the first bracket in (46) is strictly smaller than the first term. Thus, for every $\eta < 0$, $-\eta > 0$ yields strictly higher expected payoffs, so that all $\eta < 0$ are strictly dominated. To see that the optimal $\eta > 0$, note that

$$\begin{aligned} \frac{\partial V^I(\eta, W_1, 1)}{\partial \eta} = & \frac{2(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds + 2\eta(W - W_1) \int_0^\infty s \cdot f_2'(\eta s) \cdot f_1(s) ds}{K'' \left[K'^{-1} \left(f_1(0) W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds \right) \right]} \\ & 2 \int_0^\infty K'^{-1}(f_2(\eta s)(W - W_1)) f_1(s) ds + \\ & K'^{-1} \left(f_1(0) W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds \right) \cdot \\ & 2 \int_0^\infty \frac{s f_2'(\eta s)(W - W_1)}{K'' [K'^{-1}(f_2(\eta s)(W - W_1))]} f_1(s) ds \end{aligned}$$

Hence,

$$\begin{aligned}
& \left. \frac{\partial V^I(\eta, W_1, 1)}{\partial \eta} \right|_{\eta=0} = \\
& \frac{2(W - W_1) \int_0^\infty f_2(0) \cdot f_1(s) ds}{K'' [K'^{-1}(f_1(0) W_1)]} \cdot 2 \int_0^\infty K'^{-1}(f_2(0)(W - W_1)) f_1(s) ds + \\
& K'^{-1}(f_1(0) W_1) \cdot 2 \int_0^\infty \frac{s f_2'(0)(W - W_1)}{K'' [K'^{-1}(f_2(0)(W - W_1))]} f_1(s) ds = \\
& \frac{(W - W_1) f_2(0)}{K'' [K'^{-1}(f_1(0) W_1)]} \cdot 2 \int_0^\infty K'^{-1}(f_2(0)(W - W_1)) f_1(s) ds,
\end{aligned}$$

where the second equation follows from $f_2'(0) = 0$ and $\int_0^\infty f_1(s) ds = \frac{1}{2}$. As $K'' > 0$, $\left. \frac{\partial V^I(\eta, W_1, 1)}{\partial \eta} \right|_{\eta=0} > 0$ provided $W_1 < W$. Hence, $\eta > 0$ is always optimal for $W_1 < W$.

(ii) To understand the condition for $\eta = 1$, note that (45) implies that $\frac{\partial V^P(\eta, W_1, 1)}{\partial \eta} > 0$ if and only if

$$\int_0^\infty f_2(\eta s) f_1(s) ds + (\eta + 1) \int_0^\infty s f_2'(\eta s) f_1(s) ds > 0.$$

Inserting $\eta = 1$ gives the result.

8.8 Proof of Proposition 6

(i) We have

$$\begin{aligned}
& \frac{\partial V^P(\eta, W_1, 1)}{\partial W_1} = \\
& \frac{f_1(0) - 2\eta \int_0^\infty f_2(\eta s) f_1(s) ds}{K'' [(K')^{-1}(f_1(0) W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds)]} \\
& - 2 \int_0^\infty \frac{f_2(\eta s)}{K'' [(K')^{-1}(f_2(\eta s)(W - W_1))]} f_1(s) ds
\end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 V^P(\eta, W_1, 1)}{(\partial W_1)^2} &= - \frac{(f_1(0) - 2\eta \int_0^\infty f_2(\eta s) f_1(s) ds)^2}{(K''[(K')^{-1}(f_1(0)W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds)])^3}. \quad (47) \\ K''' &\left[(K')^{-1} \left(f_1(0) W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds \right) \right] \\ &- 2 \int_0^\infty \frac{(f_2(\eta s))^2}{(K''[(K')^{-1}(f_2(\eta s)(W - W_1))])^3} \cdot \\ &K''' \left[(K')^{-1}(f_2(\eta s)(W - W_1)) \right] f_1(s) ds \end{aligned}$$

$K''' \leq 0$ implies $\frac{\partial^2 V^P(\eta, W_1, 1)}{(\partial W_1)^2} > 0$. Thus, there is no interior optimum. For $W_1 = 0$ and $W_1 = W$, the principal's expected payoffs are

$$\begin{aligned} V^P(\eta, 0, 1) &= (K')^{-1} \left(2\eta W \int_0^\infty f_2(\eta s) f_1(s) ds \right) \\ &+ 2 \int_0^\infty (K')^{-1}(f_2(\eta s) W) f_1(s) ds. \\ V^P(\eta, W, 1) &= (K')^{-1}(f_1(0) W). \end{aligned}$$

Therefore

$$\begin{aligned} V^P(\eta, 0, 1) - V^P(\eta, W, 1) &= \\ &(K')^{-1} \left(2\eta W \int_0^\infty f_2(\eta s) f_1(s) ds \right) \\ &+ 2 \int_0^\infty (K')^{-1}(f_2(\eta s) W) f_1(s) ds - (K')^{-1}(f_1(0) W) \end{aligned}$$

Hence, $V^P(\eta, 0, 1) - V^P(\eta, W, 1) > 0$ if and only if (17) holds.

(ii) We have

$$\begin{aligned}
& \frac{\partial V^I(\eta, W_1, 1)}{\partial W_1} = \\
& \frac{f_1(0) - 2\eta \int_0^\infty f_2(\eta s) \cdot f_1(s) ds}{K'' [K'^{-1}(f_1(0)W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds)]} \cdot \\
& \quad 2 \int_0^\infty K'^{-1}(f_2(\eta s)(W - W_1)) f_1(s) ds \\
& - K'^{-1} \left(f_1(0)W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds \right) \cdot \\
& \quad 2 \int_0^\infty \frac{f_2(\eta s)}{K'' [K'^{-1}(f_2(\eta s)(W - W_1))]} f_1(s) ds
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\partial^2 V^I(\eta, W_1, 1)}{(\partial W_1)^2} = \\
& - \frac{(f_1(0) - 2\eta \int_0^\infty f_2(\eta s) \cdot f_1(s) ds)^2}{(K'' [K'^{-1}(f_1(0)W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds)])^3} \cdot \\
& K''' \left[K'^{-1} \left(f_1(0)W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds \right) \right] \cdot \\
& \quad 2 \int_0^\infty K'^{-1}(f_2(\eta s)(W - W_1)) f_1(s) ds \\
& \quad - 2 \int_0^\infty \frac{f_2(\eta s)}{K'' [K'^{-1}(f_2(\eta s)(W - W_1))]} f_1(s) ds \cdot \\
& \quad \frac{f_1(0) - 2\eta \int_0^\infty f_2(\eta s) \cdot f_1(s) ds}{K'' [K'^{-1}(f_1(0)W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds)]} \\
& - \frac{f_1(0) - 2\eta \int_0^\infty f_2(\eta s) \cdot f_1(s) ds}{K'' [K'^{-1}(f_1(0)W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds)]} \cdot \\
& \quad 2 \int_0^\infty \frac{f_2(\eta s)}{K'' [K'^{-1}(f_2(\eta s)(W - W_1))]} f_1(s) ds \\
& \quad - K'^{-1} \left(f_1(0)W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds \right) \cdot \\
& \quad 2 \int_0^\infty \frac{(f_2(\eta s))^2}{(K'' [(K')^{-1}(f_2(\eta s)(W - W_1))])^3} \cdot \\
& \quad K''' [(K')^{-1}(f_2(\eta s)(W - W_1))] f_1(s) ds
\end{aligned}$$

Collecting terms, we obtain

$$\begin{aligned}
& \frac{\partial^2 V^I(\eta, W_1, 1)}{(\partial W_1)^2} = \\
& - \frac{(f_1(0) - 2\eta \int_0^\infty f_2(\eta s) \cdot f_1(s) ds)^2}{(K'' [K'^{-1}(f_1(0)W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds)])^3} \cdot \\
& K''' \left[K'^{-1} \left(f_1(0)W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds \right) \right] \cdot \\
& \quad 2 \int_0^\infty K'^{-1}(f_2(\eta s)(W - W_1)) f_1(s) ds \\
& \quad - 4 \int_0^\infty \frac{f_2(\eta s)}{K'' [K'^{-1}(f_2(\eta s)(W - W_1))]} f_1(s) ds \cdot \\
& \quad \frac{f_1(0) - 2\eta \int_0^\infty f_2(\eta s) \cdot f_1(s) ds}{K'' [K'^{-1}(f_1(0)W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds)]} \\
& \quad - K'^{-1} \left(f_1(0)W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds \right) \cdot \\
& \quad 2 \int_0^\infty \frac{(f_2(\eta s))^2}{(K'' [(K')^{-1}(f_2(\eta s)(W - W_1))])^3} \cdot \\
& \quad K''' [(K')^{-1}(f_2(\eta s)(W - W_1))] f_1(s) ds
\end{aligned}$$

$K''' \leq 0$ implies $\frac{\partial^2 V^I(\eta, W_1, 1)}{(\partial W_1)^2} > 0$ if $f_1(0) - 2\eta \int_0^\infty f_2(\eta t) \cdot f_1(t) dt < 0$. Thus, there is no interior optimum. Since $V^I(\eta, W, 1) = 0$, $W_1 = W$ is never optimal. Thus, the principal always chooses $W_1 = 0$. For $f_1(0) - 2\eta \int_0^\infty f_2(\eta s) \cdot f_1(s) ds > 0$, we know that $W_1 \in [0, W)$.

8.9 Proof of Proposition 7

From (12) and (13),

$$\begin{aligned}
V^P(\eta, W_1, 0) & \equiv (K')^{-1} \left(f_1(0)W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds \right) \\
& \quad + (K')^{-1} \left(2(W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds \right)
\end{aligned}$$

We first show that $\eta < 0$ never maximizes $V^P(\eta, W_1, 0)$. Suppose $\eta_1 > 0$ and $\eta_2 = -\eta_1$. Since $f_2(\eta_2 s) = f_2(\eta_1 s)$, we have

$$\begin{aligned} V^P(\eta_1, W_1, 0) - V^P(\eta_2, W_1, 0) = \\ (K')^{-1} \left(f_1(0) W_1 + 2\eta_1 (W - W_1) \int_0^\infty f_2(\eta_1 s) f_1(s) ds \right) \\ - (K')^{-1} \left(f_1(0) W_1 - 2\eta_1 (W - W_1) \int_0^\infty f_2(\eta_1 s) f_1(s) ds \right) \end{aligned} \quad (48)$$

As $(K')^{-1}$ is strictly increasing from $K'' > 0$, the second term in (48) is strictly smaller than the first term. Consequently, for every $\eta < 0$, $-\eta > 0$ yields strictly higher expected payoffs, so that all $\eta < 0$ are strictly dominated. To see that the optimal $\eta > 0$, note that

$$\begin{aligned} \frac{\partial V^P(\eta, W_1, 0)}{\partial \eta} = & \frac{2(W - W_1) \left[\int_0^\infty f_2(\eta s) f_1(s) ds + \eta \int_0^\infty s f_2'(\eta s) f_1(s) ds \right]}{K'' \left[(K')^{-1} (f_1(0) W_1 + 2\eta (W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds) \right]} \\ & + \frac{2(W - W_1) \int_0^\infty s f_2'(\eta s) f_1(s) ds}{K'' \left[(K')^{-1} (2(W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds) \right]} \end{aligned}$$

We have

$$\begin{aligned} \left. \frac{\partial V^P(\eta, W_1, 0)}{\partial \eta} \right|_{\eta=0} = & \frac{2(W - W_1) f_2(0) \int_0^\infty f_1(s) ds}{K'' \left[(K')^{-1} f_1(0) W_1 \right]} + \\ & \frac{2(W - W_1) \int_0^\infty s f_2'(0) f_1(s) ds}{K'' \left[(K')^{-1} (2(W - W_1) \int_0^\infty f_2(0) f_1(s) ds) \right]} = \\ & \frac{(W - W_1) f_2(0)}{K'' \left[(K')^{-1} f_1(0) W_1 \right]}, \end{aligned}$$

where the second equation follows from $f_2'(0) = 0$ and $\int_0^\infty f_1(s) ds = \frac{1}{2}$. As $K'' > 0$, $\left. \frac{\partial V^P(\eta, W_1, 0)}{\partial \eta} \right|_{\eta=0} > 0$ provided $W_1 < W$. Hence, $\eta > 0$ is always optimal for $W_1 < W$.

From (12) and (13),

$$V^I(\eta, W_1, 0) = K'^{-1} \left(f_1(0) W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds \right) \cdot (K')^{-1} \left[2(W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds \right]$$

We first show that $\eta < 0$ never maximizes $V^I(\eta, W_1, 0)$. Suppose $\eta_1 > 0$ and $\eta_2 = -\eta_1$. Since $f_2(\eta_2 s) = f_2(\eta_1 s)$, we have

$$\begin{aligned} V^I(\eta_1, W_1, 0) - V^I(\eta_2, W_1, 0) = & \quad (49) \\ & \left[K'^{-1} \left(f_1(0) W_1 + 2\eta_1(W - W_1) \int_0^\infty f_2(\eta_1 s) \cdot f_1(s) ds \right) \right. \\ & \left. - K'^{-1} \left(f_1(0) W_1 - 2\eta_1(W - W_1) \int_0^\infty f_2(\eta_1 s) \cdot f_1(s) ds \right) \right] \cdot \\ & (K')^{-1} \left[2(W - W_1) \int_0^\infty f_2(\eta_1 s) f_1(s) ds \right] \end{aligned}$$

As $(K')^{-1}$ is strictly increasing by $K'' > 0$, the second term in the first bracket in (49) is strictly smaller than the first term. Consequently, for every $\eta < 0$, $-\eta > 0$ yields strictly higher expected payoffs, so that all $\eta < 0$ are strictly dominated. To see that the optimal $\eta > 0$, note that

$$\begin{aligned} \frac{\partial V^I(\eta, W_1, 0)}{\partial \eta} = & \frac{2(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds + 2\eta(W - W_1) \int_0^\infty s \cdot f_2'(\eta s) \cdot f_1(s) ds}{K'' \left[K'^{-1} \left(f_1(0) W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds \right) \right]} \cdot \\ & (K')^{-1} \left[2(W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds \right] + \\ & K'^{-1} \left(f_1(0) W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) \cdot f_1(s) ds \right) \cdot \\ & \frac{2(W - W_1) \int_0^\infty s f_2'(\eta s) f_1(s) ds}{K'' \left[(K')^{-1} \left(2(W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds \right) \right]}. \end{aligned}$$

We have

$$\begin{aligned} \frac{\partial V^I(\eta, W_1, 0)}{\partial \eta} \Big|_{\eta=0} &= \frac{2(W - W_1) \int_0^\infty f_2(0) \cdot f_1(s) ds}{K'' [K'^{-1}(f_1(0) W_1)]} \\ &\quad (K')^{-1} \left[2(W - W_1) \int_0^\infty f_2(0) f_1(s) ds \right] + \\ &\quad \frac{K'^{-1}(f_1(0) W_1) \cdot 2(W - W_1) \int_0^\infty s f_2'(0) f_1(s) ds}{K'' [(K')^{-1}(2(W - W_1) \int_0^\infty f_2(0) f_1(s) ds)]} = \\ &\quad \frac{(W - W_1) f_2(0) \cdot (K')^{-1} [2(W - W_1) f_2(0)]}{K'' [K'^{-1}(f_1(0) W_1)]} \end{aligned}$$

where the second equation follows from $f_2'(0) = 0$ and $\int_0^\infty f_1(s) ds = \frac{1}{2}$. As $K'' > 0$, $\frac{\partial V^I(\eta, W_1, 0)}{\partial \eta} \Big|_{\eta=0} > 0$ provided $W_1 < W$. Hence, $\eta > 0$ is always optimal for $W_1 < W$.

8.10 Proof of Proposition 8

We have

$$\begin{aligned} \frac{\partial V^P(\eta, W_1, 0)}{\partial W_1} &= \\ &\quad \frac{f_1(0) - 2\eta \int_0^\infty f_2(\eta s) f_1(s) ds}{K'' [(K')^{-1}(f_1(0) W_1 + 2\eta(W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds)]} \\ &\quad - \frac{2 \int_0^\infty f_2(\eta s) f_1(s) ds}{K'' [(K')^{-1}(2(W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds)]} \end{aligned}$$

and

$$\begin{aligned}
& \frac{\partial^2 V^P(\eta, W_1, 0)}{(\partial W_1)^2} = \\
& - \frac{(f_1(0) - 2\eta \int_0^\infty f_2(\eta s) f_1(s) ds)^2}{(K'' [(K')^{-1} (f_1(0) W_1 + 2\eta (W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds)])^3} \\
& K''' \left[(K')^{-1} \left(f_1(0) W_1 + 2\eta (W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds \right) \right] \\
& - \frac{(2 \int_0^\infty f_2(\eta s) f_1(s) ds)^2}{(K'' [(K')^{-1} (2(W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds)])^3} \\
& K''' \left[(K')^{-1} \left(2(W - W_1) \int_0^\infty f_2(\eta s) f_1(s) ds \right) \right]
\end{aligned} \tag{50}$$

$K''' \geq 0$ implies $\frac{\partial^2 V^P(\eta, W_1, 0)}{(\partial W_1)^2} < 0$.

(i) Thus, the principal will set $W_1 = 0$ provided

$$\begin{aligned}
\left. \frac{\partial V^P(\eta, W_1, 0)}{\partial W_1} \right|_{W_1=0} &= \frac{f_1(0) - 2\eta \int_0^\infty f_2(\eta s) f_1(s) ds}{K'' [(K')^{-1} (2\eta W \int_0^\infty f_2(\eta s) f_1(s) ds)]} \\
&\quad - \frac{2 \int_0^\infty f_2(\eta s) f_1(s) ds}{K'' [(K')^{-1} (2W \int_0^\infty f_2(\eta s) f_1(s) ds)]} \leq 0.
\end{aligned}$$

(ii) He will set $W_1 = W$ provided

$$\begin{aligned}
\left. \frac{\partial V^P(\eta, W_1, 0)}{\partial W_1} \right|_{W_1=W} &= \frac{f_1(0) - 2\eta \int_0^\infty f_2(\eta s) f_1(s) ds}{K'' [(K')^{-1} (f_1(0) W)]} \\
&\quad - \frac{2 \int_0^\infty f_2(\eta s) f_1(s) ds}{K'' [0]} \geq 0
\end{aligned}$$

8.11 Proof of Corollary 4

For E1,

$$K(e_{it}) = \frac{k}{2} (e_{it})^2 \tag{51}$$

$$f_t(s) = \frac{1}{\sigma_t \sqrt{2\pi}} \exp\left(-\frac{s^2}{2(\sigma_t)^2}\right) \tag{52}$$

We first provide some auxiliary results. From (52),

$$\begin{aligned}
f'_t(s) &= -\frac{s}{\sigma_t^3 \sqrt{2\pi}} \exp\left(-\frac{s^2}{2(\sigma_t)^2}\right) \\
f''_t(s) &= -\exp\left(-\frac{s^2}{2\sigma_t^2}\right) + \frac{s^2}{\sigma_t^2} \exp\left(-\frac{s^2}{2\sigma_t^2}\right) \\
f'''_t(s) &= \frac{s}{\sigma_t^2} \exp\left(-\frac{s^2}{2\sigma_t^2}\right) + \frac{2s}{\sigma_t^2} \exp\left(-\frac{s^2}{2\sigma_t^2}\right) - \frac{s^3}{\sigma_t^4} \exp\left(-\frac{s^2}{2\sigma_t^2}\right)
\end{aligned} \tag{53}$$

With $s = -\sigma$ (the solution to $f''_t(s) = 0$ and $f'''_t(s) < 0$) and $s = \sigma$ (the solution to $f''_t(s) = 0$ and $f'''_t(s) > 0$), (53) yields

$$\max_s f'_t(s) = \frac{1}{\sigma_t^2 \sqrt{2\pi \exp(1)}} \tag{54}$$

$$\min_s f'_t(s) = -\frac{1}{\sigma_t^2 \sqrt{2\pi \exp(1)}} \tag{55}$$

From (54) and (55),

$$f'_t(x) \leq \frac{1}{\sigma_t^2 \sqrt{2\pi \exp(1)}} \forall x \in \mathbb{R} \tag{56}$$

$$-\frac{2}{\sigma_t^2 \sqrt{2\pi \exp(1)}} \leq f'_t(s+x) - f'_t(s-x) \tag{57}$$

$$\leq \frac{2}{\sigma_t^2 \sqrt{2\pi \exp(1)}} \forall s, x \in \mathbb{R} \tag{58}$$

$$-\frac{2}{\sigma_t^2 \sqrt{2\pi \exp(1)}} \leq f'_t(s+x) + f'_t(s-x) \tag{59}$$

$$\leq \frac{2}{\sigma_t^2 \sqrt{2\pi \exp(1)}} \forall s, x \in \mathbb{R} \tag{60}$$

Furthermore, (52) implies

$$\int_0^\infty f_2(\eta s) ds = \frac{1}{2|\eta|} \quad (61)$$

$$\int_0^\infty f_2(\eta s) f_1(s) ds = \frac{1}{2\sqrt{2\pi}\sqrt{(\sigma_2)^2 + (\sigma_1)^2\eta^2}} \quad (62)$$

$$\int_0^\infty f_2(\eta s) f_2'(\eta s) ds = -\frac{1}{4\pi\eta\sigma_2^2} \quad (63)$$

From (51),

$$K''(e) = k \quad (64)$$

$$K'''(e) = 0 \quad (65)$$

$$K^{-1}(t) = \frac{t}{k} \quad (66)$$

Next, we show that the second-order conditions are fulfilled. Using (64), (3) can be written as

$$W_t f_t'(x) \leq k \forall x \in \mathbb{R} \quad (67)$$

From $W_t \leq W$ and (56),

$$W_t f_t'(x) \leq \frac{W}{\sigma_t^2 \sqrt{2\pi \exp(1)}} \quad (68)$$

(67) and (68) imply that a sufficient condition for (3) to hold is (21).

Similarly, using (64), (7) can be written as

$$\begin{aligned} & f_1'(x) W_1 + \eta W_2 \int_0^\infty f_2(\eta s) [f_1'(s+x) - f_1'(s-x)] ds \\ & + \frac{\eta W_2^2}{k} \int_0^\infty f_2(\eta s) f_2'(\eta s) [f_1'(s+x) + f_1'(s-x)] ds \leq k \forall x \in \mathbb{R} \end{aligned} \quad (69)$$

Using (56), (57) and (59), we obtain

$$f_1'(x) W_1 \leq \quad (70)$$

$$\frac{W_1}{\sigma_1^2 \sqrt{2\pi \exp(1)}} \forall x \in \mathbb{R} \quad (71)$$

$$\eta W_2 \int_0^\infty f_2(\eta s) [f_1'(s+x) - f_1'(s-x)] ds \leq \quad (72)$$

$$\frac{2}{\sigma_t^2 \sqrt{2\pi \exp(1)}} W_2 \left| \eta \int_0^\infty f_2(\eta s) ds \right| \forall x \in \mathbb{R}$$

$$\frac{\eta W_2^2}{k} \int_0^\infty f_2(\eta s) f_2'(\eta s) [f_1'(s+x) + f_1'(s-x)] ds \leq \quad (73)$$

$$\frac{2}{\sigma_t^2 \sqrt{2\pi \exp(1)}} \frac{W_2^2}{k} \left| \eta \int_0^\infty f_2(\eta s) f_2'(\eta s) ds \right| \forall x \in \mathbb{R}$$

With (70), (72) and (73), we obtain for the left-hand side of (69)

$$\begin{aligned} & f_1'(x) W_1 + \eta W_2 \int_0^\infty f_2(\eta s) [f_1'(s+x) - f_1'(s-x)] ds \\ & + \frac{\eta W_2^2}{k} \int_0^\infty f_2(\eta s) f_2'(\eta s) [f_1'(s+x) + f_1'(s-x)] ds \leq \quad (74) \\ & \frac{W_1}{\sigma_1^2 \sqrt{2\pi \exp(1)}} + \frac{2}{\sigma_1^2 \sqrt{2\pi \exp(1)}} W_2 \left| \eta \int_0^\infty f_2(\eta s) ds \right| \\ & + \frac{2}{\sigma_1^2 \sqrt{2\pi \exp(1)}} \frac{W_2^2}{k} \left| \eta \int_0^\infty f_2(\eta s) f_2'(\eta s) ds \right| \end{aligned}$$

With (61) and (63), the right-hand side of (74) can be written as

$$\begin{aligned} & \frac{W_1}{\sigma_1^2 \sqrt{2\pi \exp(1)}} + \frac{2W_2}{\sigma_1^2 \sqrt{2\pi \exp(1)}} \left| \eta \frac{1}{2|\eta|} \right| \\ & + \frac{2}{\sigma_1^2 \sqrt{2\pi \exp(1)}} \frac{W_2^2}{k} \left| -\eta \frac{1}{4\pi\eta\sigma_2^2} \right| = \end{aligned} \quad (75)$$

$$\frac{W_1}{\sigma_1^2 \sqrt{2\pi \exp(1)}} + \frac{W_2}{\sigma_1^2 \sqrt{2\pi \exp(1)}} \quad (76)$$

$$\begin{aligned} & + \frac{W_2^2}{\sigma_1^2 \sigma_2^2 (2\pi)^{\frac{3}{2}} \sqrt{\exp(1)}} \frac{1}{k} \leq \\ & \frac{W}{\sigma_1^2 \sqrt{2\pi \exp(1)}} + \frac{W^2}{\sigma_1^2 \sigma_2^2 (2\pi)^{\frac{3}{2}} \sqrt{\exp(1)}} \frac{1}{k} \end{aligned} \quad (77)$$

where the last inequality in (76) follows from $W_1 + W_2 = W$ and $W_2 \leq W$. (69), (74) and (76) thus imply that a sufficient condition for (7) to hold is (22).

With (65), Proposition 4 implies that efforts under both revelation regimes are equal in expected value. With (52), (62) and (66), (9) and (10) yield (23) and (24).

8.12 Proof of Corollary 5

With $W_2 = W - W_1$, Corollary 4 yields

$$e_1^*(\eta, W_1, W - W_1) = \frac{W_1}{k\sigma_1\sqrt{2\pi}} + \frac{\eta(W - W_1)}{k\sqrt{2\pi}\sqrt{(\sigma_2)^2 + (\sigma_1)^2\eta^2}}.$$

Thus,

$$\frac{\partial e_1^*}{\partial W_1} = \frac{1}{k\sigma_1\sqrt{2\pi}} - \frac{\eta}{k\sqrt{2\pi}\sqrt{(\sigma_2)^2 + (\sigma_1)^2\eta^2}} > 0.$$

8.13 Proof of Corollary 6

(i) With (65), Proposition 4 allows to write $V^P(\eta, W_1, 0) = V^P(\eta, W_1, 1) = V^P(\eta, W_1)$, so that $\eta^P(W_1, 0) = \eta^P(W_1, 1) = \eta^P(W_1)$ and $W_1^P(\eta, 0) =$

$W_1^P(\eta, 1) = W_1^P(\eta)$. (23) and (24) imply

$$V^P(\eta, W_1) = \frac{W_1}{k\sigma_1\sqrt{2\pi}} + \frac{(\eta+1)(W-W_1)}{k\sqrt{2\pi}\sqrt{(\sigma_2)^2 + (\sigma_1)^2\eta^2}}$$

From Proposition 5, $\eta^P(W_1) > 0$ for $W_1 < W$. The optimal solution must satisfy

$$\begin{aligned} \frac{\partial V^P(\eta, W_1)}{\partial \eta} = & \quad (78) \\ \frac{(W-W_1)}{k\sqrt{2\pi}\sqrt{(\sigma_2)^2 + (\sigma_1)^2\eta^2}} - \frac{(\eta+1)(W-W_1)(\sigma_1)^2 2\eta}{2k\sqrt{2\pi}((\sigma_2)^2 + (\sigma_1)^2\eta^2)^{\frac{3}{2}}} = 0 \end{aligned}$$

The only solution to (78) is $\eta = \left(\frac{\sigma_2}{\sigma_1}\right)^2 = \eta^P(W_1)$. With (51), (52), (62) and (66), the condition for $W_1^P(\eta, 1) = 0$ in (17) for $\eta = \left(\frac{\sigma_2}{\sigma_1}\right)^2$ can be written as

$$\frac{1}{\sigma_1\sqrt{2\pi}} < \left(2\left(\frac{\sigma_2}{\sigma_1}\right)^2 + 2\right) \frac{1}{2\sqrt{2\pi}\sqrt{(\sigma_2)^2 + (\sigma_1)^2\left(\frac{\sigma_2}{\sigma_1}\right)^4}}$$

which is equivalent to $\sigma_2 < \sqrt{(\sigma_2)^2 + (\sigma_1)^2}$. Thus, $W_1^P\left(\left(\frac{\sigma_2}{\sigma_1}\right)^2\right) = 0$.

(ii) With (65), Proposition 4 allows to write $V^I(\eta, W_1, 0) = V^I(\eta, W_1, 1) = V^I(\eta, W_1)$, so that $\eta^I(W_1, 0) = \eta^I(W_1, 1) = \eta^I(W_1)$ and $W_1^I(\eta, 0) = W_1^I(\eta, 1) = W_1^I(\eta)$. (23) and (24) imply

$$\begin{aligned} V^I(\eta, W_1) = & \left[\frac{W_1}{k\sigma_1\sqrt{2\pi}} + \frac{\eta(W-W_1)}{k\sqrt{2\pi}\sqrt{(\sigma_2)^2 + (\sigma_1)^2\eta^2}} \right] \cdot \\ & \frac{(W-W_1)}{k\sqrt{2\pi}\sqrt{(\sigma_2)^2 + (\sigma_1)^2\eta^2}} \end{aligned}$$

Thus

$$V^I(\eta, 0) = \frac{\eta W^2}{2\pi k^2 ((\sigma_2)^2 + (\sigma_1)^2 \eta^2)}$$

From Proposition 5, $\eta^I(0) > 0$. The optimal solution must satisfy

$$\begin{aligned} \frac{\partial V^I(\eta, 0)}{\partial \eta} &= \frac{W^2}{2\pi k^2 ((\sigma_2)^2 + (\sigma_1)^2 \eta^2)} \\ &\quad - \frac{\eta^2 (\sigma_1)^2 W^2}{\pi k^2 ((\sigma_2)^2 + (\sigma_1)^2 \eta^2)^2} = 0 \end{aligned} \tag{79}$$

The only positive solution to (79) is $\eta = \frac{\sigma_2}{\sigma_1} = \eta^I(0)$. Next, observe that with (52) and (62), (18) can be written as

$$\frac{1}{\sigma_1 \sqrt{2\pi}} < \frac{\eta}{\sqrt{2\pi} \sqrt{(\sigma_2)^2 + (\sigma_1)^2 \eta^2}}$$

which is equivalent to $(\sigma_2)^2 < 0$, a contradiction. Consequently, $W_1^I(\eta) > 0$ for $\eta > 0$. Since $V^I(0, 0) = 0$, we conclude $W_1^I(\eta) > 0$ for $\eta \geq 0$.