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Strategic Capital Accumulation With Singular Control

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Abstract

We present a duopoly model of strategic capital accumulation in continuous time with uncertainty, such that investment takes the form of singular control. Spot competition is of Cournot type. For this model there exists a parameterized and Pareto-rankable family of Markov perfect equilibria in symmetric strategies, according to which implicit collusion induces positive option values. However, preemption can also eliminate any option value in a limiting case corresponding to Bertrand prices.

JEL subject classification: C73, D43, D92

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1 Introduction

The aim of this work is to establish a mathematically precise framework for studying strategic capital accumulation under uncertainty. Such a model arises as a natural extension from three different perspectives that all lead to *singular control* exercised by the agents, which induces some essential formalization problems.

Capital accumulation as a game in continuous time originates from the work of Spence (1979), where firms make dynamic investment decisions to

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expand their production capacities irreversibly. Spence analyses the strategic effect of capital commitment in a deterministic world and elaborates on the crucial distinction between precommitment (or open loop) strategies and feedback (or closed loop) strategies, of which only the latter can support subgame-perfect equilibria. We add uncertainty to the model — as he suggests — to account for an important further aspect of investment. Uncertain returns induce a reluctance to invest and thus allow to abolish the artificial bound on investment rates, leading to repeated interaction in the form of singular control.

The strategic aspects of sequential irreversible investment under uncertainty have only played a limited role so far. In the extreme case of perfect competition, an individual firm's action does not influence other players' payoffs and decisions, see Baldursson and Karatzas (1997). The perfectly competitive equilibrium is linked via a social planner to the other extreme, monopoly, which benefits similarly from the lack of interaction. There is considerable work on the single agent's problem of sequential irreversible investment, see e.g. Pindyck (1988), Bertola (1998) and Riedel and Su (2011). All these instances also involve singular control. In our model, the number of players is finite and actions have a strategic effect, so this is the second line of research we extend.

With irreversible investment, the firm's opportunity to freely choose the time of investment is a perpetual real option. It is intuitive that the value of the option is strongly affected when competitors can influence the value of the underlying by their actions. The classical option value of waiting — see McDonald and Siegel (1986) or Dixit and Pindyck (1994) — is threatened under competition and the need arises to model option exercise games. While typical formulations like Huisman and Kort (1999) or Mason and Weeds (2010) assume fixed investment sizes and pose only the question of timing a single action, we determine investment sizes endogenously. Our framework is also the limiting case for repeated investment opportunities of arbitrarily small size. Since investment is allowed to take the form of singular control, its rate need not be defined even where it occurs continuously.

An early related model was formulated by Grenadier (2002). It received much attention because it connects the mentioned different lines of research, but it became also clear that one has to be very careful with the formulation of strategies. As Back and Paulsen (2009) show, it is exactly the singular nature of investment which poses the difficulties. They explain that Grenadier's results hold only for open loop strategies, which are investment plans merely contingent on exogenous shocks. Even to specify sensible feedback strategies poses severe conceptual problems.

One can formulate games of singular control with open loop strategies in

a quite general setting and still obtain qualitative results as has been done in Steg (2012), taking a different approach than Grenadier or Back and Paulsen. It turns out that the scope for strategic behaviour is rather limited, and it is not preemption what reduces option values under competition¹. Therefore, we now propose a first explicitly strategic model of singular control with closed loop strategies in a mathematically rigorous way.

After establishing the formal framework in the following, we encounter different control problems than in the open loop case and take a dynamic programming approach to develop a suitable verification theorem. It is applied to construct different classes of Markov perfect equilibria for a model with spot competition as in Grenadier (2002) to study the effect of preemption on the value of the option to delay investment. In fact, there are Markov perfect equilibria with positive option values despite perfect circumstances for preemption.

2 The model

We want to formulate a stochastic continuous-time model, in which two players strategically accumulate capital by irreversible investment. Their respective objective is to maximize the value of a profit flow depending on both capital levels and exogenous uncertainty, net of investment costs. Formally, when the capital stock processes of player i and the opponent are Q^i and Q^{-i} , the payoff of player i is given by

$$J(Q^i, Q^{-i}) \triangleq \mathbf{E} \left[\int_0^\infty e^{-rt} \Pi(X_t, Q_t^i, Q_t^{-i}) dt - \int_0^\infty e^{-rt} dQ_t^i \right], \quad (2.1)$$

with a constant positive discount rate r . Since we focus on the pure strategic effect of capital commitment, the payoffs to the players differ only through the capital stock processes. The *instantaneous revenue* function Π is further affected by an exogenous stochastic process X , which is defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$. Assume the latter satisfies the usual conditions of right-continuity and completeness and $\mathcal{F}_\infty = \mathcal{F}$. Concerning the stochastic capital stock processes Q^i and Q^{-i} , we allow the same class of processes available to the monopolist in the related irreversible investment problem. For given initial capital $q \in \mathbb{R}_+$, any feasible capital stock process has to belong to the class

$$\mathcal{A}(q) \triangleq \{Q \text{ adapted, right-continuous, nondecreasing, and } Q_0 \geq q, \mathbf{P}\text{-a.s.}\}.$$

¹Similar to static Cournot models, the externality from output expansion — that the sales price for existing output decreases — loses importance under increasing competition and accelerates aggregate investment.

Thus, we rule out foresight and capital is installed without delay. In contrast to the monopolist, who chooses a *control policy* from $\mathcal{A}(q)$, the capital stock processes here will result from the strategies of the players described below.

For the payoffs (2.1) to be well defined, we make the following

Assumption 1.

- i. $\Pi : \mathbb{R} \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$, $(x, q^i, q^{-i}) \mapsto \Pi(x, q^i, q^{-i})$ is continuous and continuously differentiable in q^i . The partial derivative Π_{q^i} increases in x and decreases in q^i , respectively.
- ii. $e^{-rt}\Pi(X_t(\omega), Q_t^i(\omega), Q_t^{-i}(\omega))$ is $\mathbf{P} \otimes dt$ -integrable for any $(Q^i, Q^{-i}) \in \mathcal{A}(q^i) \times \mathcal{A}(q^{-i})$.

Our assumption on marginal instantaneous revenue relates to the local investment incentives of each firm, and gives some structure to the state space. The profitability of investment decreases for fixed competitive output q^{-i} , which together with the monotonicity in the exogenous shock will be helpful for the emergence of action and inaction regions. If there is furthermore an adverse influence of opponent capital on marginal revenue, i.e. if Π_{q^i} decreases in q^{-i} as well, the capital stocks are *strategic substitutes*. This will frequently be true, but we do not assume it for the entire state space a priori.

2.1 Strategies

Although the processes in $\mathcal{A}(q)$ reflect the continuous revelation of uncertainty, if they were to represent action plans, they would have to be termed *open loop* strategies and would not be rich enough for our purposes. We would like to enable the players to condition their investment decisions explicitly on the evolution of the capital stocks, too. Allowing reactions to deviating investment is necessary to obtain subgame perfect equilibria. Instead of considering investment processes adapted to a broadened filtration including the capital stock histories, we take a Markovian state space approach.

However, if one tries to define investment as a function of the state of the game, the following difficulty arises. Since the investment cost is linear, we know from the monopolistic and open loop cases that investment is likely to occur not in lumps, but continuously if the shock process does not jump. Nevertheless is instantaneous investment in terms of the growth rate dQ^i unsuitable as an *action variable* if we do not artificially bound the rate. Although typical control paths in similar optimization problems are continuous, all exercise of control occurs at singular events and with an undefined

rate. This phenomenon arises when one tries to keep a diffusion X off some barrier at minimal effort, here corresponding to a price trigger strategy, for instance. Since we do not want to exclude such policies, the increments dQ^i are only meaningful in integrals as for the investment cost in (2.1).

Our earlier treatment of the open loop case, Steg (2012), hints at a possibility to reconcile dynamic strategic decision making with the required properties of the resulting capital stock processes and consistency across sub-games. There we showed that for any starting state the optimal investment policy is given by tracking the running supremum of a certain signal process L^i , once it exceeds currently installed capital. Formally,

$$Q_t^i = q_0^i \vee \sup_{0 \leq s \leq t} L_s^i, \quad t \in [0, \infty), \quad (2.2)$$

with fixed initial capital q_0^i . L^i is the *base capacity* below which the firm never wants to operate. Such base capacities, which have to be optional processes, are in principle suitable to be determined by strategies or functions of the state of the game. The decisions of the players will then be related to the base capacity. If it exceeds installed capital, the latter is adjusted by investment, otherwise the signal is ignored. This idea is formalized as follows.

Strategies prescribe *actions*. In the related theory of differential games, see Dockner et al. (2000), the possible actions of a player at a particular moment are given by the space of instantaneous control. In our case, accounting for the exogenous uncertainty, the time- t *action set* of each player is defined as U_t , the set of \mathcal{F}_t -measurable random variables, taking values in \mathbb{R}_+ almost surely. A dynamic choice of actions $\{u_t^i \in U_t | t \in [0, \infty)\}$ by player i is *feasible* if the collection forms an optional process. Then, the capital stock process with the “law of motion”

$$Q_t^i = q_0^i \vee \sup_{0 \leq s \leq t} u_s^i, \quad t \in [0, \infty), \quad (2.3)$$

is well defined and belongs to $\mathcal{A}(q_0^i)$.

With this concept, we can now define *strategies*, which are assignments of actions for all points in time t , conditional on the information available to the players including the history of investment carried out. An important subset of such closed loop strategies are *Markovian* strategies that are conditioned only on the current state of the game. Such strategies are particularly appropriate if the state represents all payoff-relevant information concerning past play, see e.g. Fudenberg and Tirole (1991). In our case the payoff functions (2.1) imply that at any moment all payoff-relevant influence of past investment is contained in the current capital levels (Q_t^1, Q_t^2) .

While one could conceive of accounting for the exogenous shock X separately², the presentation becomes clearer if we suppose it is a Markov process and include its current value X_t in the state. Then, we may focus on stationary strategies since the horizon is infinite.

A stationary Markovian strategy assigns an action for any possible state (x, q^1, q^2) of the game, independent of time t . Formally, we define a stationary Markovian strategy for player i as a measurable function $\phi^i : \mathbb{R} \times \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$. Thus, player i 's action at time t given by the Markovian strategy ϕ^i is

$$u_t^i = \phi^i(X_t, Q_t^i, Q_t^{-i}), \quad (2.4)$$

an \mathcal{F}_t -measurable random variable for given $(Q^i, Q^{-i}) \in \mathcal{A}(q^i) \times \mathcal{A}(q^{-i})$. Note that it is not clear at all at this point, whether there exist feasible capital stock processes Q^i, Q^{-i} satisfying (2.4) and (2.3) for $i = 1, 2$ simultaneously. On the other hand, there may also be a multitude of solutions.

This is a key issue in any continuous-time game, independent of the added uncertainty³, and it will become clear in our examples. We propose to resolve it by the equilibrium definition, rather than by restricting the strategy spaces. Since it depends on the particular hypothesized equilibrium whether the outcome of the game might not be uniquely defined, it would seem to require a very strong restriction to exclude all such cases a priori.

2.2 Markov perfect equilibrium

We are looking for subgame perfect equilibria in Markovian strategies. Consequently, we identify a *subgame* by a starting time $t_0 \in [0, \infty)$ and an initial state $(x, q^1, q^2) \in \mathbb{R} \times \mathbb{R}_+^2$ only. From t_0 onwards, the game evolves according to (2.3) with (2.4) and payoffs follow (2.1) with time $t = 0$ shifted to t_0 and the initial state moved to (x, q^1, q^2) , since X is by assumption Markovian.

Our notion of optimizing behaviour by the players is analogous to the differential games literature. In equilibrium, given the Markovian strategy of the opponent, player i should not be able to increase the payoff in any subgame by any feasible control path. Consider first the “subgame” starting at $t_0 = 0$ with $x \in \mathbb{R}$ fixed and let player i solve the following *verification problem*

$$\max_{u_t^i \in U_t, t \geq 0} J(Q^i, Q^{-i}) \quad (2.5)$$

² $\phi^i : \mathbb{R}_+ \times \mathbb{R}_+ \times [0, \infty) \rightarrow U_\infty$ with $\phi^i(q^i, q^{-i}, t) \in U_t$ for all $(q^i, q^{-i}) \in \mathbb{R}_+^2$ and $t \in [0, \infty)$

³Anderson (1984) addresses the multiplicity problem, while it is not explicitly mentioned in Dockner et al. (2000).

s.t.

$$\begin{aligned} X_0 &= x \\ Q_t^i &= q^i \vee \sup_{0 \leq s \leq t} u_s^i \in \mathcal{N}(q^i) \\ Q_t^{-i} &= q^{-i} \vee \sup_{0 \leq s \leq t} \phi^{-i}(X_s, Q_s^{-i}, Q_s^i) \end{aligned}$$

In this problem, we can identify any control $\{u_t^i | t \in [0, \infty)\}$ satisfying the second constraint by the generated capital process Q^i itself. A *feasible* control for problem (2.5) is one that satisfies all constraints and by which $Q^{-i} \in \mathcal{N}(q^{-i})$ is uniquely (\mathbf{P} -a.s.) determined. Otherwise, the value of this problem would not be clear. The existence of feasible controls depends of course on the particular function ϕ^{-i} . A control is *optimal* for this problem if it is maximal among all feasible controls.

Note that in this optimization problem, player i has the same controls available as in the open loop problem in Steg (2012). This is however not a restriction but in fact gives the player the greatest conceivable power. With the reactions of the opponent specified by ϕ^{-i} , player i can in (2.5) perfectly control the entire evolution of the “subgame”, without having to worry how to implement the desired outcome by a Markovian strategy. Player i can for instance perfectly preempt the opponent in (2.5), as will be illustrated below, without even an ϵ -margin.

Also note that because of our Markovian assumption on X and the stationarity of the strategies ϕ^i , a subgame starting at any time $t_0 \in [0, \infty)$ is fully characterized by its initial state (x, q^1, q^2) . So, the verification problem analogous to (2.5) for the subgame beginning in t_0 is in fact of the same form as (2.5) with the appropriate initial state. Then, if we allow player i to optimize in *any* subgame by solving the related problem (2.5), this endows the player also with the greatest conceivable flexibility.

Summing up, if we require for a subgame perfect equilibrium in Markov strategies, called *Markov perfect equilibrium*, that each player i fares in any subgame as well as in the related verification problem (2.5), the players could not improve by any other closed loop strategy. The equilibrium would still persist with richer strategy spaces.

Definition 2.1. The pair (ϕ^1, ϕ^2) of Markovian strategies is a Markov perfect equilibrium for initial capital stocks $(q_0^1, q_0^2) \in \mathbb{R}_+^2$ if for each state $(x, q^1, q^2) \in \mathbb{R} \times \mathbb{R}_+^2$ with $q^i \geq q_0^i$ there exist a solution $(\tilde{Q}^1, \tilde{Q}^2) \in \mathcal{N}(q^1) \times \mathcal{N}(q^2)$ to

$$\begin{aligned} Q_t^1 &= q^1 \vee \sup_{0 \leq s \leq t} \phi^1(X_s, Q_s^1, Q_s^2) \\ Q_t^2 &= q^2 \vee \sup_{0 \leq s \leq t} \phi^2(X_s, Q_s^2, Q_s^1), \quad t \in [0, \infty) \end{aligned} \tag{2.6}$$

where $X_0 = x$, \mathbf{P} -a.s., and a pair of optimal controls $\{u_t^1 | t \in [0, \infty)\}$ and $\{u_t^2 | t \in [0, \infty)\}$ for problem (2.5) with initial state (x, q^1, q^2) yielding payoffs $J(\tilde{Q}^1, \tilde{Q}^2)$ and $J(\tilde{Q}^2, \tilde{Q}^1)$, respectively.

In the equilibrium definition, we do not ask for a unique solution of the combination of Markov strategies (2.6), which is still a drawback and hoped to be resolved in further research. But the chosen solution $(\tilde{Q}^1, \tilde{Q}^2)$ has to be optimal for both players simultaneously in the strong notion above. None of the players has at any moment or state of the game an incentive to employ any different control.

To identify any equilibria, we need to address the central optimization problems (2.5). These are singular control problems, but not of the monotone follower type as in the open loop case. Owing to the argument of the Markovian strategy ϕ^{-i} , there is a stronger path dependence and the methods employed in Steg (2012) are not applicable⁴. Now, for planning ahead, not only the capital stock at a certain future time matters from that point onwards, but also how capital has evolved until then. The decision to delay investment thus obtains a new aspect and we have to take an alternative approach to account for it.

3 A verification theorem

We aim to establish a verification theorem for the optimization problems (2.5). We already argued that for a given Markovian strategy ϕ^{-i} and a Markov process X , the value of the problem depends only on the initial state (x, q^1, q^2) and not on the starting time, set to zero in (2.5). With our Assumption 1, the value function

$$V^*(x, q^i, q^{-i}) \triangleq \sup_{Q^i \in \mathcal{N}(q^i)} J(Q^i, Q^{-i}) \quad (3.1)$$

s.t.

$$\begin{aligned} X_0 &= x \\ Q_t^{-i} &= q^{-i} \vee \sup_{0 \leq s \leq t} \phi^{-i}(X_s, Q_s^{-i}, Q_s^i) \end{aligned}$$

is well defined, provided ϕ^{-i} is sufficiently regular such that the additional feasibility constraint for the “controls” Q^i to uniquely determine Q^{-i} is satisfied.

⁴In particular, we cannot switch from singular control to optimal stopping by the Fubini theorem anymore.

As suggested above, we here replaced the control sequences $\{u_t^i | t \in [0, \infty)\}$ by the generated capital processes.

The stationarity of the value function V^* motivates a dynamic programming approach. Specifically, it holds for any $Q^i \in \mathcal{A}(q_0^i)$, with $Q^{-i} \in \mathcal{A}(q_0^{-i})$ generated by ϕ^{-i} and any almost surely finite $(\mathcal{T}_t)_{t \geq 0}$ -stopping time τ

$$V^*(X_0, q_0^i, q_0^{-i}) \geq \mathbf{E} \left[\int_0^\tau e^{-rt} \Pi(X_t, Q_t^i, Q_t^{-i}) dt - \int_0^\tau e^{-rt} dQ_t^i + V^*(X_\tau, Q_\tau^i, Q_\tau^{-i}) \right]. \quad (3.2)$$

Consequently, the argument of the expectation is a super-martingale for any feasible Q^i and the route is to identify such a process.

Suppose now that X is an Itô process, i.e. solves the stochastic differential equation

$$dX_t = \mu_t dt + \sigma_t dB_t, \quad t \in [0, \infty) \quad (3.3)$$

with $X_0 = x_0 \in \mathbb{R}$, \mathbf{P} -a.s., for a Brownian motion B on our filtered probability space and appropriate⁵ drift and variance processes μ, σ . Then, for all feasible capital stock processes $(Q^1, Q^2) \in \mathcal{A}(q^1) \times \mathcal{A}(q^2)$, the state process is a semi-martingale, because the components Q^1 and Q^2 are monotone, adapted processes, i.e. of finite variation. Consequently, we may attempt to “construct” sufficiently smooth functions V and verify by Itô’s lemma whether they coincide with the value function V^* in (3.1).

However, to identify a Markov perfect equilibrium, we need to solve (3.1) for all possible states and check whether there exists a solution $(\tilde{Q}^1, \tilde{Q}^2)$ to the combination of equilibrium strategies (ϕ^1, ϕ^2) as formalized in Definition 2.1, which actually attains the respective value. This procedure would be strongly simplified if we could apply the sought verification theorem somehow to Markovian strategies directly. To facilitate such an approach, we exploit the properties of local investment incentives following Assumption 1. They help to identify appropriate classes of best replies, that will eventually admit a Markovian representation as well.

Specifically, note that player i can undertake an initial discrete investment of size $\xi_0 > 0$. Thus, $V^*(x, q^i, q^{-i}) \geq V^*(x, q^i + \xi_0, q^{-i}) - \xi_0$, where the player behaves optimally after the investment. The optimal investment policy from state $(x, q^i + \xi_0, q^{-i})$ may require a further discrete investment, then the estimate holds with equality, but the jump ξ_0 may also have been too large, i.e. unprofitable. Since we assumed that instantaneous marginal revenue decreases in q^i , it might be optimal for fixed x and q^{-i} to make an

⁵ μ and σ predictable, $\mu \in L^1(\mathbf{P} \otimes dt), \sigma \in L^2(\mathbf{P} \otimes dt)$

initial investment whenever q^i is below a certain value, and otherwise not. This critical value might further depend on x and q^{-i} in the same direction as marginal revenue, i.e. increase in x and decrease in q^{-i} . Formally, this hypothesis corresponds to a Markovian strategy with the properties $\phi_{q^i}^i = 0$, $\phi_x^i > 0$, and $\phi_{q^{-i}}^i \leq 0$ in case of differentiability. Then, the inverse

$$\bar{X}^i(q^i, q^{-i}) \triangleq \sup\{x \in \mathbb{R} \mid q^i \geq \phi^i(x, q^i, q^{-i})\} \quad (3.4)$$

is well defined and satisfies

$$\begin{aligned} \bar{X}_{q^i}^i &> 0, \\ \bar{X}_{q^{-i}}^i &\geq 0 \end{aligned} \quad (3.5)$$

(where the set is non-empty) and

$$\lim_{q^i \rightarrow \infty} \bar{X}^i(q^i, q^{-i}) = \infty \quad (q^{-i} \in \mathbb{R}_+). \quad (3.6)$$

Changing perspective, we can for any \mathcal{C}^1 function $\bar{X}^i : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ with the properties (3.5) and (3.6) define a corresponding Markov strategy

$$\phi^i(x, q^{-i}) \triangleq \sup\{q \in \mathbb{R}_+ \mid x \geq \bar{X}^i(q, q^{-i})\} \vee 0 \quad (3.7)$$

with the argued properties. Here, we neglected the irrelevant argument of ϕ^i and dominated the supremum of the empty set, $-\infty$, by 0 for later use.

3.1 Reflection strategies

We will call strategies of the type (3.7) with the properties (3.5) and (3.6) *reflection strategies*. They prescribe to keep the state outside the “forbidden” region $\{x > \bar{X}^i(q^i, q^{-i})\}$ with minimal effort, like controls in obstacle problems. Since X has almost surely continuous paths, this policy involves only an initial discrete investment to bring the state onto the boundary of the forbidden region if necessary. Afterwards, the mentioned continuous singular control is exercised to reflect the state whenever X approaches the boundary \bar{X}^i .

Now suppose the opponent of player i uses a reflection strategy with \bar{X}^{-i} satisfying (3.5) and (3.6). Our verification theorem will specify conditions under which a particular reflection strategy \bar{X}^i is a best reply for player i . At this point, we face the problem that the two considered strategies might not uniquely define the capital processes Q^1 and Q^2 for any initial state, for instance if the boundaries are functions of the sum of the capital stocks. Thus, we have to pick a particular solution Q^i to be used as control in the

verification problem (2.5), resp. (3.1). The solution we generally select is where player i acts as the *leader*.

Consider first the discrete initial investments for the state $(x_0, q_0^1, q_0^2) \in \mathbb{R} \times \mathbb{R}_+^2$. As the leader, player i first adjusts the capital stock to

$$Q_0^i = q_0^i \vee \phi^i(x_0, q_0^{-i}). \quad (3.8)$$

Then, the opponent's capital stock moves to

$$Q_0^{-i} = q_0^{-i} \vee \phi^{-i}(x_0, q_0^i \vee \phi^i(x_0, q_0^{-i})), \quad (3.9)$$

which is well defined. Now note that with these initial investments, the state is no longer in any of the forbidden regions, i.e.

$$x_0 \leq \bar{X}^i(Q_0^i, Q_0^{-i}) \wedge \bar{X}^{-i}(Q_0^{-i}, Q_0^i). \quad (3.10)$$

The only critical step here is when player $-i$ does invest, i.e. when

$$dQ_0^{-i} > 0 \Leftrightarrow \bar{X}^{-i}(q_0^{-i}, Q_0^i) < x_0 = \bar{X}^i(Q_0^i, q_0^{-i}). \quad (3.11)$$

But then the investment induces $\bar{X}^{-i}(Q_0^{-i}, Q_0^i) = x_0$ and $\bar{X}^i(Q_0^i, Q_0^{-i}) \geq x_0$ because of (3.5).

Outside the joint forbidden region, we still assume that player i acts as the leader in the sense that if player i chooses to invest at the same boundary $\bar{X}^i(q^i, q^{-i}) = \bar{X}^{-i}(q^{-i}, q^i)$ in a certain part of the state space, there usually is a strict incentive for preemption. Such perfect preemption with zero margin is feasible in problem (2.5) by the solution of the following *Skorohod*-type problem.

Problem 3.1. Given $i \in \{1, 2\}$, two reflection boundaries \bar{X}^1 and \bar{X}^2 which satisfy (3.5) and (3.6) and a starting state $(x_0, q_0^1, q_0^2) \in \mathbb{R} \times \mathbb{R}_+^2$, find two processes $Q^1 \in \mathcal{A}(q_0^1)$ and $Q^2 \in \mathcal{A}(q_0^2)$ such that

$$\left. \begin{aligned} Q_0^i &= q_0^i \vee \phi^i(x_0, q_0^{-i}) \\ Q_0^{-i} &= q_0^{-i} \vee \phi^{-i}(x_0, q_0^i \vee \phi^i(x_0, q_0^{-i})) \\ X_t &\leq \bar{X}^1(Q_t^1, Q_t^2) \wedge \bar{X}^2(Q_t^2, Q_t^1), \quad t \in [0, \infty) \\ \int_0^\infty (1 - \mathbf{1}_{\{X_t \geq \bar{X}^i(Q_t^i, Q_t^{-i})\}}) dQ_t^i &= 0 \\ \int_0^\infty (1 - \mathbf{1}_{\{\bar{X}^{-i}(Q_t^{-i}, Q_t^i) \leq X_t < \bar{X}^i(Q_t^i, Q_t^{-i})\}}) dQ_t^{-i} &= 0 \end{aligned} \right\} \quad \mathbf{P}\text{-a.s.} \quad (3.12)$$

The sought capital processes keep the state outside the joint forbidden region over the entire time interval, almost surely. The investment needed to do so is minimal, since it only occurs *on* the boundary.

Assume there exists a unique solution to this problem⁶, which will be verified in the particular cases discussed below. Then, player i undertakes all the investments to reflect the state from the joint forbidden region, except where the boundary \bar{X}^i *strictly* exceeds the minimum of the two. Where the boundaries coincide, player $-i$'s investment is completely preempted by the “leader” i .

Now we are in a position to state our verification theorem for reflection strategies with assignment of a leader.

3.2 Verification theorem

Assume now X is a geometric Brownian motion, i.e. solves the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dB_t, \quad t \in [0, \infty)$$

with *constant*, real μ and σ . We will only consider initial values $X_0 = x_0 \geq 0$, so for any $t \in [0, \infty)$, $X_t \in \mathbb{R}_+$, \mathbf{P} -a.s.

For the usual notation, introduce the *infinitesimal generator* \mathcal{L}_x of the process X , which applied to any \mathcal{C}^2 function $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto f(x)$ yields

$$\mathcal{L}_x f = \mu x f_x + \frac{1}{2} \sigma^2 x^2 f_{xx}.$$

In the following verification theorem, we want to identify a function V of the state that equals the payoff from a particular solution to Problem 3.1 at the given initial state, $J(Q^i, Q^{-i})$. Outside the joint forbidden region, i.e. absent any investment, the payoff evolves like an asset whose price is a function of X and which generates a dividend flow Π . At starting states inside the forbidden region, player i is the leader of initial investment, bringing the state on the boundary \bar{X}^i , at a cost equal to the size of the jump. Only if this still exceeds \bar{X}^{-i} will the opponent make an *anticipated* investment, which will not affect the value of player i 's payoff.

Theorem 3.2. *Let \bar{X}^1 and \bar{X}^2 satisfying (3.5) and (3.6) be given and assume (Q^i, Q^{-i}) solve Problem 3.1 for initial state (x_0, q_0^1, q_0^2) .*

⁶For the classical Skorohod problem, see El Karoui and Karatzas (1991); Karatzas and Shreve (1988); Skorohod (1961).

Suppose there exists a function $V : \mathbb{R}_+^3 \rightarrow \mathbb{R}$, $(x, q^i, q^{-i}) \mapsto V(x, q^i, q^{-i})$ that is of class $\mathcal{C}^{1,1,1}$ and satisfies

$$\lim_{T \rightarrow \infty} \mathbf{E}[e^{-rT} V(X_T, Q_T^i, Q_T^{-i})] = 0. \quad (3.13)$$

If

$$\cdot \text{ on } \{x \leq \bar{X}^i(q^i, q^{-i}) \wedge \bar{X}^{-i}(q^{-i}, q^i)\} : V \text{ is class } \mathcal{C}^{2,1,1}, -rV + \Pi + \mathcal{L}_x V = 0$$

$$\cdot \text{ on } \{x > \bar{X}^i(q^i, q^{-i})\} : V(x, q^i, q^{-i}) = V(x, \phi^i(x, q^{-i}), q^{-i}) - \phi^i(x, q^{-i}) + q^i$$

$$\cdot \text{ on } \{\bar{X}^{-i}(q^{-i}, q^i) < x \leq \bar{X}^i(q^i, q^{-i})\} : V(x, q^i, q^{-i}) = V(x, q^i, \phi^{-i}(x, q^i))$$

then $V(x_0, q_0^1, q_0^2) = J(Q^i, Q^{-i})$.

If furthermore

$$\begin{aligned} \cdot V \text{ is class } \mathcal{C}^{2,1,1} & \quad \text{on } \{x \leq \bar{X}^{-i}(q^{-i}, q^i)\} \\ \cdot V_{q^i} \leq 1 & \quad \text{on } \{x \leq \bar{X}^i(q^i, q^{-i}) \wedge \bar{X}^{-i}(q^{-i}, q^i)\} \\ \cdot V_{q^{-i}}(\bar{X}^i(q^i, q^{-i}), q^i, q^{-i}) \leq 0 & \quad \text{on } \{\bar{X}^i(q^i, q^{-i}) \leq \bar{X}^{-i}(q^{-i}, q^i)\} \\ \cdot -rV + \Pi + \mathcal{L}_x V \leq 0 & \quad \text{on } \{\bar{X}^i(q^i, q^{-i}) < x \leq \bar{X}^{-i}(q^{-i}, q^i)\} \end{aligned}$$

then $V(x_0, q_0^1, q_0^2) \geq J(Q^i, Q^{-i})$ for any feasible $(Q^i, Q^{-i}) \in \mathcal{A}(q_0^i) \times \mathcal{A}(q_0^{-i})$ in (2.5) for which (3.13) holds.

$V(x_0, q_0^1, q_0^2) = V^*(x_0, q_0^1, q_0^2)$ only if

$$\begin{aligned} \cdot V_{q^i} \leq 1 & \quad \text{on } \{x \leq \bar{X}^i(q^i, q^{-i})\} \\ \cdot -rV + \Pi + \mathcal{L}_x V \leq 0 & \quad \text{on } \{x \leq \bar{X}^{-i}(q^{-i}, q^i)\} \end{aligned}$$

The proof is in the Appendix. The sufficient conditions given in the verification theorem are quite constructive. For a given pair of reflection strategies, resp. boundaries \bar{X}^1 and \bar{X}^2 , we can try to construct the associated

V , by solving the partial differential equation, subject to the constraint that the extension to the other regions happens in a differentiable way. If the optimality conditions are satisfied, the verification problems (2.5) are solved for all initial states for which the Skorohod Problem 3.1 with \bar{X}^1 and \bar{X}^2 has a unique solution. This comes very close to our equilibrium definition and enables us to determine Markov perfect equilibria quite systematically in the following.

4 Bertrand equilibrium

The example that we discuss from now on is the revenue specification of Grenadier (2002). Suppose, the firms produce a homogeneous good at full capacity and sell it on a common market, facing inverse demand with constant elasticity. The price is multiplicatively affected by the exogenous shock X , our geometric Brownian motion defined above. With zero variable cost, the revenue function for firm i is then

$$\Pi(x, q^i, q^{-i}) = xP(q^i + q^{-i})q^i = x(q^i + q^{-i})^{-\frac{1}{\alpha}}q^i. \quad (4.1)$$

Assume $\alpha > 1$ to conform to Assumption 1. Regarding the integrability requirement, we anticipate a result of the subsequent section, where we will see that the monopolist's optimal payoff is finite iff $\alpha < \beta$. The latter is a function of the remaining parameters and will be presented soon. The expected *revenue* of any player in the game is now nonnegative and dominated by that of the monopolist, since competitive output can only decrease the price.

In this section, we begin with a simple type of reflection strategies, where a firm invests whenever the price $X_t P(Q_t^i + Q_t^{-i})$ rises above a certain constant threshold. Such policies will lead to the most commonly conjectured closed loop equilibrium, Bertrand quantities. Its elaboration is useful to illustrate some concepts and to derive some general results employed in the following, more involved cases.

Consequently, assume player i 's opponent uses such a fixed price level to trigger investment, i.e.

$$\bar{X}^{-i}(q^{-i}, q^i) = \frac{p^{-i}}{P(q^i + q^{-i})} \quad (4.2)$$

with $p^{-i} \in \mathbb{R}_+$. From player i 's point of view this means that independently of $Q^i \in \mathcal{N}(q_0^i)$, the price $X_t P(Q_t^i + Q_t^{-i})$ will never exceed p^{-i} for any $t > 0$.

If i does not invest, the price will be reflected at this barrier. The problem is now how to preempt the opponent optimally, if at all.

Whenever player i invests, the additional net revenue is of course countered by the externality of lowering the price for existing output, as always in Cournot competition. This becomes obvious in marginal revenue $x(P(q^i + q^{-i}) + q^i P'(q^i + q^{-i}))$, where the derivative is negative.

In the cases of monopoly, oligopoly with open loop strategies, and perfect competition, the optimal investment strategies can be determined by deciding when to install marginal capital units in order to start the associated marginal revenue flow. For each unit, there is an option to delay investment and the optimal exercise time can be determined independently by optimal stopping. In particular we showed in Steg (2012) that the value of the option to delay marginal investment is closely related to the opportunity cost.

In the present case, player i 's investment also influences the capital stock of the opponent and due to the running suprema in (2.5), resp. (3.1), some path-dependence arises and we cannot treat the marginal capital units independently, anymore. The opportunity cost principle is now only applicable subject to *full* preemption (when $Q^{-i} \equiv q_0^{-i}$), respectively over intervals in which the opponent's investment boundary is not reached. This will be illustrated below.

We begin the study of our present example with the question when it is profitable for player i to cause the price reflection by own investment. In fact, since the price barrier here is constant, the decision will always be the same whenever the boundary is reached. Thus, we aim to determine the values of always preempting, as well as never investing, with the help of Theorem 3.2.

First, we can solve the partial differential equation which V has to satisfy off the forbidden region. The general solution is polynomial.

$$V(x, q^i, q^{-i}) = A(q^i, q^{-i})x + B(q^i, q^{-i})x^\beta \quad (4.3)$$

Here, β is the positive root of the typical quadratic equation⁷ and given by

$$\beta = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}. \quad (4.4)$$

Note that $\beta > 1 \Leftrightarrow r > \mu$, which is necessary for our assumption $\beta > \alpha$ to hold.

⁷We neglected the corresponding negative root as further exponent in V , which would otherwise diverge to positive or negative infinity when x approaches zero.

Furthermore must the first coefficient satisfy

$$A(q^i, q^{-i}) = \frac{1}{r - \mu} P(q^i + q^{-i}) q^i. \quad (4.5)$$

As a consequence, the first term of V is necessarily equal to the net present value of the revenue flow, were the current capacities *fixed forever*. This observation holds independently of the considered boundaries \bar{X}^1 and \bar{X}^2 .

The latter come into play by the boundary conditions for the partial derivatives of V in Theorem 3.2. We have to match the coefficient function B to \bar{X}^1 and \bar{X}^2 by these boundary conditions.

Begin in the current example with the case of never investing, i.e. $\bar{X}^i \equiv \infty$. So, since $\bar{X}^i(q^i, q^{-i}) > \bar{X}^{-i}(q^{-i}, q^i)$, V must satisfy (cf. (A.2))

$$\begin{aligned} V_{q^{-i}}\left(\frac{p^{-i}}{P(q^i + q^{-i})}, q^i, q^{-i}\right) &= 0 \\ \Leftrightarrow B_{q^{-i}} &= -A_{q^{-i}}(p^{-i})^{1-\beta} P(q^i + q^{-i})^{\beta-1} \\ &= \frac{(p^{-i})^{1-\beta}}{\alpha(r - \mu)} q^i (q^i + q^{-i})^{-\frac{\beta}{\alpha}-1}. \end{aligned} \quad (4.6)$$

The last expression can be integrated to obtain

$$\begin{aligned} B(q^i, q^{-i}) &= -\frac{(p^{-i})^{1-\beta}}{\beta(r - \mu)} q^i (q^i + q^{-i})^{-\frac{\beta}{\alpha}} + C(q^i) \\ &= -\frac{p^{-i}}{\beta(r - \mu)} q^i \left(\frac{P(q^i + q^{-i})}{p^{-i}} \right)^{\beta} + C(q^i). \end{aligned} \quad (4.7)$$

Using this coefficient B , we can define by

$$\begin{aligned} V^\infty(x, q^i, q^{-i}) &\triangleq \frac{p^{-i}}{r - \mu} q^i \frac{xP(q^i + q^{-i} \vee \phi^{-i}(x, q^i))}{p^{-i}} \\ &\quad - \frac{p^{-i}}{\beta(r - \mu)} q^i \left(\frac{xP(q^i + q^{-i} \vee \phi^{-i}(x, q^i))}{p^{-i}} \right)^{\beta} \end{aligned} \quad (4.8)$$

a function satisfying the first set of sufficient conditions and hypothesis of Theorem 3.2. The latter is true because the prices xP here are bounded above by p^{-i} and xV_x is of the same order as V . The corresponding solutions to Problem 3.1 are

$$Q^i \equiv q_0^i \text{ and } Q^{-i} = q_0^{-i} \vee \left(\sup_{0 \leq s \leq t} (X_s/p^{-i})^\alpha - q_0^i \right)_{t \geq 0}.$$

The integration constant $C(q^i)$ has been set to zero, since V^∞ represents the net present value of selling the constant output flow q^i at a diffusion price,

reflected at the barrier p^{-i} , which cannot indefinitely increase nor decrease in the initial shock value x . Also note that generally any constant coefficient component

$$C \cdot x^\beta$$

of V will drop out when we apply Itô's lemma in (A.3).

Now consider the other case and let

$$\bar{X}^i = \frac{p^{-i}}{P(q^i + q^{-i})},$$

i.e. player i preempts the investment of $-i$ at the identical boundary by implementing the capital stock process

$$Q^i = q_0^i \vee \left(\sup_{0 \leq s \leq t} (X_s / p^{-i})^\alpha - q_0^{-i} \right)_{t \geq 0}$$

so that $Q^{-i} \equiv q_0^{-i}$, which together again solve Problem 3.1.

Then, since $\bar{X}^i(q^i, q^{-i}) \leq \bar{X}^{-i}(q^{-i}, q^i)$, the relevant boundary condition for V is (cf. (A.1))

$$V_{q^i} \left(\frac{p^{-i}}{P(q^i + q^{-i})}, q^i, q^{-i} \right) = 1$$

$$\Leftrightarrow B_{q^i} = -(p^{-i})^{-\beta} \left(q^i \left(\frac{p^{-i}(\alpha-1)}{(r-\mu)\alpha} - 1 \right) + q^{-i} \left(\frac{p^{-i}}{r-\mu} - 1 \right) \right) (q^i + q^{-i})^{-\frac{\beta}{\alpha}-1}.$$

This is again one of the rare cases in which one can explicitly integrate for B . Neglecting the integration constant for the same reason as before, we arrive at

$$\frac{B^{p^{-i}}(q^i, q^{-i})}{(P(q^i + q^{-i}))^\beta} = \frac{\alpha}{\beta-\alpha} (p^{-i})^{-\beta} \left(q^i \left(\frac{p^{-i}(\alpha-1)}{(r-\mu)\alpha} - 1 \right) + q^{-i} \left(\frac{p^{-i}(\beta-1)}{(r-\mu)\beta} - 1 \right) \right).$$

With this particular coefficient function, the value of preempting at p^{-i} from a given initial state is completely determined by⁸

$$V^{p^{-i}}(x, q^i, q^{-i}) \triangleq \begin{cases} A(q^i, q^{-i})x + B^{p^{-i}}(q^i, q^{-i})x^\beta & \text{if } x \leq p^{-i}/P(q^i + q^{-i}) \\ V^{p^{-i}}(x, \phi^i(x, q^{-i}), q^{-i}) - \phi^i(x, q^{-i}) + q^i & \text{else.} \end{cases}$$

Now we can compare the values of both policies. Preempting is more profitable than never investing iff $V^{p^{-i}} \geq V^\infty$. On $\{q^i \geq \phi^i(x, q^{-i})\}$ this is equivalent to

$$\frac{\alpha}{\beta-\alpha} \left(q^i \left(\frac{p^{-i}(\alpha-1)}{(r-\mu)\alpha} - 1 \right) + q^{-i} \left(\frac{p^{-i}(\beta-1)}{(r-\mu)\beta} - 1 \right) \right) + q^i \frac{p^{-i}}{\beta(r-\mu)} \geq 0$$

⁸At the price boundary, $V^{p^{-i}}$ is linear in q^i . We clarify below that the boundedness condition is satisfied.

$$\begin{aligned} &\Leftrightarrow \left(\frac{p^{-i}(\beta-1)}{(r-\mu)\beta} - 1 \right) \left(\frac{\alpha}{\beta-\alpha} q^i + q^{-i} \right) \geq 0 \\ &\Leftrightarrow p^{-i} \geq \frac{\beta(r-\mu)}{(\beta-1)} \triangleq p^*. \end{aligned}$$

Then, $V^{p^{-i}} \geq V^\infty$ also on $\{q^i < \phi^i(x, q^{-i})\}$, since in this region, $V_{q^i}^{p^{-i}} = 1 \leq V_{q^i}^\infty = p^{-i}/p^*$.

p^* is a quite important quantity, it is precisely the *Bertrand price*, which we know from the case of perfect competition. If the price is reflected at this barrier, the net present value of a marginal capital unit equals one at the boundary, its cost. Consequently, the option to delay investment is valueless.

In principle, we already know now that in the only Markov perfect equilibrium with a constant reflection price both players invest at the Bertrand price, where each is just indifferent. However, we want to formally prove this finding by completing the consistent application of Theorem 3.2 and checking our equilibrium definition.

Specifically, we have only determined when preemption at p^{-i} is *superior* to remaining passive. For optimality, we need to verify the further sufficient conditions.

In particular, for concluding that our candidate function V is really the value function defined in (3.1), we need to verify that it satisfies the boundedness condition for all relevant controls. This first problem can be easily tackled because the expected revenue from any capital process is finite by assumption and we only need to consider processes with finite investment cost. In this case integration by parts yields

$$\mathbf{E} \left[\int_0^\infty e^{-rt} dQ_t \right] < \infty \Rightarrow \lim_{T \rightarrow \infty} \mathbf{E} \left[e^{-rT} Q_T \right] = 0 \quad (4.9)$$

and it suffices to establish a linear bound on V for arbitrary capital processes. Then,

Lemma 4.1. *For any $Q^i \in \mathcal{A}(q^i)$ with finite investment cost and $q^{-i} \in \mathbb{R}_+$,*

$$\lim_{T \rightarrow \infty} \mathbf{E} \left[e^{-rT} V^{p^{-i}}(X_T, Q_T^i, q^{-i}) \right] = 0.$$

The investment cost of reflecting the price at any constant barrier is finite.

The proof is given in the Appendix.

By Lemma 4.1, we may now check the second set of sufficient conditions in Theorem 3.2 to verify when either strategy is optimal⁹ in (2.5).

⁹One could also restrict the search to the class of processes never exceeding the Bertrand quantity. Lemma A.1 in the Appendix states that such a cap is profitable and is proven by the optimal stopping approach of Steg (2012).

Let us again begin with $\bar{X}^i \equiv \infty$. All but the second sufficient condition hold by construction. Since $V_{q^i}^\infty$ is increasing in x , the condition is satisfied iff

$$\begin{aligned} V_{q^i}^\infty\left(\frac{p^{-i}}{P(q^i + q^{-i})}, q^i, q^{-i}\right) &\leq 1 \\ \Leftrightarrow \frac{p^{-i}}{(r - \mu)} \frac{(\beta - 1)}{\beta} &\leq 1 \\ \Leftrightarrow p^{-i} &\leq p^*. \end{aligned} \quad (4.10)$$

We may as expected conclude that at any constant reflection barrier lower than Bertrand it is optimal for player i to abstain from investment.

The corresponding condition for $\bar{X}^i = p^{-i}/P(q^i + q^{-i})$ can be verified by an important general result. For this, note that whenever $\bar{X}^i \leq \bar{X}^{-i}$, the coefficient function B needs to satisfy the boundary condition

$$\begin{aligned} V_{q^i}(\bar{X}^i(q^i, q^{-i}), q^i, q^{-i}) &= 1 \\ \Leftrightarrow B_{q^i} &= (1 - A_{q^i} \bar{X}^i(q^i, q^{-i})) \bar{X}^i(q^i, q^{-i})^{-\beta}. \end{aligned} \quad (4.11)$$

We want to answer the question in which cases the necessary optimality condition for V_{q^i} is compatible with (4.11).

$$\begin{aligned} V_{q^i} &\leq 1, \quad \forall x \in [0, \bar{X}^i(q^i, q^{-i})] \\ \Leftrightarrow B_{q^i} &\leq (1 - A_{q^i} x) x^{-\beta}, \quad \forall x \in [0, \bar{X}^i(q^i, q^{-i})] \end{aligned} \quad (4.12)$$

The last condition can only hold if the right hand side is not increasing in x (in the given interval), since equality is attained at the upper bound by (4.11). It is nonincreasing iff

$$x \leq \frac{\beta}{\beta - 1} (A_{q^i})^{-1} = p^* \frac{q^i + q^{-i}}{\frac{\alpha-1}{\alpha} q^i + q^{-i}} (P(q^i + q^{-i}))^{-1} \triangleq \frac{\bar{p}(q^i, q^{-i})}{P(q^i + q^{-i})}. \quad (4.13)$$

Thus, when $\bar{X}^i \leq \bar{X}^{-i}$, it satisfies the optimality condition if and only if $\bar{X}^i \leq \bar{p}(q^i, q^{-i})/P(q^i + q^{-i})$. The latter function is not only important because of this property, it also happens to be the *myopic* price trigger discussed in detail below. For future reference note also

$$\bar{X}^i \leq \frac{\bar{p}(q^i, q^{-i})}{P(q^i + q^{-i})} \Leftrightarrow V_{q^i x}(\bar{X}^i, q^i, q^{-i}) \geq 0. \quad (4.14)$$

Concerning the present example, we thus only have to verify whether $p^{-i} \leq \bar{p}(q^i, q^{-i})$, which is satisfied iff

$$q^i \geq q^{-i} \frac{p^{-i} - p^*}{p^* - \frac{\alpha-1}{\alpha} p^{-i}} \geq 0. \quad (4.15)$$

Both inequalities must hold, so if $p^{-i} \geq p^* \cdot \alpha / (\alpha - 1)$, it cannot be the optimal boundary (which will be lower). Furthermore, we also consider subgames off the equilibrium path — at least with capital stocks not strictly below the initial levels —, so (4.15) has to be satisfied by all $q^i \geq q_0^i$ and $q^{-i} \geq q_0^{-i}$, and does so only if $p^{-i} \leq p^*$.

It remains to check under which conditions $V_{q^{-i}}^{p^{-i}} \leq 0$ at the investment boundary. A short calculation shows

$$V_{q^{-i}}^{p^{-i}}(x, \phi^i(x, q^{-i}), q^{-i}) = \frac{x}{p^{-i}} \left(1 - \frac{p^{-i}}{p^*}\right) P(q^i + q^{-i}) \leq 0 \Leftrightarrow p^{-i} \geq p^*.$$

Consequently, preempting at the fixed price level is optimal in all subgames iff¹⁰ $p^{-i} = p^*$, i.e. when both players are just indifferent to invest at the boundary.

In fact, if both players use $\bar{X}^i = p^* / P(q^i + q^{-i})$, we can select any pair of processes $(Q^1, Q^2) \in \mathcal{A}(q_0^1) \times \mathcal{A}(q_0^2)$ that jointly reflect the price at p^* to comply with our equilibrium Definition 2.1. Let us summarize this result.

Proposition 4.2. *For all initial capital levels $(q_0^1, q_0^2) \in \mathbb{R}_+^2$, the pair of Markovian strategies (ϕ^B, ϕ^B) , where*

$$\phi^B(x, q) \triangleq \left(\frac{x}{p^*}\right)^\alpha - q,$$

is a Markov perfect equilibrium for the game with revenue function (4.1). The equilibrium value of firm i at state $(x, q^i, q^{-i}) \in \mathbb{R}_+^3$ is given by the function

$$V^B(x, q^i, q^{-i}) \triangleq \begin{cases} q^i & \text{if } xP(q^i + q^{-i}) \geq p^*, \\ \frac{\beta}{\beta-1} q^i \frac{xP(q^i + q^{-i})}{p^*} - \frac{1}{\beta-1} q^i \left(\frac{xP(q^i + q^{-i})}{p^*}\right)^\beta & \text{else.} \end{cases}$$

We observe that the value of each firm equals its current capital stock in the forbidden region. The additional revenue flow from any investment in this region has present value one, identical to its cost, and thus does not affect firm value. Consequently, the option value of waiting completely disappears. Any profitable investment is immediately exploited, like under perfect competition.

5 Myopic investment

We saw in the previous section that it might not be an optimal reply to wait until the price reaches the opponent's investment boundary. In this section

¹⁰We did not prove that $V_{q^{-i}}^{p^{-i}} \leq 0$ is necessary, but we showed $V^\infty > V^{p^{-i}} \Leftrightarrow p^{-i} < p^*$.

we will take a look at earlier investment and elaborate on the associated optimality conditions. The results will have important implications for the existence of further equilibria.

5.1 The myopic investor

For concreteness, begin with the assumption that player i 's opponent uses the “reflection” strategy $X^{-i} \equiv \infty$, which was shown to be optimal when the price never exceeds p^* . In this case, player i can act like a monopolist, taking the fixed competitive output as given, so the optimal strategy is not very difficult to determine. Nevertheless, the situation is of intrinsic interest, since we showed in Steg (2012) that the best reply to the current, fixed capital levels is the optimal investment policy in any open loop equilibrium under very general conditions. Such investment behaviour is called *myopic* and was already discussed by Leahy in his derivation of a perfectly competitive equilibrium Leahy (1993). The principle is widely known since then and we would like to know which role it is playing in our setting.

Let us try to identify an optimal myopic reflection boundary $X^m(q^i, q^{-i})$ with the help of Theorem 3.2, by constructing the myopic value function V^m . From the previous section, we know that necessarily on $\{x \leq X^m(q^i, q^{-i})\}$

$$V^m(x, q^i, q^{-i}) = \frac{1}{r - \mu} q^i x P(q^i + q^{-i}) + B^m(q^i, q^{-i}) x^\beta,$$

where we have to determine B^m by the boundary conditions. Furthermore, we know from (4.14) that the necessary optimality conditions can only hold if $\bar{X}^m \leq \bar{p}(q^i, q^{-i})/P(q^i + q^{-i})$.

Now consider the second necessary condition of the theorem for optimality, which we only need to check for $\{\bar{X}^i(q^i, q^{-i}) < x \leq \bar{X}^{-i}(q^{-i}, q^i)\}$, since it will hold by construction at smaller x . The notation is kept general for the moment since the intended result is, too.

In the given region, $V(x, q^i, q^{-i}) = V(x, \phi^i(x, q^{-i}), q^{-i}) - \phi^i(x, q^{-i}) + q^i$. This implies for the first partial derivative required to evaluate $\mathcal{L}_x V$:

$$\begin{aligned} V_x(x, q^i, q^{-i}) &= \partial_x (V(x, \phi^i(x, q^{-i}), q^{-i}) - \phi^i(x, q^{-i})) \\ &= V_x(x, \phi^i(x, q^{-i}), q^{-i}), \end{aligned} \tag{5.1}$$

where the last line follows from (A.1). Differentiating once more yields

$$\begin{aligned} V_{xx}(x, q^i, q^{-i}) &= \partial_x (V_x(x, \phi^i(x, q^{-i}), q^{-i})) \\ &= V_{xx}(x, \phi^i(x, q^{-i}), q^{-i}) + V_{xq^i}(x, \phi^i(x, q^{-i}), q^{-i}) \phi_x^i(x, q^{-i}). \end{aligned} \tag{5.2}$$

By these formulae, V, V_x , and V_{xx} in the given region are — apart from some correction terms — all evaluated at $(x, \phi^i(x, q^{-i}), q^{-i})$, i.e. at an argument for which the partial differential equation is satisfied by construction. This observation admits the following simplification on $\{\bar{X}^i(q^i, q^{-i}) < x \leq \bar{X}^{-i}(q^{-i}, q^i)\}$:

$$\begin{aligned} -rV + \Pi + \mathcal{L}_x V = & r\phi^i(x, q^{-i}) - rq^i \\ & + \Pi(x, q^i, q^{-i}) - \Pi(x, \phi^i(x, q^{-i}), q^{-i}) \\ & + \frac{1}{2}\sigma^2 x^2 V_{xq^i}(x, \phi^i(x, q^{-i}), q^{-i}) \phi_x^i(x, q^{-i}). \end{aligned} \quad (5.3)$$

Then, if q^i approaches $\phi^i(x, q^{-i})$, the only term on the right hand side that remains is the last. Since $\phi_x^i > 0$ (corresponding to $\bar{X}_{q^i}^i > 0$ and necessary for a well defined reflection strategy), we conclude that the necessary optimality condition can only be satisfied if

$$V_{xq^i}(x, \phi^i(x, q^{-i}), q^{-i}) \leq 0. \quad (5.4)$$

Combined with the first necessary condition (4.14), we must have equality. Equivalently, the optimal myopic investment boundary¹¹ is completely determined by

$$\bar{X}^m(q^i, q^{-i}) = \frac{\bar{p}(q^i, q^{-i})}{P(q^i + q^{-i})}, \quad (5.5)$$

as claimed in the previous section. Requiring $V_{xq^i} = 0$ at the investment boundary is actually the “smooth pasting condition” which is often treated like an abstract, universal optimality condition.

As the general conclusion, whenever player i considers to invest strictly before the opponent, at a boundary admitting a sufficiently smooth and bounded V , it must happen at the myopic boundary, and this is the only boundary at which we will encounter smooth pasting.

Continuing with the particular case $\bar{X}^{-i} \equiv \infty$ and $\bar{X}^i = \bar{X}^m$, we still need to verify for optimality that (5.3) is nonpositive in the entire given region, not only near the boundary. But this follows in fact from the equivalence

$$\partial_{q^i}(\Pi(x, q^i, q^{-i}) - rq^i) \geq 0 \quad (5.6)$$

$$\Leftrightarrow x \geq r(P(q^i + q^{-i}) + q^i P'(q^i + q^{-i}))^{-1} = \frac{r}{p^*} \bar{X}^m,$$

which holds on $\{x \geq \bar{X}^m\}$ since our assumption $r > \mu$ implies $r < p^*$. The only sufficient condition for optimality left unanswered is the third item, which is however irrelevant since $dQ^{-i} \equiv 0$.

¹¹It is indeed a proper reflection boundary with $\bar{X}_{q^i}^m > 0$ and $\lim_{q^i \rightarrow \infty} \bar{X}^m = \infty$.

While the necessary conditions have already fixed X^m and simultaneously ensure that the associated value function will satisfy all sufficient conditions, we need of course the coefficient B^m to see that such V^m indeed exists, and to check the boundedness condition for V^m in the theorem's hypothesis.

Unfortunately, though the present case seems even simpler because i acts like a monopolist, it is not possible to integrate explicitly for B^m as in the previous section. As in later instances, we have to cope with its definition via an integral.

The determining boundary condition (A.1) simplifies in this case to

$$\begin{aligned} B_{q^i}^m(q^i, q^{-i}) &= -\frac{1}{\beta-1}(\bar{X}^m(q^i, q^{-i}))^{-\beta} \\ &= -\frac{1}{\beta-1}(p^*)^{-\beta}(P(q^i + q^{-i}) + q^i P'(q^i + q^{-i}))^\beta \\ &= -\frac{1}{\beta-1}(p^*)^{-\beta}\left(\frac{\alpha-1}{\alpha}q^i + q^{-i}\right)^\beta (q^i + q^{-i})^{-\frac{\beta}{\alpha}-\beta}. \end{aligned} \quad (5.7)$$

Based on this partial derivative, we obtain the following result.

Proposition 5.1. *Let $\bar{X}^i = \bar{X}^m$ given by (5.5), $\bar{X}^{-i} = \infty$, and define $\phi^m(x, q^{-i}) \triangleq \sup\{q \in \mathbb{R}_+ | x \geq \bar{X}^m(q, q^{-i})\} \vee 0$. Then, for any $(x, q^i, q^{-i}) \in \mathbb{R}_+^3$,*

$$V^m(x, q^i, q^{-i}) \triangleq \begin{cases} \frac{\beta}{\beta-1}q^i \frac{xP(q^i+q^{-i})}{p^*} + B^m(q^i, q^{-i})x^\beta & \text{if } x \leq \bar{X}^m(q^i, q^{-i}) \\ V^m(x, \phi^m(x, q^{-i}), q^{-i}) - \phi^m(x, q^{-i}) + q^i & \text{else,} \end{cases}$$

with

$$B^m(q^i, q^{-i}) \triangleq - \int_{q^i}^{\infty} B_{q^i}^m(q, q^{-i}) dq$$

satisfies the hypothesis of Theorem 3.2.

$Q_t^i = q^i \vee \sup_{0 \leq s \leq t} \phi^m(X_s, q^{-i})$ is optimal for any $(x, q^i, q^{-i}) \in \mathbb{R}_+^3$ in problem (2.5) with $\phi^{-i} \equiv 0$.

Proof. Concerning the boundedness condition for arbitrary $Q^i \in \mathcal{N}(q^i)$ with finite cost, one can repeat the proof of Lemma 4.1 with the following estimates:

$$\frac{xP(q^i \vee \phi^m + q^{-i})}{p^*} \leq \frac{\bar{p}(q^i \vee \phi^m, q^{-i})}{p^*} \leq \frac{\alpha}{\alpha-1}$$

and

$$\begin{aligned} -B_{q^i}^m &\leq \frac{1}{\beta-1}(p^*)^{-\beta}(P(q^i + q^{-i}))^\beta \\ \Rightarrow 0 \leq B^m(q^i \vee \phi^m, q^{-i})x^\beta &\leq \frac{\alpha}{\beta-\alpha} \frac{1}{\beta-1} (q^i \vee \phi^m + q^{-i}) \left(\frac{\alpha}{\alpha-1}\right)^\beta. \end{aligned}$$

□

B^m actually can be calculated explicitly in the special case $q^{-i} = 0$, true monopoly. Then,

$$V^m(x, q^i, 0) = \begin{cases} q^i + \frac{\beta}{(\alpha-1)(\beta-\alpha)} \left(\frac{\alpha-1}{\alpha} \frac{x}{p^*} \right)^\alpha & \text{if } x > \bar{X}^m(q^i, 0) \\ \frac{\beta}{\beta-1} q^i \frac{x P(q^i)}{p^*} + \frac{1}{\beta-1} \frac{\alpha}{\beta-\alpha} q^i \left(\frac{\alpha-1}{\alpha} \right)^\beta \left(\frac{x P(q^i)}{p^*} \right)^\beta & \text{else,} \end{cases}$$

which is well defined and finite for $\alpha < \beta$ as claimed before.

5.2 Playing against a myopic investor

In the preceding, we identified the central importance of the myopic investment boundary in any potential equilibrium with two differing reflection strategies, so the natural next step to take is the complementary point of view. Which reflection strategies can be best replies to a myopically investing firm? Thus, suppose the opponent of player i uses the myopic boundary in part of the state space, $\bar{X}^{-i} = \bar{X}^m$, and this is indeed strictly smaller than \bar{X}^i .

Then, the relevant boundary condition to construct player i 's value function V is

$$\begin{aligned} V_{q^{-i}} \left(\frac{\bar{p}(q^{-i}, q^i)}{P(q^i + q^{-i})}, q^i, q^{-i} \right) &= 0 \\ \Leftrightarrow B_{q^{-i}}(q^i, q^{-i}) &= \frac{\beta}{\alpha(\beta-1)} (p^*)^{-\beta} q^i (q^i + q^{-i})^{-\frac{\beta}{\alpha}-\beta} \left(q^i + \frac{\alpha-1}{\alpha} q^{-i} \right)^{\beta-1} \\ &\triangleq B_{q^{-i}}^{\text{tm}}(q^i, q^{-i}). \end{aligned} \quad (5.8)$$

We denote the right hand side by $B_{q^{-i}}^{\text{tm}}$, since player i *tolerates* myopic investment at these points.

If we are able to determine the coefficient B from the preceding equation, the necessary optimality condition is

$$\begin{aligned} V_{q^i} \left(\frac{\bar{p}(q^{-i}, q^i)}{P(q^i + q^{-i})}, q^i, q^{-i} \right) &\leq 1 \\ \Leftrightarrow B_{q^i}(q^i, q^{-i}) &\leq \frac{1}{\alpha(\beta-1)} (p^*)^{-\beta} (q^i + q^{-i})^{-\frac{\beta}{\alpha}-\beta} \left(q^i + \frac{\alpha-1}{\alpha} q^{-i} \right)^{\beta-1} \\ &\quad \cdot ((\beta - \alpha)q^i - (\beta + \alpha - 1)q^{-i}) \\ &\triangleq B_{q^i}^{\text{pm}}(q^i, q^{-i}). \end{aligned} \quad (5.9)$$

If equality holds, this is the relevant boundary condition if player i *preempts* the opponent's myopic investment.

In fact, if player i considered that it might only be optimal in part of the state space not to intervene when the opponent invests at \bar{X}^{-i} , the transition

to the preemption regime has to occur continuously for a proper reflection strategy. At those points, equality must hold in (5.9).

Since the myopic boundary is strictly increasing in the first argument, the preempting decision for player i with q^i fixed is likely to be monotone in q^{-i} . The smaller the latter, the smaller is the expected return from a preemption investment. On the other hand, we saw that player i optimally has to invest no later than at the *own* myopic boundary. From $\bar{p}(q^i, q^{-i}) \leq \bar{p}(q^{-i}, q^i) \Leftrightarrow q^i \leq q^{-i}$ it is clear that player i can only await and tolerate the opponent's investment when having a higher capital stock. Consequently, suppose there exists

$$\bar{q}(q^i) \triangleq \inf\{q \in \mathbb{R}_+ | \bar{X}^i(q^i, q) \leq \bar{p}(q, q^i)/P(q + q^i)\} \leq q^i.$$

The corresponding V can only be continuously differentiable at the transition if $B_{q^i}(q^i, \bar{q}(q^i)) = B_{q^i}^{\text{pm}}(q^i, \bar{q}(q^i))$. This enables us to consider the optimality condition (5.9), although we only know the partial derivative $B_{q^{-i}}^{\text{tm}}$ for $q^{-i} < \bar{q}(q^i)$.

With

$$B_{q^i}(q^i, q^{-i}) = B_{q^i}^{\text{pm}}(q^i, \bar{q}(q^i)) - \int_{q^{-i}}^{\bar{q}(q^i)} B_{q^i q^{-i}}^{\text{tm}}(q^i, q) dq,$$

and

$$B_{q^i}^{\text{pm}}(q^i, q^{-i}) = B_{q^i}^{\text{pm}}(q^i, \bar{q}(q^i)) - \int_{q^{-i}}^{\bar{q}(q^i)} B_{q^i q^{-i}}^{\text{pm}}(q^i, q) dq,$$

(5.9) is on $\{q^{-i} < \bar{q}(q^i)\}$ equivalent to

$$\int_{q^{-i}}^{\bar{q}(q^i)} B_{q^i q^{-i}}^{\text{pm}}(q^i, q) dq \leq \int_{q^{-i}}^{\bar{q}(q^i)} B_{q^i q^{-i}}^{\text{tm}}(q^i, q) dq.$$

One can show¹² that the relation between the two integrands is very clear

$$\begin{aligned} B_{q^{-i} q^i}^{\text{tm}} &= \frac{\beta}{\alpha^2(\beta-1)} (p^*)^{-\beta} (q^i + q^{-i})^{-\frac{\beta}{\alpha} - \beta - 1} \left(q^i + \frac{\alpha-1}{\alpha} q^{-i} \right)^{\beta-2} \\ &\quad \cdot \left(-\beta(q^i)^2 + (\beta/\alpha + \alpha - 1) q^i q^{-i} + (\alpha-1)(q^{-i})^2 \right) \\ B_{q^i q^{-i}}^{\text{pm}} &= \frac{\beta}{\alpha^2(\beta-1)} (p^*)^{-\beta} (q^i + q^{-i})^{-\frac{\beta}{\alpha} - \beta - 1} \left(q^i + \frac{\alpha-1}{\alpha} q^{-i} \right)^{\beta-2} \\ &\quad \cdot \left(-(2\beta-1)(q^i)^2 + (\beta/\alpha + \alpha + \beta - 2) q^i q^{-i} + (\alpha-1) \frac{(\beta + \alpha - 1)}{\alpha} (q^{-i})^2 \right) \\ B_{q^{-i} q^i}^{\text{tm}} &\geq B_{q^i q^{-i}}^{\text{pm}} \Leftrightarrow (q^i)^2 - q^i q^{-i} - \frac{\alpha-1}{\alpha} (q^{-i})^2 \geq 0 \end{aligned}$$

cut,

$$B_{q^i q^{-i}}^{\text{pm}}(q^i, q^{-i}) \leq B_{q^i q^{-i}}^{\text{tm}}(q^i, q^{-i}) \Leftrightarrow \frac{q^i}{q^{-i}} \geq \left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\alpha - 1}{\alpha}} \right) \triangleq D > 1.$$

This implies that if $\bar{q}(q^i)$ really is the upper boundary of an interval for q^{-i} , in which $\bar{X}^i(q^i, q^{-i}) > \bar{X}^m(q^{-i}, q^i)$, this can only be optimal if $\bar{q}(q^i) \leq D^{-1} \cdot q^i$. Put differently, not preempting the myopic investor can only be optimal for player i on $\{q^i \geq D \cdot q^{-i}\}$, when having *sufficiently more* capital than the opponent.

By a similar argument, we can derive a complementary condition for optimally preempting a myopic investor. In a preemption region, the boundary condition for V is equality in (5.9), i.e.

$$B_{q^i}(q^i, q^{-i}) = B_{q^i}^{\text{pm}}(q^i, q^{-i}).$$

Correspondingly, (5.8) turns into the (sufficient) optimality condition

$$B_{q^{-i}}(q^i, q^{-i}) \leq B_{q^{-i}}^{\text{tm}}(q^i, q^{-i}). \quad (5.10)$$

We know that player i optimally invests at the own myopic boundary if this is below the opponent's, i.e. whenever $q^i \leq q^{-i}$. So suppose that for given q^{-i} , player i stops preempting the opponent at

$$\hat{q}(q^{-i}) \triangleq \inf\{q \in \mathbb{R}_+ | \bar{X}^i(q, q^{-i}) > \bar{p}(q^{-i}, q)/P(q + q^{-i})\} \geq q^{-i},$$

and becomes passive. Analogous to the above, (5.10) is then equivalent to

$$\int_{q^i}^{\hat{q}(q^{-i})} B_{q^{-i} q^i}^{\text{tm}}(q, q^{-i}) dq \leq \int_{q^i}^{\hat{q}(q^{-i})} B_{q^{-i} q^i}^{\text{pm}}(q, q^{-i}) dq$$

on $\{q^i < \hat{q}(q^{-i})\}$. It can only be satisfied if $\hat{q}(q^{-i}) \leq D \cdot q^{-i}$, respectively on $\{q^i \leq D \cdot q^{-i}\}$, when player i 's capital is not *too much* larger.

5.3 Equilibrium failure

In Steg (2012), we observed that in any open loop equilibrium the investment behaviour is as follows. The smaller firm invests myopically until having caught up to the other firm. If jumps occur, either only the smaller firm jumps to a capital level not exceeding the opponent's, or both jump to the same level of mutual best myopic replies. We will now determine Markovian strategies that generate exactly these capital processes in any subgame.

In the previous section, we have already exploited the fact that the smaller firm's myopic investment boundary is strictly lower than the larger firm's, $\bar{p}(q^i, q^{-i}) \leq \bar{p}(q^{-i}, q^i) \Leftrightarrow q^i \leq q^{-i}$. But $\bar{X}_{q^{-i}}^m \geq 0$ iff $\alpha \cdot q^{-i} \geq q^i$, so we have to adjust the strategies to ensure that the initial jumps are clearly resolved as in Subsection 3.1. Therefore, note that the *symmetric* myopic capital level is well defined by

$$q^s(x) \triangleq \sup\{q \in \mathbb{R}_+ | x \geq \frac{\bar{p}(q, q)}{P(2q)}\} \vee 0, \quad (5.11)$$

since $\bar{p}(q, q)/P(2q)$ is strictly increasing in q . Whenever there is a simultaneous jump, both firms have to settle at this value. The appropriate myopic Markov strategies are then given by

$$\tilde{\phi}^m(x, q^{-i}) \triangleq \phi^m(x, q^{-i} \vee q^s(x)) = \phi^m(x, q^{-i}) \wedge q^s(x) \quad (5.12)$$

and have the required properties $\tilde{\phi}_{q^{-i}}^m \leq 0$ and $\tilde{\phi}_x^m > 0$ for all $(x, q^{-i}) \in \mathbb{R}_+^2$. In fact, if both players use these strategies, there is a unique solution to the state equation. For any initial state $(x_0, q_0^1, q_0^2) \in \mathbb{R}_+^3$, $Q_0^i = q_0^i \vee \tilde{\phi}^m(x_0, q_0^{-i})$ is uniquely determined for both players, independent of assigning a leader. The state is then outside the forbidden region, $Q_0^i \geq \tilde{\phi}^m(x_0, Q_0^{-i})$, while no player *jumped* across $q^s(x_0)$. However, $Q_0^1 \vee Q_0^2 \geq q^s(x_0)$, and either the smaller firm now expands using the myopic signal ϕ^m , or both are equally sized and simultaneously track $q^s(x)$.

In order to construct a smooth function \tilde{V}^m when both players use $\tilde{\phi}^m$, we again have to find the coefficient \tilde{B}^m which ensures that all boundary conditions are satisfied. For this, we have to distinguish the regimes in which player i 's capital stock is smaller or larger than the opponent's.

Begin with $q^i \leq q^{-i}$. Then, player i invests at the myopic boundary and the related boundary condition is the same as (5.7). Thus, on $\{0 \leq q^i \leq q^{-i}\}$,

$$\tilde{B}^m(q^i, q^{-i}) = \tilde{B}^m(q^{-i}, q^{-i}) - \int_{q^i}^{q^{-i}} B_{q^i}^m(q, q^{-i}) dq. \quad (5.13)$$

This implies for the partial derivatives at the regime boundary

$$\begin{aligned} \partial_{q^i}^- \tilde{B}^m(q, q) &= B_{q^i}^m(q, q) \quad \text{and} \\ \partial_{q^{-i}}^+ \tilde{B}^m(q, q) &= \partial_q \tilde{B}^m(q, q) - B_{q^i}^m(q, q). \end{aligned} \quad (5.14)$$

Now consider the other regime, when i tolerates the myopic investment by the opponent. The related boundary condition here is the same as (5.8). Thus, on $\{0 \leq q^{-i} < q^i\}$,

$$\tilde{B}^m(q^i, q^{-i}) = \tilde{B}^m(q^i, q^i) - \int_{q^{-i}}^{q^i} B_{q^{-i}}^{tm}(q^i, q) dq. \quad (5.15)$$

This implies similarly for the partial derivatives at the regime boundary

$$\begin{aligned}\partial_{q^{-i}}^- \tilde{B}^m(q, q) &= B_{q^{-i}}^{tm}(q, q) \quad \text{and} \\ \partial_{q^i}^+ \tilde{B}^m(q, q) &= \partial_q \tilde{B}^m(q, q) - B_{q^{-i}}^{tm}(q, q).\end{aligned}\tag{5.16}$$

Compare (5.14) and (5.16). For \tilde{B}^m to be continuously differentiable, we must have

$$\partial_q \tilde{B}^m(q, q) = B_{q^i}^m(q, q) + B_{q^{-i}}^{tm}(q, q).$$

This equation in one variable can be integrated to find that

$$\tilde{B}^m(q, q) = -\frac{1}{\beta-1}(p^*)^{-\beta}q^{-\frac{\beta}{\alpha}+1}2^{-\frac{\beta}{\alpha}-\beta}\left(\frac{\beta+1-2\alpha}{\beta-\alpha}\right)\left(\frac{2\alpha-1}{\alpha}\right)^{\beta-1}, \tag{5.17}$$

which together with (5.13) and (5.15) determines \tilde{B}^m on all of \mathbb{R}_+^2 . Now,

$$\begin{aligned}\tilde{V}^m(x, q^i, q^{-i}) &\triangleq \\ \frac{\beta}{\beta-1}q^i \frac{xP(q^i + q^{-i})}{p^*} + \tilde{B}^m(q^i, q^{-i})x^\beta &\quad \text{if } \begin{cases} q^i \geq \tilde{\phi}^m(x, q^{-i}) \\ q^{-i} \geq \tilde{\phi}^m(x, q^i) \end{cases} \\ \tilde{V}^m(x, \tilde{\phi}^m(x, q^{-i}), q^{-i}) - \tilde{\phi}^m(x, q^{-i}) + q^i &\quad \text{if } \begin{cases} q^i < \tilde{\phi}^m(x, q^{-i}) \\ q^{-i} \geq \tilde{\phi}^m(x, q^i) \end{cases} \\ \tilde{V}^m(x, q^i, \tilde{\phi}^m(x, q^i)) &\quad \text{if } \begin{cases} q^i \geq \tilde{\phi}^m(x, q^{-i}) \\ q^{-i} < \tilde{\phi}^m(x, q^i) \end{cases} \\ \tilde{V}^m(x, q^s(x), q^s(x)) - q^s(x) + q^i &\quad \text{if } \begin{cases} q^i < \tilde{\phi}^m(x, q^{-i}) \\ q^{-i} < \tilde{\phi}^m(x, q^i) \end{cases}\end{aligned}\tag{5.18}$$

satisfies the first set of sufficient conditions in Theorem 3.2. Note that in particular, because we have matched (5.14) and (5.16),

$$\begin{aligned}\tilde{V}_{q^i}^m\left(\frac{\bar{p}(q, q)}{P(2q)}, q, q\right) &= 1 \\ \tilde{V}_{q^{-i}}^m\left(\frac{\bar{p}(q, q)}{P(2q)}, q, q\right) &= 0\end{aligned}\tag{5.19}$$

when simultaneous investment occurs.

Proposition 5.2. *Define $\tilde{X}^m(q^i, q^{-i}) \triangleq \sup\{x \in \mathbb{R} | q^i \geq \tilde{\phi}^m(x, q^{-i})\}$ and let (Q^1, Q^2) be the solution of Problem 3.1 if $\bar{X}^1 = \bar{X}^2 = \tilde{X}^m$, with initial state $(x_0, q_0^1, q_0^2) \in \mathbb{R}_+^3$. Then,*

$$\tilde{V}^m(x_0, q_0^i, q_0^{-i}) = J(Q^i, Q^{-i}).$$

Yet, $(\tilde{\phi}^m, \tilde{\phi}^m)$ are not a Markov perfect equilibrium.

\tilde{V}^m satisfies the hypothesis of Theorem 3.2 by a very similar argument as in the proof of Proposition 5.1. However, \tilde{V}^m does not satisfy the necessary conditions for optimality, because the strategies allow the smaller opponent to catch up gradually. We saw in the previous subsection that this cannot be optimal on $\{q^i < D \cdot q^{-i}\}$, with D strictly greater than one. Once the capital levels are within this distance, tolerating further investment is suboptimal. Consequently, the Markovian strategies generating the open loop equilibrium processes are *not* a Markov perfect equilibrium.

6 Collusive equilibria

The previous section tells us that there exists no Markov perfect equilibrium in which the players have different investment boundaries whenever their capital stocks differ. In these cases, the smaller firm necessarily invests myopically, but the larger firm has a strict incentive to preempt before the two have equal capital stocks. Consequently, in any equilibrium, the players must use the same investment boundary over part of the state space, also for some heterogeneous capital levels. An example is of course the earlier obtained Bertrand equilibrium with a shared price trigger.

On the other hand, it is clear from the equilibrium definition that at least one of the firms must be indifferent to invest at the common boundary. We can express this requirement in terms of the value function V .

So, suppose the players use the same reflection boundary \bar{X} over part of the state space. At the respective own reflection boundary, V always has to satisfy the boundary condition (4.11), $V_{q^i}(\bar{X}, q^i, q^{-i}) = 1$. Expanded, it is equivalent to

$$B_{q^i} = \bar{X}^{-\beta} - \frac{\beta}{\beta-1}(p^*)^{-1} \left(\frac{\alpha-1}{\alpha} q^i + q^{-i} \right) (q^i + q^{-i})^{-\frac{1}{\alpha}-1} \bar{X}^{1-\beta} \triangleq B_{q^i}^p. \quad (6.1)$$

We denote the right hand side by $B_{q^i}^p$, since this is the condition associated with investment, resp. preemption.

The intended indifference condition is then

$$V_{q^{-i}}(\bar{X}, q^i, q^{-i}) = 0, \quad (6.2)$$

i.e. the value of the strategy is not affected if the opponent invests at the same boundary. In the previous case, myopic investment, these two conditions only held simultaneously for each player when both have equal capital stocks, cf. (5.19). We can also expand the latter condition to

$$B_{q^{-i}} = \frac{1}{\beta-1}(p^*)^{-1} q^i \frac{\beta}{\alpha} (q^i + q^{-i})^{-\frac{1}{\alpha}-1} \bar{X}^{1-\beta} \triangleq B_{q^{-i}}^t. \quad (6.3)$$

Similarly as before, $B_{q^{-i}}^t$ stands for tolerating investment.

Now, we can turn the approach taken so far around and try to identify a reflection boundary \bar{X} that is consistent with (6.1) and (6.3) simultaneously.

Specifically, since $B_{q^i q^{-i}} = B_{q^{-i} q^i}$ must hold, we obtain¹³ a partial differential equation for \bar{X} :

$$\bar{X}^{-1} p^* (q^i + q^{-i})^{\frac{1}{\alpha} + 1} \bar{X}_{q^{-i}} - \left(\frac{\alpha-1}{\alpha} q^i + q^{-i} \right) \bar{X}_{q^{-i}} - \frac{1}{\alpha} q^i \bar{X}_{q^i} = 0. \quad (6.4)$$

If we require that both player be indifferent at the sought reflection boundary, it is easy to see¹⁴ that the only solution of this PDE is the Bertrand price trigger $\bar{X}(q^i, q^{-i}) = p^* (q^i + q^{-i})^{\frac{1}{\alpha}}$.

However, it is sufficient for an equilibrium that only one player is indifferent, say who has more capital installed. One can narrow down further solutions of (6.4) by reflecting that a firm might only care about the observed price in the investment decision, and that the price of indifference depends only on the own installed capital. Correspondingly, we look for solutions of the functional form

$$\bar{X}(q^i, q^{-i}) = f(q^i) (q^i + q^{-i})^{\frac{1}{\alpha}}. \quad (6.5)$$

In this case, we calculate

$$\begin{aligned} \bar{X}_{q^i} &= (q^i + q^{-i})^{\frac{1}{\alpha} - 1} \left((q^i + q^{-i}) f' + \frac{1}{\alpha} f \right) \quad \text{and} \\ \bar{X}_{q^{-i}} &= (q^i + q^{-i})^{\frac{1}{\alpha} - 1} \frac{1}{\alpha} f. \end{aligned}$$

(6.4) now dramatically simplifies to

$$f + q^i f' = p^*,$$

the general solution of which is

$$f(q^i) = p^* + c \cdot (q^i)^{-1}. \quad (6.6)$$

Since investment below the Bertrand price cannot be optimal, we only admit constants $c \in [0, \infty)$.

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$$\begin{aligned} \partial_{q^i} B_{q^{-i}}^t &= \frac{1}{\beta-1} (p^*)^{-1} \frac{\beta}{\alpha} \bar{X}^{-\beta} (q^i + q^{-i})^{-\frac{1}{\alpha} - 2} \left(\left(-\frac{1}{\alpha} q^i + q^{-i} \right) \bar{X} - (\beta-1) q^i (q^i + q^{-i}) \bar{X}_{q^i} \right) \\ \partial_{q^{-i}} B_{q^i}^p &= -\beta \bar{X}^{-\beta-1} \bar{X}_{q^{-i}} - \frac{\beta}{\beta-1} (p^*)^{-1} \bar{X}^{-\beta} (q^i + q^{-i})^{-\frac{1}{\alpha} - 2} \\ &\quad \cdot \left(-\frac{1}{\alpha} \left(-\frac{1}{\alpha} q^i + q^{-i} \right) \bar{X} - (\beta-1) \left(\frac{\alpha-1}{\alpha} q^i + q^{-i} \right) (q^i + q^{-i}) \bar{X}_{q^{-i}} \right) \end{aligned}$$

¹⁴Switch roles by swapping $i, -i$ and take the difference of the two equations.

With all candidates other than the Bertrand equilibrium, i.e. whenever $c > 0$, indeed only player i can be indifferent. The question which player to make indifferent is easily answered, if we check whether (6.5) defines proper reflection strategies. In fact, $\bar{X}_{q^{-i}}$ is nonnegative since f is, but $\bar{X}_{q^i} > 0$ iff

$$(q^i + q^{-i})f' + \frac{1}{\alpha}f > 0$$

$$\Leftrightarrow p^*(q^i)^2 - (\alpha - 1)cq^i - \alpha cq^{-i} > 0.$$

This only holds for a wide range of capital levels if $q^i \geq q^{-i}$ and is then implied by

$$q^i > c \frac{2\alpha - 1}{p^*}. \quad (6.7)$$

Consequently, if we define the symmetric investment boundary

$$\bar{X}^c(q^i, q^{-i}) \triangleq (p^* + c \cdot (q^i \vee q^{-i})^{-1})(q^i + q^{-i})^{\frac{1}{\alpha}}, \quad (6.8)$$

it is a proper reflection boundary on $(c \frac{2\alpha-1}{p^*}, \infty)^2$ with $\bar{X}_{q^i}^c > 0$ and $\bar{X}_{q^{-i}}^c > 0$.

Furthermore, we can apply Theorem 3.2 to show that \bar{X}^c is a mutual best reply.

Proposition 6.1. *Let $\bar{X}^1 = \bar{X}^2 = \bar{X}^c$ given by (6.8) with fixed $c \in \mathbb{R}_+$, and define $\phi^c(x, q^{-i}) \triangleq \sup\{q \in \mathbb{R}_+ | x \geq \bar{X}^c(q, q^{-i})\} \vee 0$. Further fix $(x_0, q_0^1, q_0^2) \in \mathbb{R}_+ \times (c \frac{2\alpha-1}{p^*}, \infty)^2$ and set $Q^i = (q_0^i \vee \sup_{0 \leq s \leq t} \phi^c(X_s, q_0^{-i}))_{t \geq 0}$ and $Q^{-i} \equiv q_0^{-i}$.*

Define for any $(x, q^1, q^2) \in \mathbb{R}_+ \times (c \frac{2\alpha-1}{p^}, \infty)^2$*

$$V^c(x, q^i, q^{-i}) \triangleq \begin{cases} \frac{\beta}{\beta-1} q^i \frac{x P(q^i + q^{-i})}{p^*} + B^c(q^i, q^{-i}) x^\beta & \text{if } x \leq \bar{X}^c(q^i, q^{-i}) \\ V^c(x, \phi^c(x, q^{-i}), q^{-i}) - \phi^c(x, q^{-i}) + q^i & \text{else,} \end{cases}$$

where

$$B^c(q^i, q^{-i}) \triangleq - \int_{q^i}^{\infty} B_{q^i}^c(q, q^{-i}) dq, \quad (6.9)$$

and

$$B_{q^i}^c(q^i, q^{-i}) \triangleq \left(1 - \frac{\beta}{\beta-1} (p^*)^{-1} \left(\frac{\alpha-1}{\alpha} q^i + q^{-i}\right) (q^i + q^{-i})^{-\frac{1}{\alpha}-1} \bar{X}^c\right) (\bar{X}^c)^{-\beta}. \quad (6.10)$$

Then, $V^c(x_0, q_0^i, q_0^{-i}) = J(Q^i, Q^{-i}) = V^*(x_0, q_0^i, q_0^{-i})$.

The proof is in the Appendix. In the proposition, we selected the processes Q^i and Q^{-i} by solving Problem 3.1 and determined the solution to the verification problems (2.5). The outcome depends on which player is the

leader, because there is full preemption. As the involved investment occurs at higher prices than Bertrand, the payoffs to the players also differ. Nevertheless, (ϕ^c, ϕ^c) are a Markov perfect equilibrium, because we can select feasible capital stock processes that are a Pareto improvement compared to full preemption as in the proposition. We exploit the indifference of the respective larger firm.

Theorem 6.2. *For any $c \in \mathbb{R}_+$, (ϕ^c, ϕ^c) as defined in Proposition 6.1 is a Markov perfect equilibrium for initial capital levels $(q_0^1, q_0^2) \in (c^{\frac{2\alpha-1}{p^*}}, \infty)^2$.*

The proof is in the Appendix. Theorem 6.2 answers the “open question” posed in Back and Paulsen (2009) whether there exist any other subgame perfect equilibria of the game with reflection strategies besides the Bertrand equilibrium.

Our additional class of equilibria is driven by the fact that preemptive investment of the larger firm lowers the *price* at which the smaller firm invests. The smaller firm does not face this externality and it also has the greater local investment incentive, based on marginal revenue. Thus, it is able to set a *dynamic* investment price boundary above Bertrand which leaves the opponent indifferent. Note that the investment price boundary has to decline gradually in equilibrium, since we saw in Section 4 that preemption is otherwise the dominant strategy.

With the present example we falsified the frequent conjecture that preemption concerns completely eliminate option values under arbitrarily divisible investment. The friction implied by uncertain returns and the decreasing marginal revenue effect do enable more collusive outcomes with quantities less than those enforcing the Bertrand price. We consequently opt for calling these *collusive* equilibria.

7 Conclusion

In this work we focused on the strategic effect of capital commitment with arbitrarily divisible, irreversible investment under dynamic uncertainty. In order to be able to verify formally whether ideal circumstances for preemption necessarily eliminate all option values, we proposed a duopoly model with closed loop strategies. Using a state space representation, we enabled the players to make investment decisions by choosing desired capital levels, which are relevant only when exceeding installed capital.

This approach is inspired by the signal processes arising in the open loop case and allowed to avoid artificial bounds on the speed of capital adjustment — so singular control was admissible —, while ensuring consistency

across subgames. Since one encounters the typical difficulty of continuous-time games, that a priori natural strategies do not uniquely determine the course of the state, particular attention has to be paid to an appropriate equilibrium concept. We introduced a strong optimality notion by requiring that there exists a solution in capital processes, such that no player has an incentive to choose any other control process at any state, even if unilateral perfect preemption was made feasible by hypothesis.

Our equilibrium definition has been complemented by establishing a verification theorem. It serves to solve the optimal control problems arising in equilibrium verification in the presence of Markovian strategies and geometric Brownian motion as exogenous shock process. We subsequently applied the theorem to the example in which firms face an inverse demand with constant elasticity, in order to derive Markov perfect equilibria.

The simplest instance has been the arguably expected equilibrium, showing perfectly competitive investment. It results when firms use a constant price to trigger investment. Then, implied by stationarity, the decision whether to preempt at the threshold is always the same. By repeated investment opportunities and rent equalization, firms are always indifferent when investing and make zero expected profits.

The observation that firms invest (or threaten to) at an identical threshold is an important aspect of any equilibrium. Whenever the thresholds differ, we could show that one firm necessarily invests myopically to behave optimally. Then, it can only be optimal for the opponent to refrain from preemption when having sufficiently more capital already installed. On the other hand, when having too much capital, preemption is definitely unprofitable.

Consequently, there is potential for equilibria without full preemption. While those Markovian strategies generating the open loop equilibrium processes for any starting state are not eligible, we identified a particular class of Markov perfect equilibria with positive option values.

As the crucial component, a dynamic output good price to trigger investment allows for collusive behaviour. The respective larger firm is kept indifferent in these equilibria when the opponent invests. By refraining from preemption, the larger firm allows both players to obtain the highest possible returns, given the equilibrium Markovian strategies, since any investment by the larger firm reduces the (common) investment boundary. Neither player has an incentive to deviate from the Pareto optimal solution in capital processes. When they have equal capital stocks, simultaneous investment occurs. One observes collusion similar to the simultaneous investment equilibria in the real option exercise games.

Since the well-known technical problem of choosing an outcome has been approached only exogenously, continuing research concentrates on endoge-

nous solutions.

A Appendix

Proof of Theorem 3.2

Suppose there exists V satisfying the first set of sufficient conditions.

Note the frequently used equivalence

$$x = \bar{X}^i(q^i, q^{-i}) \Leftrightarrow q^i = \phi^i(x, q^{-i}).$$

Note also that ϕ^i and ϕ^{-i} are continuously differentiable in x and q^{-i} and q^i , respectively, since \bar{X}^i and \bar{X}^{-i} are. We calculate the following partial derivatives.

On $\{x > \bar{X}^i(q^i, q^{-i})\}$:

$$\begin{aligned} V_{q^i} &= 1 = V_{q^i}(x, \phi^i(x, q^{-i}), q^{-i}) \quad (V_{q^i} \text{ continuous}) \\ V_{q^{-i}} &= \partial_{q^{-i}}(V(x, \phi^i(x, q^{-i}), q^{-i}) - \phi^i(x, q^{-i})) \\ &= V_{q^{-i}}(x, \phi^i(x, q^{-i}), q^{-i}) \end{aligned} \tag{A.1}$$

To obtain the last line, we already used the first result.

On $\{\bar{X}^{-i}(q^{-i}, q^i) < x \leq \bar{X}^i(q^i, q^{-i})\}$:

$$\begin{aligned} V_{q^{-i}} &= 0 = V_{q^{-i}}(x, q^i, \phi^{-i}(x, q^i)) \quad (V_{q^{-i}} \text{ continuous}) \\ V_{q^i} &= \partial_{q^i}(V(x, q^i, \phi^{-i}(x, q^i))) \\ &= V_{q^i}(x, q^i, \phi^{-i}(x, q^i)) \end{aligned} \tag{A.2}$$

Again the first line implies the last.

Now consider an initial state (x_0, q_0^1, q_0^2) such that $X_0 = x_0 \leq \bar{X}^1(q_0^1, q_0^2) \wedge \bar{X}^2(q_0^2, q_0^1)$. Then, the paths of the semi-martingale (X, Q^1, Q^2) , where the capital processes are the hypothesized solution to Problem 3.1, satisfy $X_t \leq \bar{X}^1(Q_t^1, Q_t^2) \wedge \bar{X}^2(Q_t^2, Q_t^1)$ for all $t \in [0, \infty)$ \mathbf{P} -a.s.

Since V is class $\mathcal{C}^{2,1,1}$ on $\{x \leq \bar{X}^i(q^i, q^{-i}) \wedge \bar{X}^{-i}(q^{-i}, q^i)\}$, we can apply Itô's lemma to obtain for arbitrary $T \in [0, \infty)$

$$\begin{aligned}
e^{-rT}V(X_T, Q_T^i, Q_T^{-i}) - V(X_0, Q_0^i, Q_0^{-i}) = & \\
& \int_0^T e^{-rt}(-rV(X_t, Q_t^i, Q_t^{-i}) + \mathcal{L}_x V(X_t, Q_t^i, Q_t^{-i})) dt \\
& + \int_0^T e^{-rt} \mu X_t V_x(X_t, Q_t^i, Q_t^{-i}) dB_t \\
& + \int_0^T e^{-rt} V_{q^i}(X_t, Q_t^i, Q_t^{-i}) dQ_t^{i,c} \\
& + \int_0^T e^{-rt} V_{q^{-i}}(X_t, Q_t^i, Q_t^{-i}) dQ_t^{-i} \\
& + \sum_{t \leq T} e^{-rt} \Delta V(X_t, Q_t^i, Q_t^{-i}), \quad \mathbf{P}\text{-a.s.},
\end{aligned} \tag{A.3}$$

where $Q^{i,c}$ is the continuous part of Q^i and the sum is over the jumps of Q^i up to T . Note that the presently discussed Q^i and Q^{-i} are continuous, but we will later allow for jumps of Q^i alone.

The second integral is a martingale, since for all $(Q^i, Q^{-i}) \in \mathcal{A}(q^i) \times \mathcal{A}(q^{-i})$ and $T \in \mathbb{R}_+$, $(\mathbf{1}_{\{t \leq T\}} e^{-rt} \mu X_t V_x(X_t, Q_t^i, Q_t^{-i}))_{t \geq 0} \in L^2(\mathbf{P} \otimes dt)$ by continuity of V_x . It disappears when we now take expectations. We also rearrange and subtract the payoff stream including investment costs up to T on both sides.

$$\begin{aligned}
V(X_0, Q_0^i, Q_0^{-i}) - \mathbf{E} \left[\int_0^T e^{-rt} \Pi(X_t, Q_t^i, Q_t^{-i}) dt - \int_0^T e^{-rt} dQ_t^i \right] = & \\
\mathbf{E} \left[- \int_0^T e^{-rt} (-rV(X_t, Q_t^i, Q_t^{-i}) + \Pi(X_t, Q_t^i, Q_t^{-i}) + \mathcal{L}_x V(X_t, Q_t^i, Q_t^{-i})) dt \right. & \\
- \int_0^T e^{-rt} (V_{q^i}(X_t, Q_t^i, Q_t^{-i}) - 1) dQ_t^{i,c} & \\
- \sum_{t \leq T} e^{-rt} (\Delta V(X_t, Q_t^i, Q_t^{-i}) - \Delta Q_t^i) & \\
- \int_0^T e^{-rt} V_{q^{-i}}(X_t, Q_t^i, Q_t^{-i}) dQ_t^{-i} & \\
\left. + \mathbf{E} \left[e^{-rT} V(X_T, Q_T^i, Q_T^{-i}) \right] \right] &
\end{aligned} \tag{A.4}$$

All integrals and the sum on the right hand side are zero by the sufficient conditions. $dQ^i > 0$ only if $X_t \geq \bar{X}^i(Q_t^i, Q_t^{-i})$ and then $V_{q^i} = 1$. Similarly,

$dQ^{-i} > 0$ only if $\bar{X}^{-i}(Q_t^{-i}, Q_t^i) \leq X_t < \bar{X}^i(Q_t^i, Q_t^{-i})$ and then $V_{q^{-i}} = 0$. Finally, the last term goes to zero by hypothesis if we let T go to ∞ , so

$$V(x_0, q_0^i, q_0^{-i}) - J(Q^i, Q^{-i}) = 0, \quad (\text{A.5})$$

since there was no initial jump. However, for initial states (x_0, q_0^1, q_0^2) inside the forbidden region, the equality still holds since the jumps occur while $V_{q^i} = 1$ and $V_{q^{-i}} = 0$.

For the next claim, consider the additional set of sufficient conditions. Note that for any $Q^i \in \mathcal{N}(q_0^i)$, Q^{-i} given by ϕ^{-i} solves the Skorohod problem

$$\begin{aligned} X_t &\leq \bar{X}^{-i}(Q_t^{-i}, Q_t^i), \quad t \in [0, \infty) \\ \int_0^\infty (1 - \mathbf{1}_{\{X_t \geq \bar{X}^{-i}(Q_t^{-i}, Q_t^i)\}}) dQ_t^{-i} &= 0, \end{aligned} \quad (\text{A.6})$$

P-a.s., since no jump of Q^i can move the state into $-i$'s forbidden region, see (3.5). Thus, it is sufficient that V is class $\mathcal{C}^{2,1,1}$ on $\{x \leq \bar{X}^{-i}(q^{-i}, q^i)\}$ to apply Itô's lemma on that region. Equation (A.4) remains valid for arbitrary $Q^i \in \mathcal{N}(q_0^i)$ after the initial jumps, where $X_0 \leq \bar{X}^{-i}(Q_0^{-i}, Q_0^i)$ follows for all dQ_0^i .

The second given condition implies that $V_{q^i} \leq 1$ on \mathbb{R}_+^3 , since in the region $\{\bar{X}^{-i}(q^{-i}, q^i) < x \leq \bar{X}^i(q^i, q^{-i})\}$ it is evaluated at the lower boundary, see (A.2).

The third condition similarly implies $V_{q^{-i}} \leq 0$ on $\{x \geq \bar{X}^{-i}(q^{-i}, q^i)\}$ (the only region where dQ^{-i} can be strictly positive), since on $\{x \geq \bar{X}^i(q^i, q^{-i})\}$, $V_{q^{-i}}$ it is evaluated at the boundary, see (A.1).

The last condition implies $-rV + \Pi + \mathcal{L}_x V \leq 0$ on $\{x \leq \bar{X}^{-i}(q^{-i}, q^i)\}$, to which the state is constrained after initial jumps.

Together, the conditions imply that the integrals and sum on the right hand side of (A.4) are nonnegative. Letting again T go to ∞ , we obtain

$$V(x_0, q_0^i, q_0^{-i}) - J(Q^i, Q^{-i}) \geq 0. \quad (\text{A.7})$$

A similar remark regarding initial states inside the forbidden region applies and we conclude that there is no feasible capital stock process for player i with a payoff dominating V .

Now suppose the first necessary condition is violated, i.e. there exists a state such that $x < \bar{X}^i(q^i, q^{-i})$, where the strategy prescribes no investment, but where $V_{q^i} > 1$. Then, by continuity of the derivative, there exists $\epsilon > 0$ such that $V_{q^i}(x, q, q^{-i}) > 1$ for all $q \in [q^i, q^i + \epsilon]$. Then, the payoff from an ϵ -investment, followed by pursuing the given reflection strategy, is $V(x, q^i + \epsilon, q^{-i}) - \epsilon > V(x, q^i, q^{-i})$.

Similarly, if the second necessary condition is violated, there exists a state $x \leq \bar{X}^{-i}(q^{-i}, q^i)$ where $-rV + \Pi + \mathcal{L}_x V > 0$, which can only happen where $\bar{X}^{-i}(q^{-i}, q^i) < \bar{X}^i(q^i, q^{-i})$. Since V there is twice continuously differentiable in x , there exists $\epsilon > 0$ such that $-rV + \Pi + \mathcal{L}_x V > 0$ for all $x' \in [x - \epsilon, x]$. Then, for all initial states (x', q^i, q^{-i}) , the capital process $(q^i \vee \mathbf{1}_{\{t \geq \tau_\epsilon\}} Q^i)$, where $\tau_\epsilon = \inf\{t \geq 0 | X_t \notin (x - \epsilon, x)\}$ yields a higher payoff than Q^i by the Itô-formula, since $dQ^{-i} = 0$ before τ_ϵ . \square

Proof of Lemma 4.1

We want to prove that

$$\lim_{T \rightarrow \infty} \mathbf{E}[e^{-rT} V(X_T, Q_T^i, q^{-i})] = 0$$

for arbitrary $Q^i \in \mathcal{A}(0)$ with finite cost.

The value function candidate is given by

$$V(x, q^i, q^{-i}) = \begin{cases} \frac{\beta}{\beta-1} q^i \frac{x P(q^i + q^{-i})}{p^*} + B(q^i, q^{-i}) x^\beta & \text{if } q^i \geq \phi^i(x, q^{-i}) \\ V(x, \phi^i, q^{-i}) - \phi^i + q^i & \text{else.} \end{cases}$$

We hide the arguments of ϕ^i in the following.

First, we derive joint bounds for all terms but Bx^β . Since the reflection price is not less than the Bertrand price,

$$\begin{aligned} x P(\phi^i + q^{-i}) &\geq p^* \\ \Rightarrow \frac{\beta}{\beta-1} \phi^i \frac{x P(\phi^i + q^{-i})}{p^*} - \phi^i &\geq 0. \end{aligned}$$

For an upper bound, note that the price term is in both cases bounded by a constant:

$$\frac{x P(q^i \vee \phi^i + q^{-i})}{p^*} \leq \frac{p^{-i}}{p^*}. \quad (\text{A.8})$$

Thus, we can estimate both cases of V simultaneously by

$$V(x, q^i, q^{-i}) \in \left(0, \frac{\beta}{\beta-1} \frac{p^{-i}}{p^*} (q^i \vee \phi^i)\right) + B(q^i \vee \phi^i, q^{-i}) x^\beta \quad (\text{A.9})$$

The remaining term can also be estimated by using the price bound (A.8). In the present case:

$$|B^{p^{-i}}(q^i \vee \phi^i, q^{-i})x^\beta| \leq \frac{\alpha}{\beta - \alpha} \left| (q^i \vee \phi^i) \left(\frac{p^{-i}(\alpha - 1)}{(r - \mu)\alpha} - 1 \right) + q^{-i} \left(\frac{p^{-i}}{p^*} - 1 \right) \right|$$

Thus, V is bounded by a linear function of $q^i \vee \phi^i$.

Since ϕ^i never pushes the price below the Bertrand price, the Bertrand quantity (particularly neglecting competitive output) is an upper bound for it:

$$xP(\phi^i + q^{-i}) \geq p^* \Rightarrow \phi^i \leq \left(\frac{x}{p^*} \right)^\alpha - q^{-i} \leq \left(\frac{x}{p^*} \right)^\alpha.$$

This proves the first claim, since $\lim_{T \rightarrow \infty} \mathbf{E}[e^{-rT} X_T^\alpha] = 0$ for $\alpha < \beta$ and $\lim_{T \rightarrow \infty} \mathbf{E}[e^{-rT} Q_T^i] = 0$ by hypothesis.

For the second claim, note that the capital process resulting from the Bertrand reflection strategy $\bar{X}^i = p^*/P(q^i + q^{-i})$ is $Q_t^i = q_0^i \vee (X_t^*/p^*)^\alpha - q_0^{-i}$ with $X_t^* \triangleq \sup_{0 \leq s \leq t} X_s$. Consequently, the investment cost is bounded if the following holds

$$\mathbf{E} \left[\int_0^\infty e^{-rt} (X_t^*)^\alpha dt \right] = \frac{\beta}{\beta - \alpha} \in \mathbb{R}_+ \Leftrightarrow \alpha < \beta, \quad (\text{A.10})$$

cf. Riedel and Su (2011); the left hand side equals

$$\mathbf{E} [(X_{\tau(r)}^*)^\alpha] = \frac{\Psi^{\alpha Y}(r)}{\Psi^{\alpha Y}(r) - 1},$$

where $\tau(r)$ is an independent, exponentially distributed time, and $\Psi^{\alpha Y}(r)$ is the Laplace exponent of the process αY at r . In our case, $X_t^\alpha = x_0^\alpha e^{\alpha Y_t}$, i.e. $Y_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t \Rightarrow \Psi^{\alpha Y}(r) = \frac{\beta}{\alpha}$. \square

Lemma A.1

Lemma A.1. *Let $(q^i, q^{-i}) \in \mathbb{R}_+^2$ and $\bar{X}^{-i} \geq p^*/P(q^i + q^{-i})$ satisfying (3.5) be given. Suppose $Q^i \in \mathcal{N}(q^i)$ and $Q^{-i} = q^{-i} \vee (\sup_{0 \leq s \leq t} \phi^{-i}(X_s, Q_s^i))_{t \geq 0} \in \mathcal{N}(q^{-i})$.*

Define the cumulative Bertrand quantity

$$Q^B \triangleq q^i \vee \left(\sup_{0 \leq s \leq t} (X_s/p^*)^\alpha - q^{-i} \right)_{t \geq 0},$$

as well as the capped capital process

$$\hat{Q}^i \triangleq Q^i \wedge Q^B$$

and the resulting

$$\hat{Q}^{-i} \triangleq q^{-i} \vee \left(\sup_{0 \leq s \leq t} \phi^{-i}(X_s, \hat{Q}_s^i) \right)_{t \geq 0}.$$

Then,

$$J(Q^i, Q^{-i}) - J(\hat{Q}^i, \hat{Q}^{-i}) \leq 0.$$

Proof. Define the stopping times

$$\tau^B \triangleq \inf\{t \geq 0 | Q_t^i > Q_t^B\} \quad \text{and} \quad \hat{\tau}^B \triangleq \inf\{t \geq \tau^B | Q_t^B \geq Q_t^i\}$$

and note that $Q_t^{-i} = \hat{Q}_t^{-i} = Q_{\tau^B}^{-i}$ for $t \in [\tau^B, \hat{\tau}^B]$.

This allows to use Fubini's theorem as in Steg (2012) to obtain

$$\begin{aligned} & \mathbf{E} \left[\int_{\tau^B}^{\hat{\tau}^B} e^{-rt} \Pi(X_t, Q_t^i, Q_t^{-i}) dt - \int_{\tau^B}^{\hat{\tau}^B} e^{-rt} dQ_t^i \right] \\ & - \mathbf{E} \left[\int_{\tau^B}^{\hat{\tau}^B} e^{-rt} \Pi(X_t, Q_t^B, \hat{Q}_t^{-i}) dt - \int_{\tau^B}^{\hat{\tau}^B} e^{-rt} dQ_t^B \right] \\ & = \int_0^\infty \mathbf{E} \left[\mathbf{1}_{\{Q_{\tau^B}^i \leq l \leq Q_{\hat{\tau}^B}^i\}} \int_{\tau^{Q^i}(l)}^{\tau^{Q^B}(l)} e^{-rt} (\Pi_{Q^i}(X_t, l, Q_{\tau^B}^{-i}) - r) dt \right] dl, \end{aligned}$$

where $\tau^{Q^i}(l) \triangleq \inf\{t \geq 0 | Q_t^i \geq l\}$ and $\tau^{Q^B}(l)$ analogously.

Now note that in the random interval $[\tau^{Q^i}(l), \tau^{Q^B}(l)]$,

$$\Pi_{Q^i}(X_t, l, Q_{\tau^B}^{-i}) \leq X_t P(l + Q_{\tau^B}^{-i}) \leq p^* X_t,$$

so that for any l , stopping at $\tau^{Q^i}(l)$ immediately is optimal for maximizing the expectation with respect to stopping times $\tau \geq \tau^{Q^i}(l)$. Consequently, for any l , the expectation is nonpositive and this implies the same for the payoff difference over $[\tau^B, \hat{\tau}^B]$. \square

Proof of Proposition 6.1

Since $\bar{X}^1 = \bar{X}^2$, it is easy to see that Q^i, Q^{-i} solve Problem 3.1 for initial state (x_0, q_0^1, q_0^2) . It remains to show that V^c satisfies the hypothesis and sufficient conditions of Theorem 3.2.

Note that B^c in (6.9) is well defined, since for fixed q^{-i} , $B_{q^i}^c$ goes to zero at speed $(q^i)^{-\frac{\beta}{\alpha}}$ when q^i gets large. One can indeed show by repeating the

proof of Lemma 4.1 that V^c also satisfies the boundedness condition, just using different bounds for B^c and the reflection price:

$$\frac{xP(q^i \vee \phi^c(x, q^{-i}) + q^{-i})}{p^*} \leq \frac{2\alpha}{2\alpha - 1}$$

and

$$\begin{aligned} -\frac{\beta}{\beta - 1} \left(\frac{P(q^i + q^{-i})}{p^*} \right)^\beta &\leq B_{q^i}^c \leq \left(\frac{P(q^i + q^{-i})}{p^*} \right)^\beta \\ \Rightarrow \left| B^c(q^i \vee \phi^c, q^{-i})x^\beta \right| &\leq \frac{\alpha}{\beta - \alpha} \frac{\beta}{\beta - 1} (q^i \vee \phi^c + q^{-i}) \left(\frac{2\alpha}{2\alpha - 1} \right)^\beta. \end{aligned}$$

Note finally that the process Q^{-i} , which results from the Markovian strategy ϕ^c given arbitrary $Q^i \in \mathcal{A}(0)$, is dominated by the Bertrand quantity and thus has finite investment cost.

All sufficient conditions are actually satisfied by construction, except for the two sufficient conditions relating to the partial derivatives of V^c . The easier one, $V_{q^i}^c \leq 1$ is equivalent to (4.14) as we have shown. Suppose wlog $q^{-i} \geq q^i$. Then,

$$p^* + c(q^{-i})^{-1} \leq \bar{p}(q^i, q^{-i})$$

is satisfied for all $q^{-i} \geq q^i$ iff $q^i \geq c \frac{2\alpha-1}{p^*}$, the restriction we already encountered. Note that $\bar{p}(q^i, q^{-i}) \leq \bar{p}(q^{-i}, q^i)$.

The other sufficient condition is $V_{q^{-i}}(\bar{X}^c, q^i, q^{-i}) \leq 0$. It is equivalent to $B_{q^{-i}}^c \leq B_{q^{-i}}^t$, cf. (6.3) using \bar{X}^c . For the player with larger capital stock, it holds with equality by construction. Still supposing $q^{-i} \geq q^i$, we can thus also write for player i

$$B_{q^{-i}}^c(q^i, q^{-i}) = B_{q^{-i}}^t(q^{-i}, q^{-i}) - \int_{q^i}^{q^{-i}} B_{q^i q^{-i}}^c(q, q^{-i}) dq,$$

and compare to

$$B_{q^{-i}}^t(q^i, q^{-i}) = B_{q^{-i}}^t(q^{-i}, q^{-i}) - \int_{q^i}^{q^{-i}} B_{q^{-i} q^i}^t(q, q^{-i}) dq.$$

For concreteness,

$$B_{q^{-i}}^t(q^i, q^{-i}) = \frac{1}{\beta-1} (p^*)^{-1} q^i \frac{\beta}{\alpha} (q^i + q^{-i})^{-\frac{\beta}{\alpha}-1} (p^* + c(q^{-i})^{-1})^{1-\beta}$$

and

$$\begin{aligned} B_{q^i}^c(q^i, q^{-i}) &= \frac{1}{\beta-1} (p^*)^{-1} (q^i + q^{-i})^{-\frac{\beta}{\alpha}-1} (p^* + c(q^{-i})^{-1})^{-\beta} \\ &\quad \cdot \left(p^*(\beta-1)(q^i + q^{-i}) - \beta \left(\frac{\alpha-1}{\alpha} q^i + q^{-i} \right) (p^* + c(q^{-i})^{-1}) \right). \end{aligned}$$

Then, a lengthy calculation yields that

$$\begin{aligned} B_{q^i q^{-i}}^c &\geq B_{q^{-i} q^i}^t \\ \Leftrightarrow q^{-i} &\geq c \frac{\alpha - 1}{p^*}. \end{aligned}$$

This is a weaker restriction than already imposed and thus $B_{q^{-i}}^c \leq B_{q^{-i}}^t$ for all $q^{-i} \geq q^i \geq c \frac{2\alpha-1}{p^*}$. Consequently, $V_{q^{-i}}(\bar{X}^c, q^i, q^{-i}) \leq 0$. \square

Proof of Theorem 6.2

Introduce the symmetric capital levels which are just on the investment boundary by

$$Q^s(x) \triangleq \sup\{q \in \mathbb{R}_+ \mid x \geq \bar{X}^c(q, q)\} \vee 0, \quad (\text{A.11})$$

which is well defined for $x \in \mathbb{R}_+$ because

$$\partial_q \bar{X}^c(q, q) = 2^{\frac{1}{\alpha}} q^{\frac{1}{\alpha}-1} \left(\frac{1}{\alpha} p^* - \frac{\alpha-1}{\alpha} c q^{-1} \right) \quad (\text{A.12})$$

is strictly positive for all $q > c \frac{\alpha-1}{p^*}$.

Q^s will generate the capital processes where the firms grow jointly in equilibrium. We have to show that this is not to the disadvantage of any firm.

Begin with an initial state in the forbidden region, i.e. $x_0 > \bar{X}^c(q^i, q^{-i})$, which requires a jump.

On $\{x > \bar{X}^c(q^i, q^{-i})\}$, player i jumps when being the leader and by our definition,

$$V^c(x, q^i, q^{-i}) = V^c(x, \phi^c(x, q^{-i}), q^{-i}) - \phi^c(x, q^{-i}) + q^i.$$

Consequently, in this region, $V_{q^i}^c = 1$ and

$$\begin{aligned} V_{q^{-i}}^c(x, q^i, q^{-i}) &= V_{q^{-i}}^c(x, \phi^c(x, q^{-i}), q^{-i}) \\ &= 0 \quad \text{if } \phi^c(x, q^{-i}) \geq q^{-i}, \end{aligned} \quad (\text{A.13})$$

where the latter holds by construction, cf. (6.2).

Now we dictate a different investment for player i in two parts of the forbidden region.

If, on the one hand, $\{\bar{X}^c(q^i, q^{-i}) < x \leq \bar{X}^c(q^i, q^i)\}$, this is equivalent to $Q^s(x) \leq q^i < \phi^c(x, q^{-i})$.

The monotonicity of \bar{X}^c then implies $q^i \geq \phi^c(x, q^i)$. Further, by symmetry of \bar{X}^c , we always have

$$q^i = \phi^c(x, \phi^c(x, q^i)).$$

So, for all $q \in [q^{-i}, \phi^c(x, q^i)]$, (A.13) holds, including the second line. It follows

$$V^c(x, q^i, q^{-i}) = V^c(x, q^i, \phi^c(x, q^i)),$$

i.e. player i is indifferent if we let $-i$ jump to bring the state onto the boundary of the forbidden region.

If, on the other hand, $\{x \geq \bar{X}^c(q^i, q^i) \vee \bar{X}^c(q^{-i}, q^{-i})\}$, this is equivalent to $q^i \vee q^{-i} \leq Q^s(x)$.

This time, the monotonicity of \bar{X}^c implies that for all $q^{-i} \leq Q^s(x)$, $\phi^c(x, q^{-i}) \geq q^{-i}$. We apply the second line of (A.13) once more to obtain

$$V^c(x, q^i, q^{-i}) = V^c(x, q^i, Q^s(x)) = V^c(x, Q^s(x), Q^s(x)) - Q^s(x) + q^i$$

in this region. Thus, player i is indifferent if we allow both to jump to $Q^s(x)$.

Now consider reflection investment at the boundary. Then, by definition, $V_{q^i}^c(\bar{X}^c, q^i, q^{-i}) = 1$. We also constructed \bar{X}^c such that a player is indifferent to invest at the boundary if the opponent does not have strictly more capital installed, i.e. if $q^i \geq q^{-i}$, $V_{q^{-i}}^c(\bar{X}^c, q^i, q^{-i}) = 0$. Consequently, we can choose capital processes such that only the smaller firm invests, or both invest simultaneously.

Such processes are indeed feasible.

Select $Q^{c,i} \in \mathcal{N}(q_0^i)$ that satisfies for all $t \in [0, \infty)$ **P**-a.s.,

$$\begin{aligned} Q_t^{c,i} = q_0^i \vee & \left(\mathbf{1}_{\{q_0^i \vee \sup_{0 \leq s \leq t} \phi^c(X_s, q_0^{-i}) < q_0^{-i}\}} \sup_{0 \leq s \leq t} \phi^c(X_s, q_0^{-i}) \right. \\ & \left. + \mathbf{1}_{\{q_0^i \vee \sup_{0 \leq s \leq t} \phi^c(X_s, q_0^{-i}) \geq q_0^{-i}\}} \sup_{0 \leq s \leq t} Q^s(X_s) \right). \end{aligned}$$

The larger firm starts tracking $Q^s(X_t)$ from the beginning and the smaller firm, i.e. with $q_0^i < q_0^{-i}$, switches when

$$\begin{aligned} \phi^c(X_s, q_0^{-i}) \geq q_0^{-i} & \Leftrightarrow X_s \geq \bar{X}^c(q_0^{-i}, q_0^{-i}) \\ & \Leftrightarrow Q^s(X_s) \geq q_0^{-i}. \end{aligned}$$

This also implies that the processes solve (2.6):

$$Q_t^{c,i} = q_0^i \vee \sup_{0 \leq s \leq t} \phi^c(X_s, Q_s^{c,-i}).$$

Now we can perform the estimation in the proof of Theorem 3.2 with equality holding, to find that

$$V^c(x_0, q_0^i, q_0^{-i}) = J(Q^{c,i}, Q^{c,-i}) = J(Q^i, Q^{-i}).$$

□

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