

Ott, Marion; Beck, Marissa

Conference Paper

Incentives for Overbidding in Minimum-Revenue Core-Selecting Auctions

Beiträge zur Jahrestagung des Vereins für Socialpolitik 2013: Wettbewerbspolitik und
Regulierung in einer globalen Wirtschaftsordnung - Session: Auctions and Licensing, No.
F16-V3

Provided in Cooperation with:

Verein für Socialpolitik / German Economic Association

Suggested Citation: Ott, Marion; Beck, Marissa (2013) : Incentives for Overbidding in Minimum-
Revenue Core-Selecting Auctions, Beiträge zur Jahrestagung des Vereins für Socialpolitik
2013: Wettbewerbspolitik und Regulierung in einer globalen Wirtschaftsordnung - Session:
Auctions and Licensing, No. F16-V3

This Version is available at:

<http://hdl.handle.net/10419/79946>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen
Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle
Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich
machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen
(insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten,
gelten abweichend von diesen Nutzungsbedingungen die in der dort
genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

*Documents in EconStor may be saved and copied for your
personal and scholarly purposes.*

*You are not to copy documents for public or commercial
purposes, to exhibit the documents publicly, to make them
publicly available on the internet, or to distribute or otherwise
use the documents in public.*

*If the documents have been made available under an Open
Content Licence (especially Creative Commons Licences), you
may exercise further usage rights as specified in the indicated
licence.*

Incentives for Overbidding in Minimum-Revenue Core-Selecting Auctions

Marissa Beck^a, Marion Ott^b

^a*Department of Economics, Stanford University, USA*

^b*School of Business and Economics, RWTH Aachen University, Germany*

Abstract

We find new equilibria of minimum-revenue core-selecting (MRCS) auctions that, in contrast to previously identified equilibria, involve *overbidding* – bidding more than one’s true value for some packages of goods. With full information, every MRCS auction in every possible setting has equilibria with overbidding and these equilibria have different properties than the previously known equilibria with bid shading. Namely, they can lead to strictly higher revenues for the seller and larger price differences among bidders. Previous studies of MRCS games with incomplete information assumed restricted strategy spaces that prevented overbidding. In this paper, we allow bidders access to their complete strategy sets and show that, in some settings, overbidding occurs in all Bayesian equilibria in undominated strategies. In a simple setting with independent private values, equilibrium strategies of a particular set of MRCS auctions employ a mixture of bid shading and overbidding. These new equilibria improve expected efficiency relative to equilibria with restricted strategy spaces and lead to higher expected revenues than those from the Vickrey package auction. A second incomplete-information setting demonstrates that equilibria with overbidding can be in some sense unique. In this setting, every Bayesian equilibrium in undominated strategies of every MRCS auction has at least one bidder who overbids and there is no bid shading on winning packages. Overbidding eliminates the threshold problem, leading to an efficient assignment and payoffs that are in the core with respect to the true values.

Contents

1	Introduction	3
2	Model	7
3	Full Information	13
4	Incomplete Information	16
4.1	Independent Private Values	16
4.2	Partially Informed Bidders	25
5	Conclusion	29
Appendix A	Independent Private Values	32
Appendix A.1	Bidder 1's Expected Payoff and Equilibrium Bids	32
Appendix A.2	Payoff Vectors in the Core $\mathcal{C}(v)$	35
Appendix A.3	Lowering Payments Drives Overbidding	36
Appendix A.4	Restricting the Message Space Decreases Revenues	38
Appendix A.5	Distributions That Favor the Vickrey Auction	40
Appendix B	Proof of Theorem 5	42

1. Introduction

Core-selecting auctions have been used or proposed recently in a variety of settings, from radio spectrum in the UK to airport landing slots in New York City. Part of the appeal of these auctions is that, with respect to reported values (i.e., bids), they implement efficient assignments while assuring competitive revenues – no group of bidders would want to offer the seller a higher total amount. This corrects one of the weaknesses of the Vickrey package auction (or VCG mechanism), which may result in much lower revenue than that of core-selecting auctions, given the same set of bids.¹ The most common core-selecting auctions, both studied and implemented, are minimum-revenue core-selecting (MRCS) auctions. They select an assignment and payments that give the seller the lowest revenue within the core of the reported values.² These MRCS auctions implement the VCG outcome whenever it is in the core and minimize total incentives for bidders to misreport their values within the class of core-selecting auctions.³ However, these auctions are not strategy proof and so we should not expect truthful bidding.

Not much is known about the equilibria of core-selecting auctions, especially with incomplete information, and they exhibit some peculiar outcomes in practice. The recent Swiss spectrum auction for new mobile frequencies raised over \$1 billion in revenues from three winning telecommunications companies. However, the prices paid by the bidders varied significantly. Sunrise Communications paid more than three times as much as Orange for a similar amount of spectrum (approximately \$528 million versus \$169 million). Should we assume the difference in prices reflects an underlying difference in values? Or could strategic bidding explain these high revenues and the large difference in prices paid by winning bidders?

This paper finds new equilibria with *overbidding* – bidding more than one’s true value

¹See Ausubel and Milgrom (2006).

²In general, many payment rules give the seller the same revenue (with bidders paying different amounts for the same items) and thus many possible rules fit into this category.

³See Day and Milgrom (2008) and Day and Raghavan (2007).

– that can explain the price differences and high revenues in the Swiss auction without any asymmetries in the underlying values. Previous literature in both complete- and incomplete-information settings has found only equilibria in which bidders shade their bids relative to their true values. Compared to these equilibria with bid shading, our newly identified equilibria with overbidding produce higher revenues for the seller and larger price differences among bidders. They can also lead to improved expected efficiency in settings with incomplete information. We consider equilibria in undominated strategies and prove that, in contrast to the VCG auction, overbidding is not a dominated strategy. The incentives to overbid depend on the information structure, but we show that equilibria with overbidding exist both with full information and with independent private values. We also demonstrate that, in some examples, every undominated equilibrium of every MRCS auction has the properties that some bidder overbids and no bidder engages in bid shading, so the overbidding phenomenon cannot be ruled out by clever equilibrium selection.

In the identified equilibria, bidders overbid on packages that they do not want to win in order to raise the prices paid by the other bidders and, thereby, lower their own prices. This incentive arises from the payment rules employed by MRCS auctions. Any winning bidder must pay at least the opportunity cost his presence places on the other bidders. By overbidding on packages he does not win, a bidder increases the reported opportunity cost caused by the other bidders' presence and increases their prices. If the overbidding does not change the minimum revenue within the reported core, then it does not affect the total payment by bidders, so increasing his competitors' prices means he pays a lower price. Of course, depending on the information he has, a bidder risks overbidding by too much and winning the package on which he overbid.

Our equilibrium analysis treats three information structures: full information, independent private values, and a special setting with partially informed bidders. Our results from each setting and their relation to the previous literature are as follows.

In a general full-information setting, Day and Milgrom (2008) and Day and Raghavan (2007) showed that every MRCS auction has *truncation* equilibria in which all bids are

(weakly) less than the true valuations. These equilibria lead to efficient assignments and payoffs that are in the core with respect to the true values. We show that in addition to these truncation equilibria, every MRCS auction has full-information equilibria in which all bids are (weakly) *higher* than the true values, with at least one of these inequalities being strict. These equilibria with overbidding also lead to an efficient assignment and payoffs that are in the core with respect to the true values, but they can give the seller much larger revenues and can lead to larger differences among bidder payoffs. In fact, the seller can capture all of the gains from trade.

With incomplete information, overbidding creates the risk of winning an unwanted package but it still occurs in equilibrium. We study the same simple setting on which the previous literature has focused. This setting has three bidders and two items for sale. There are two “local” bidders, each interested in only one of the two items, and a “global” bidder, interested only in the bundle of both items. This information is common knowledge, but the exact value each bidder has for their desired package is private knowledge. Due to the dimensionality of even this simple setup, the literature has only considered strategies in which bidders do not place bids for the items they do not value. So each local bidder bids only for his desired item and the global bidder places a positive bid only for the package of both items. With this restriction, Goeree and Lien (2009) and Ausubel and Baranov (2010) find only equilibria in which bids are (at least weakly) below the true values and, because of the lower bids, these equilibria may involve inefficient assignments and low revenues.⁴ However, we will show that local bidders prefer to place bids for unwanted items to drive up the other local bidder’s price. To the best of our knowledge, this analysis is the first that studies the full set of strategies available to bidders.

Even with the simplified setting described above, solving for equilibria when values are private and bidders can place bids on all packages can be quite complex. We introduce a

⁴Sano (2011) also studied this setting, finding a necessary and sufficient condition for truthful reporting to be a weakly dominant strategy and showing that this condition is rarely satisfied.

class of payment rules (i.e., particular MRCS auctions) for which we can analytically solve for all of the Bayesian equilibria in undominated strategies. Overbidding exists in all of these equilibria. One of the local bidders places a bid for the bundle of both items that exceeds the true valuation when the true valuation is high. This local bidder still shades his bid for his desired item, but his bids imply a positive demand for his unwanted item because, for all realized values, he bids strictly more for the bundle of both items than for his individual desired item. Comparing expected revenues, we find that the unique equilibrium outcome of the MRCS auctions has an advantage over both the truthful-bidding outcome of the Vickrey auction and the equilibrium outcome of hypothetical MRCS auctions in which bidders are restricted to bidding only on their desired packages.

Incentives for overbidding depend on the information structure, so we look at one additional, extreme information structure to highlight how bid shading can be more costly than overbidding. Namely, we suppose the local bidders know each others' values but that they do not know the global bidder's value. We continue to assume the global bidder knows only his own value. Under these informational assumptions, in *every* Bayesian equilibrium in undominated strategies of *every* MRCS auction at least one bidder bids strictly above his true value for at least one package and no winning bid from any bidder is below the true value. The overbidding always shows an implied demand for an unwanted item (either through a positive bid made directly on that item or through a bid for the package of both items that is higher than the bid for the individual desired item). Overbidding in this manner allows bidders to overcome the threshold problem – that the local bidders must coordinate their bids to beat the threshold set by the global bidder – and so the assignment is efficient. The equilibrium payoffs are in the core with respect to the true values and the revenue (both expected and realized) is weakly higher than that of the Vickrey auction and weakly higher than if bidders reported truthfully in the same MRCS auction. Moreover, whenever the local bidders win the auctioned items, at least one of them pays his full value. This can create a large asymmetry between the local bidders, with one paying the maximum (his full value) and the other paying the minimum necessary to win his desired

item.

In all of our equilibria with overbidding, the higher bids translate to higher revenues, despite the fact that MRCS auctions can have revenues that are non-monotonic in bids.⁵ However, overbidding that occurs in our equilibria does increase revenues relative to previously discovered equilibria with bid shading and the higher the degree of overbidding, the higher the revenues.⁶ In the special setting we consider with partially informed bidders, this means that Bayesian equilibria in undominated strategies of any MRCS auction can give the seller more revenue than the minimum revenue in the true core and that the equilibrium outcome need not be bidder optimal (unlike the outcomes of the truncation equilibria in the full information setting). In this special setting and also in the setting with independent private values, equilibria in undominated strategies in which bidders *only* shade their bids do not exist. This means the higher revenues resulting from overbidding can be a unique outcome of MRCS auctions.

The paper proceeds as follows. Section 2 describes the model, core-selecting auctions, and the class of payment rules we use to analyze the setting with independent values. Section 3 explores overbidding with full information. Section 4.1 contains the Bayesian analysis with independent private values. Section 4.2 provides the results for our third information structure and Section 5 concludes.

2. Model

We consider a setting with one seller, whom we denoted 0, and a set of bidders $N = \{1, \dots, n\}$. The seller owns a set of goods $K = \{1, \dots, k\}$, which may be purchased

⁵See Lamy (2010) and Beck and Ott (2009).

⁶Overbidding can increase the prices paid by competitors and such an increase can occur without a corresponding decrease in one's own price. If the auction is not an isolated interaction but part of an ongoing relationship between bidders, they may prefer to punish their competitors in order to better their own position in the future of the industry. This possibility is not unique to MRCS auctions. Our equilibria with full and partial information are also equilibria of the Vickrey package auction (albeit in dominated strategies).

individually or in packages. He has zero value for all items and packages of items, so his payoff is $\pi_0 = \sum_{i=1}^n p_i$ where p_i is bidder i 's payment. Bidder i 's payoff is $\pi_i = v_{i,x_i} - p_i$ if he wins a bundle x_i of value v_{i,x_i} and pays p_i and is zero if he does not win any package. (In all core-selecting auctions, losing bidders make zero payments.) Each item can be sold only once and a feasible assignment of packages to bidders is $x = (x_1, \dots, x_n) \in X$ where $X = \{(x_1, \dots, x_n) | x_i \in 2^K \text{ and } x_i \cap x_j = \emptyset \forall i \neq j\}$.

The core $\mathcal{C}(v)$ of a cooperative game consists of all feasible *payoffs* that are not blocked by any coalition (i.e., each group receives at least as much as it could achieve on its own so that it could not deviate and make all of its members better off). In our setting, since the seller owns all items, any group that does not include the seller cannot generate any value. Any coalition that contains the seller must receive at least what it could get from trading among itself. Therefore, the core consists of all payoff vectors $\pi = (\pi_0, \pi_1, \dots, \pi_n)$ that satisfy the following constraints (where the first is feasibility and the rest assure no coalition can block the payoffs):

$$\begin{aligned} \pi_0 + \sum_{i \in N} \pi_i &\leq \max_{x \in X} \sum_{i \in N} v_{i,x_i} \\ \pi_i &\geq 0 \quad \forall i \in N \\ \pi_0 + \sum_{i \in S} \pi_i &\geq \max_{x \in X} \sum_{i \in S} v_{i,x_i} \quad \forall S \subseteq N \end{aligned}$$

A core-selecting auction is direct mechanism (inducing a non-cooperative game among bidders) that maps bids on packages of goods to assignments and payments such that payoffs are in the core with respect to the bids. For any package $y \in 2^K$, denote by $b_{i,y}$ bidder i 's bid for the bundle y and assume $b_{i,\emptyset} = 0$. Let $b_i = (b_{i,y})_{y \in 2^K}$ denote the vector of bidder i 's bids and $b = (b_1, \dots, b_n)$. To simplify notation, we will denote the (reported) maximum value generated by the seller and a group of bidders $S \subseteq N$:

$$w(S) = \max_{x \in X} \sum_{i \in S} b_{i,x_i}$$

and $w(\emptyset) \equiv 0$. A MRCS auction selects an assignment and payments that minimize the seller's revenue while ensuring that payoffs are in the core $\mathcal{C}(b)$ with respect to bids b . The auction mechanism computes the vector of payoffs $\pi^b = (\pi_0^b, \pi_1^b, \dots, \pi_n^b)$ as if the bids were the true values, so $\pi_i^b = b_{i,y} - p_i$ if i wins package y and pays price p_i and $\pi_0^b = \pi_0 = \sum_{i \in N} p_i$. Therefore, any MRCS auction chooses payoffs that solve:

$$\begin{aligned} & \min \pi_0 \\ \text{s.t. } & \pi_0 + \sum_{i \in N} \pi_i^b \leq w(N) \\ & \pi_i^b \geq 0 \quad \forall i \in N \\ & \pi_0 + \sum_{i \in S} \pi_i^b \geq w(S) \quad \forall S \subseteq N \end{aligned}$$

Additionally, we assume that a bidder's payment $p_i(b)$ must be continuous in $b_{i,K}$ for all $b_{i,K} \leq \max_{y \neq K} b_{i,y}$.⁷

To translate these restrictions on the payoffs into restrictions on the assignment and payments, notice that the first and last inequalities necessitate that the assignment $x^*(b) = (x_1^*(b), \dots, x_n^*(b))$ be optimal with respect to the bids: $x^*(b) \in \arg \max_{x \in X} \sum_{i \in N} b_{i,x_i}$. This optimal assignment may not be unique, in which case any such assignment can be chosen. If this optimal assignment is value-maximizing (i.e., $x^*(b) = x^*(v)$) we call it efficient because in our game such an assignment comes with efficient payoffs. We can rewrite the

⁷This guarantees the existence of equilibria. Low bids for K (those that are below bids for smaller packages $y \neq K$), do not affect the assignment nor any of the core constraints, so a MRCS auction could use this extra degree of freedom to determine prices in such a way that a best response does not exist.

linear program in terms of payments instead of payoffs:

$$\begin{aligned}
& \min_{(p_1, \dots, p_n)} \sum_{i \in N} p_i & (1) \\
\text{s.t. } & b_{i, x_i^*(b)} \geq p_i \quad \forall i \in N \\
& \sum_{i \in N} p_i \geq 0 \\
& \sum_{j \notin S} p_j \geq w(S) - \sum_{i \in S} b_{i, x_i^*(b)} \quad \forall S \subset N
\end{aligned}$$

Note that losing bidders pay zero: $x_i^*(b) = \emptyset \implies p_i = 0$.

The Vickrey auction results in the same assignment, that which is optimal with respect to the reported values. However, payments in any MRCS auction are bounded below by those in the Vickrey auction with the same reported values (*Vickrey payments*): $p_i^V(b) = w(N \setminus \{i\}) - \sum_{j \in N \setminus \{i\}} b_{j, x_j^*(b)}$ for all $i \in N$. We say that bidder i 's *Vickrey constraint* $p_i \geq p_i^V(b)$ is binding if his payment in a MRCS auction equals $p_i^V(b)$. Notice that bidder i 's bids influence his Vickrey constraint through their impact on the optimal assignment.

The minimum revenue problem in (1) always has a continuum of solutions when the Vickrey payoffs are not in the core, so an additional *payment rule* is necessary to select an outcome of the auction. Among these rules are the Vickrey-nearest rule (Cramton and Day, 2008), reference rules (Erdil and Klemperer, 2010), and the nearest bids rule (Ausubel and Baranov, 2010). The complexity of solving for equilibria when bidders can place bids on all packages has previously prevented analysis of equilibria in incomplete information environments. In Section 4.1, we will use a favored-bidder payment rule that allows us to explicitly derive equilibria and isolate bidding incentives.

Definition 1 (Favored-Bidder Payment Rule). *A payment rule is a favored-bidder payment rule if there exists a bidder i that pays his Vickrey payment p_i^V for every realization of bids.*⁸

⁸Note that Day and Milgrom (2010, forthcoming) use this term differently, calling a bidder who wins an item the in case of a tie “favored.”

It is always possible to favor a particular bidder while solving the minimum core revenue problem in a setting with two goods.⁹ In general, this is not a complete specification of a payment rule because it does not pin down the payments of the other bidders. We will show that all payment rules that favor our chosen bidder will lead to the same equilibrium outcome, so our results apply to all such rules. Note that a favored-bidder payment rule is asymmetric and non-anonymous. It must favor the same bidder for every realization of bids.¹⁰ We call this bidder favored because the Vickrey payment is the lowest possible payment a bidder can obtain in any MRCS auction for a given b .

Due to the possible multiplicity of optimal assignments, we must also define a *tie-breaking rule* for our MRCS auctions. Equilibria in MRCS auctions will in general depend on the tie-breaking rule because strategic bidding can provoke ties (even with independent private values). Moreover, the correct tie-breaking rule may be necessary for best responses to exist in continuous games and hence, necessary to sustain any equilibrium. We assume a tie-breaking rule that randomizes among the optimal assignments with the maximal number of winning bidders. The equilibria we discuss throughout the paper will rely on our choice of tie-breaking rule and we will assume throughout the rest of the paper that this tie-breaking rule holds (even when we do not explicitly reference it).

In Section 4, we will look at incomplete information in a simple setting with three bidders, 1, 2, and 3, and one seller. The seller owns two goods, labeled A and B , which may be purchased individually or as a bundle.

Bidders 1 and 2 are “local” bidders, interested in only one item. Bidder 1 wants good A , has no value for good B , and values the bundle AB the same as the good A . Bidder 2 wants good B , has no value for good A , and values the bundle AB the same as the good B . Bidder 3 is a “global” bidder, interested only in the bundle AB , and has no value for

⁹Even with more than two goods, it is always possible to favor a particular bidder in a core-selecting auction, but the revenue might not be minimal.

¹⁰Alternatively, we would get the same results from a bidder-symmetric and anonymous *favored-bundle rule*: the bidder who wins bundle B pays his Vickrey payment. Bidder 2, who wants to win item B in our simple setting, would then also have a weakly dominant strategy to tell the truth.

either item alone. Table 1 illustrates these preferences. The seller has zero value for all packages and thus we ignore him in the table.

	A	B	AB
Bidder 1	v_1	0	v_1
Bidder 2	0	v_2	v_2
Bidder 3	0	0	v_3
$v_1, v_2 \sim U(0, 1) \quad v_3 \sim U(0, 2)$			

The structure of their preferences is common knowledge among the bidders. The values are independently drawn from uniform distributions: the distributions of v_1 and v_2 have support $[0,1]$ and that of v_3 has support $[0,2]$. We consider two informational settings. In Section 4.1, all three bidders' values are private information whereas in Section 4.2, both bidders 1 and 2 are informed about v_1 and v_2 .¹¹ Our results would be the same if the values v_1 and v_2 were private information but perfectly correlated, so that each local bidder could deduce the other's value upon learning his own.

In our formulation, the results of the Vickrey auction are in the core $\mathcal{C}(v)$ if and only if the global bidder wins both items (i.e., if $v_3 > v_1 + v_2$), which occurs with probability $1/2$.¹² So the setting is balanced in the sense that the cases in which the Vickrey auction generates competitive revenue (bidder 3 wins and revenues equal the sum of bidder 1 and 2's bids) and where it performs poorly (if bidders 1 and 2 win then revenue is less than bidder 3's bid) occur with equal probability. Furthermore, this is the setting analyzed by Goeree and Lien (2009) and Ausubel and Baranov (2010), so we can compare our results

¹¹In Section 4.2, we actually assume v_1 and v_2 are drawn from uniform distributions on $(0,1)$. The solution if they are drawn from uniform distributions on $[0,1]$ is the same except that additional equilibrium bids arise when $v_1 = v_2 = 1$ and when one (and only one) of the two bidders has a zero valuation. In the first case, the local bidders know their combined value must be at least as large as that of the global bidder. This leads to equilibria in the which they bid far above their true values. However, these extra equilibria lead to exactly the same outcomes as we discuss in Section 4.2. In the second case, when one local bidder has a zero valuation and the other doesn't, there are extra equilibria because a multiplicity of bids made by the non-zero bidder all lead to the same outcome (again, the same outcome as we discuss in Section 4.2).

¹²In Appendix A.5, we vary this probability to show how our results change in a less balanced setting.

to theirs.

Thus, the game we analyze is as follows. First, bidders simultaneously submit non-negative bids for *all* packages of items in K . This differs from Goeree and Lien (2009) and Ausubel and Baranov (2010) who, in the setting with three bidders, allow only one-dimensional bids (i.e., they imposed the following restrictions: $b_1 = b_{1A}$, $b_2 = b_{2B}$, $b_3 = b_{3AB}$).¹³ After the bidders submit their bids, a MRCS auction is run with the tie-breaking rule described above.

3. Full Information

We begin by considering full-information equilibria of MRCS auctions. In this section, we consider a general setting with any number of bidders, any number of goods, and any possible valuations for those goods. For every MRCS auction, there exist equilibria with overbidding. To prove this, we will make use of the following lemma.

Lemma 1. *Given other players' bids, the highest possible payoff for bidder i in the core $\mathcal{C}(b)$ is $v_{i,x_i^*(b)} - p_i^V(b)$. This expression is maximized when bidder i bids truthfully.¹⁴*

Proof: Ausubel and Milgrom (2002) prove this fact when the other players bid their true values (Theorem 5). The same argument works for any arbitrary profile of other players' bids: $v_i \in \arg \max_{b_i} v_{i,x_i^*(b_i,b_{-i})} - p_i^V(b_i,b_{-i})$ by the well-known fact that bidder i maximizes his Vickrey payoff $v_{i,x_i^*(b_i,b_{-i})} - p_i^V(b_i,b_{-i})$ by bidding truthfully. ■

Thus, a bidder who receives his Vickrey payoff given the others' bids has no profitable deviation. Now we are ready to state the theorem about the existence of equilibria with overbidding.

¹³Technically, to make this assumption comparable to our setting in which package auctions take bids of full dimensionality as input, you must make some assumption about the bids for other packages (e.g., that they are all zero).

¹⁴This means that a truthful bid maximizes the highest available payoff in the core for any particular bidder, but in general this will not be the payoff received in a MRCS auction because the core constraints may prevent all bidders from simultaneously receiving their highest payoffs.

Theorem 1. *Take any set of bidders $N = \{1, \dots, n\}$, any set of goods K , and any values for those goods $v_i : 2^K \rightarrow \mathbb{R}_+$ for all $i \in N$. Then, for any $m \in [1, n]$, the following bids form a Nash equilibrium of all MRCS auctions.*

$$\begin{aligned} b_{i,y} &= v_{i,y} \quad \forall i \in N, \forall y \in 2^K \setminus \{K\} \\ b_{i,K} &= \max_{x \in X} \sum_{i \in N} v_{i,x_i} \quad \forall i \leq m \\ b_{i,K} &= v_{i,K} \quad \forall i > m \end{aligned}$$

The equilibrium payoffs are in the core $\mathcal{C}(v)$ with respect to the true values and the revenues are at least as large as those from the Vickrey auction with truthful bidding.

Proof: By Lemma 1, given any b_{-i} , the highest possible payoff for bidder i in any core-selecting auction is $v_{i,x_i^*(v_i, b_{-i})} - p_i^V(v_i, b_{-i})$. We will show that all bidders are necessarily achieving this payoff in any MRCS auction. Given the equilibrium bids, any bidder $i > m$ has a Vickrey payment of $p_i^V(b) = p_i^V(v_i, b_{-i}) = w(N \setminus \{i\}) - \sum_{j \in N \setminus \{i\}} b_{j,x_j^*(v_i, b_{-i})} = v_{i,x_i^*(v_i, b_{-i})}$. Thus, given the other players' bids, his maximum payoff in any core-selecting auction is 0. But a core-selecting auction must choose an individually rational payoff given the reported bids. So a bidder $i > m$ can never pay more than $v_{i,x_i^*(v_i, b_{-i})}$. Thus he must pay exactly $v_{i,x_i^*(v_i, b_{-i})}$ and receive his highest potential payoff of zero.

If $m > 1$, then all bidders $i \leq m$ also have a Vickrey payoff of zero by the same argument. They cannot receive a payoff less than zero, so they must receive their Vickrey payoff and have no profitable deviations.

If $m = 1$ and given that every other bidder receives an imputed payoff of zero, the only relevant core constraint is $\pi_1 \leq w(N) - w(N \setminus \{1\}) = \pi_1^V$. Therefore, any MRCS auction must, in order to minimize revenue, give bidder 1 exactly his Vickrey payoff given bids: $v_{1,x_1^*(b_1, v_{-1})} - p_1^V(b_1, v_{-1}) = v_{1,x_1^*(v)} - p_1^V(v)$ because bidder 1's altered bid for the global package does not change the optimal assignment or his Vickrey payment.

The payoffs in these Nash equilibria are in the core with respect to the true values $\mathcal{C}(v)$. If $m > 1$, every bidder has a payoff of zero and the seller receives the total surplus. This satisfies all of the core constraints. If $m = 1$, we use the facts that $x_i^*(b) = x_i^*(v) \equiv x_i^*$, $\pi_1 = \pi_1^V$, $\pi_i = 0$, and $p_i = v_{i,x_i^*}$ for all $i \in N \setminus \{1\}$ to see that $\pi_0 + \pi_1 + \sum_{i \in S} \pi_i = (\sum_{i \in N \setminus \{1\}} v_{i,x_i^*} + p_1^V(v)) + (v_{1,x_1^*} - p_1^V(v)) + 0 = w(N) \geq w(S \cup \{1\})$ for all $S \subseteq N \setminus \{1\}$. Similarly, for all $S \subseteq N \setminus \{1\}$, $\pi_0 + \sum_{i \in S} \pi_i = \sum_{i \in N \setminus \{1\}} v_{i,x_i^*} + w(N \setminus \{1\}) - \sum_{i \in N \setminus \{1\}} v_{i,x_i^*} = w(N \setminus \{1\}) \geq w(S)$.

Payoffs in the core $\mathcal{C}(v)$ always imply a revenue that is weakly higher than the revenue from the Vickrey auction with truthful bidding. If the Vickrey payoffs are in the core, they correspond to the minimum revenue point. \blacksquare

This theorem is actually true of every bidder-optimal core-selecting auction, not only of MRCS auctions. Furthermore, these are not the only equilibria with overbidding. Bidders can overbid on losing packages other than K and bidders can also combine bid shading with overbidding. In the equilibria in Theorem 1, any number of bidders can overbid. Thus, we can have anything from a very asymmetric outcome in which a single bidder (bidder 1) pays the minimum possible payment while the rest pay the maximum possible (their full values for the packages they win) to a more symmetric outcome in which *all* bidders pay their full value and the seller captures the entire gain from trade.

The intuition for these equilibria with overbidding is as follows. A bidder has two ways to decrease the price he pays. One is to reduce his bids so that his individual rationality constraint holds – he cannot be forced to pay more than his winning bid. This corresponds to the truncation equilibria discovered by Day and Milgrom (2008). The other way to decrease his price is to increase the price paid by the other bidders. Any minimum-revenue auction fixes the sum of the payments made by the bidders, but does not specify the proportion paid by each individual. Increasing the price paid by the other bidders can thereby decrease a bidder’s own price. Overbidding on the right packages, namely those which imply a demand for the items won by the other bidders, does exactly that. By overbidding on the package containing all of the items, a bidder creates an implied demand

for the packages won by the other bidders, which in turn forces those bidders to pay higher prices. In the equilibria in Theorem 1, the truthful bidders must pay their full value for their winning packages because every bidder $i \leq m$ reports that he would be willing to pay that much and they must outbid him to win.

These equilibria with overbidding lead to some of the same conclusions as the truncation equilibria, namely that the assignment maximizes the true total value and the payoffs are in the core with respect to the true values. However, they can differ markedly in the revenues generated for the seller and the asymmetry of bidder outcomes. Truncation equilibria lead to outcomes in the bidder optimal part of the core $C(v)$ whereas our equilibria with overbidding can give the seller the entire gains from trade. Our equilibria can also result in one bidder receiving his Vickrey payoff while all of the others receive payoffs of zero. Whenever these payoffs are not in the bidder optimal part of the core, the difference in prices paid between bidder 1 and the other bidders is strictly larger than in any truncation equilibrium.

4. Incomplete Information

We next explore Bayesian equilibria in two settings with incomplete information, one with completely private values and the other in which the local bidders know each other's values. We will show that the incentives for overbidding do not disappear when there is uncertainty about the other bidders' values. In fact, the right kind of uncertainty can help to eliminate equilibria in which bidders shade their bids, as we will see in Section 4.2. We begin by looking at the most standard incomplete information setting, that in which bidders have independent private values.

4.1. Independent Private Values

A Bayesian equilibrium is a profile of strategies $b_i(v_i) = (b_{iA}(v_i), b_{iB}(v_i), b_{iAB}(v_i))$ for all i and v_i such that each bidder is maximizing his expected payoff at every possible value v_i , given the other bidders' strategies. In this section, we study MRCS auctions with

favored-bidder-2 payment rules. These rules simplify the analysis of Bayesian equilibria because they make truth-telling a weakly dominant strategy for bidder 2.

Theorem 2. *A favored bidder in a core-selecting auction with private values has a weakly dominant strategy to bid truthfully.*¹⁵

Proof: A favored bidder always pays his Vickrey payment and wins the total-value maximizing package (given the bids), so he faces exactly the same payoff as a function of his bid as he would in the Vickrey auction. The result follows directly from this fact. ■

Bidder 2 is favored in our MRCS auctions, so in any equilibrium in undominated strategies he bids truthfully: $b_2^*(v_2) = (0, v_2, v_2)$. Whenever he wins B , he pays $p_2 = \max\{b_{1A}(v_1) + b_{3B}(v_3), b_{1B}(v_1) + b_{3A}(v_3), b_{1AB}(v_1), b_{3AB}(v_3)\} - b_{1,x_1^*(b)}(v_1) - b_{3,x_3^*(b)}(v_3)$.

Global bidders, who value only the bundle of all items (the *global package*), will also bid truthfully, but for a different reason.

Theorem 3. *A global bidder in a MRCS auction with private values has a weakly dominant strategy to bid truthfully.*

Proof: By definition, a global bidder g has a positive value only for the global package. He doesn't want to win any smaller bundle because his payment is weakly positive in any core-selecting auction. Therefore, the only reason to bid a positive amount on a smaller bundle is to decrease his payment when he wins the global package. If he wins the global package, all other bidders lose and pay nothing. Thus, the only relevant constraint on his payment in (1) is $p_g \geq w(N \setminus \{g\})$ and is binding. But then his payment $p_g = w(N \setminus \{g\})$ does not depend on his own bids, so he has no reason to bid a positive amount on any smaller package. Therefore, he can only win the global package. His assignment and payment are the same as in the Vickrey auction, so it is a weakly dominant strategy to tell the truth. ■

¹⁵Note that this is true for all core-selecting auctions with a favored bidder, not only for MRCS auctions.

Thus, in any equilibrium in undominated strategies of our MRCS auctions, the global bidder 3 also bids truthfully: $b_3^*(v_3) = (0, 0, v_3)$. Whenever he wins AB , he pays $p_3 = \max\{b_{1A}(v_1) + b_{2B}(v_2), b_{1B}(v_1) + b_{2A}(v_2), b_{1AB}(v_1), b_{2AB}(v_2)\}$. Otherwise, he pays nothing.

We will denote bidder 1's bids by $b_A(v_1)$, $b_B(v_1)$, and $b_{AB}(v_1)$ in what follows. Table 2 summarizes the results so far.

Table 2: Equilibrium Bids

	A	B	AB
$b_1(v_1)$	$b_A(v_1)$	$b_B(v_1)$	$b_{AB}(v_1)$
$b_2(v_2)$	0	v_2	v_2
$b_3(v_3)$	0	0	v_3

Given that bidders 2 and 3 report their values truthfully, bidder 1 strictly prefers bids with $b_B(v_1) < b_{AB}(v_1)$ for all v_1 (see Appendix A.1). When $b_B(v_1) < b_{AB}(v_1)$ for all v_1 , $b_B(v_1)$ does not affect any assignment or price. It remains to determine $b_A(v_1)$ and $b_{AB}(v_1)$.

We modify our tie-breaking rule slightly for this section only to ensure bidder 1 has a best response to the other bidders reporting truthfully. In the case of a tie, we still favor the assignment with the maximal number of bidders but, when that does not resolve the tie, we next choose assignments in which bidder 1 wins package A . If this still does not resolve the tie, it is broken randomly.¹⁶

With our tie-breaking rule, three possible assignments occur with positive probability in the auction: (a) bidder 3 wins AB (if $v_3 > \max\{b_{AB}(v_1), b_A(v_1) + v_2\}$), (b) bidder 1 wins AB (if $b_{AB}(v_1) > \max\{v_3, b_A(v_1) + v_2\}$), or (c) bidders 1 and 2 win A and B , respectively (if $b_A(v_1) + v_2 \geq \max\{b_{AB}(v_1), v_3\}$). The corresponding MRCS optimization problems in

¹⁶Note that most ties are zero probability events. This tie-breaking rule prevents a discontinuity in bidder 1's expected payoff function when he bids zero for A .

these three cases are:

$$\begin{aligned}
(a) \quad & \min_p p_3 \\
& \text{s.t. } p_1 = 0, \quad p_2 = 0, \quad p_3 \geq \max\{b_{AB}(v_1), b_A(v_1) + v_2\} \\
(b) \quad & \min_p p_1 \\
& \text{s.t. } p_1 \geq \max\{v_2, v_3\}, \quad p_2 = 0, \quad p_3 = 0 \\
(c) \quad & \min_p (p_1 + p_2) \\
& \text{s.t. } p_1 + p_2 \geq v_3 \\
& p_1 \geq \max\{v_2, v_3\} - v_2 \\
& p_2 = \max\{b_A(v_1), b_{AB}(v_1), v_3\} - b_A(v_1) \\
& p_3 = 0
\end{aligned}$$

If bidder 3 wins, he pays $p_3 = \max\{b_{AB}(v_1), b_A(v_1) + v_2\}$ and the other bidders pay nothing. If bidder 1 wins AB , he pays $p_1 = \max\{v_2, v_3\}$ and the other bidders pay nothing. Whenever bidder 2 wins, he wins B and bidder 1 wins A . The payments are $p_3 = 0$, $p_2 = \max\{b_A(v_1), b_{AB}(v_1), v_3\} - b_A(v_1)$, and

$$p_1 = \max\{v_3 - p_2, \max\{v_3 - v_2, 0\}\} = \max\{\min\{v_3, v_3 - b_{AB}(v_1) + b_A(v_1), b_A(v_1)\}, 0\}. \quad (2)$$

Note that bidder 1 can decrease his payment when he wins A by bidding low for A and high for AB , as long as his bids still satisfy $b_A(v_1) + v_2 \geq \max\{b_{AB}(v_1), v_3\}$.

The unique bidding functions for packages A and AB that maximize bidder 1's expected payoff (see Appendix A.1) are

$$b_A^*(v_1) = \begin{cases} 0 & \text{if } 0 \leq v_1 \leq 2 - 2\sqrt{6}/3 \\ z - (2 - \sqrt{1 - 6z + 6v_1})/3 & \text{if } 2 - 2\sqrt{6}/3 < v_1 \leq 1 \end{cases}$$

$$b_{AB}^*(v_1) = \begin{cases} v_1/2 & \text{if } 0 \leq v_1 \leq 2 - 2\sqrt{6}/3 \\ z & \text{if } 2 - 2\sqrt{6}/3 < v_1 \leq 1 \end{cases}$$

where z is the unique solution to

$$0 = 12v_1 - 15z - 1 + (9z - 1 - 3v_1)\sqrt{1 - 6z + 6v_1}$$

$$z \geq \max \left\{ (1 - \sqrt{6v_1 - 2})/3, v_1 - 1/2 \right\}$$

The bidding functions are also given explicitly in Appendix A.1.

Figure 1 illustrates the bidding functions $b_A^*(v_1)$ and $b_{AB}^*(v_1)$. We summarize the resulting equilibria in the following theorem.

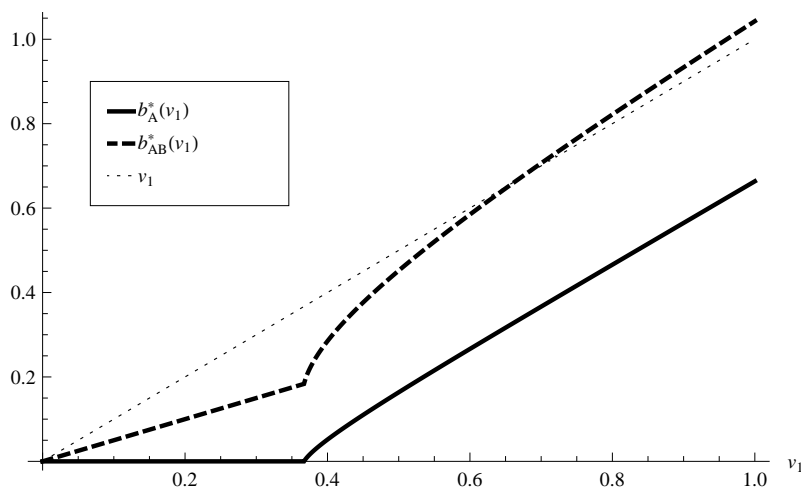


Figure 1: Equilibrium Bidding Functions $b_A^*(v_1)$ and $b_{AB}^*(v_1)$

Theorem 4. *Consider any favored-bidder-2 MRCS auction with three bidders who have*

values for two items as given in Table 1 and a tie-breaking rule that favors bidder 1 winning A . The following are all of the Bayesian equilibria in pure strategies in which bidders 2 and 3 play undominated strategies:

- $b_1^*(v_1) = (b_A^*(v_1), b_B^*(v_1), b_{AB}^*(v_1))$ for any $b_B^*(v_1) \in [0, b_{AB}^*(v_1))$
- $b_2^*(v_2) = (0, v_2, v_2)$
- $b_3^*(v_3) = (0, 0, v_3)$

Since bidder 1 has unique bidding functions for A and AB that maximize his expected payoff, the only such equilibria in mixed strategies involve a randomized choice of $b_B^*(v_1)$ with maximum support $[0, b_{AB}^*(v_1))$.

Corollary 1. *For every realization of values, there is a unique outcome (assignment and payments) of Bayesian equilibria in undominated strategies.*¹⁷

Proof: Consider the equilibria in Theorem 4. Bidders 2 and 3 have unique undominated strategies. Given those strategies, all of bidder 1's equilibrium strategies lead to the same assignment and payments as $b_1(v_1) = (b_A^*(v_1), 0, b_{AB}^*(v_1))$. The strategies $b_1^*(v_1)$ and mixtures thereof are his set of best response bids to $b_2^*(v_2)$ and $b_3^*(v_3)$. At least one strategy in this set must be undominated. Namely, $b_1^*(v_1) = (b_A^*(v_1), 0, b_{AB}^*(v_1))$ is undominated. First, it cannot be dominated by any bid (or mixture of bids) that is not a best response to the truthful bidding by bidders 2 and 3. So it will be undominated as long as no bid (or mixture of bids) $b'_1(v_1) = (b_A^*(v_1), x, b_{AB}^*(v_1))$ with $x \in (0, b_{AB}^*(v_1))$ dominates it.

¹⁷The outcomes are as follows:

- If $b_A^*(v_1) + v_2 \geq \max\{b_{AB}^*(v_1), v_3\}$, then bidder 1 wins A , bidder 2 wins B , $p_1 = \max\{\min\{v_3 - b_{AB}^*(v_1) + b_A^*(v_1), b_A^*(v_1)\}, 0\}$, $p_2 = \max\{b_{AB}^*(v_1), v_3\} + b_A^*(v_1)$ and $p_3 = 0$.
- If $b_{AB}^*(v_1) > \max\{b_A^*(v_1) + v_2, v_3\}$, then bidder 1 wins AB , $p_1 = \max\{v_2, v_3\}$ and $p_2 = p_3 = 0$.
- If $v_3 > \max\{b_A^*(v_1) + v_2, b_{AB}^*(v_1)\}$, then bidder 3 wins AB , $p_1 = p_2 = 0$, and $p_3 = v_3 - \max\{b_{AB}^*(v_1), b_A^*(v_1) + v_2\}$.
- If $v_3 = b_{AB}^*(v_1) > b_A^*(v_1) + v_2$, then one of the two preceding outcomes is chosen randomly.

However, b_1^* cannot be dominated by b'_1 because for every v_1 and every $x \in (0, b_{AB}^*(v_1))$, there are strategies for bidders 2 and 3 such that, for every realization of values v_2 and v_3 , bidder 1 wins B and pays at least $p_1 = \varepsilon > 0$. This results in a negative expected payoff while he would have received a payoff of zero if he had bid b_1^* . An example of such bids is $b_2(v_2) = (\beta, 0, \beta + \varepsilon)$ and $b_3(v_3) = (0, 0, 0)$, where $\beta \geq b_{1AB}^*(v_1)$ and $0 < \varepsilon < x$. ■

Corollary 2. *Equilibrium bids may be above true values.*

The bid $b_{AB}^*(v_1)$ exceeds v_1 when $v_1 > 2/3$. To profit from lower prices when he wins A , bidder 1 bids above his value for AB , thereby risking a negative payoff from winning AB at a price $p_1 > v_1$.

The kinks in bidder 1's bidding functions occur because of the binding nonnegativity constraint on bids when $v_1 \leq 2 - \frac{2}{3}\sqrt{6}$. He would prefer to bid less than zero for package A . He is willing to bid zero because our tie-breaking rule ensures he will still win package A when $v_2 \geq v_3$. The two general characteristics of his bids are $b_A(v_1) < v_1$ and $b_{AB}(v_1) > b_A(v_1)$. He bids below his true value for package A because it reduces his payment when he wins that package. However, it also reduces his chance of winning package A . Bidding $b_{AB}(v_1) > b_A(v_1)$ mitigates this effect, giving him a second chance to get his desired item as part of package AB . Bidding $b_{AB}(v_1) > b_A(v_1)$ has another beneficial effect – it reduces bidder 1's payment when he wins package A . The separation between his bids for A and AB could also be considered overbidding because it implies a demand for item B , which raises the price for B and thereby lowers the price bidder 1 pays for A . This effect only matters when $b_A(v_1) > 0$ because otherwise he pays nothing for A regardless of $b_{AB}(v_1)$. Bidding too much for AB also has a downside because packages AB and A provide the same value but the payment for AB is higher with probability one.

Table 3 compares equilibrium revenues, payoffs, and efficiency (expected total value of the items traded) for the Vickrey and favored-bidder-2 MRCS auctions. The values for the MRCS auctions are approximated (weighted averages of 1001 equidistant values of v_1).

Table 3: Comparing Revenue, Payoffs, and Efficiency in the Vickrey and Favored-Bidder-2 MRCS Auctions

	Vickrey	MRCS	MRCS/Vickrey
$E[\pi_0]$	0.583	0.596 [#]	102.2 %
$E[\pi_1]$	0.208	0.154 [#]	74.1 %
$E[\pi_2]$	0.208	0.093 [#]	44.5 %
$E[\pi_3]$	0.292	0.418 [#]	143.2 %
$E[\sum_{i=0}^3 \pi_i]$	1.292	1.261 [#]	97.6 %
$\text{Prob}\{\pi \in \text{Core}\}$	0.50	0.308 [#]	61.6 %

[#] Approximated

Expected revenues in any favored-bidder-2 MRCS auction are higher than those in the Vickrey auction. The revenue advantage of these MRCS auctions stems mainly from the realizations where bidders 1 and 2 win packages A and B , respectively. The Vickrey auction is usually better for the seller if bidder 3 wins because $b_A^*(v_1) < v_1$.¹⁸ However, the probability of *zero revenue* in the Vickrey auction is 1/6 whereas it is approximately zero in the MRCS auction. The maximum revenue in the Vickrey auction is 2 while that in the MRCS auction is $b_A^*(1) + 1 = 1.66$ (both are zero probability events).

Figure 2 displays the expected revenue in the Vickrey and favored-bidder-2 MRCS auctions in equilibrium for different values of v_1 . The MRCS auctions outperform the Vickrey auction in expectation for high values of v_1 and vice versa for low v_1 . For higher v_1 , the probability that bidder 3 wins decreases and the zero revenue outcomes of the Vickrey auction, which occur when $v_1 \geq v_3$ and $v_2 \geq v_3$, become more likely.

The revenue ranking between the Vickrey and favored-bidder-2 MRCS auctions clearly depends on the distributions of values.¹⁹ However, the key here is that the overbidding created by allowing bidders to place bids on all packages increases the revenue relative to the restricted setting.²⁰

¹⁸This does not hold if $b_{AB}^*(v_1) > v_1 + v_2$.

¹⁹Ausubel and Baranov (2010) showed that varying the distributional assumptions changes the revenue ranking in their setting with restricted bids. We show the same for our setting in Appendix A.5.

²⁰See Appendix A.4.

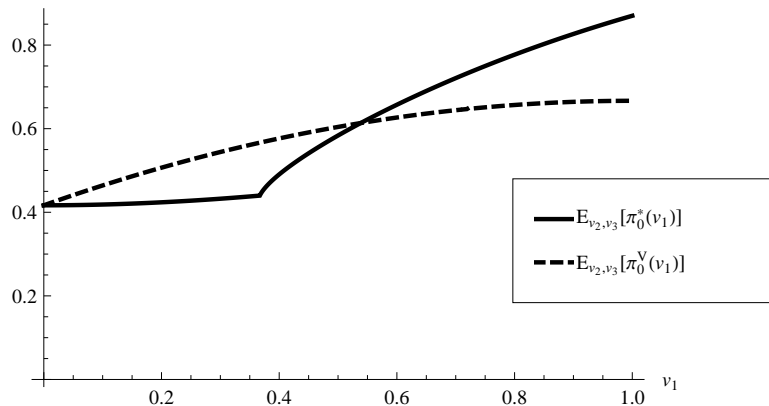


Figure 2: Expected Revenue in the Vickrey and Favored-Bidder-2 MRCS Auctions as a Function of v_1

Both bidders 1 and 2 receive lower expected payoffs in our MRCS auction than in the Vickrey auction, while bidder 3 gets a higher expected payoff. He profits from the competition for low prices between bidders 1 and 2.

In equilibrium, bidder 2 has a *lower expected payoff* than bidder 1 even though the payment rule favors bidder 2. If all bidders reported their values truthfully, the expected payoffs of the seller and bidders 1, 2, and 3 would be 0.708, 0.083, 0.208, and 0.292, respectively. Bidder 3 profits from the distortion in bidder 1's bid both by winning with higher probability and paying lower prices on average. The strategic bids increase bidder 1's payoff by more than 60% compared to his payoff from truthful bidding in the MRCS auction.²¹ However, bidder 2 suffers. As a consequence of bidder 1's strategic bids, his expected payoff decreases by 45%. Not surprisingly, the strategic bids also reduce the efficiency of the allocation compared to that in the Vickrey auction.

The probability that the payoff vector from the Vickrey auction is in the core with respect to the true values $\mathcal{C}(v)$ is higher than this probability for the favored-bidder-2 MRCS auction (0.5 and 0.308, respectively).²² The values for which the Vickrey payoffs

²¹Note that his maximum payoff in the MRCS auction is bounded above by his equilibrium payoff in the Vickrey auction because bidders 2 and 3 bid truthfully.

²²See Appendix A.2 for an analysis.

are in $\mathcal{C}(v)$ are almost the opposite of those for which our MRCS payoffs are in $\mathcal{C}(v)$. The Vickrey payoffs are in the core if and only if bidder 3 wins AB .²³ The MRCS payoffs are in the core if bidders 1 and 2 win their desired items and in a small subset of cases where bidder 3 wins. The MRCS payoffs are never in the core when bidder 1 wins AB because the assignment is not optimal. When bidder 3 wins AB , the assignment is not always optimal and the price is often below $v_1 + v_2$. Thus, if the MRCS payoffs are not in $\mathcal{C}(v)$, then the blocking coalition usually consists of the seller and bidders 1 and 2, even though bidder 1's low bid caused the outcome not to be in the core. Note the difference: if the Vickrey payoffs are not in the core, the blocking coalition consists of the seller and bidder 3, who reported a higher value than the sum of the winners' prices.

4.2. Partially Informed Bidders

So far we have only analyzed one set of MRCS auctions under incomplete information. What happens in auctions with different payment rules? Incentives for overbidding may depend on the information structure, so we now choose an extreme one to highlight how shading bids can be costly compared to overbidding and to show that overbidding can not only occur under *all* payment rules but also in *every* Bayesian equilibria in undominated strategies for all of these payment rules.

Suppose bidders 1 and 2 know each other's values (and their own values), but they do not know bidder 3's value. Bidder 3 knows only his own value. Then, for bidders 1 and 2, shading bids is risky because the sum of their bids may not meet the unknown threshold (v_3) necessary to win their preferred items. Overbidding comes with no such risk. They can always force each other to pay their full values without worrying about winning the wrong package. The freedom to place arbitrary bids on all packages allows bidders 1 and 2 to overcome the threshold problem through overbidding.²⁴

²³See Goeree and Lien (2009, Lemma 1) and Appendix A.2.

²⁴This is in sharp contrast to the results in Ausubel and Baranov (2010). They study a case in which the local bidders have perfectly correlated values, so they essentially know each other's values. However, for

Theorem 5. *Consider a setting with three bidders who have values for two items as given in Table 1. For every MRCS auction that breaks ties by randomizing over optimal assignments that maximize the number of bidders, there always exists at least one Bayesian equilibrium in undominated strategies and every Bayesian equilibrium in undominated strategies satisfies the following properties:*

- (1) *every winning bid is truthful*
- (2) *at least one bidder places a bid strictly above his true value*
- (3) *no bidder places a bid strictly below his value for any package, unless that bid cannot affect the assignment*
- (4) *the assignment is efficient*
- (5) *the revenue is at least as large as that from the Vickrey auction for every realization of values*
- (6) *the payoffs are in the core with respect to the true values $\mathcal{C}(v)$*
- (7) *the equilibrium is an ex-post equilibrium*

The proof follows from a series of lemmas in Appendix B. First, we narrow down the set of equilibria by applying Theorem 3 and restricting the global bidder to his dominant strategy of bidding truthfully. We also prove that for either local bidder, it is weakly dominated to bid zero on his desired item. Then we characterize the possible Bayesian equilibria under these restrictions. The resulting strategy profiles are Bayesian equilibria for every MRCS auction, but not necessarily in undominated strategies. Whether any particular strategy is undominated depends on the exact MRCS auction rules, so the exact set of Bayesian equilibria in undominated strategies will too. Therefore, we prove that, for every MRCS auction, there exists at least one equilibrium in the following set that is an equilibrium in undominated strategies. Moreover, all of the Bayesian equilibria in

many of the MRCS auctions they study, the equilibria with their restricted bids involve bid shading and inefficient outcomes.

undominated strategies for every MRCS auction belongs to the following set:

$$b_1(v_1, v_2) = (v_1, b_{1B}(v_1, v_2), b_{1AB}(v_1, v_2))$$

$$b_2(v_1, v_2) = (b_{2A}(v_1, v_2), v_2, b_{2AB}(v_1, v_2))$$

$$b_3(v_3) = (0, 0, v_3)$$

where

$$\max\{b_{1AB}(v_1, v_2), b_{2AB}(v_1, v_2)\} = v_1 + v_2$$

$$b_{1B}(v_1, v_2) + b_{2A}(v_1, v_2) < v_1 + v_2.$$

In each of these equilibria, bids for desired packages are truthful and there is almost no bid shading.^{25,26} At least one of the local bidders overbids for some package. He does this to force the other bidder to pay his full value whenever he wins, regardless of the bid placed by bidder 3. This means the local bidder doing the overbidding will only need to pay the residual necessary to beat bidder 3: $\max\{v_3 - v_i, 0\}$ where v_i is the other local bidder's value.

A common problem in combinatorial auctions is the threshold problem, in which bidders have trouble coordinating their bids because they want to free ride off of each others' payments. In our MRCS auctions without any bidding restrictions, the local bidders can free ride by overbidding on losing items, thereby driving up each other's prices. The

²⁵For some payment rules, there may be an equilibrium in undominated strategies in which the local bids are $b_1(v_1, v_2) = (v_1, 0, v_1 + v_2)$ and $b_2(v_1, v_2) = (0, v_2, x)$ such that $x < v_2$. However, the reduced bid for AB leads to the same assignment as bid $(0, v_2, v_2)$. For familiar payment rules, such as the Vickrey-nearest rule, it leads to the same prices as well, so the two bids are equivalent.

²⁶Among all tie-breaking rules that maximize the number of winners, there is one in which alternative Bayesian equilibria in undominated strategies arise for some payment rules. If we had instead chosen the tie-breaking rule that gives item A to bidder 1 and item B to bidder 2 whenever possible, it would be undominated under some payment rules for the local bidders to bid zero for their desired items. This relies critically on the fact that local bidders could win their desired item with probability one when bidding zero and necessarily pay a higher price if they raised their bid. Then there could be inefficient equilibria that reduce to a single item auction for the global package.

overbidding on AB represents an implied demand for the other bidder's desired item, forcing him to pay a higher price. The ability to drive up a competitor's price without distorting the bids for the optimal winning packages eliminates the coordination problem and restores efficiency.

If bidders 1 and 2 win, at least one and, depending on the equilibrium, maybe both of them pay their full value. This can lead to very high revenues or to an extremely asymmetric outcome among bidders. The revenues are strictly greater than those from truthful bidding in the Vickrey auction with probability one when the local bidders win. When the global bidder wins, the revenues are the same as those from the Vickrey auction with truthful bidding. So the equilibria in undominated strategies in the MRCS auctions lead to strictly higher expected revenues and weakly higher realized revenues than those in the Vickrey auction. However, all of the equilibria list above are also equilibria of the Vickrey auction, just in dominated strategies.

These results do not depend on the exact distribution of the bidders' values, but only on their support. The derivation of these equilibria and the proof of Theorem 5 hold whenever the local bidders have values drawn from the open interval $(0,1)$ and the global bidder's value is drawn from $[0,2]$. The equilibria we listed above remain equilibria when we expand the support of the local bidder values to the closed interval $[0,1]$, however, some additional equilibria arise at the end points. The additional equilibria lead to the same outcomes as the ones we list, so they still have an efficient assignment, revenue at least as large as that from the Vickrey auction, and payoffs in the core with respect to the true values, and they are ex-post equilibria. However, the actual bids can be slightly different. When $v_1 = v_2 = 1$, the bidders know v_3 cannot be strictly higher than the sum of their values, so they can bid more than their true values for their desired items. When either $v_1 = 0$ or $v_2 = 0$, there will not be any overbidding because the bidder with a zero valuation cannot pay more than zero (his full value) regardless of the other bids.

Theorem 5 would also hold if we increased the number of bidders while holding the number of goods constant. We could also increase the number of items and, as long as the

number of local bidders grew with the number of items, the theorem would still apply.

A special feature of this information structure – that price can be manipulated without affecting the assignment – makes overbidding particularly advantageous compared to bid shading. This structure might be of particular interest in spectrum auctions, in which the MRCS auction usually follows an ascending price round. Ausubel and Cramton (2011) showed that particular revealed-preference eligibility rules, which restrict bids in the core-selecting round, can actually prevent the bidders from being able to change the assignment that was provisionally winning at the end of the clock stage. In this case, bidders could place higher losing bids without the risk of actually winning those packages. So incentives might be similar to those in the equilibria in this section.

5. Conclusion

We have found new equilibria of MRCS auctions with overbidding in a variety of informational settings. These equilibria stem from a common incentive to raise other bidders' payments so that one's own payment may decrease. In certain settings, every equilibrium in undominated strategies of every MRCS auction has the properties that some bidder overbids and no bidder engages in bid shading for winning packages. Therefore, the overbidding phenomenon cannot be ruled out by clever equilibrium selection. Since bidders want to overbid on losing packages to drive up their competitor's prices, the overbidding does not cause a non-monotonic decrease in revenue. Instead, overbidding leads to higher revenues and possibly asymmetric payoffs among bidders. Also, because overbidding helps bidders overcome the threshold problem when there is incomplete information, it improves expected efficiency and can result in the efficient assignment. Therefore, it increases the total surplus to be split between buyers and the seller.

Though we use favored-bidder-2 MRCS auctions in the independent private values setting for tractability, the incentives for overbidding do not disappear under other payment rules. We have shown that overbidding occurs with full and partial information under all

payment rules, including Vickrey-nearest, reference rules, and the nearest bid rule. With more uncertainty, a bidder must balance incentives for both shading and overbidding. We cannot say whether an equilibrium of the Vickrey-nearest rule in the independent-private-values setting would have bids above the true values, but the incentives to create false demand for an unwanted item – by separating the bids for the desired item and global bundle – do not disappear.

Strategic overbidding requires less sophistication than strategic shading. To overbid optimally, a bidder needs only know the total value of the winning assignment. To truncate optimally, a bidder needs to not only know the value of the winning assignment but also what the value of the winning assignment would be if he were not present. Also, a (full information) equilibrium with overbidding can be reached with just one strategic bidder. A truncation equilibrium may require multiple strategic bidders coordinating their truncations. Therefore, the equilibria with overbidding may actually be easier to find in practice. This might be of special concern in the design of spectrum auctions, which usually involve an ascending auction, followed by a minimum-revenue core-selecting sealed-bid round. If the bidders can use the ascending piece to learn about the value of the winning bids, they enter the supplementary MRCS auction equipped with the knowledge needed to strategically overbid.

References

- Ausubel, L. M., Baranov, O. V., 2010. Core-selecting auctions with incomplete information. Working Paper, University of Maryland, preliminary.
- Ausubel, L. M., Cramton, P., 2011. Activity rules for the combinatorial clock auction. Discussion Paper, University of Maryland.
- Ausubel, L. M., Milgrom, P. R., 2002. Ascending auctions with package bidding. *Frontiers of Theoretical Economics* 1 (1), 1–42.

- Ausubel, L. M., Milgrom, P. R., 2006. The lovely but lonely Vickrey auction. In: Cramton, P., Shoham, Y., Steinberg, R. (Eds.), *Combinatorial Auctions*. MIT Press, Cambridge, Massachusetts.
- Beck, M., Ott, M., 2009. Revenue monotonicity in core-selecting package auctions. Working paper.
- Cramton, P., Day, R., 2008. The quadratic core-selecting payment rule for combinatorial auctions. Working Paper, University of Maryland.
- Day, R., Milgrom, P., 2010, forthcoming. Optimal incentives in core-selecting auctions. *Handbook of Market Design*, Oxford University Press.
- Day, R., Milgrom, P. R., 2008. Core-selecting package auctions. *International Journal of Game Theory* 36 (3), 393–407.
- Day, R., Raghavan, S., 2007. Fair payments for efficient allocations in public sector combinatorial auctions. *Management Science* 53 (9), 1389–1406.
- Erdil, A., Klemperer, P., 2010. A new payment rule for core-selecting package auctions. *Journal of the European Economic Association* 8 (2–3), 537–547.
- Goeree, J. K., Lien, Y., 2009. On the impossibility of core-selecting auctions. Working Paper, University of Zurich.
- Lamy, L., 2010. Core-selecting package auctions: A comment on revenue monotonicity. *International Journal of Game Theory* 39 (3), 503–510.
- Salonen, H., 1996. On the existence of undominated Nash equilibria in normal form games. *Games and Economic Behavior* 14, 208–219.
- Sano, R., 2011. Incentives in core-selecting auctions with single-minded bidders. *Games and Economic Behavior* 72, 602–606.

Appendix A. Independent Private Values

Appendix A.1. Bidder 1's Expected Payoff and Equilibrium Bids

First we show that $2 \geq b_A(v_1) + 1 \geq b_{AB}(v_1) \geq b_A(v_1) \geq 0$ and $b_{AB}(v_1) > b_B(v_1)$ for all v_1 in any Bayesian equilibrium in undominated strategies. Then we maximize bidder 1's expected payoff subject to those constraints to find his equilibrium bids.

All bids $b_{AB}(v_1) \leq b_A(v_1)$ give bidder 1 the same payoff, so we can concentrate on $b_{AB}(v_1) \geq b_A(v_1)$.

Given that bidders 2 and 3 report their values truthfully, bidder 1 strictly prefers bids with $b_B(v_1) < b_{AB}(v_1)$ for all v_1 . As long as $b_B(v_1) < b_{AB}(v_1)$, $b_B(v_1)$ does not affect bidder 1's expected payoff. To rule out $b_B(v_1) \geq b_{AB}(v_1)$, note that any bid $b_1(v_1) = (b_A(v_1), b_B(v_1), b_{AB}(v_1))$ with $b_B(v_1) \geq b_{AB}(v_1)$ yields a strictly lower expected payoff than the bid $b'_1(v_1)$ with $b'_A(v_1) = b_A(v_1)$, $b'_B(v_1) = 0$, and $b'_{AB}(v_1) = b_{AB}(v_1)$. With these two bids $b_1(v_1)$ and $b'_1(v_1)$, bidder 1 wins package A against the same realizations of v_2 and v_3 and pays the same price for it. When bidder 1 does not win package A , he wins package B with bid $b_1(v_1)$ (or he wins B and AB with equal probability if $b_B(v_1) = b_{AB}(v_1)$) and package AB with $b'_1(v_1)$, but pays the same price regardless of which he wins. Thus, winning B yields a strictly lower payoff because it has no value. This occurs with positive probability when bidders 2 and 3 bid truthfully, and so $b'_1(v_1)$ is strictly better for bidder 1.

Given that bidders 2 and 3 report their true values, that $b_{AB}(v_1) \geq b_A(v_1)$ and that $b_{AB}(v_1) > b_B(v_1)$, any bid $b_1(v_1) = (b_A(v_1), b_B(v_1), b_{AB}(v_1))$ with $b_A(v_1) > 1$ yields a lower expected payoff than the bid $b'_1(v_1) = (1, \max\{b_B(v_1) - (b_A(v_1) - 1), 0\}, b_{AB}(v_1) - (b_A(v_1) - 1))$. We will show that for every realization of v_2 and v_3 , bidder 1's ex-post payoff is weakly higher with $b'_1(v_1)$ and that it is strictly higher for some realizations that occur with positive probability. Therefore, equilibrium bids have $b_1(v_1) \leq 1$ for all v_1 . Bidder 1 cannot win package B because bidders 2 and 3 bid zero for A , so the remaining possibilities are as follows.

Case 1: Bidder 1 wins nothing with $b_1(v_1)$ (i.e., $v_3 > \max\{b_{AB}(v_1), b_A(v_1) + v_2\}$). Then he also wins nothing with $b'_1(v_1)$ and his payoff is the same.

Case 2: Bidder 1 wins AB with $b_1(v_1)$ (i.e., $b_{AB}(v_1) > \max\{v_3, b_A(v_1) + v_2\}$). If $b_{AB}(v_1) - b_A(v_1) + 1 > v_3$, then he also wins AB with $b'_1(v_1)$ and pays $p'_1 = \max\{v_2, v_3\} = p_1$, so his payoff is the same. If $v_3 > b_{AB}(v_1) - b_A(v_1) + 1$, then he loses and receives a payoff of zero. His payoff with $b_1(v_1)$ is $v_1 - \max\{v_2, v_3\} \leq v_1 - v_3 < 0$ because $v_3 > b_{AB}(v_1) - b_A(v_1) + 1 \geq 1$.

Case 3: Bidder 1 wins A with $b_1(v_1)$ (i.e., $b_A(v_1) + v_2 \geq \max\{v_3, b_{AB}(v_1)\}$). If $1 + v_2 > v_3$, then he also wins A with $b'_1(v_1)$ and pays $p'_1 = \max\{\min\{v_3 + 1 - b_{AB}(v_1) + b_A(v_1) - 1, 1\}, 0\} \leq p_1 = \max\{\min\{v_3 - b_{AB}(v_1) + b_A(v_1), b_A(v_1)\}, 0\}$ (see Equation 2 and note that $b_A(v_1) \leq b_{AB}(v_1)$). If $v_3 > 1 + v_2$, then he loses and receives a payoff of zero. With bid $b_1(v_1)$, his payoff $\pi_1 = v_1 - p_1 = v_1 - \max\{\min\{b_A(v_1), v_3 + b_A(v_1) - b_{AB}(v_1)\}, 0\} \leq v_1 - (v_3 - v_2) < v_1 - 1 \leq 0$.

For any $b_{AB}(v_1) > 2$ and $b_A(v_1) \leq 1$, bidder 1's expected payoff is $v_1 - 13/12$, which is negative. Therefore, bidding $b_{AB}(v_1) \leq 2$ yields a strictly higher payoff because in doing so bidder 1 can always choose to receive a payoff of zero.

If $2 \geq b_{AB}(v_1) \geq b_A(v_1) + 1$, the probability that bidder 1 wins package A is zero. His expected payoff is $(6b_{AB}(v_1)v_1 - 3b_{AB}(v_1)^2 - 1)/12$, which is decreasing in $b_{AB}(v_1)$ when $b_{AB}(v_1) \geq b_A(v_1) + 1$. Therefore, bidder 1 would prefer to bid $b_{AB}(v_1) = b_A(v_1) + 1$ and we can conclude $b_A(v_1) + 1 \geq b_{AB}(v_1)$ for all v_1 .

Thus, $2 \geq b_A(v_1) + 1 \geq b_{AB}(v_1) \geq b_A(v_1) \geq 0$ for all v_1 in equilibrium and bidder 1's

expected payoff is:²⁷

$$\begin{aligned}
E_{V_2, V_3}[\pi_1(v_1)] &:= E[\pi_1(v_1) | b_2(v_2) = (0, v_2, v_2), b_3(v_3) = (0, 0, v_3)] \\
&= v_1 (1 + b_{AB}(v_1)^2 - 2b_A(v_1)b_{AB}(v_1) + b_A(v_1)^2 + 2b_A(v_1)) / 4 \quad (\text{A.1}) \\
&\quad + (-4b_{AB}(v_1)^3 + 3b_A(v_1)b_{AB}(v_1)^2 + 6b_A(v_1)^2b_{AB}(v_1)) / 12 \\
&\quad + (6b_A(v_1)b_{AB}(v_1) - 5b_A(v_1)^3 - 9b_A(v_1)^2 - 3b_A(v_1)) / 12.
\end{aligned}$$

Maximizing bidder 1's expected payoff (A.1) with respect to $b_A(v_1)$ and $b_{AB}(v_1)$ under the constraints $2 \geq b_A(v_1) + 1 \geq b_{AB}(v_1) \geq b_A(v_1) \geq 0$ gives us the equilibrium bidding functions.

The explicit formulas are:

$$\begin{aligned}
b_A^*(v_1) &= \left(-41 + 45v_1 + \frac{1540 + 1440v_1 + 1296v_1^2}{2^{4/3}x^{1/3}} + \frac{x^{1/3}}{2^{2/3}} \right) / 81 \\
b_{AB}^*(v_1) &= \begin{cases} \left(1 + 3b_A(v_1) - \sqrt{2}\sqrt{-1 + 3v_1 - 3b_A(v_1)} \right) / 3 & \text{if } 2 - 2\sqrt{6}/3 \leq v_1 < 2/3 \\ \left(1 + 3b_A(v_1) + \sqrt{2}\sqrt{-1 + 3v_1 - 3b_A(v_1)} \right) / 3 & \text{if } 2/3 \leq v_1 \leq 1. \end{cases}
\end{aligned}$$

$$\begin{aligned}
x &:= -35600 - 44820v_1 - 15795v_1^2 + 23328v_1^3 \\
&\quad + 405\sqrt{3}\sqrt{720 + 1280v_1 - 3184v_1^2 - 10776v_1^3 - 11781v_1^4 - 5184v_1^5}
\end{aligned}$$

We have found bidder 1's unique best-response bidding functions for packages A and AB , and have excluded bids for B that satisfy $b_B(v_1) \geq b_{AB}(v_1)$. Given $b_2^*(v_2)$ and $b_3^*(v_3)$, all bids for B that satisfy $b_B(v_1) < b_{AB}(v_1)$ combined with $b_A^*(v_1)$ and $b_{AB}^*(v_1)$ are best responses and yield the same assignment and payments.

²⁷We denote the random variable of bidder i 's value by V_i .

Appendix A.2. Payoff Vectors in the Core $\mathcal{C}(v)$

When is the payoff vector selected by the MRCS auctions or the Vickrey auction in the core with respect to the true values $\mathcal{C}(v)$?

Assume that v_3 is drawn from the uniform distribution on $[0, k]$ with $k \geq 2$. For $k = 2$ we get the case analyzed in Section 4.1. The equilibrium analysis for $k > 2$ and independent values is given in Appendix A.5.

For the MRCS auctions, we have to consider three cases: (i) bidder 1 wins AB , (ii) bidder 3 wins AB , and (iii) bidders 1 and 2 win A and B , respectively.

Case (i): If bidder 1 wins AB , the assignment is not optimal because $v_1 + v_2 \geq v_1$ and, therefore, the allocation is not in the core.

Case (ii): If bidder 3 wins AB , his payment is $p_3 = \max\{b_A(v_1) + v_2, b_{AB}(v_1)\}$. The core constraint requires that $p_3 \geq v_1 + v_2$. Since $v_1 > b_A(v_1)$, p_3 satisfies the constraint only if $v_3 \geq b_{AB}(v_1) \geq v_1 + v_2$. This occurs with probability $\int_{b_{AB}(v_1)}^k \int_0^{\max\{b_{AB}(v_1) - v_1, 0\}} 1/k dv_2 dv_3 = \max\{b_{AB}(v_1) - v_1, 0\} - b_{AB}(v_1) \max\{b_{AB}(v_1) - v_1, 0\}/k$ for given a v_1 , or approximately $0.008 - 0.007/k$ in expectation.

Case (iii): If bidder 1 wins A and bidder 2 wins B , the constraints in the auction and in the true core are, respectively:

$$p_1 + p_2 \geq v_3 \quad (\text{A.2}) \quad p_1 + p_2 \geq v_3 \quad (\text{A.5})$$

$$p_1 \geq \max\{v_2, v_3\} - v_2 \quad (\text{A.3}) \quad p_1 \geq \max\{v_2, v_3\} - v_2 \quad (\text{A.6})$$

$$p_2 \geq \max\{b_{AB}(v_1), v_3\} - b_A(v_1) \quad (\text{A.4}) \quad p_2 \geq \max\{v_1, v_3\} - v_1 \quad (\text{A.7})$$

Indeed, the auction payments determined by constraints (A.2), (A.3), and (A.4) satisfy the core constraints (A.5), (A.6), and (A.7), so the payoff vector is in the core with respect to the true values. The fulfillment of (A.5) and (A.6) is trivial. Comparing (A.4) and (A.7), we find that

$$p_2 \geq \max\{b_{AB}(v_1), v_3\} - b_A(v_1) \geq \max\{v_1, v_3\} - v_1$$

because $\max\{b_{AB}(v_1), v_3\} - b_A(v_1) + v_1 = \max\{v_1 + (b_{AB}(v_1) - b_A(v_1)), v_3 + (v_1 - b_A(v_1))\} \geq \max\{v_1, v_3\}$ since $b_A(v_1) < v_1$ and $b_A(v_1) \leq b_{AB}(v_1)$. Given v_1 , the probability of this case is $\text{Prob}\{b_A(v_1) + v_2 \geq \max\{b_{AB}(v_1), v_3\}\} = \int_{b_{AB}(v_1) - b_A(v_1)}^1 \int_0^{v_2 + b_A(v_1)} 1/k \, dv_3 \, dv_2 = (b_A(v_1)^2 + 2b_A(v_1) - b_{AB}(v_1)^2 + 1)/(2k)$, which is approximately $0.607/k$ in expectation.

So, combining the three cases, the total probability that a payoff vector from a favored-bidder-2 MRCS auction is in the true core is approximately $0.008 + 0.600/k$, which is 0.308 for $k = 2$.

In the Vickrey auction, either bidder 3 wins AB or bidders 1 and 2 win A and B , respectively.²⁸ The assignment is always optimal. The payoff vector is always in the core if bidder 3 wins: $p_3 = v_1 + v_2$. If bidders 1 and 2 win, the payoff vector is not in the core. Payments are $p_1 = \max\{v_2, v_3\} - v_2$ and $p_2 = \max\{v_1, v_3\} - v_1$, so

$$\begin{aligned} p_1 + p_2 &= \max\{v_2, v_3\} - v_2 + \max\{v_1, v_3\} - v_1 \\ &= \max\{v_2 + v_1, v_2 + v_3, v_3 + v_1, 2v_3\} - (v_1 + v_2) \leq v_3 \end{aligned}$$

because $v_1 + v_2 \geq v_3$ if they win. The probability that the payoff vector is in the core is $\text{Prob}(v_1 + v_2 < v_3) = \int_0^1 \int_0^1 \int_{v_1 + v_2}^k 1/k \, dv_3 \, dv_2 \, dv_1 = 1 - 1/k$, which is 0.5 for $k = 2$.

Appendix A.3. Lowering Payments Drives Overbidding

In the independent private values setting from Section 4.1, the higher bid for AB (in comparison with $b_{1AB} \leq b_{1A}$) has three effects for bidder 1: it decreases his chance of winning A (which he prefers to winning AB because his payment is weakly lower), it increases his chance of winning a bundle of value v_1 , and it weakly decreases his payment if he wins A . We analyze the same setting but where local bidders have no value for AB : $v_{1AB} = v_{2AB} = 0$. Overbidding occurs in all Bayesian equilibria in undominated strategies. We conclude that decreasing the payment is the main driver behind overbidding.

²⁸Goeree and Lien (2009) first proved when the Vickrey payoffs are in the core in their Lemma 1.

For bidders 2 and 3, truthful bidding is a weakly dominant strategy by Theorems 2 and 3. From the arguments in Appendix A.1, we know that bidder 1's best response must satisfy $2 \geq 1 + b_A(v_1) \geq b_{AB}(v_1) \geq b_A(v_1) \geq 0$ and $b_B(v_1) < b_{AB}(v_1)$. Given these constraints, bidder 1's expected payoff is:

$$\begin{aligned} E_{V_2, V_3}[\pi_1(v_1)] &:= E[\pi_1(v_1) | b_2(v_2) = (0, V_2, V_2), b_3(v_3) = (0, 0, V_3)] \\ &= v_1 (1 - b_{AB}(v_1)^2 + b_A(v_1)^2 + 2b_A(v_1)) / 4 \\ &\quad + (-4b_{AB}(v_1)^3 + 3b_A(v_1)b_{AB}(v_1)^2 + 6b_A(v_1)^2b_{AB}(v_1)) / 12 \\ &\quad + (6b_A(v_1)b_{AB}(v_1) - 5b_A(v_1)^3 - 9b_A(v_1)^2 - 3b_A(v_1)) / 12. \end{aligned}$$

His equilibrium bidding functions for A and AB are:

$$b_A^*(v_1) = \begin{cases} 0 & \text{for } 0 \leq v_1 < 1/2, \\ \left(-28 + 48v_1 + 9v_1^2 + (2 + 3v_1)\sqrt{-20 + 36v_1 + 9v_1^2} \right) / 108 & \text{for } 1/2 \leq v_1 < 1 \end{cases}$$

$$b_{AB}^*(v_1) = \begin{cases} 0 & \text{for } 0 \leq v_1 < 1/2 \\ \left(-4 - 6v_1 + 9v_1^2 + (8 + 3v_1)\sqrt{-20 + 36v_1 + 9v_1^2} \right) / 108 & \text{for } 1/2 \leq v_1 \leq 1. \end{cases}$$

The equilibrium bid $b_{1B}^*(v_1)$ can be anything less than the bid for AB because it will not affect the auction outcome. Thus, strategy profiles with $b_1^*(v_1) = (b_{1A}^*(v_1), b_{1B}^*(v_1), b_{1AB}^*(v_1))$ (and mixtures thereof), $b_2^*(v_2) = (0, v_2, 0)$, and $b_3^*(v_3) = (0, 0, v_3)$ are all of the Bayesian equilibria in undominated strategies.

Figure A.3 displays $b_{1A}^*(v_1)$ and $b_{1AB}^*(v_1)$. Obviously, bidder 1 bids much less than when $v_{1AB} = v_{1A}$ (compared to Figure 1).

Note 1. *Unvalued bundles may receive positive bids in equilibrium.*

Surprisingly, bidder 1 submits positive bids for AB when $v_1 > 1/2$. The difference between his bid for AB and its value (in this case, zero) is even larger than when $v_{1AB} = v_{1A} = v_1$ for high v_1 .

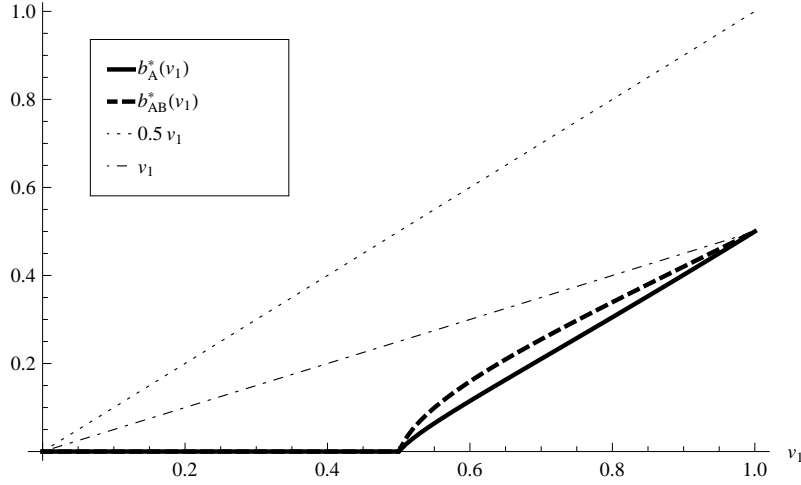


Figure A.3: Equilibrium Bidding Functions $b_A^*(v_1)$ and $b_{AB}^*(v_1)$ When $v_{1AB} = 0$ and $v_{2AB} = 0$

Revenue, payoffs, and efficiency are summarized in Table A.4. The results for the Vickrey auction are the same when $v_{1AB} = v_{2AB} = 0$ as when $v_{1AB} = v_{1A}$ and $v_{2AB} = v_{2A}$ (see Table 3).

Table A.4: Revenue, Payoffs, and Efficiency in Favored-Bidder-2 MRCS Auctions When $v_{1AB} = v_{2AB} = 0$

	MRCS	MRCS/Vickrey
$E[\pi_0]$	0.503 [#]	86.2 %
$E[\pi_1]$	0.136	65.1 %
$E[\pi_2]$	0.114 [#]	54.6 %
$E[\pi_3]$	0.497 [#]	170.5 %
$E[\sum_{i=0}^3 \pi_i]$	1.249	96.7 %

[#] Approximated (weighted averages of 1001 equidistant values of v_1)

Appendix A.4. Restricting the Message Space Decreases Revenues

In Section 4.1, we calculate expected revenue and efficiency that is higher than the results in previous literature. To better understand whether the expanded message space or the payment rule is the main driver of these differences, we restrict the message space, requiring $b_A(v_1) \geq b_{AB}(v_1)$ and $b_{2AB}(v_2) \leq b_{2B}(v_2)$, and continue to use the favored-bidder

2 payment rule.²⁹

Again, in the auction with favored bidder 2, bidders 2 and 3 bid truthfully. For bidder 1, outcomes are identical for all bids $b_{AB}(v_1) \leq b_A(v_1)$ (bidder 1 never wins AB and $b_{AB}(v_1)$ never appears in any calculation of prices), so we concentrate on $b_A(v_1) = b_{AB}(v_1)$. Bidder 1 maximizes his expected payoff $E_{V_2, V_3}[\pi_1(v_1)] = (v_1(1 + 2b_A(v_1)) - b_A(v_1) - b_A(v_1)^2) / 4$ with $b_A(v_1) = b_{AB}(v_1) = \max\{0, v_1 - 1/2\}$. Expected revenues, payoffs, and efficiency are summarized in Table A.5.

Table A.5: Revenue, Payoffs, and Efficiency in the Favored-Bidder-2 MRCS Auctions if $b_A(v_1) \geq b_{AB}(v_1)$.

	MRCS	MRCS/Vickrey
$E[\pi_0]$	0.5	85.7 %
$E[\pi_1]$	0.135	65.0 %
$E[\pi_2]$	0.115	55.0 %
$E[\pi_3]$	0.5	171.4 %
$E[\sum_{i=0}^3 \pi_i]$	1.25	96.8 %
$\text{Prob}\{\pi \in \text{Core}\}$	0.313	62.5 %

Compared with the results from Section 4.1, restricting the message space lowers expected revenue and efficiency. The resulting expected revenue is 85.7% of the expected Vickrey revenue instead of 102.2% and expected efficiency drops from 97.6% to 96.8%. This weak performance of the favored-bidder-2 MRCS auction with the restricted message space might indicate that restricting the message space with other payment rules would have similar negative effects.³⁰

Second, restricting the message space allows us to compare the performance of our payment rule with that of different payment rules studied in the literature. The expected

²⁹This restriction on bids is one of the possible assumptions underlying the results of Ausubel and Baranov (2010) and Goeree and Lien (2009).

³⁰Note that efficiency increases when the message space is restricted under the somewhat extreme assumption that local bidders have no value for the bundle AB ($v_{iAB} = 0$, see Section Appendix A.3). For the restricted message space, the optimal bidding functions and payoffs are identical when $v_{1AB} = v_{1A}$ and $v_{2AB} = v_{2B}$ and when $v_{1AB} = v_{2AB} = 0$. In both cases, expected efficiency is 96.8%, which is slightly higher than the expected efficiency of 96.7% that results from the expanded message space and $v_{1AB} = v_{2AB} = 0$.

revenue using the favored-bidder-2 payment rule is the same as that with the nearest bid rule (85.7% of Vickrey revenue) but is lower than that with the Vickrey-nearest rule (91.3%) and that with the proxy auction rule (91.9%). Expected efficiency is lower than with the Vickrey-nearest rule (98% efficiency).³¹ This is due to the strong bid shading of bidder 1 ($\max\{0, v_1 - 1/2\}$ vs. e.g. $\max\{0, v_i - (3 - 2\sqrt{2})\}$, $i = 1, 2$ for the Vickrey-nearest rule). Also, the probability that the allocation is in the core is lower than with the Vickrey-nearest rule (0.313 vs. 0.343). So, although the favored-bidder-2 payment rule ensures that only one bidder has an incentive to lie, it performs worse than other MRCS payment rules when the message space is restricted. The consequences for expected revenue and efficiency are stronger than when both local bidders lie about their values. However, these comparisons hold for the restricted message space and the relative performance might be different if bidders bid for all packages.

Appendix A.5. Distributions That Favor the Vickrey Auction

In Section 4.1, we found that favored-bidder-2 MRCS auctions have a revenue advantage over the Vickrey auction. Now we show that, under different distributional assumptions, the revenue ranking may be reversed.³²

One way to reverse the revenue ranking is to increase the likelihood of the cases in which the Vickrey auction generates higher revenue (the cases in which bidder 3 wins). Let bidder 3's value for AB be uniformly distributed on $[0, k]$ for $k \geq 2$. Assume k is

³¹The expected revenues are taken from Ausubel and Baranov (2010) and the expected efficiency comes from Goeree and Lien (2009). Since Ausubel and Baranov (2010) use a different measure we cannot compare the efficiency of nearest bid rule and the proxy auction rule with that of the favored-bidder rule.

³²Ausubel and Baranov (2010) also find that, in their setting with one-dimensional bids, the revenue ranking depends on the value distributions. They show this by varying α in the cumulative distribution function $F(v_i) = v_i^\alpha$ on $[0, 1]$ for the local bidders' values.

common knowledge. Bidder 1's expected payoff is now:³³

$$\begin{aligned}
E_{V_2, V_3}[\pi_1(v_1)] &:= E[\pi_1(v_1) | b_2(v_2) = (0, V_2, V_2), b_3(v_3) = (0, 0, V_3)] \\
&= v_1 (1 + b_{AB}(v_1)^2 - 2b_A(v_1)b_{AB}(v_1) + b_A(v_1)^2 + 2b_A(v_1)) / (2k) \\
&\quad + (-4b_{AB}(v_1)^3 + 3b_A(v_1)b_{AB}(v_1)^2 + 6b_A(v_1)^2b_{AB}(v_1)) / (6k) \\
&\quad + (6b_A(v_1)b_{AB}(v_1) - 5b_A(v_1)^3 - 9b_A(v_1)^2 - 3b_A(v_1)) / (6k)
\end{aligned} \tag{A.8}$$

His optimal bid is independent of k . Since bidders 2 and 3 bid truthfully in equilibrium, all three bidders have the same bidding functions regardless of k . However, for higher k it is more likely that bidder 3 wins. Table A.6 summarizes the results.

Table A.6: Comparing Revenue, Payoffs, and Efficiency in the Vickrey and Favored-Bidder-2 MRCS Auctions When $v_3 \sim U[0, k]$

	Vickrey	MRCS
$E[\pi_0]$	$(12k - 10)/(12k) \approx 1 - .833/k$	$.756 - .318/k^\#$
$E[\pi_1]$	$5/(12k) \approx .417/k$	$.309/k^\#$
$E[\pi_2]$	$5/(12k) \approx .417/k$	$.185/k^\#$
$E[\pi_3]$	$(6k^2 - 12k + 7)/(12k) \approx .5k - 1 + .583/k$	$.5k - .756 + .347/k^\#$
$E[\sum_{i=0}^3 \pi_i]$	$(6k^2 + 7)/(12k) \approx .5k + .583/k$	$.5k + .522/k^\#$
$\text{Prob}\{\pi \in \text{Core}\}$	$1 - 1/k$	$0.008 + 0.600/k^\#$

[#] Approximated (weighted average of 1001 equidistant values of v_1)

As k increases, the revenue advantage of the favored-bidder-2 MRCS auctions disappears and the expected revenue in the Vickrey auction is higher. The probability of zero revenue in the Vickrey auction, which is $1/(3k)$, decreases in k . Maximum and minimum revenues in the favored-bidder-2 MRCS auctions are $b_A^*(1) + 1 = 1.66$ and 0, respectively, for all $k \geq 2$. Of course, the Vickrey auction has a higher expected efficiency for all k .

Also, the probability that the payoff vector in Vickrey auction is in the core with respect to the true values is always higher than this probability for the MRCS auction ($1 - 1/k$

³³This assumes $2 \geq b_A(v_1) + 1 \geq b_{AB}(v_1) \geq b_A(v_1) \geq 0$ and $b_{AB}(v_1) > b_B(v_1)$ for all v_1 . The other cases can be excluded as in Appendix A.1.

versus approximately $0.008 + 0.600/k$). The difference increases in k .

As k increases, the Vickrey auction becomes more advantageous. The probability that bidder 3 wins and, therefore, that it generates competitive revenues, increases. Bidder 1 bids identically for all k but the higher the k , the less he gains from deviating from the truth ($0.309/k - 1/(6k) = 0.142/k$). So, for large k , he may not find it worthwhile to bid strategically.

Appendix B. Proof of Theorem 5

Lemma 2. *In every MRCS auction, if strategy b_i is undominated for local bidder i then $b_i(0, v_j) = (0, 0, 0)$ where j denotes the other local bidder.*

Proof: For $v_i = 0$ and any possible $v_j \in [0, 1]$, $b_i(0, v_j) = (0, 0, 0)$ yields the maximum possible payoff of zero against every profile of others' bids b_{-i} . If $b_i(0, v_j) \neq (0, 0, 0)$ then there exists some profile of the others' bids such that bidder i wins an item at a positive price. Thus, against said profile, his expected payoff conditional on having value $v_i = 0$ is negative. Since this is strictly worse than bidding $(0, 0, 0)$, it contradicts the strategy being undominated. ■

Lemma 3. *In every MRCS auction, every strategy $b_1(v_1, v_2) = (v_1, 0, v_1 + k_1(v_1, v_2))$ with $k_1(v_1, v_2) \in (0, v_2]$ for all $v_1 \in (0, 1]$ and all $v_2 \in (0, 1]$ can be dominated only by a mixture of strategies $b_1^m(v_1, v_2) = (v_1, 0, y_1^m(v_1, v_2))$ such that $y_1^m(v_1, v_2) < v_1 + k_1(v_1, v_2)$ for all m , all $v_1 \in (0, 1]$, and all $v_2 \in (0, 1]$.*

Similarly, in every MRCS auction, every strategy $b_2(v_1, v_2) = (0, v_2, v_2 + k_2(v_1, v_2))$ with $k_2(v_1, v_2) \in (0, v_1]$ for all $v_1 \in (0, 1]$ and all $v_2 \in (0, 1]$ can be dominated only by a mixture of strategies $b_2^m(v_1, v_2) = (0, v_2, y_2^m(v_1, v_2))$ such that $y_2^m(v_1, v_2) < v_2 + k_2(v_1, v_2)$ for all m , all $v_1 \in (0, 1]$, and all $v_2 \in (0, 1]$.

Proof: We will prove this only for bidder 1. The proof for bidder 2 is analogous.

For the remainder of the proof, fix $(\hat{v}_1, \hat{v}_2) \in (0, 1] \times (0, 1]$ and $k \in (0, \hat{v}_2]$. Then $b_1(\hat{v}_1, \hat{v}_2) = (\hat{v}_1, 0, \hat{v}_1 + k)$ is a best response (for every payment rule) to the strategies $b'_2(v_1, v_2) = (0, k, 0)$, $b'_3(v_3) = (0, 0, k + \alpha)$ for $\alpha \in (0, \hat{v}_1)$. With this bid, bidder 1 gets his highest possible payoff $\pi_1 = \hat{v}_1 - \alpha = \pi_1^V$ because he wins A ($\hat{v}_1 + k > k + \alpha$) and $p_1 = \alpha$ is the unique solution to minimizing $p_1 + p_2$ such that $p_1 + p_2 \geq k + \alpha$, $\hat{v}_1 \geq p_1 \geq \alpha$, and $k \geq p_2 \geq k$.

Every pure strategy that is chosen with strictly positive probability in a strategy that weakly dominates $b_1(v_1, v_2) = (v_1, 0, v_1 + k)$ must also be a best response to all these strategies $b'_2(v_1, v_2) = (0, k, 0)$, $b'_3(v_3) = (0, 0, k + \alpha)$. (Any mixture that puts positive probability on a bid that is not a best response to one of these strategies cannot dominate $b_1(v_1, v_2) = (v_1, 0, v_1 + k)$ because its expected payoff against the respective profile must be strictly lower than the best-response payoff.)

Consider potential other best responses to the strategies $b'_2(v_1, v_2) = (0, k, 0)$, $b'_3(v_3) = (0, 0, k + \alpha)$. A bid with which bidder 1 wins AB results in $\pi_1 = \hat{v}_1 - (k + \alpha) < \hat{v}_1 - \alpha$. If bidder 1 wins nothing or B , his maximum payoff is $0 < \hat{v}_1 - \alpha$. Thus, it remains to consider alternative bids for bidder 1 that win A at price $p_1 = \alpha$. To win A , his bid must be such that $b_{1A} + k \geq \max\{k + \alpha, b_{1B}, b_{1AB}\}$, or equivalently, $b_{1A} \geq \max\{\alpha, b_{1B} - k, b_{1AB} - k\}$. So potential alternate best responses are of the form (b_{1A}, x, y) with $b_{1A} \geq \alpha$, $x \leq k + b_{1A}$, $y \leq k + b_{1A}$. These inequalities must hold for every $\alpha \in (0, \hat{v}_1)$, so we must have $b_{1A} \geq \hat{v}_1$. (If $b_{1A} < \hat{v}_1$, then there exists some α such that $b_{1A} < \alpha < \hat{v}_1$, a contradiction.)

$(\hat{v}_1, 0, \hat{v}_1 + k)$ is a best response (for every payment rule) to $b''_2(v_1, v_2) = (0, k, k)$, $b''_3(v_3) = (0, 0, k + \beta)$ for $\beta > \hat{v}_1$. Bidder 1 gets a payoff of zero. Consider one of the remaining candidates with $b_{1A} > \hat{v}_1$. For this bid, there exists β such that bidder 1 wins A and pays $p_1 \geq \beta > \hat{v}_1$ so that $\pi_1 < 0$. Thus, only bids $b_1 = (\hat{v}_1, x, y)$ with $x \leq k + \hat{v}_1$ and $y \leq k + \hat{v}_1$ remain to be considered.

$(\hat{v}_1, 0, \hat{v}_1 + k)$ is a best response (for every payment rule) to $b'''_2(v_1, v_2) = (k, k, k)$, $b'''_3(v_3) = (0, 0, k + \alpha)$, $\alpha \in (0, \hat{v}_1)$ but bids (\hat{v}_1, x, y) , $x > \hat{v}_1$ are not. The best response results in winning A at $p_1 = \alpha$ and payoff $\pi_1 = \hat{v}_1 - \alpha > 0$. The other bids result in winning B at

$p_1 \geq \alpha$ and $\pi_1 < 0$.

$(\hat{v}_1, 0, \hat{v}_1 + k)$ is a best response (for every payment rule) to $b_2'''(v_1, v_2) = (\hat{v}_1 + k, 0, \hat{v}_1 + k)$, $b_3'''(v_3) = (0, 0, \hat{v}_1 + k + \varepsilon)$ for all $x \in (0, \hat{v}_1]$ but (\hat{v}_1, x, y) is not for ε such that $0 < \varepsilon < x$ (and such an ε exists for every $x \in (0, \hat{v}_1]$). The best response results in a payoff of zero. Each of the second set of bids results in winning B for $p_1 \geq \varepsilon > 0$ and payoff $\pi_1 < 0$.

Thus, the only possible candidates remaining for being part of a mixture that dominates $b_{1A}(\hat{v}_1, \hat{v}_2) = (\hat{v}_1, 0, \hat{v}_1 + k)$ are bids $(\hat{v}_1, 0, y)$ with $y < \hat{v}_1 + k$. The argument can be repeated for each $(\hat{v}_1, \hat{v}_2) \in (0, 1] \times (0, 1]$ and for each $k \in (0, \hat{v}_2]$. ■

Lemma 4. *In every MRCS auction, there exists an undominated strategy for bidder 1 of the form $b_1(v_1, v_2) = (v_1, 0, y(v_1, v_2))$ such that $y(v_1, v_2) \in [0, v_1]$ for all $v_1 \in [0, 1]$ and all $v_2 \in [0, 1]$.*

Similarly, in every MRCS auction, there exists an undominated strategy for bidder 2 of the form $b_2(v_1, v_2) = (0, v_2, y(v_1, v_2))$ such that $y(v_1, v_2) \in [0, v_2]$ for all $v_1 \in [0, 1]$ and all $v_2 \in [0, 1]$.

Proof: First we will prove that for bidder 1, bids in the set $T_1(v_1, v_2) \equiv \{b_1(v_1, v_2) = (v_1, 0, y(v_1, v_2)) \mid y(v_1, v_2) \in [0, v_1]\}$ cannot be dominated by any mixture of bids outside of this set. The proof for bidder 2 is analogous.

The bid $b_1(0, v_2) = (0, 0, 0)$ is undominated by Lemma 2.

For the remainder of the proof, consider any fixed $(\hat{v}_1, \hat{v}_2) \in (0, 1] \times [0, 1]$. Any bid $b_1 \in T_1(\hat{v}_1, \hat{v}_2)$ is a best response to the strategies $b_2(v_1, v_2) = (0, \alpha, \alpha)$, $b_3(v_3) = (0, \alpha, \alpha + \beta)$ for all $\alpha > 0$ and all $\beta \in [0, \hat{v}_1]$. Suppose there is some mixture of strategies that dominates b_1 . Let b'_1 be a strategy that is played with positive probability in that mixture. Then $b'_1(\hat{v}_1, \hat{v}_2)$ must also be a best response to these strategies for all $\alpha > 0$ and all $\beta \in [0, \hat{v}_1]$. This means that $b'_1(\hat{v}_1, \hat{v}_2)$ must win A at price β . (Winning B leads to a payoff of at most zero, which is less than $\hat{v}_1 - \beta$ and winning AB leads to a payoff of $\hat{v}_1 - \alpha - \beta$, which is also less than $\hat{v}_1 - \beta$.)

Against $b_2(v_1, v_2) = (0, \alpha, \alpha)$, $b_3(v_3) = (0, \alpha, \alpha + \beta)$, any bid $b'_1(\hat{v}_1, \hat{v}_2)$ that wins A does so at a price of β . So any bid such that $b'_{1A}(\hat{v}_1, \hat{v}_2) + \alpha \geq \max\{b'_{1B}(\hat{v}_1, \hat{v}_2), b'_{1AB}(\hat{v}_1, \hat{v}_2), \alpha + \beta\}$ is a best response. This inequality must hold for all $\alpha > 0$ and all $\beta \in [0, \hat{v}_1)$, which implies $b'_{1A}(\hat{v}_1, \hat{v}_2) \geq \max\{b'_{1B}(\hat{v}_1, \hat{v}_2), b'_{1AB}(\hat{v}_1, \hat{v}_2), \hat{v}_1\}$.

Any bid $b_1 \in T_1(\hat{v}_1, \hat{v}_2)$ is also a best response to the bids $b_2(v_1, v_2) = (0, 0, 0)$, $b_3(v_3) = (0, 0, \hat{v}_1 + \varepsilon)$ for all $\varepsilon > 0$ because it loses and any bid that won A or AB would have to pay more than \hat{v}_1 for it. However, any bid $b'_1(\hat{v}_1, \hat{v}_2)$ with $b'_{1A}(\hat{v}_1, \hat{v}_2) > \hat{v}_1$ does strictly worse when $\varepsilon \in (0, b'_{1A}(\hat{v}_1, \hat{v}_2) - \hat{v}_1)$ because it wins A at a price of $\hat{v}_1 + \varepsilon$. So no such bid can be part of a mixture that dominates any $b_1 \in T_1(\hat{v}_1, \hat{v}_2)$.

Thus, the remaining candidates are $b'_1(\hat{v}_1, \hat{v}_2) = (\hat{v}_1, b'_{1B}(\hat{v}_1), b'_{1AB}(\hat{v}_1))$ such that $\hat{v}_1 \geq \max\{b'_{1B}(\hat{v}_1, \hat{v}_2), b'_{1AB}(\hat{v}_1, \hat{v}_2)\}$. Any bid $b_1 \in T_1(\hat{v}_1, \hat{v}_2)$ is a best response to the bids $b_2(v_2) = (\hat{v}_1, 0, \hat{v}_1)$, $b_3(v_3) = (0, 0, \hat{v}_1 + \varepsilon)$ for all $\varepsilon > 0$ because it loses and any bid that won A or AB would have to pay more than \hat{v}_1 for it. However, any bid $b'_1(\hat{v}_1, \hat{v}_2)$ with $b'_{1B}(\hat{v}_1, \hat{v}_2) > 0$ does strictly worse when $b'_{1B}(\hat{v}_1, \hat{v}_2) > \varepsilon > 0$ because it wins B at the price ε . So no such bid can be part of a mixture that dominates b_1 .

Thus, we have eliminated all bids not in $T_1(\hat{v}_1, \hat{v}_2)$, so any bid in $T_1(\hat{v}_1, \hat{v}_2)$ cannot be dominated by any mixture that includes bids from outside of this set.

Next, consider the modified game where the strategy of player one is restricted to bids $b_1(v_1, v_2) = (v_1, 0, y)$ such that $y \in [0, v_1]$. If player one has an undominated strategy in this game then that strategy will be undominated in the larger game. This game has a compact strategy set and a payoff function for player 1 that is upper-hemicontinuous in his choice variable y . (The payment function is upper-hemicontinuous in y by assumption and the assignment function is not affected by any $y \in [0, 1]$.) Thus, by Proposition 0 in Salonen (1996), player 1 has an undominated strategy. ■

Lemma 5. *In every MRCS auction, for every $(v_1, v_2) \in (0, 1] \times (0, 1]$, either bidder 1 has an undominated strategy b_1 such that $b_1(v_1, v_2) = (v_1, 0, v_1 + v_2)$ or bidder 2 has an undominated strategy b_2 such that $b_2(v_1, v_2) = (0, v_2, v_1 + v_2)$.*

Proof: Fix some $(v_1, v_2) \in (0, 1] \times (0, 1]$. Suppose to the contrary that every strategy of bidder 1 that includes $b_1(v_1, v_2) = (v_1, 0, v_1 + v_2)$ is dominated and every strategy of bidder 2 that includes $b_2(v_1, v_2) = (0, v_2, v_1 + v_2)$ is dominated. Consider the mixtures of strategies that dominate these strategies. By Lemma 3, each mixture must contain a strategy such that $b'_1(v_1, v_2) = (v_1, 0, y_1)$ and $b'_2(v_1, v_2) = (0, v_2, y_2)$, respectively, for some $y_1 < v_1 + v_2$ and $y_2 < v_1 + v_2$. Then consider the profile of bids $((v_1, 0, y_1), (0, v_2, y_2), (0, 0, \max\{v_1, v_2, y_1, y_2\}))$. Bidder 1 wins A and bidder 2 wins B . One of them must pay more than their Vickrey price because in any core-selecting auction, $p_1 + p_2 \geq b_{3AB} > (b_{3AB} - v_2) + (b_{3AB} - v_1) = p_1^V + p_2^V$. Assume the bidder that pays more than his Vickrey price is bidder 1. Note that the bid $b_1(v_1, v_2) = (v_1, 0, v_1 + v_2)$ is a best response to strategies $b'_2(v_1, v_2) = (0, v_2, y_2), b_3(v) = (0, 0, \max\{v_1, v_2, y_1, y_2\})$ because bidder 1 pays p_1^V . The bid $b'_1(v_1, v_2) = (v_1, 0, y_1)$ isn't a best response because bidder 1 pays a strictly higher price. This contradicts $b'_1(v_1, v_2) = (v_1, 0, y_1)$ being part of a mixture that dominates $b_1(v_1, v_2) = (v_1, 0, v_1 + v_2)$. ■

Lemma 6. *For a local bidder in any MRCS auction, a strategy in which he bids zero for his desired item is a dominated strategy.*

Proof: We will prove this only for bidder 1. The proof for bidder 2 is analogous. Namely, we will show that every bid $b_1(v_1, v_2) = (0, b_{1B}(v_1, v_2), b_{1AB}(v_1, v_2))$ is dominated by $b'_1(v_1, v_2) = (\min\{b_{1AB}(v_1, v_2), v_1/4\}, 0, b_{1AB}(v_1, v_2))$ for all $(v_1, v_2) \in (0, 1] \times [0, 1]$. We will assume $b_{1AB} > 0$ because a bid of $(0, b_{1B}(v_1, v_2), 0)$ when bidder 1 has a positive value $v_1 > 0$ is clearly dominated by a truthful bid.

Moving from b_1 to b'_1 , the bid for A has increased, the bid for B has (weakly) decreased and the bid for AB is the same. This means that if bidder 1 was winning B or AB with b_1 , he must win either the same package or A with b'_1 . If he was winning A he must continue to win A . And if he was losing he must continue to lose or win A .

If he was winning B with b_1 and continues to win B with b'_1 , his payoff goes from being non-positive to being zero. If he switches to winning A with b'_1 , his payoff is at least

$\frac{3}{4}v_1 > 0$. Thus, he is better off with b'_1 .

If he was winning AB with b_1 and continues to win AB with b'_1 , his payoff stays the same ($v_1 - w(23)$). If he switches to winning A with b'_1 , he receives the same benefit and pays weakly less because his bid for A is no larger than his original bid for AB . Thus, he is better off with b'_1 .

If he was losing with b_1 and continues to lose with b'_1 , his payoff stays the same (zero). If he switches to winning A with b'_1 , his payoff is at least $\frac{3}{4}v_1 > 0$. Thus, he is better off with b'_1 .

If he was winning A with b_1 , he was doing so with probability at most a half because his zero bid must be tied with at least one other bidder. His payoff is at most $\frac{1}{2}v_1$. He must then win A with probability one with bid b'_1 . This gives him a payoff of at least $\frac{3}{4}v_1 > \frac{1}{2}v_1$. Thus, he is better off with b'_1 . ■

Lemma 7. *Any Bayesian equilibrium in undominated strategies of any MRCS auction must belong to the following set of strategy profiles for all $(v_1, v_2) \in (0, 1)^2$:*

$$b_1(v_1, v_2) = (v_1, b_{1B}(v_1, v_2), b_{1AB}(v_1, v_2))$$

$$b_2(v_1, v_2) = (b_{2A}(v_1, v_2), v_2, b_{2AB}(v_1, v_2))$$

$$b_3(v_3) = (0, 0, v_3)$$

$$\max\{b_{1AB}(v_1, v_2), b_{2AB}(v_1, v_2)\} = v_1 + v_2$$

$$b_{1B}(v_1, v_2) + b_{2A}(v_1, v_2) < v_1 + v_2.$$

Proof: By Theorem 3, the global bidder 3 has a weakly dominant strategy to bid truthfully in any MRCS auction. By lemma 6, neither local bidder can use a strategy in which they bid zero for their desired item when their true value is nonzero, so $b_{1A}(v_1, v_2) > 0$ for all $v_1 > 0$ and $b_{2B}(v_1, v_2) > 0$ for all $v_2 > 0$. When $v_i = 0$ for local bidder i , his only undominated bid is $b_i(v_1, v_2) = (0, 0, 0)$ by lemma 2. Thus, we will only consider positive

values in the rest of the proof.

Given that the global bidder bids truthfully and both local bidders bid a nonzero amount for their desired item whenever their value is greater than zero, we will show the following. In any Bayesian equilibrium in undominated strategies, (1) if bidders 1 and 2 win anything, then they always win A and B , respectively, (2) for $v \in (0, 1)$, bidders 1 and 2 always bid truthfully for their desired individual items, and (3) either bidder 1 or 2 always pays their full bid when they win. The strategy profiles listed in the lemma follow directly from these statements.

This amounts to proving the following steps:

- (1) $b_{1A}(v_1, v_2) + b_{2B}(v_1, v_2) \geq \max\{b_{1AB}(v_1, v_2), b_{2AB}(v_1, v_2)\}$ and $b_{1A}(v_1, v_2) + b_{2B}(v_1, v_2) > b_{1B}(v_1, v_2) + b_{2A}(v_1, v_2)$ for all $(v_1, v_2) \in (0, 1)^2$
- (2) $b_{1A}(v_1, v_2) = v_1$ and $b_{2B}(v_1, v_2) = v_2$ for all $(v_1, v_2) \in (0, 1)^2$
- (3) $\max\{b_{1AB}(v_1, v_2), b_{2AB}(v_1, v_2)\} = v_1 + v_2$ for all $v_1, v_2 \in (0, 1)^2$.

If (1), (2), or (3) are not fulfilled, a local bidder can increase his expected payoff (where the expectation is taken over v_3) by deviating unilaterally.

Step 1: $b_{1A}(v_1, v_2) + b_{2B}(v_1, v_2) \geq \max\{b_{1B}(v_1, v_2) + b_{2A}(v_1, v_2), b_{1AB}(v_1, v_2), b_{2AB}(v_1, v_2)\}$ for all $(v_1, v_2) \in (0, 1)^2$

Assume this inequality did not hold. Then we must be in one of the following three cases:

Case 1 $b_{1AB}(v_1, v_2) = \max\{b_{1B}(v_1, v_2) + b_{2A}(v_1, v_2), b_{1AB}(v_1, v_2), b_{2AB}(v_1, v_2)\} > b_{1A}(v_1, v_2) + b_{2B}(v_1, v_2)$. Since $b_{2B}(v_1, v_2) > 0$, bidder 1 has a strictly profitable deviation to $b'_1(v_1, v_2) = (b_{1AB}(v_1, v_2) - b_{2B}(v_1, v_2), b_{1B}(v_1, v_2), b_{1AB}(v_1, v_2))$. Bidder 1 wins against the same realizations of v_3 , but pays a strictly lower price because he now wins A instead of AB : $p'_1 = w(23) - b_{2B}(v_1, v_2) < w(23) = p_1$.

Case 2 $b_{2AB}(v_1, v_2) = \max\{b_{1B}(v_1, v_2) + b_{2A}(v_1, v_2), b_{1AB}(v_1, v_2), b_{2AB}(v_1, v_2)\} > b_{1A}(v_1, v_2) + b_{2B}(v_1, v_2)$. Since $b_{1A}(v_1, v_2) > 0$, this case can be ruled out by the same arguments as in case 1.

Case 3 $b_{1B}(v_1, v_2) + b_{2A}(v_1, v_2) = \max\{b_{1B}(v_1, v_2) + b_{2A}(v_1, v_2), b_{1AB}(v_1, v_2), b_{2AB}(v_1, v_2)\} \geq b_{1A}(v_1, v_2) + b_{2B}(v_1, v_2)$. This implies that either $b_{1B}(v_1, v_2) > 0$ or $b_{2A}(v_1, v_2) > 0$. For any realization of v_3 , the lower bound on $p_1 + p_2$ is v_3 . Thus, if $b_{1B}(v_1, v_2) + b_{2A}(v_1, v_2) > v_3 > 0$, at least one of the two local bidders has to pay a positive price for an item that he does not value. He has a profitable deviation in bidding zero for the item he does not value.

Step 2: $b_{1A}(v_1, v_2) = v_1$ and $b_{2B}(v_1, v_2) = v_2$ for all $(v_1, v_2) \in (0, 1)^2$ Again assume this were not true. Then we must be in one of the following cases.

Case 1 $b_{1A}(v_1, v_2) > v_1, b_{2B}(v_1, v_2) \leq v_2$. Consider the alternate bid $b'_1(v_1, v_2) = (v_1, 0, v_1 + b_{2B}(v_1, v_2))$. This bid ensures that bidder 1 pays the lowest possible price (his Vickrey price), given that bidder 3 bids truthfully. Thus, it gives a weakly higher payoff than any other bid. Moreover, there is a positive probability that $b_{1A}(v_1, v_2) + b_{2B}(v_1, v_2) > v_3 > v_1 + b_{2B}(v_1, v_2)$ because $\bar{v}_3 = 2 > b_{2B}(v_1, v_2) + v_1$. Against these realizations of v_3 , and given that (1) holds, $b'_1(v_1, v_2)$ yields a strictly higher payoff (of zero) because if bidder 1 wins, his price is $p_1 \geq v_3 - b_{2B}(v_1, v_2) > v_1$. So $b'_1(v_1, v_2)$ yields a strictly higher expected payoff and is a profitable deviation.

Case 2 $b_{1A}(v_1, v_2) \leq v_1, b_{2B}(v_1, v_2) > v_2$. This case can be similarly excluded.

Case 3 $b_{1A}(v_1, v_2) > v_1, b_{2B}(v_1, v_2) > v_2$. In this case and given that (1) holds, the local bidders win and at least one of them pays more than their value when the realization of v_3 is such that $b_{1A}(v_1, v_2) + b_{2B}(v_1, v_2) \geq v_3 > v_1 + v_2$. This occurs with positive probability. The bidder who pays more than their value – bidder 1 for example – can strictly improve by deviating to $b'_1(v_1, v_2) = (v_1, 0, v_1 + b_{2B}(v_1, v_2))$, which always results in the Vickrey payoff and never in a strictly negative payoff.

Case 4 $b_{1A}(v_1, v_2) \leq v_1, b_{2B}(v_1, v_2) \leq v_2$. Again, consider the alternate bid $b'_1(v_1, v_2) = (v_1, 0, v_1 + b_{2B}(v_1, v_2))$. When $b_{1A}(v_1, v_2) + b_{2B}(v_1, v_2) < v_3 < v_1 + b_{2B}(v_1, v_2)$ and

given that (1) holds, it yields a strictly higher payoff. This occurs with positive probability, so $b'_1(v_1, v_2)$ yields strictly higher expected profits.

Step 3: $\max\{b_{1AB}(v_1, v_2), b_{2AB}(v_1, v_2)\} = v_1 + v_2$ for all $v_1, v_2 \in (0, 1)^2$

Given that $b_{1A}(v_1, v_2) = v_1$ and $b_{2B}(v_1, v_2) = v_2$, either of the above bids ensure that either bidder 1 or bidder 2 pays the lowest price possible, given the other players' bids. If $b_{1AB}(v_1, v_2) = v_1 + v_2$, the core constraints require $v_2 = b_{2B}(v_1, v_2) \geq p_2 \geq w(\{1, 3\}) - b_{1A}(v_1, v_2) = v_1 + v_2 - v_1 = v_2$, so $p_2 = v_2$. Then $p_1 = \max\{v_3 - v_2, w(\{2, 3\}) - v_2\} = w(\{2, 3\}) - v_2 = p_1^V$. A similar calculation shows that $p_2 = p_2^V$ when $b_{2AB}(v_1, v_2) = v_1 + v_2$. Next, note that any higher bids would violate (1). Suppose instead that $\max\{b_{1AB}(v_1, v_2), b_{2AB}(v_1, v_2)\} < v_1 + v_2$. Then, when $v_3 > w(\{2, 3\}) - b_{2B}(v_1, v_2) + w(\{1, 3\}) - b_{1A}(v_1, v_2) = p_1^V + p_2^V$, either bidder 1 or bidder 2 must pay more than his Vickrey price. Thus, either bidder 1 or bidder 2 has a profitable deviation. (Note that there exist v_3 such that $b_{1A}(v_1, v_2) + b_{2B}(v_1, v_2) > v_3 > w(\{2, 3\}) - b_{2B}(v_1, v_2) + w(\{1, 3\}) - b_{1A}(v_1, v_2)$, so that the local bidders win and the above inequality is satisfied.)

■

Lemma 8. *For every MRCS auction, there exists at least one Bayesian equilibrium in undominated strategies.*

Proof: It is straightforward to verify that every strategy profile listed in lemma 7 is indeed a Bayesian equilibrium. It remains to show that some profile in that set contains only undominated strategies.

For every $(v_1, v_2) \in (0, 1)^2$, lemma 5 tells us that either bidder 1 has an undominated strategy b_1 such that $b_1(v_1, v_2) = (v_1, 0, v_1 + v_2)$ or bidder 2 has an undominated strategy b_2 such that $b_2(v_1, v_2) = (0, v_2, v_1 + v_2)$. Then by lemma 4, the other bidder (use bidder 1 for example) has an undominated strategy $b_1(v_1, v_2) = (v_1, 0, y(v_1, v_2))$ such that $y(v_1, v_2) \in [0, v_1]$. Together with bidder 3's truthful bid, this strategy profile is one of those listed in lemma 7.

■

Theorem 5. *For every MRCS auction, there always exists at least one Bayesian equilibrium in undominated strategies and every Bayesian equilibrium in undominated strategies satisfies the following properties:*

- (1) *every winning bid is truthful*
- (2) *at least one bidder places a bid strictly above his true value*
- (3) *no bidder places a bid strictly below his value for any package, unless that bid cannot affect the assignment*
- (4) *the assignment is efficient*
- (5) *the revenue is at least as large as that from the Vickrey auction for every realization of values*
- (6) *the payoffs are in the core with respect to the true values $\mathcal{C}(v)$*
- (7) *the equilibrium is an ex-post equilibrium*

Proof: By Lemma 8, there exists at least one Bayesian equilibrium in undominated strategies of every MRCS auction. By Lemma 7: (1) every bidder bids truthfully for his desired item and that item is the only package he can win, (2) at least one bidder places a bid strictly above his true value (namely, the bid of $v_1 + v_2$ by one of the local bidders), and (3) only one bid per equilibrium strategy profile could be below the true value. That bid is (potentially) a local bidder's bid for AB . As long as he bids his true value for his desired item, a lower bid on AB does not affect the assignment. Moreover, the assignment maximizes the total reported value because all three bidders place truthful bids for their desired items. The bids of $v_1 + v_2$ cannot be winning due to the tie-breaking rule, which chooses an assignment that maximizes the number of winners.

The revenue is at least as large as that from the Vickrey auction because winning bids are truthful and the core prices for each bidder are bounded below by the Vickrey price $p^V(v)$ by Lemma 1.

The payoffs are also in the core with respect to the true values because only the optimal assignment (with respect to the true values) can be winning and prices are sufficiently high.

In particular, if bidder 3 wins then the only binding constraint in both $C(b)$ and $C(v)$ is $p_3 \geq v_1 + v_2$. If bidders 1 and 2 win, then the relevant constraints for $C(b)$ are:

$$\begin{aligned} p_1 + p_2 &\geq v_3 \\ p_1 &\geq \max\{b_{2A}, v_2, b_{2AB}, v_3\} - v_2 \\ p_2 &\geq \max\{v_1, b_{1B}, b_{1AB}, v_3\} - v_1. \end{aligned}$$

These constraints are more restrictive than the constraints for $C(v)$, which are $p_1 + p_2 \geq v_3$, $p_1 \geq \max\{v_2, v_3\} - v_2$ and $p_2 \geq \max\{v_1, v_3\} - v_1$. Therefore, payments in our equilibria satisfy the constraints for $C(v)$.

The equilibria are ex-post equilibria for two reasons. First, bidder 3 has a dominant strategy to bid truthfully, so he can never regret his bid. It is a best-response regardless of the other players' actions. Second, the only incomplete information from the perspective of bidders 1 and 2 is v_3 , so their strategies already optimize against the other local bidder. Regardless of the realization of bidder 3's bid, bidders 1 and 2 are playing best responses. The bidder that receives a payoff of zero can only lose if he lowers his bid and cannot pay less when winning. The other local bidder pays the lowest possible price (since the other pays his full value) for his desired item, so he has no incentive to deviate. Therefore, the equilibria are ex-post equilibria. ■