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The Value of Information in Asymmetric All-Pay Auctions

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Abstract This paper analyzes a two-player all-pay auction with incomplete information. More precisely, one bidder is uncertain about the size of the initial advantage of his rival modeled as a head start in the auction.

I derive the unique Bayesian Nash equilibrium outcome for a large class of cumulative distribution functions over head starts. In equilibrium, the stronger player generates an informational rent if and only if his head start distribution is not stochastically dominated by a uniform distribution. This result introduces a new perspective on lobbying contests and procurement contests.

JEL classification: C72, C73

Keywords: All-pay contests, head starts, asymmetric information

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1 Introduction

In many contests, one competitor has an initial advantage over her rivals. For instance, an incumbent firm possesses more knowledge about a new task when it is related to their previous work. In a contest for job promotion, a worker with a good reputation has an advantage over her competitors. Most representatives favor companies from their own district in public procurement. For an excellent motivation and many other examples of contests in which a participant has an initial advantage, see Siegel (2012).

In recent literature, these initial advantages are often modeled as deterministic head starts in all-pay contests (see, e.g., Konrad, 2002, 2004, Kirkegaard, 2012, and Siegel, 2012). In the above mentioned applications, however, a contestant is typically not perfectly informed about the size of the head start of a rival. For instance, a lobbyist might know that the politician who he is trying to influence comes from the same district as another lobbyist. Yet, he can only guess by how much this lowers his prospects in the contest.

This paper provides a simple contest model which captures asymmetric information about head starts. More precisely, I assume that one player knows the size of her head start, while her competitor only knows a probability distribution over the size of possible head starts.

I fully characterize the unique Bayesian Nash equilibrium outcome of the game for any generic strictly increasing and continuous distribution over head starts in Proposition 1 and 2. I compare this equilibrium outcome to the equilibrium outcome for public information about the size of the head start—the payoff difference between both settings is the *informational rent*. Theorem 1 shows that the competitor with a head start generates a positive informational rent if and only if her head start distribution is not stochastically dominated by a uniform distribution. Hence, to exploit her private information, a competitor needs to be sufficiently “strong”.

In contrast to this result, I show that for uncertain valuations or costs, the “stronger” player always gets an informational rent by characterizing the unique equilibrium for this case in Proposition 3. Intuitively, for different

realizations of head starts, the stronger player keeps the same local indifference condition, while his local indifference condition changes for different valuations.

Yet, the lack of local changes in behavior for different head starts entails the possibility of global changes in behavior, i.e., the occurrence of a bid bifurcation. Above the bifurcation distribution, the weaker player bids zero instead of engaging in a fierce competition; the stronger player wins with probability 1. Such an equilibrium does not exist for incomplete information about valuations.

2 The Model

I consider a model with two players $i = 1, 2$ who have a common valuation v for an object; v is henceforth normalized to 1. Player 1 enjoys a head start which is drawn from a commonly known, continuous distribution function H . H is strictly increasing on its support $[\underline{\theta}, \bar{\theta}]$, where $0 \leq \underline{\theta} < 1$ and $H(0) = 0$. Player 1 observes θ , while player 2 only knows the distribution H .

Both players simultaneously submit bids $b_i \in \mathbb{R}_+^0$ and pay costs $c(b_i) = b_i$. A mixed strategy of player 1 assigns a probability distribution over bids to each possible draw from H . For player 2, a mixed strategy is simply a probability distribution over bids.

The *score* of a player is the sum of his head start and his bid, i.e., $s_1 = b_1 + \theta$ and $s_2 = b_2$. Player 1 wins the object if $s_1 \geq s_2$; otherwise player 2 wins.¹ Both players maximize expected payoffs which I denote by $\Pi_1(\theta, b_1) = \Pi_1(s_1)$ and $\Pi_2(b_2)$. Each player's strategy induces a distribution over scores which I denote by F_i .

3 The Equilibrium

As a benchmark, I briefly review the two-player equilibrium in the absence of uncertainty, which is an example in Siegel (2012), Section 3.2. Note that

¹The tie-breaking rule is irrelevant for any equilibrium distribution I derive, but the above specification facilitates some proofs.

it contains the equilibrium without a head start as a special case.

Lemma 1. *Assume player 1 has a deterministic head start of $\theta \in [0, 1)$. In the unique Nash equilibrium distribution over scores, player 2 places an atom of size θ at 0 and player 1 places an atom of size θ at θ . Both players randomize uniformly with density 1 on $(\theta, 1]$. The expected payoff of player 1 is θ ; the expected payoff of player 2 is 0.*

In the following, I discuss how results differ if the size of the head start is private information. To do so, I first construct a Bayesian Nash equilibrium of the game. The following proposition derives equilibrium scoring distributions and a corresponding bidding strategy for player 1; the mixed strategy of player 2 randomizes according to the distribution function $F_2(x)$, since his score is equal to his bid.

Proposition 1. *There exists a Bayesian Nash equilibrium such that $F_1(x) = \min\{H(x), x\}$ and $F_2(x) = \int_0^1 \mathbf{1}_{\{H(y) < y\}} dy + \int_0^x \mathbf{1}_{\{H(y) \geq y\}} dy$. An equilibrium strategy of player 1 is $b_1(\theta) = \max\{H(\theta) - \theta, 0\}$.*

Proof. The proof proceeds in two steps. I first show that $b_1(\theta)$ induces the scoring distribution F_1 . In a second step, I show that no player wants to deviate from the derived strategy.

Step 1: Since H is continuous and strictly increasing on its support, it is also invertible on its support. Thus, I obtain

$$\begin{aligned} F_1(x) &= \mathbb{P}(s_1 \leq x) = \mathbb{P}(\theta + b_1(\theta) \leq x) = \mathbb{P}(\max\{H(\theta), \theta\} \leq x) \\ &= \mathbb{P}(\{\theta \leq x\} \cap \{\theta \leq H^{-1}(x)\}) = \mathbb{P}(\theta \leq \min\{x, H^{-1}(x)\}) \\ &= \begin{cases} H(x) & \text{if } x \leq H^{-1}(x) \\ H(H^{-1}(x)) = x & \text{if } x > H^{-1}(x) \end{cases} \\ &= \min\{x, H(x)\}. \end{aligned}$$

Step 2: (i) For any two points x, y such that $x > y$, $F_2(x) - F_2(y) \leq x - y$. Hence,

$$\Pi_1(\theta, b_1) = F_2(\theta + b_1) - b_1 \leq F_2(\theta) = \Pi_1(\theta, 0),$$

i.e., it is a best response to bid zero for each type θ of player 1.

Assume $H(\theta) > \theta$ and $x \in [\theta, H(\theta)]$. Then $H(x) \geq H(\theta) \geq x$. As $H(x) \geq x$ for all $x \in [\theta, H(\theta)]$, $F_2(H(\theta)) - F_2(\theta) = H(\theta) - \theta$. Hence, $\Pi_1(\theta, H(\theta) - \theta) = \Pi_1(\theta, 0)$, i.e., it is a best response to bid $H(\theta) - \theta$ if $H(\theta) > \theta$.

(ii) By continuity of $F_1(x)$, I obtain $\Pi_2(b_2) = F_1(b_2) - b_2$. Hence, $\Pi_2(b_2) = 0$ whenever $F_1(b_2) = b_2$ and $\Pi_2(b_2) < 0$ otherwise. Thus, any bid contained in the randomization of player 2 is a best response.

□

The following proposition establishes uniqueness of the Bayesian Nash equilibrium which I have constructed in the previous proposition under a weak genericity assumption. The proof relies on a sequence of lemmas that are now commonplace in uniqueness proofs for static game theory with a continuous state space.

Proposition 2. *Assume $H(\theta) = \theta$ on a set of measure zero. In any Bayesian Nash equilibrium, players use strategies which induce the distributions F_1 and F_2 from Proposition 1.*

For head start distributions which are comparable to the uniform distribution on $[0, 1]$ in terms of first-order stochastic dominance, I obtain the following corollaries:

Corollary 1. *Assume $H(\theta) > \theta$ for all $\theta \in (0, 1)$. In the unique Bayesian Nash equilibrium outcome of the game, each player's score is distributed uniformly on $[0, 1]$, i.e., $F_i(x) = \min\{x, 1\}$ for $i = 1, 2$.*

Corollary 2. *Assume $H(\theta) < \theta$ for all $\theta \in (0, 1)$. In the unique Bayesian Nash equilibrium outcome, $F_1(x) = H(x)$ and $F_2(x) = 1$ for all x , i.e., both players always bid 0.*

In the unique equilibrium outcome of the 2-player all-pay auction without a head start (valuation $v = 1$ and costs $c(b_i) = b_i$), both players also randomize their score uniformly on $[0, 1]$ (see Lemma 1). Hence, by Corollary

1, a random head start does not change the equilibrium score distribution if the head start distribution is stochastically dominated by the uniform distribution on $[0, 1]$. On the other hand, if the head start distribution exhibits first-order stochastic dominance over the uniform distribution on $[0, 1]$, both players bid zero and the player with a head start wins with probability one.

In particular, the bidding behavior of player 2 exhibits a *bid bifurcation* at $H(\theta) = \theta$, i.e., a qualitative change in behavior.² To get an intuition, consider the family of distribution functions $H_\epsilon(\theta) = (1 + \epsilon)\theta$. For any $\epsilon < 0$, player 2 bids zero with probability 1. For any $\epsilon > 0$, however, player 2 randomizes his bid uniformly over the unit interval. Therefore, the payoff of player 1 is discontinuous in his head start distribution.

This discontinuity has implications for the value player 1 attaches to having private information about his head start. To be more precise, I compare the (ex-ante) difference in expected payoffs for the equilibria I have obtained under private information to the benchmark of public information. Throughout this analysis, I assume that in the public information case, both players observe the draw of player 1. I refer to the (ex-ante) difference in expected equilibrium payoffs as the *informational rent*.

Theorem 1. *Player 1 generates an informational rent if she is sufficiently “strong” ex-ante, i.e., $H(\theta) < \theta$ for some $\theta \in (0, 1)$. However, if $H(\theta) > \theta$ for all $\theta \in (0, 1)$, player 1 does not obtain an informational rent. The informational rent of player 2 is always 0.*

Thus, private information about head starts is only beneficial if a player is perceived as sufficiently strong ex-ante. This finding is in contrast to the existing literature on all-pay auctions with incomplete information in which the stronger player always benefits from having private information. I give a more detailed explanation for this new feature after the analysis of the corresponding model with uncertainty about valuations.

²For other results about bid bifurcation in all-pay auctions, see Noussair and Silver (2006) and Parreiras and Rubinchik (2010).

4 Uncertainty About Valuations

In this section, I contrast the main results of this paper with the corresponding results for an all-pay auction in which player 1 has a higher valuation than player 2.³ More precisely, player 2 has a valuation of v_2 (henceforth normalized to one), while player 1 draws his valuation v_1 from a commonly known, continuous distribution H which is strictly increasing on its connected support $[1, \bar{v}_1]$. As before, players compete in an all-pay auction with costs $c(b_i) = b_i$.⁴ No player has a head start.

For the benchmark case in which both players know the realization v_1 , Hillman and Samet (1987) obtain the following result:

Lemma 2 (Hillman and Samet, 1987). *In the unique Nash equilibrium, $F_1(x) = \min\{x, 1\}$ and $F_2(x) = \min\{\frac{v_1-1}{v_1} + \frac{x}{v_1}, 1\}$. Player 1 obtains a profit of $v_1 - 1$, player 2 obtains a profit of 0.*

In the following, I consider the private information case. I impose the following smoothness assumption:

Assumption 1. *If $F_2(x)$ is continuous, then it is also differentiable.*

This assumption greatly facilitates the proof for uniqueness of the equilibrium in the following proposition, since it allows me use the first-order condition of player 1's optimization problem.

Proposition 3. *In any Bayesian Nash equilibrium, $F_1(x) = \min\{x, 1\}$ and*

$$F_2(x) = \begin{cases} 1 - \int_0^1 \frac{1}{H^{-1}(y)} dy + \int_0^x \frac{1}{H^{-1}(y)} dy & \text{if } x \leq 1, \\ 1 & \text{otherwise.} \end{cases}$$

The unique equilibrium strategy of player 1 is $b_1(v_1) = H(v_1)$. Player 1 obtains an informational rent. The informational rent of player 2 is zero.

³See, e.g., Amann and Leininger (1996), Moldovanu and Sela (2001), and Kirkegaard (2012) for results on two-sided incomplete information.

⁴To determine their best responses, players only care about the relationship between valuation-cost ratio and winning probability. Hence, a higher valuation of a player is strategically equivalent to a lower (marginal) cost.

Intuitively, player 2 uses a randomization which makes it optimal for each type v_1 of player 1 to bid $H(v_1)$. Every type $v_1 < \bar{v}_1$ makes a strictly higher profit than in the benchmark case of public information. Thus, differing from the analysis for uncertain head starts, player 1 can always use his private information to his advantage.

5 Discussion of the Main Results

Uncertainty about head starts leads to qualitatively different predictions than uncertainty about valuations. From a technical point of view, the local indifference condition of the stronger player changes depending on his valuation. As the equilibrium distribution of the weaker player has to ensure optimality for each type of the stronger player, this change always leads to an informational rent for uncertain valuations. On the other hand, the local indifference condition is the same for all types of stronger player for uncertainty about head starts. Hence, the weaker player uses the same density as in the public information case on each interval of his support.

Yet, the lack of local changes of behavior for uncertainty about head starts may lead to a global change, i.e., the occurrence of a bid bifurcation. In this case, an ϵ -increase in the head start distribution can almost double the expected profits of the stronger player. Intuitively, the weaker player competes fiercely up to a certain threshold of disadvantage above which he resigns.

From an empirical point of view, an outside observer would not be able to tell whether a player had a head start if the head start distribution is low. On the other hand, a non-zero bid of both players entails that the head start distribution is not too high, i.e., not stochastically dominated by a uniform distribution on $[0, 1]$. Hence, observing the equilibrium outcome allows an outside observer to (partially) learn about the information structure.

Apart from this possibility of deducing the information structure from the equilibrium play, the closed-form solution of the equilibrium also answers several questions about gains from hiding or acquiring information. These questions arise naturally in several applications.

In a lobbying contest, for instance, an out-of-state company would like to know how beneficial it is for her to learn her opponents advantage; the home company would want to know how much hiding this information is worth. In (public) procurement, firms are also eager to understand possible benefits from learning or hiding the size of the advantage an incumbent has. The results of this paper show the subtle difference between incomplete information about previous work effort which can be used for a new task (head start) and a better technology (costs or valuation).

6 An Extension: Negative Head Starts

In this extension, I reverse the informational assumptions, i.e., the disadvantaged player now possesses private information. For instance, one might think of a procurement contest in which an incumbent builds upon her innovation from a previous contest. In this case, a new competitor knows the quality of the incumbent's innovation, but the incumbent is not perfectly informed how difficult it is for the new competitor to replicate the innovation.

In the model, the head start distribution of player 1 is continuous and strictly increasing on its connected support $[\underline{\theta}, 0]$. As before, player 1 observes his draw, while player 2 only knows the distribution H .

Denote by θ^* all points such that $H(\theta) = |\theta|$. In the appendix (Lemma 6), I show that θ^* is unique. The point θ^* turns out to be crucial for the equilibrium construction:

Proposition 4. *In any Bayesian Nash equilibrium*

$$F_1(x) = \begin{cases} H(x) & \text{if } x \leq \theta^*, \\ H(\theta^*) & \text{if } \theta^* < x \leq 0, \\ H(\theta^*) + x & \text{if } 1 - H(\theta^*) \geq x > 0, \\ 1 & \text{if } 1 - H(\theta^*) < x, \end{cases}$$

and

$$F_2(x) = \begin{cases} 0 & \text{if } x < 0, \\ H(\theta^*) + x & \text{if } 1 - H(\theta^*) \geq x \geq 0, \\ 1 & \text{if } 1 - H(\theta^*) < x. \end{cases}$$

An equilibrium strategy of player 1 is

$$b_1(\theta) = \begin{cases} H(\theta) - H(\theta^*) - \theta & \text{if } \theta > \theta^* \\ 0 & \text{if } \theta \leq \theta^*. \end{cases}$$

In equilibrium, all high types of player 1 (above θ^*) make strictly positive expected profits, while the low types make zero profits. Hence, player 1 is better off if his type remains private information compared to the public information case in which he makes zero profits independently of his type (see Lemma 1).

For high types of player 1, the profit of player 2 is zero which is lower than in the public information case. If the type of player 1 is slightly below or equal to θ^* , however, the profit of player 2 is higher in the private information case. For public information, player 2 cannot credibly commit to play the same strategy which deters entrance of all types of player 1 below θ^* in equilibrium, since player 1 bidding zero would give player 2 an incentive to reduce his own bid to zero. Put differently, the profit of player 2 against certain types is higher under private information since private information is similar to a commitment device. The sign of player 2's informational rent depends on the shape of the head start distribution.

7 Appendix

In order to prove Proposition 2, I derive a sequence of auxiliary results that have to hold *in any (Bayesian Nash) equilibrium*. Recall for the derivation that ties are broken in favor of player 1.

Lemma 3. *In any equilibrium, F_2 is continuous, i.e., player 2 does not place an atom at any $x > 0$.*

Proof. Assume to the contrary that there exists an $x > 0$ such that player 2 places an atom of size k at x . Hence, $F_2(x) - k \geq F_2(y)$ for all $y < x$. In this case, player 1 never chooses a score $s_1 \in [x - \frac{k}{2}, x)$, since

$$\Pi_1(\theta, x - \theta) = F_2(x) - (x - \theta) > F_2(x) - k - (x - \frac{k}{2} - \theta) \geq \Pi_1(s_1).$$

This implies that is not optimal for player 2 to bid $b_2 = x$, since a strictly lower bid of $x - \frac{k}{2}$ yields him the same winning probability which contradicts the initial assumption. More formally,

$$\Pi_2(x) = F_1(x - \frac{k}{2}) - x < F_1(x - \frac{k}{2}) - x + \frac{k}{2} = \Pi_2(x - \frac{k}{2}).$$

□

Lemma 4. *If player 2 randomizes with a positive density f_2 over an interval $I = [a, b]$, then $f_2(x) \leq 1$ on I .*

Proof. Assume player 2 randomizes with positive density on I . By optimality for player 2, every bid $x \in I$ leads to the same payoff which implies $f_1(x) = 1$ for all $x \in I$. Hence, $f_2(x) \leq 1$, since player 1 would strictly prefer to increase his score s_1 on an interval $(c, d) \subset I$ if $f_2(x) > 1$ for all $x \in (c, d)$; this contradicts optimality of player 1's randomization on I . □

Lemma 5. *Player 2 obtains a payoff of zero.*

Proof. As any bid above 1 leads to a negative payoff for player 2, $F_2(1) = 1$ in any equilibrium. Moreover, player 2 places no mass points above 0 by Lemma 3 and $f_2(x) \leq 1$ by Lemma 4. As $F_2(1) = F_2(0) + \int_0^1 f_2(x)dx = 1$, player 2 either places a mass point at zero or he randomizes uniformly with $f_2(x) = 1$ on the interval $[0, 1]$. Since all equilibrium bids yield the same payoff for player 2, he makes a payoff of $\Pi_2(0) = 0$. □

Proof of Proposition 2. Case 1: Assume $H(\theta) < \theta$. Then, by continuity of H , there exists an $\epsilon > 0$ such that player 2 does not bid $b_2 \in [\theta, \theta + \epsilon]$, because

$$\Pi_2(\theta) \leq F_1(\theta) - \theta \leq H(\theta) - \theta < 0.$$

Since $f_2(y) \leq 1$ for all y by Lemma 4 and $f_2(y) = 0$ for $y \in [x, x + \epsilon]$, I obtain

$$\Pi_1(x, 0) = F_2(x) > F_2(b_1 + x) - b_1 = \Pi_2(b_1 + x)$$

for all $b_1 > 0$. Hence, player 1 bids 0 if $H(\theta) < \theta$.

Case 2: Assume $H(a) > a$. By the zero-profit condition for player 2 (Lemma 5), $F_1(x) \leq x$ for all x . By contradiction, assume there exists an $\epsilon > 0$ such that $F_1(x) < x$ for all $x \in [a, a + \epsilon]$. Then $f_2(x) = 0$ for $x \in [a, a + \epsilon]$. Recall that $f_2(x) \leq 1$ for all x by Lemma 4. Hence, if player 1 has a head start $\theta \leq a$, she does not choose a score above θ , because

$$\Pi_1(\theta, a - \theta) = F_2(a) - (a - \theta) > F_2(a + \epsilon) - (a + \epsilon - \theta) = \Pi_1(\theta, a - \theta + \epsilon).$$

for all $\epsilon > 0$. Hence, $F_1(a) = H_1(a) > a$ which contradicts $F_1(x) < x$ for all $x \in [a, a + \epsilon]$.

Since $H(\theta) = \theta$ on a set of measure zero, the bidding strategy in this case has no impact on the scoring distribution. Hence, the scoring distribution of player 1 is unique.

Whenever the score of player 1 for some type $\theta \leq a$ is above a , then $f_2(x) = 1$ for all $x \in (\theta, \theta + b_1)$ to ensure optimality for player 1. This condition is satisfied whenever $H(a) > a = F_1(a)$. Together with $f_2(x) = 0$ whenever $H(x) < x$, this uniquely determines the distribution of player 2. \square

Proof of Theorem 1. If $H(\theta) > \theta$ for all $\theta \in (0, 1)$, player 2 randomizes uniformly on $(0, 1)$ by Corollary 2. Thus, the expected payoff for player 1 is equal to the size of his head start θ for all θ , which equals the payoff from Lemma 1.

If $H(\theta) < \theta$ on some interval (a, b) , then $F_2(x) > x$ for all $x \in (a, b)$ by Proposition 1 and 2. Hence, for any $\theta \in (a, b)$, a bid of 0 yields player 1 a payoff which is strictly higher than her payoff under public information. Moreover, for all $\theta < 1$, player 1 can ensure her public information payoff of θ by bidding $1 - \theta$. Hence, player 1 obtains a strictly higher expected payoff under private information.

The result for player 2 follows immediately from Lemma 1 and Lemma 5. \square

Proof of Proposition 3. In the first step, I show that the bidding strategy $b_1(v_1) = H(v_1)$ induces the uniform distribution on $[0, 1]$:

$$F_1(x) = \mathbb{P}(b_1(v_1) \leq x) = \mathbb{P}(v_1 \leq b_1^{-1}(x)) = \min\{H(H^{-1}(x)), 1\} = \min\{x, 1\}.$$

The next step establishes uniqueness of player 1's optimal bidding strategy. Player 2 never bids above 1 in equilibrium, since this would entail a negative payoff. Thus, player 1 can guarantee himself a payoff of $v_1 - 1$ by bidding 1. A similar argument to that in the proof of Lemma 3 shows that F_1 and F_2 are continuous with $F_1(0) = 0$. A similar argument to that in the proof Lemma 4 with reversed player roles entails $f_1(x) \leq 1$.⁵ As player 1 places no mass point and $F_1(1) = 1$, this implies $F_1(1) = \int_0^1 f_1(x)dx = 1$. Thus, $F_1(x) = \min\{x, 1\}$. To obtain uniqueness of the strategy of player 1, it remains to show that his bid is increasing in his valuation in any equilibrium. This, however, is straightforward, since a higher bid entails the same additional cost for each type, but increases the winning probability and thereby the gain of a type with a higher valuation.

The next step establishes uniqueness of player 2's equilibrium distribution. Given $F_1(x)$, any bid in $[0, 1]$ is a best response for player 2, and, thus, $F_2(x)$ is a best response. It remains to show that $F_2(x)$ is the only distribution function which makes the bidding strategy of player 1 optimal. Consider player 1's the optimization problem of choosing the $b_1(v_1)$ which maximizes $v_1 F_2(b_1(v_1)) - b_1(v_1)$. By Assumption 1 and continuity, F_2 is differentiable. The first-order condition yields $\frac{1}{v_1} = f_2(b_1(v_1))$ for all $v_1 \in [1, \bar{v}_1]$. Hence, since H is invertible on player 1's support, $f_2(x) = \frac{1}{b_1^{-1}(x)} = \frac{1}{H^{-1}(x)}$. Imposing the boundary condition $F_2(1) = 1$, the differential equation yields

$$F_2(x) = 1 - \int_0^1 \frac{1}{H^{-1}(x)} dx + \int_0^x \frac{1}{H^{-1}(y)} dy$$

⁵Details on both steps are available upon request.

as the unique solution. The second-order condition shows that the bid $b_1(v_1) = H(v_1)$ is indeed the maximizer for all $v_1 \in [1, \bar{v}_1]$.

Player 1 obtains an informational rent in equilibrium, since bidding $b_1(v_1) = 1$ would yield him the public information payoff $v_1 - 1$, but any equilibrium bid yields a strictly higher payoff for all types $v_1 \in [1, \bar{v}_1]$. Player 2 obtains a payoff of 0 under both informational assumptions. □

Lemma 6. *There exists a unique point θ^* such that $H(\theta) = |\theta|$.*

Proof. Obviously, $H(\underline{\theta}) - |\underline{\theta}| = -|\underline{\theta}| < 0$, $H(0) - |0| = 1 > 0$, $H(\theta)$ is continuous and strictly monotone increasing and $|\theta|$ is continuous and strictly monotone decreasing for $\theta \in (\underline{\theta}, 0)$. Hence, $H(\theta) - |\theta|$ is continuous and strictly monotone increasing for $\theta \in (\underline{\theta}, 0)$. By the Intermediate Value Theorem, there exists a unique point θ^* for which $H(\theta) - |\theta| = 0$, i.e., $H(\theta) = |\theta|$. □

Proof of Proposition 4. As in the proof of Proposition 1, I first show that the strategy of player 1 leads to the distribution $F_1(x)$. In the second step, I establish optimality.

Step 1: If $\theta \leq \theta^*$, player 1 bids zero, i.e., $F_1(x) = H(x)$. If $\theta > \theta^*$, for $x \in (0, 1 - H(\theta^*))$, I obtain

$$F_1(x) = \mathbb{P}(H(\theta) - H(\theta^*) \leq x) = \mathbb{P}(\theta \leq H^{-1}(x + H(\theta^*))) = x + H(\theta^*).$$

Since F_1 is a distribution, I obtain $F_1(x) = H(\theta^*)$ for $x \in (\theta^*, 0]$ and $F_1(x) = 1$ for $x \geq 1 - H(\theta^*)$ which completes the derivation of $F_1(x)$.

Step 2: Given the distribution of player 2, $\Pi_1(\theta, b_1) \leq 0$ for all b_1 if $\theta \leq \theta^*$. If $\theta > \theta^*$, any bid in $[-\theta, -\theta + 1 - H(\theta^*)]$ is a best response. Thus, the strategy of player 1 is a best response to F_2 .

Given the distribution of player 1, $\Pi_2(b_2) = F_1(b_1) - b_1 = H(\theta^*)$ for all $x \in [0, 1 - H(\theta^*)]$ and strictly lower otherwise. Hence, F_2 is a best response to F_1 .

The uniqueness proof is similar to the proof of Proposition 2. Details are available upon request. □

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