

Meier, Martin; Schipper, Burkhard C.

Working Paper

Speculative trade under unawareness: The infinite case

Working Paper, No. 10-2

Provided in Cooperation with:

University of California Davis, Department of Economics

Suggested Citation: Meier, Martin; Schipper, Burkhard C. (2013) : Speculative trade under unawareness: The infinite case, Working Paper, No. 10-2, University of California, Department of Economics, Davis, CA

This Version is available at:

<https://hdl.handle.net/10419/79672>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.



DEPARTMENT OF ECONOMICS

Working Paper Series

Speculative Trade under Unawareness: The Infinite Case

Martin Meier
Institute of Advanced Studies, Vienna
Burkhard Schipper
U.C. Davis

April 18, 2013

Paper # 10-2

We generalize the ``No-speculative-trade" theorem for finite unawareness belief structures in Heifetz, Meier, and Schipper (2013) to the infinite case.

Department of Economics
One Shields Avenue
Davis, CA 95616
(530)752-0741

http://www.econ.ucdavis.edu/working_search.cfm

SPECULATIVE TRADE UNDER UNAWARENESS: THE INFINITE CASE*

Martin Meier[†]

Burkhard C. Schipper[‡]

This draft: April 18, 2013

First draft: March 1, 2007

Abstract

We prove a “no-speculative-trade” theorem under unawareness for the infinite case. This generalizes the result for the finite case by Heifetz, Meier, and Schipper (2013).

Keywords: Awareness, unawareness, speculation, trade, agreement, common prior, common certainty.

JEL-Classifications: C70, C72, D53, D80, D82, D84.

*We are very grateful for helpful comments by two anonymous referees. Martin acknowledges financial support from the Spanish Ministerio de Educación y Ciencia via a Ramon y Cajal Fellowship and Research Grant SEJ2004-07861, and the Fundación Ramón Areces. Burkhard received financial support from the NSF SES-0647811.

[†]Institut für Höhere Studien (IHS), Vienna. Email: martin.meier@ihs.ac.at

[‡]Department of Economics, University of California, Davis. Email: bcschipper@ucdavis.edu

1 Introduction

Unawareness refers to the lack of conception rather than to the lack of information. It is natural to presume that asymmetric unawareness may lead to speculative trade. Indeed, Heifetz, Meier, and Schipper (2013) present a simple example of speculation under unawareness in which there is common certainty of willingness to trade but agents have a strict preference to trade despite the existence of a common prior.¹ This is impossible in standard state-space structures with a common prior. In standard “no-speculative-trade” theorems, if there is common certainty of willingness to trade, then agents are necessarily indifferent to trade (Milgrom and Stokey, 1982). Somewhat surprisingly, Heifetz, Meier, and Schipper (2013) also prove a “no-speculative-trade” result according to which under a common prior there cannot be common certainty of strict preference to trade. This means that arbitrarily small transaction costs rule out speculation under asymmetric unawareness. The “no-speculative-trade” result in Heifetz, Meier, and Schipper (2013) was stated for finite unawareness belief structures. In this note we generalize the result to infinite unawareness belief structures. Such a generalization is relevant since the space of underlying uncertainties may be large. Especially if it is large, agents may be unaware of some of them. Moreover, the generalization serves as a robustness check for our “no-speculative-trade” result for finite unawareness belief structures. It shows that the result in Heifetz, Meier, and Schipper (2013) is not an artefact of the finiteness assumption but holds more generally. The topological unawareness belief structure introduced in this paper to prove our result may be of independent interest for other applications.

Board and Chung (2013) present a different model of unawareness, in which unawareness is about “objects” rather than events. Intuitively, we model an agent’s unawareness of events like “penicillium rubens has antibiotic properties” while in their model the agent’s corresponding unawareness would be about “penicillium rubens”. Board and Chung (2013) also prove a “no-speculative-trade” result for finite spaces. This suggests that the “no-speculative-trade” result in Heifetz, Meier, and Schipper (2013) remains robust if unawareness of events is replaced by unawareness of objects. Further, Board and Chung (2013) study awareness of unawareness of objects. This allows them to show that the “no-speculative-trade” result obtains both under “living in denial” and “living under paranoia”, where former refers to the situation in which every agent is certain that there is nothing they are unaware of while latter refer to the situation in which every agent is certain that there is something they are unaware of. Grant and Quiggin (2013)

¹This example is a probabilistic version of the speculation example in Heifetz, Meier, and Schipper (2006).

discuss the speculate trade example of Heifetz, Meier, and Schipper (2006) in the context of awareness of unawareness. The authors argue that agents should induce from previous experiences of becoming aware and from differences in awareness across agents that they themselves could be unaware of something. This awareness of unawareness may be coupled with a version of a precautionary principle which may make them reluctant to engage in speculative trade.

The paper is organized as follows. The next section introduces topological unawareness belief structures. The general “no-speculative-trade” theorem is stated in Section 3. Finally, Section 4 contains the proof of the theorem.

2 Topological Unawareness Belief Structures

We consider an unawareness belief structure as defined in Heifetz, Meier, and Schipper (2013) but with additional topological properties.

2.1 Compact Hausdorff State-Spaces

Let $\mathcal{S} = \{S_\alpha\}_{\alpha \in \mathcal{A}}$ be a complete lattice of disjoint *state-spaces*, with respect to the partial order \preceq on \mathcal{S} . If S_α and S_β are such that $S_\alpha \succeq S_\beta$ we say that S_α is more expressive than S_β . (\mathcal{S}, \preceq) is well-founded, that is, every non-empty subset $\mathcal{X} \subseteq \mathcal{S}$ contains a \preceq -minimal element. (That is, there is a $S' \in \mathcal{X}$ such that for all $S \in \mathcal{X}$: if $S \preceq S'$, then $S = S'$.) Each state-space $S \in \mathcal{S}$ is a non-empty compact Hausdorff space with a Borel σ -field \mathcal{F}_S . Denote by $\Omega = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$ the union of these spaces. Ω is endowed with the disjoint-union topology: $O \subseteq \Omega$ is open if and only if $O \cap S$ is open in S for all $S \in \mathcal{S}$.

2.2 Continuous Projections

For every S and S' such that $S' \succeq S$, there is a continuous surjective projection $r_S^{S'} : S' \rightarrow S$, where r_S^S is the identity. Note that the cardinality of S is smaller than or equal to the cardinality of S' . We require the projections to commute: if $S'' \succeq S' \succeq S$ then $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$. If $\omega \in S'$, denote $\omega_S = r_S^{S'}(\omega)$. If $D \subseteq S'$, denote $D_S = \{\omega_S : \omega \in D\}$.

2.3 Events

Denote $g(S) = \{S' : S' \succeq S\}$. For $D \subseteq S$, denote $D^\uparrow = \bigcup_{S' \in g(S)} (r_S^{S'})^{-1}(D)$.

An *event* is a pair (E, S) , where $E = D^\uparrow$ with $D \subseteq S$, where $S \in \mathcal{S}$. D is called the *base* and S the *base-space* of (E, S) , denoted by $S(E)$. If $E \neq \emptyset$, then S is uniquely determined by E and, abusing notation, we write E for (E, S) . Otherwise, we write \emptyset^S for (\emptyset, S) . Note that not every subset of Ω is an event.

Let Σ be the set of *measurable events* of Ω , i.e., D^\uparrow such that $D \in \mathcal{F}_S$, for some state-space $S \in \mathcal{S}$. Note that unless \mathcal{S} is a singleton, Σ is not an algebra because it contains distinct \emptyset^S for all $S \in \mathcal{S}$.

2.4 Negation

If (D^\uparrow, S) is an event where $D \subseteq S$, the negation $\neg(D^\uparrow, S)$ of (D^\uparrow, S) is defined by $\neg(D^\uparrow, S) := ((S \setminus D)^\uparrow, S)$. Note, that by this definition, the negation of a (measurable) event is a (measurable) event. Abusing notation, we write $\neg D^\uparrow := (S \setminus D)^\uparrow$. Note that by our notational convention, we have $\neg S^\uparrow = \emptyset^S$ and $\neg \emptyset^S = S^\uparrow$, for each space $S \in \mathcal{S}$. $\neg D^\uparrow$ is typically a proper subset of the complement $\Omega \setminus D^\uparrow$. That is, $(S \setminus D)^\uparrow \subsetneq \Omega \setminus D^\uparrow$.

2.5 Conjunction and Disjunction

If $\left\{ (D_\lambda^\uparrow, S_\lambda) \right\}_{\lambda \in L}$ is a finite or countable collection of events (with $D_\lambda \subseteq S_\lambda$, for $\lambda \in L$), their conjunction $\bigwedge_{\lambda \in L} (D_\lambda^\uparrow, S_\lambda)$ is defined by $\bigwedge_{\lambda \in L} (D_\lambda^\uparrow, S_\lambda) := \left(\left(\bigcap_{\lambda \in L} D_\lambda^\uparrow \right), \sup_{\lambda \in L} S_\lambda \right)$. Note, that since \mathcal{S} is a *complete* lattice, $\sup_{\lambda \in L} S_\lambda$ exists. If $S = \sup_{\lambda \in L} S_\lambda$, then we have $\left(\bigcap_{\lambda \in L} D_\lambda^\uparrow \right) = \left(\bigcap_{\lambda \in L} \left((r_{S_\lambda}^S)^{-1}(D_\lambda) \right) \right)^\uparrow$. Again, abusing notation, we write $\bigwedge_{\lambda \in L} D_\lambda^\uparrow := \bigcap_{\lambda \in L} D_\lambda^\uparrow$ (we will therefore use the conjunction symbol \wedge and the intersection symbol \cap interchangeably).

We define the relation \subseteq between events (E, S) and (F, S') , by $(E, S) \subseteq (F, S')$ if and only if $E \subseteq F$ as sets *and* $S' \preceq S$. If $E \neq \emptyset$, we have that $(E, S) \subseteq (F, S')$ if and only if $E \subseteq F$ as sets. Note however that for $E = \emptyset^S$ we have $(E, S) \subseteq (F, S')$ if and only if $S' \preceq S$. Hence we can write $E \subseteq F$ instead of $(E, S) \subseteq (F, S')$ as long as we keep in mind that in the case of $E = \emptyset^S$ we have $\emptyset^S \subseteq F$ if and only if $S \succeq S(F)$. It follows from these definitions that for events E and F , $E \subseteq F$ is equivalent to $\neg F \subseteq \neg E$ only when E and F have the same base, i.e., $S(E) = S(F)$.

The disjunction of $\left\{ D_\lambda^\uparrow \right\}_{\lambda \in L}$ is defined by the de Morgan law $\bigvee_{\lambda \in L} D_\lambda^\uparrow := \neg \left(\bigwedge_{\lambda \in L} \neg (D_\lambda^\uparrow) \right)$. Typically $\bigvee_{\lambda \in L} D_\lambda^\uparrow \subsetneq \bigcup_{\lambda \in L} D_\lambda^\uparrow$, and if all D_λ are nonempty we have that $\bigvee_{\lambda \in L} D_\lambda^\uparrow = \bigcup_{\lambda \in L} D_\lambda^\uparrow$ holds if and only if all the D_λ^\uparrow have the same base-space. Note, that by these

definitions, the conjunction and disjunction of (at most countably many measurable) events is a (measurable) event.

Apart from the topological conditions and the well-foundedness assumption, the event-structure outlined so far is analogous to Heifetz, Meier, and Schipper (2006, 2008, 2013).

2.6 Regular Borel Probability Measures

Here and in what follows, the term 'events' always means measurable events in Σ unless otherwise stated.

For each $S \in \mathcal{S}$, $\Delta(S)$ is the set of regular Borel probability measures on (S, \mathcal{F}_S) . We consider this set itself as a measurable space which is endowed with the topology of weak convergence.²

$\bigcup_{S \in \mathcal{S}} \Delta(S)$ is endowed with the disjoint-union topology: $O_\Delta \subseteq \bigcup_{S \in \mathcal{S}} \Delta(S)$ is open if and only if $O_\Delta \cap \Delta(S)$ is open in $\Delta(S)$ for all $S \in \mathcal{S}$.

Note that although each S and each $\Delta(S)$ are compact, if \mathcal{S} is infinite, Ω and $\bigcup_{S \in \mathcal{S}} \Delta(S)$ are *not* compact.

2.7 Marginals

For a probability measure $\mu \in \Delta(S')$, the marginal $\mu|_S$ of μ on $S \preceq S'$ is defined by

$$\mu|_S(D) := \mu\left(\left(r_S^{S'}\right)^{-1}(D)\right), \quad D \in \mathcal{F}_S.$$

Let S_μ be the space on which μ is a probability measure. Whenever $S_\mu \succeq S(E)$ then we abuse notation slightly and write

$$\mu(E) = \mu(E \cap S_\mu).$$

If $S(E) \not\succeq S_\mu$, then we say that $\mu(E)$ is undefined.

²The topology of weak convergence is generated by the sub-basis of sets $\{\mu \in \Delta(S) : \mu(O) > r\}$ where $O \subseteq S$ is open and $r \in \mathbb{R}$ (see e.g. Billingsley (1968), appendix III). This topology coincides with the weak* topology when S is normal (and in particular compact and/or metric); the weakest topology for which the mapping

$$\mu \longrightarrow \int_S f d\mu$$

is continuous for every continuous real-valued function f on S (see Heifetz, 2006, Fn. 3).

2.8 Continuous Type Mappings

Let I be a nonempty finite or countable set of individuals. For every individual, each state gives rise to a probabilistic belief over states in some space.

Definition 1 *For each individual $i \in I$ there is a continuous type mapping $t_i : \Omega \rightarrow \bigcup_{\alpha \in \mathcal{A}} \Delta(S_\alpha)$. We require the type mapping t_i to satisfy the following properties:*

- (0) *Confinement: If $\omega \in S'$ then $t_i(\omega) \in \Delta(S)$ for some $S \preceq S'$.*
- (1) *If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \Delta(S)$ then $t_i(\omega_{S'}) = t_i(\omega)$.*
- (2) *If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \Delta(S')$ then $t_i(\omega_S) = t_i(\omega)|_S$.*
- (3) *If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega_{S'}) \in \Delta(S)$ then $S_{t_i(\omega)} \succeq S$.*

Define³

$$Ben_i(\omega) := \left\{ \omega' \in \Omega : t_i(\omega')|_{S_{t_i(\omega)}} = t_i(\omega) \right\}.$$

This is the set of states at which individual i 's type or the marginal thereof coincides with her type at ω . Such sets are events in our structure:

Remark 1 *For any $\omega \in \Omega$, $Ben_i(\omega)$ is an $S_{t_i(\omega)}$ -based event, which is not necessarily measurable.⁴*

Assumption 1 *If $Ben_i(\omega) \subseteq E$, for a measurable event E , then $t_i(\omega)(E) = 1$.*

This assumption implies introspection (Property (va) in Proposition 4 in Heifetz, Meier, and Schipper, 2013). Note, that if $Ben_i(\omega)$ is measurable, then Assumption 1 implies $t_i(\omega)(Ben_i(\omega)) = 1$.

Definition 2 *We denote by $\underline{\Omega} := \left\langle \mathcal{S}, \left(r_{S_\beta}^{S_\alpha} \right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I} \right\rangle$ a topological unawareness belief structure.*

Topological unawareness belief structures are analogous to unawareness belief structures in Heifetz, Meier, and Schipper (2013) except for the additional topological properties and the well-foundedness assumption. See Heifetz, Meier, and Schipper (2013) for further details and interpretations.

³The name “Ben” is chosen analogously to the “ken” in knowledge structures.

⁴Even in a standard type-space, if the σ -algebra is not countably generated, then the set of states where a player is of a certain type might not be measurable.

2.9 Belief, Common Certainty, and Awareness

For $i \in I$, $p \in [0, 1]$ and an event E , the p -belief operator is defined by

$$B_i^p(E) := \{\omega \in \Omega : t_i(\omega)(E) \geq p\},$$

if there is a state ω such that $t_i(\omega)(E) \geq p$, and by

$$B_i^p(E) := \emptyset^{S(E)}$$

otherwise. The mutual p -belief operator on events is defined by

$$B^p(E) = \bigcap_{i \in I} B_i^p(E).$$

The common certainty operator on events is defined by

$$CB^1(E) = \bigcap_{n=1}^{\infty} (B^1)^n(E).$$

These are standard definitions (e.g. see Monderer and Samet, 1989) adapted to our unawareness structures.

As in Heifetz, Meier, and Schipper (2013) we define for every $i \in I$ the awareness operator

$$A_i(E) := \{\omega \in \Omega : t_i(\omega) \in \Delta(S) \text{ for some } S \succeq S(E)\},$$

for every event E , if there is a state ω such that $t_i(\omega) \in \Delta(S)$ with $S \succeq S(E)$, and by

$$A_i(E) := \emptyset^{S(E)}$$

otherwise.

In Heifetz, Meier, and Schipper (2013, Proposition 1 and 2) we show that $A_i(E)$, $B_i^p(E)$, $B^p(E)$, and $CB^1(E)$ are all $S(E)$ -based measurable events. We also show in Heifetz, Meier, and Schipper (2013, Proposition 4) that standard properties of belief obtain. Moreover, in Heifetz, Meier, and Schipper (2013, Proposition 5) we show “standard” properties of awareness. One of those properties is *weak necessitation*, i.e., for any event $E \in \Sigma$, $A_i(E) = B_i^1(S(E)^\uparrow)$. This property will be used later in the proof.

Definition 3 *An event E is evident if for each $i \in I$, $E \subseteq B_i^1(E)$.*

Proposition 1 *For every event $F \in \Sigma$:*

(i) $CB^1(F)$ is evident, that is $CB^1(F) \subseteq B_i^1(CB^1(F))$ for all $i \in I$.

(ii) There exists an evident event E such that $\omega \in E$ and $E \subseteq B_i^1(F)$ for all $i \in I$, if and only if $\omega \in CB^1(F)$.

The proof is analogous to Proposition 3 in Monderer and Samet (1989) for a standard state-space and thus omitted.

3 A Generalized “No-speculative-trade” Theorem

Definition 4 (Prior) A prior for player i is a system of probability measures $P_i = (P_i^S)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ such that

1. The system is projective: If $S' \preceq S$ then the marginal of P_i^S on S' is $P_i^{S'}$. (That is, if $E \in \Sigma$ is an event whose base-space $S(E)$ is lower or equal to S' , then $P_i^S(E) = P_i^{S'}(E)$.)
2. Each probability measure P_i^S is a convex combination of i 's beliefs in S : for every event $E \in \Sigma$ such that $S(E) \preceq S$,

$$P_i^S(E \cap S \cap A_i(E)) = \int_{S \cap A_i(E)} t_i(\cdot)(E) dP_i^S(\cdot). \quad (1)$$

We call any probability measure $\mu_i \in \Delta(S)$ satisfying equation (1) in place of P_i^S a prior of player i on S .

Definition 5 (Common Prior) $P = (P^S)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ (resp. $P^S \in \Delta(S)$) is a common prior (resp. a common prior on S) if P (resp. P^S) is a prior (resp. a prior on S) for every player $i \in I$.

Denote by $[t_i(\omega)] := \{\omega' \in \Omega : t_i(\omega') = t_i(\omega)\}$.

Definition 6 A common prior $P = (P^S)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ (resp. a common prior P^S on S) is positive if and only if for all $i \in I$ and $\omega \in \Omega$: if $t_i(\omega) \in \Delta(S')$, then $P^S([t_i(\omega)] \cap S')^\uparrow \cap S) > 0$ for all $S \succeq S'$ (resp. for S).

Note that by Lemma 3 below, $[t_i(\omega)] \cap S' \in \mathcal{F}_{S'}$.

Recall Remark 7 in Heifetz, Meier, and Schipper (2013) according to which if \hat{S} is the upmost state-space in the lattice \mathcal{S} , and $(P_i^S)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ is a tuple of probability measures, then $(P_i^S)_{S \in \mathcal{S}}$ is a prior for player i if and only if $P_i^{\hat{S}}$ is a prior for player i on \hat{S} and P_i^S is the marginal of $P_i^{\hat{S}}$ for every $S \in \mathcal{S}$.

Speculative trade between agents could occur at a state if it is common certainty that agents form different expectations of a random variable (e.g. stock returns).

Definition 7 *Let x_1 and x_2 be real numbers and v a continuous random variable⁵ on Ω . Define the sets $E_1^{\leq x_1} := \left\{ \omega \in \Omega : \int_{S_{t_1(\omega)}} v(\cdot) d(t_1(\omega))(\cdot) \leq x_1 \right\}$ and $E_2^{\geq x_2} := \left\{ \omega \in \Omega : \int_{S_{t_2(\omega)}} v(\cdot) d(t_2(\omega))(\cdot) \geq x_2 \right\}$. We say that at ω , conditional on his information, player 1 (resp. player 2) believes that the expectation of v is weakly below x_1 (resp. weakly above x_2) if and only if $\omega \in E_1^{\leq x_1}$ (resp. $\omega \in E_2^{\geq x_2}$).*

Since we endowed Ω with the disjoint union topology, we have that a function $v : \Omega \rightarrow \mathbb{R}$ is continuous if and only if for each $S \in \mathcal{S}$ the restriction of v to S is a continuous function from S to \mathbb{R} .

Note that the sets $E_1^{\leq x_1}$ or $E_2^{\geq x_2}$ may not be events in our unawareness belief structure, because $v(\omega) \neq v(\omega_S)$ is allowed, for $\omega \in S' \succ S$. Yet, we can define p -belief, mutual p -belief, and common certainty for measurable subsets of Ω , and show that the properties stated in Heifetz, Meier, and Schipper (2013, Propositions 4 and 6) obtain as well. The proofs are analogous and thus omitted.

Theorem 1 *Let $\underline{\Omega}$ be a topological unawareness belief structure and P a positive common prior. Then there is no state $\tilde{\omega} \in \Omega$, continuous random variable $v : \Omega \rightarrow \mathbb{R}$, and*

⁵We thank an anonymous referee for encouraging us to discuss to what extent the assumption of a continuous random variable would preclude situations in which for instance the “returns from investing in eurozone sovereign debt seems to depend discontinuously on the question of whether the eurozone stays intact”. We view continuity more as a technical assumption. The intuitive notion of continuity referred to in the eurozone example does not necessarily correspond to the mathematical notion of continuity. To see this, note that since we do not assume that state-spaces are connected, we allow for a states-space to be a disjoint union of two open subsets, call them A and B , with A standing for the event that the eurozone stays intact while B corresponding to the event that the eurozone falls apart. Then a random variable, which restricted to A is continuous and larger than x while restricted to B being also continuous but smaller than y with $x > y$, is a continuous random variable although it depends “discontinuously” (i.e., more precisely it “jumps”) on the event of whether or not the eurozone stays intact or not.

$x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ such that: at $\tilde{\omega}$ it is common certainty that conditional on her information, player 1 believes that the expectation of v is weakly below x_1 and, conditional on his information, player 2 believes that the expectation of v is weakly above x_2 .

This general “no-speculative-trade” theorem implies our “no-speculative-trade” theorem for finite unawareness belief structures (Heifetz, Meier, and Schipper, 2013).

We should clarify how the topological assumptions allow us to generalize the “no-speculative-trade” theorem for finite unawareness belief structures in Heifetz, Meier, and Schipper (2013) to the infinite case. One assumption imposed on topological unawareness structures is that the complete lattice of spaces is well-founded. This assumption is used in an important step of the proof. If the set of states in which there is common certainty that the first player’s expectation is strictly above x and the second player’s expectations is weakly below x is nonempty, we need to find a minimal state-space such that the common certainty event restricted to this space is nonempty. Assuming that the lattice of spaces is well-founded allows us to find such a minimal space. Otherwise, it could be the case that agent 1 may believe that agent 2 is less aware and believes that agent 1 is even less aware and believes that agent 2 is even less aware and believes ... The assumption of state-spaces to be compact Hausdorff helps us to deal with the difficulties involved with infinite spaces. For instance, we need to have a well-defined notion of support for probability measures like common priors even in the infinite case. The support of a probability measure is the smallest closed set that has probability 1. Without appropriate topological assumptions, there may be no notion of a “smallest” set that has probability 1.

On the feasibility of a converse to Theorem 1, we note that Heifetz, Meier, and Schipper (2013) show by example that the converse of the “no-speculative-trade” theorem does not hold even in the finite case.

4 Proof of Theorem 1

4.1 Preliminary Definitions and Results

We define $G \subseteq \Omega$ to be a *measurable set* if and only if for all $S \in \mathcal{S}$, $G \cap S \in \mathcal{F}_S$. The collection of measurable sets forms a sigma-algebra on Ω .

Let $\underline{\Omega}$ be an unawareness belief structure. As in Heifetz, Meier, and Schipper (2009, Section 2.13), we define the *flattened type-space* associated with the unawareness belief

structure $\underline{\Omega}$ by

$$F(\underline{\Omega}) := \langle \Omega, \mathcal{F}, (t_i^F)_{i \in I} \rangle,$$

where Ω is the union of all state-spaces in the unawareness belief structure $\underline{\Omega}$, \mathcal{F} is the collection of all measurable sets in $\underline{\Omega}$, and $t_i^F : \Omega \longrightarrow \Delta(\Omega)$ is defined by

$$t_i^F(\omega)(E) := \begin{cases} t_i(\omega)(E \cap S_{t_i(\omega)}) & \text{if } E \cap S_{t_i(\omega)} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

The definition of the belief operator as well as standard properties of belief and Proposition 1 can be extended to measurable subsets of Ω . The proofs are analogous and thus omitted.

In this article, a *standard topological type space* (see for instance, Heifetz, 2006) is a compact Hausdorff space Ω such that for every individual $i \in I$ there is a continuous type mapping $t_i : \Omega \longrightarrow \Delta(\Omega)$ from Ω to the space of regular Borel probability measures $\Delta(\Omega)$ endowed with the topology of weak convergence.

Let $\underline{\Omega}$ be a topological unawareness belief structure and P a positive common prior. For the proof of the theorem, we have to show that there is no evident measurable set $E \in \mathcal{F}$ such that $\tilde{\omega} \in E$ and

$$\int_{\Omega} v(\cdot) d(t_1(\omega))(\cdot) \leq x_1 < x_2 \leq \int_{\Omega} v(\cdot) d(t_2(\omega))(\cdot)$$

for all $\omega \in E$.

We need the following lemmata:

Lemma 1 *Let $\underline{\Omega}$ be a topological unawareness belief structure, $v : \Omega \longrightarrow \mathbb{R}$ be a continuous random variable, and $x \in \mathbb{R}$. Then $\{\omega \in \Omega : \int_{\Omega} v(\cdot) d(t_i(\omega))(\cdot) \geq x\}$ and $\{\omega \in \Omega : \int_{\Omega} v(\cdot) d(t_i(\omega))(\cdot) \leq x\}$ are closed subsets of Ω .⁶*

PROOF OF THE LEMMA. Since for every $S \in \mathcal{S}$, the topology on $\Delta(S)$ coincides with the weak* topology and since in particular, $v : S \longrightarrow \mathbb{R}$ is continuous, $\{\mu \in \Delta(S) : \int_S v(\cdot) d\mu(\cdot) < x\}$ is open in $\Delta(S)$. Hence $\{\nu \in \bigcup_{S \in \mathcal{S}} \Delta(S) : \int_S v(\cdot) d\nu(\cdot) < x\}$ is open in $\bigcup_{S \in \mathcal{S}} \Delta(S)$.

By the continuity of $t_i : \Omega \longrightarrow \bigcup_{S \in \mathcal{S}} \Delta(S)$, it follows that $\{\omega \in \Omega : \int_{\Omega} v(\cdot) d(t_i(\omega))(\cdot) < x\}$ is open in Ω and hence its relative complement with respect to Ω , $\{\omega \in \Omega : \int_{\Omega} v(\cdot) d(t_i(\omega))(\cdot) \geq x\}$ is closed in Ω . \square

⁶Note that we abuse notation and write $\int_{\Omega} v(\cdot) d(t_i(\omega))(\cdot)$ instead of $\int_{S_{t_i(\omega)}} v(\cdot) d(t_i(\omega))(\cdot)$.

Lemma 2 *Let $\underline{\Omega}$ be a topological unawareness belief structure. Let E be a closed subset of Ω . Then $CB^1(E)$ is a closed subset of Ω .*

PROOF OF THE LEMMA. The relative complement of E with respect to Ω , $\Omega \setminus E$, is open, and hence for every $S \in \mathcal{S}$, $(\Omega \setminus E) \cap S = S \setminus (E \cap S)$ is open in S . Therefore $\{\mu \in \Delta(S) : \mu(S \setminus (E \cap S)) > 0\}$ is open. It follows that $\bigcup_{S \in \mathcal{S}} \{\mu \in \Delta(S) : \mu(S \setminus (E \cap S)) > 0\}$ is open. Hence for every $i \in I$, $\{\omega \in \Omega : t_i(\omega) \in \bigcup_{S \in \mathcal{S}} \{\mu \in \Delta(S) : \mu(S \setminus (E \cap S)) > 0\}\}$ is open. It follows that its relative complement with respect to Ω , $B_i^1(E) = \{\omega \in \Omega : t_i(\omega) \in \bigcup_{S \in \mathcal{S}} \{\mu \in \Delta(S) : \mu(E \cap S) = 1\}\}$ is closed. Since an arbitrary intersection of closed sets is closed, the Lemma follows by induction. \square

Lemma 3 *Let $\underline{\Omega}$ be a topological unawareness belief structure. Then for every $\omega \in \Omega$, every state-space $S \in \mathcal{S}$ and every player $i \in I$, the set $\{\omega' \in \Omega : t_i(\omega') = t_i(\omega)\} \cap S$ is closed in S .*

PROOF OF THE LEMMA. Since $\Delta(S_{t_i(\omega)})$ is the set of regular Borel probability measures on $S_{t_i(\omega)}$ endowed with the topology of weak convergence, $\{t_i(\omega)\}$ is closed in $\Delta(S_{t_i(\omega)})$, and hence $\{t_i(\omega)\}$ is closed in $\bigcup_{S \in \mathcal{S}} \Delta(S)$. Therefore, by continuity of t_i , $t_i^{-1}(\{t_i(\omega)\}) = [t_i(\omega)]$ is closed in Ω . Hence, $[t_i(\omega)] \cap S$ is closed in S . \square

Lemma 4 *Let $\underline{\Omega}$ be a topological unawareness belief structure. Let P^S be a positive (common) prior on the state-space S , and let $\omega \in S$ be such that $t_i(\omega) \in \Delta(S)$. Then, for every $E \in \mathcal{F}_S$, we do have $t_i(\omega)(E) = t_i(\omega)(E \cap [t_i(\omega)]) = \frac{P^S(E \cap [t_i(\omega)])}{P^S(S \cap [t_i(\omega)])}$.*

PROOF. We have $t_i(\omega)(S \cap [t_i(\omega)]) = 1$ and hence $t_i(\omega)(E) = t_i(\omega)(E \cap S \cap [t_i(\omega)]) = t_i(\omega)(E \cap [t_i(\omega)])$. Since P^S is positive, we do have $P^S(S \cap [t_i(\omega)]) > 0$.

Since $S((E \cap [t_i(\omega)])^\uparrow) = S$ and since $\omega' \in [t_i(\omega)]$ implies $t_i(\omega') \in \Delta(S)$, we do have $(E \cap [t_i(\omega)])^\uparrow \cap A_i((E \cap [t_i(\omega)])^\uparrow) = (E \cap [t_i(\omega)])^\uparrow$. We also have $(S \cap [t_i(\omega)])^\uparrow \subseteq A_i(S^\uparrow) = A_i((E \cap [t_i(\omega)])^\uparrow)$. The last equality follows from weak necessitation. We have - by the definition of a common prior - the following (with our abuse of notation):

$$\begin{aligned} P^S(E \cap [t_i(\omega)]) &= \int_{S \cap A_i((E \cap [t_i(\omega)])^\uparrow)} t_i(\cdot)(E \cap [t_i(\omega)]) dP^S(\cdot) \\ &= \int_{S \cap [t_i(\omega)]} t_i(\cdot)(E \cap [t_i(\omega)]) dP^S(\cdot) \\ &\quad + \int_{(S \cap A_i(S^\uparrow)) \setminus (S \cap [t_i(\omega)])} t_i(\cdot)(E \cap [t_i(\omega)]) dP^S(\cdot) \end{aligned}$$

But if $\omega' \in (S \cap A_i((E \cap [t_i(\omega)]^\dagger)) \setminus (S \cap [t_i(\omega)]))$, then $t_i(\omega')(E \cap [t_i(\omega)]) = 0$, and hence, we have

$$\begin{aligned} P^S(E \cap [t_i(\omega)]) &= \int_{S \cap [t_i(\omega)]} t_i(\cdot)(E \cap [t_i(\omega)]) dP^S(\cdot) \\ &= t_i(\omega)(E \cap [t_i(\omega)]) \int_{S \cap [t_i(\omega)]} 1 dP^S(\cdot) \\ &= t_i(\omega)(E \cap [t_i(\omega)]) P^S(S \cap [t_i(\omega)]). \end{aligned}$$

Since $P^S(S \cap [t_i(\omega)]) > 0$, it follows that $t_i(\omega)(E \cap [t_i(\omega)]) = \frac{P^S(E \cap [t_i(\omega)])}{P^S(S \cap [t_i(\omega)])}$. \square

4.2 Proof of the Theorem

Suppose by contradiction that there are $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and a continuous random variable $v : \Omega \rightarrow \mathbb{R}$ such that $CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \neq \emptyset$, where

$$\begin{aligned} E_1^{\leq x_1} &:= \left\{ \omega \in \Omega : \int_{S_{t_1(\omega)}} v(\cdot) d(t_1(\omega))(\cdot) \leq x_1 \right\}, \text{ and} \\ E_2^{\geq x_2} &:= \left\{ \omega \in \Omega : \int_{S_{t_2(\omega)}} v(\cdot) d(t_2(\omega))(\cdot) \geq x_2 \right\}. \end{aligned}$$

Let S be a \preceq -minimal state-space with the property that $S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \neq \emptyset$.

By standard properties of beliefs, we have $CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \subseteq B_i^1(CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))$ for $i = 1, 2$. This implies that for each $\omega \in S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ and $i = 1, 2$, we have $t_i(\omega)(CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) = 1$, which by the minimality of S implies that $t_i(\omega) \in \Delta(S)$ and $t_i(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) = 1$.

By Lemma 2, $S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ is closed in S . Therefore it is easy to verify that if flattened, $F(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))$, that is $S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ with the induced structure, is a standard topological type-space (as in Heifetz, 2006), since for each $\omega \in S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$, we have $t_i(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) = 1$ for $i = 1, 2$.

Since P^S is a positive prior on S , we have that $P^S(S \cap [t_i(\omega)]) > 0$, for each $\omega \in S$.

For $\omega \in S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ we also have $t_i(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap [t_i(\omega)]) = 1$, and by Lemma 4, we have $t_i(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap [t_i(\omega)]) = \frac{P^S(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap [t_i(\omega)])}{P^S(S \cap [t_i(\omega)])}$.

Hence, since $P^S(S \cap [t_i(\omega)]) > 0$, it follows that $P^S(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap [t_i(\omega)]) = P^S(S \cap [t_i(\omega)]) > 0$. It follows that $P^S(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) > 0$. Therefore it is easy to check that $\frac{P^S(\cdot)}{P^S(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))}$ is a common prior on $F(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))$.

Claim: Let $\omega \in CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap S$. Then $\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_1(\omega))(\cdot) \leq x_1$ and $\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_2(\omega))(\cdot) \geq x_2$.

We prove the second inequality, the first is analogous to the second one. We know already that $t_2(\omega) \in \Delta(S)$. By the definitions $\omega \in S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ implies $\omega \in S \cap B_2^1(E_2^{\geq x_2})$, and therefore $t_2(\omega)([t_2(\omega)] \cap E_2^{\geq x_2} \cap S) = 1$. It follows that $[t_2(\omega)] \cap E_2^{\geq x_2} \cap S$ is non-empty. Let $\omega' \in [t_2(\omega)] \cap E_2^{\geq x_2} \cap S$. Then we have $\int_S v(\cdot) d(t_2(\omega'))(\cdot) \geq x_2$. But we have $t_2(\omega) = t_2(\omega')$ and therefore $\int_S v(\cdot) d(t_2(\omega))(\cdot) \geq x_2$.

Since S is compact and $v : S \rightarrow \mathbb{R}$ is continuous, there is a $\bar{v} \in \mathbb{R}$ such that $|v(\tilde{\omega})| \leq \bar{v}$ for all $\tilde{\omega} \in S$.

Since $t_2(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) = 1$, we have

$$\begin{aligned} \left| \int_{S \setminus (S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))} v(\cdot) d(t_2(\omega))(\cdot) \right| &\leq \bar{v} \int_{S \setminus (S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))} 1 d(t_2(\omega))(\cdot) \\ &= \bar{v} \quad t_2(\omega)(S \setminus (S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))) \\ &= 0. \end{aligned}$$

Hence, we have

$$\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_2(\omega))(\cdot) = \int_S v(\cdot) d(t_2(\omega))(\cdot) \geq x_2$$

and this finishes the proof of the claim.

It follows that we have found a standard topological type-space $S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ in the sense of Heifetz (2006) with a common prior and a continuous random variable $v : S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \rightarrow \mathbb{R}$ such that

$$\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_1(\omega))(\cdot) \leq x_1 < x_2 \leq \int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_2(\omega))(\cdot).$$

Note that if we replace $v(\cdot)$ by $v(\cdot) - \frac{x_1 + x_2}{2}$, we get

$$\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) - \frac{x_1 + x_2}{2} d(t_1(\omega))(\cdot) < 0 < \int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) - \frac{x_1 + x_2}{2} d(t_2(\omega))(\cdot).$$

But this is a contradiction to Feinberg's (2000) Theorem (Proposition 1 in Heifetz, 2006). Hence this completes the proof of the theorem. \square

References

- [1] Billingsley, P. (1968). *Convergence of probability measures*, John Wiley & Sons.

- [2] Board, O. and K.S. Chung (2011). Object-based unawareness: Theory and applications, mimeo.
- [3] Feinberg, Y. (2000). Characterizing common priors in form of posteriors, *Journal of Economic Theory* **91**, 127-179.
- [4] Grant, S. and J. Quiggin (2013). Inductive reasoning about unawareness, *Economic Theory*, forthcoming.
- [5] Heifetz, A. (2006). The positive foundation of the common prior assumption, *Games and Economic Behavior* **56**, 105-120.
- [6] Heifetz, A., Meier, M., and B.C. Schipper (2013). Unawareness, beliefs, and speculative trade, *Games and Economic Behavior* **77**, 100–121.
- [7] Heifetz, A., Meier, M., and B.C. Schipper (2009). Unawareness, beliefs, and speculative trade, mimeo.
- [8] Heifetz, A., Meier, M., and B.C. Schipper (2008). A canonical model for interactive unawareness, *Games and Economic Behavior* **62**, 304-324.
- [9] Heifetz, A., Meier, M., and B.C. Schipper (2006). Interactive unawareness, *Journal of Economic Theory* **130**, 78-94.
- [10] Milgrom, P. and N. Stokey (1982). Information, trade and common knowledge, *Journal of Economic Theory* **26**, 17-27.
- [11] Monderer, D. and D. Samet (1989). Approximating common knowledge with common beliefs, *Games and Economic Behavior* **1**, 170-190.