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# Monetary Policy Shocks in an Economy with Segmented Markets

#### Filippo Occhino\*

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#### Abstract

After a contractionary monetary policy shock, aggregate output decreases over time with a trough after a year and a half, while the real interest rate increases immediately, and remains high for about three quarters. A central step in the explanation is obtaining a persistent increase in the real interest rate, holding aggregate output constant. I study an endowment economy with segmented markets, where, as in the U.S. economy, monetary policy is set in terms of a short-term nominal interest rate, and I show that the real interest rate increases sizeably for up to one year. The shock has a liquidity effect, moving money and interest rates in opposite directions. The endogenous processes for the money growth rate and the real interest rate are strongly serially correlated and close to their empirical counterparts. The more segmented are markets, the stronger and more persistent are the effects of monetary policy shocks, and the higher is the serial correlation of the processes for the money growth rate and the real interest rate. Economies where the intertemporal elasticity of substitution is low exhibit the same qualitative behavior as economies where the market segmentation is high.

*Keywords:* segmented markets, limited participation, monetary policy shocks, real interest rate, persistence, liquidity effect.

JEL Classification Number: E52.

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## 1 Introduction

The VAR literature documents the short-run effects of unanticipated changes in the stance of monetary policy: After a contractionary monetary policy shock, aggregate output decreases over time with a trough after a year and a half, the nominal interest rate increases immediately and remains high for about three quarters, and the price level either does not respond or declines over time<sup>1</sup>. The Fisher equation, stating that the real interest rate is approximately equal to the nominal interest rate minus the expected inflation rate, implies that the real interest rate also increases immediately, and remains high for about three quarters. Both the delayed decrease in aggregate output and the immediate increase in the real interest rate are central features that monetary models must be able to account for. The behavior of aggregate output is, of course, of primary interest in itself, while the behavior of the real interest rate is crucial in most explanations of the transmission mechanism of monetary policy, where any component of the aggregate demand decreases following a persistent increase in the real interest rate<sup>2</sup>.

Standard representative agent models where current consumption only affects current period utility cannot account for the previous evidence. The Euler equations of the representative agent imply a positive relationship between the real interest rate and the growth rate of aggregate consumption. Hence, whenever aggregate output and consumption decrease over time, the real interest rate must be low, not high. Alternatively, whenever the real interest rate remains high for several quarters, aggregate output and consumption must increase, not decrease, over time, so the trough in their response to a contractionary monetary policy shock must occur immediately, not after a year and a half. For instance, the sticky prices framework models the fact that firms adjust prices only infrequently, and can replicate the smooth behavior of the price level. By modelling the nominal interest rate as the exogenous instrument of monetary policy, it can successfully replicate the effects of a monetary policy shock on the real interest rate, but it cannot replicate its persistent and delayed effect on aggregate output<sup>3</sup>.

A central step in the explanation must be, then, abandoning the representative agent framework and explaining why, holding aggregate output constant, the real interest rate increases immediately after a contractionary monetary policy shock, and remains high for several quarters. The limited participation framework, originating from the works of Grossman and Weiss (1983) and Rotemberg (1984), successfully explains the immediate increase in the real interest rate by modelling the fact that households trade bonds only infrequently, so only a subset of households trades bonds at each moment in time. When a contractionary monetary policy shock hits the economy, it can only be absorbed by the subset of

<sup>&</sup>lt;sup>1</sup>The evidence is robust with respect to different ways of identifying a monetary policy shock—Gordon and Leeper (1994), Strongin (1995), Leeper, Sims and Zha (1996), Bernanke and Mihov (1998) and Christiano, Eichenbaum and Evans (1999) document these stylized facts using different identification schemes.

<sup>&</sup>lt;sup>2</sup>Bernanke and Gertler (1995) survey the literature on the credit channel explanation, which does not work through changes in the real interest rate.

<sup>&</sup>lt;sup>3</sup>Clarida, Gali' and Gertler (1999) survey the literature on the sticky prices framework as a model of monetary economies. Chari, Kehoe and McGrattan (1996) point out the persistence problem, and show that the response of aggregate output to a monetary policy shock is neither persistent nor hump-shaped.

households trading bonds at that moment in time, call them the *participants* in the bond market. Then, the percentage decrease in the participants' money demand must be larger than the percentage decrease in aggregate money supply. If money supply and prices are proportional, the participants' real money demand must decrease. The equilibrium real interest rate, then, must increase to discourage the participants' consumption and real money demand<sup>4</sup>. In limited participation models, however, all households invest in bonds and eventually trade bonds, so the real effects are bound to be short-lived. For instance, in the benchmark limited participation economy of Lucas (1990), monetary policy shocks have real effects only in the impact period, even in the case that the monetary policy process is serially correlated. Christiano and Eichenbaum (1992), Alvarez and Atkeson (1997) and Alvarez, Atkeson and Kehoe (1999) propose economies where the real effects last longer, but how to obtain persistent real effects remains the open and central issue.

Data about bond market participation suggest the use of a different framework. The 1995 Survey of Consumer Finances shows that part of the households does not invest in bonds at all<sup>5</sup>. The percentage of households investing in certificates of deposits is 14.1; in savings bonds 22.9; in bonds 3; in mutual funds 12; in life insurance 31.4<sup>6</sup>. The fact that the percentages are stable over time leads to conjecture that, as an approximation and in the short-run, the *same* households do not invest in bonds. This is what should be expected in the presence of fixed costs of investing in bonds, as Mulligan and Sala-i-Martin (2000) show. In the presence of such costs, the choice of investing depends on the wealth to be invested and the nominal interest rate. If the nominal interest rate is stable, only rich households choose to invest. And the 1995 Survey of Consumer Finances shows that, indeed, the percentage of households investing in each category of assets sharply increases with income, which, in turn, is positively correlated with wealth.

The previous evidence motivates the study of monetary economies where part of the households never invests in bonds<sup>7</sup>. In such economies with *segmented markets*, as Lucas, Alvarez and Weber (2001) label them, monetary policy affects the distribution of money across households and the real allocation even when perfectly anticipated<sup>8</sup>, and the real effects of a monetary policy shock last for several periods. With regard to the effects on the real interest rate, the money growth rate and the investors' consumption co-vary for the same reason why, in limited participation models, the money growth rate and the participants' consumption co-vary in the impact period of a monetary policy shock. When the monetary policy process is positively serially correlated, after a contractionary shock, the money growth rate remains low for several periods. The investors' consumption, then, remains low for several periods, and increases over time to return to its stationary level. The investors' Euler equations imply, then, that the real interest rate increases immediately and remains

<sup>&</sup>lt;sup>4</sup>Grossman and Weiss (1983) and Lucas (1990) focus on the effects of monetary policy shocks on the real interest rate in limited participation endowment economies.

<sup>&</sup>lt;sup>5</sup>See Table 5 B of Kennickell, Starr-McCluer and Sunden (1995).

<sup>&</sup>lt;sup>6</sup>Mulligan and Sala-i-Martin (2000) and Mankiw (2000) cite related evidence.

<sup>&</sup>lt;sup>7</sup>Alvarez and Atkeson (1996), myself in Occhino (2000) and Lucas, Alvarez and Weber (2001) adopt this framework for the study of monetary policy, while Mankiw (2000) uses it to study fiscal policy issues.

 $<sup>^{8}</sup>$ Cochrane (1998) points out that anticipated changes in the monetary policy variable seem to have real effects.

high for several periods.

In this paper, I study an endowment economy with segmented markets, where monetary policy is a first-order Markov process for the nominal interest rate<sup>9</sup>. Modelling monetary policy in terms of a short-term nominal interest rate is of primary importance for two reasons. First of all, this assumption best models the operating procedure of most monetary authorities. In the U.S. case, in particular, the Federal Reserve announces a target for the federal funds rate, that is the overnight rate that private banks charge each other for non-borrowed reserves, and then tries to reach that target affecting the aggregate supply of non-borrowed reserves through open market operations, i.e. through the purchase and sale of government securities. Also, Bernanke and Blinder (1992) show that the federal funds rate is an excellent indicator of the stance of monetary policy, so innovations to the federal funds rate can be identified with monetary policy shocks.

As documented in the following sections, the modelling strategy that I adopted is successful along several important dimensions. A contractionary monetary policy shock has a sizeable and persistent effect on the real interest rate: When the serial correlation of the nominal interest rate is .9 as in the U.S. economy, after a 10 basis points unanticipated increase in the nominal interest rate, the real interest rate increases by 20 basis points, and remains high for up to one year. The shock has a liquidity effect, moving money and interest rates in opposite directions. The endogenous processes for the money growth rate and the real interest rate are strongly serially correlated and close to their empirical counterparts. The money growth rate, in particular, is very close to a first-order autoregressive process with a serial correlation of .5<sup>10</sup>. Finally, the more segmented are markets, the stronger and more persistent is the effect of monetary policy shocks, and the higher is the serial correlation of the processes for the money growth rate and the real interest rate.

The rest of the paper is organized as follows. In section 2, I describe the economy and define the equilibrium. The equilibrium behavior of the economy and the effects of monetary policy shocks are characterized analytically in section 3, and numerically in section 4. Section 5 studies economies where the intertemporal elasticity of substitution is low, and shows that they exhibit the same qualitative behavior as economies where the market segmentation is high. Section 6 concludes.

### 2 The economy

Let us consider an endowment economy populated with two types of households, I and N. There are  $\omega > 0$  households of type I, and  $1 - \omega > 0$  households of type N, and households of the same type are identical. There is a single non-storable good, and the preferences of households of type i over their present and future stochastic consumption  $\{c_{i,t}\}_{t=0}^{\infty}, c_{i,t} > 0$ 

<sup>&</sup>lt;sup>9</sup>Alvarez and Atkeson (1996) and Lucas, Alvarez and Weber (2001) consider the case where monetary policy is set in terms of the money growth rate. In the second chapter of Occhino (2000), I model monetary policy as a first-order Markov process for the ratio of the bond supply to aggregate nominal assets.

<sup>&</sup>lt;sup>10</sup>Christiano, Eichenbaum and Evans (1998) document that the M2 growth rate is well approximated by an AR(1) process with serial correlation of .5.

all t, are

$$E\left\{\sum_{t=0}^{\infty}\beta^t\log(c_{i,t})\right\},\$$

where  $\beta \in (0, 1)$ , and the expectation is conditional on their information in period zero. The more general case of constant elasticity of substitution preferences is considered in section 5.

In each period, a goods market session opens where goods are traded for money, which is a dollar-denominated, durable asset. Households are subject to a cash-in-advance constraint— Households of type *i* can only consume goods purchased with the cash balances  $M_{i,t} > 0$  available at the beginning of the goods market session:

$$p_t c_{i,t} \leq M_{i,t},$$

where  $p_t > 0$  is the goods price. Let  $M_t \equiv \omega M_{I,t} + (1 - \omega) M_{N,t}$  be the money supply in period t.

Before the goods market session, households of type *i* receive a constant endowment  $y_i > 0$ . It is convenient to define the aggregate endowment  $y \equiv \omega y_I + (1 - \omega)y_N$ , and the investors' share of aggregate endowment  $\lambda \equiv \omega y_I/y$ . We will notice that, as in the economies of Alvarez and Atkeson (1996) and Lucas, Alvarez and Weber (2001), the aggregate behavior of the economy only depends on the investors' share of aggregate endowment  $\lambda$ , and neither the percentage of investors  $\omega$  nor the ratio of their individual endowment  $y_I$  to the aggregate endowment y play independent roles. The endowment cannot be consumed and must be sold in the goods market. The goods market equilibrium condition is then

$$\omega c_{I,t} + (1-\omega)c_{N,t} = y.$$

The receipts from the sale are only available at the end of the session, and can only be used to buy goods in the following periods. At the end of the session, the cash balances of households of type *i* are the sum of their initial cash balances  $M_{i,t}$ , minus their consumption expenditure  $p_t c_{i,t}$ , plus the receipts  $p_t y_i$  from the sale of their endowment.

In each period, before the goods market session, a bond market session opens where one-period nominal bonds and money are traded. A bond is a claim to one dollar at the end of the period. To model the fact that only part of the households invests in bonds, I assume that only households of type I, the *investors*, access the bond market. Investors cannot issue bonds, and they can purchase  $B_{I,t} \ge 0$  bonds with the cash balances  $A_{I,t} > 0$  available at the beginning of the period, so

$$q_t B_{I,t} \le A_{I,t},$$

where  $0 < q_t < 1$  is the bond price. Since the goods market session follows the bond market session, the investors' cash balances  $M_{I,t}$  at the beginning of the goods market session are equal to  $A_{I,t} - q_t B_{I,t}$ . The non-investors' cash balances  $M_{N,t}$  at the beginning of the goods market session are, of course, the same as their cash balances  $A_{N,t} > 0$  at the beginning of the period.

A monetary authority also enters the bond market, issues  $B_t > 0$  bonds, and redeems them at the end of the period. The bond market equilibrium condition is then

$$\omega B_{I,t} = B_t$$

Let  $D_t \equiv q_t B_t$  be the value of the bond supply in period t.

The only source of uncertainty in this economy is the behavior of the monetary authority. I assume that the monetary authority announces the bond price  $q_t$  at the beginning of the period, and stands ready to issue any number of bonds to clear the market at that price. I model monetary policy as an exogenous stochastic process for the bond price, and let the bond supply and the money supply be determined endogenously. Specifically, I assume that the bond price  $q_t$  follows a first-order Markov process with transition function

$$P(q, A) = Prob\{q_{t+1} \in A | q_t = q\}, q \in \mathbf{Q}, A \in \mathcal{B}(\mathbf{Q}),$$

where  $\mathbf{Q} \subset (0, 1)$  is a Borel set, and  $\mathcal{B}(\mathbf{Q})$  denotes the Borel subsets of  $\mathbf{Q}$ .

At the end of the period, all bonds are redeemed and the investors receive  $B_{I,t}$  dollars. Their cash balances  $A_{I,t+1}$  at the beginning of the following period are then

$$A_{I,t+1} = A_{I,t} - q_t B_{I,t} - p_t c_{I,t} + p_t y_I + B_{I,t}.$$

The non-investors cash balances are instead

$$A_{N,t+1} = A_{N,t} - p_t c_{I,t} + p_t y_N.$$

As in Lucas and Stokey (1987) and Lucas (1990), it is convenient to normalize all nominal variables dated period t dividing them by the aggregate cash balances  $A_t \equiv \omega A_{I,t} + (1 - \omega)A_{N,t} > 0$  at the beginning of the period:

$$b_{I,t} \equiv \frac{B_{I,t}}{A_t} , a_{I,t} \equiv \frac{A_{I,t}}{A_t} , a_{N,t} \equiv \frac{A_{N,t}}{A_t} , b_t \equiv \frac{B_t}{A_t} , d_t \equiv \frac{D_t}{A_t} , m_t \equiv \frac{M_t}{A_t} , p_t \equiv \frac{P_t}{A_t}$$

Notice that  $\omega a_{I,t} + (1 - \omega)a_{N,t} \equiv 1$ , all t, by definition. Also,

$$A_{t+1} \equiv \omega A_{I,t+1} + (1-\omega)A_{N,t+1}$$
  
=  $\omega [A_{I,t} - q_t B_{I,t} - p_t c_{I,t} + p_t y_I + B_{I,t}] + (1-\omega)[A_{N,t} - p_t c_{I,t} + p_t y_N]$   
=  $A_t - D_t + D_t/q_t$   
=  $[1 + d_t/q_t - d_t]A_t$ ,

where the second equality follows from the bond market and goods market equilibrium conditions, and from the definition of  $A_t$ . Notice that other ways of normalizing nominal variables are possible and might be useful. For instance, one can divide all nominal variables by the investors' cash balances  $\omega A_{I,t}$ . The equilibria do not depend, of course, on the choice of normalization. However, that choice can help in finding and characterizing the equilibria. In particular, all the following existence and characterization results depend on the choice of normalization adopted.

I restrict attention to stationary equilibria, where all aggregate real variables and all aggregate normalized nominal variables depend in a time-invariant way on only two state variables. The first state variable is the bond price  $q_t$ , whose evolution is entirely exogenous. The second state variable is an indicator of the distribution of cash balances between the two

types of households at the beginning of each period, and the evolution of this second state variable must be endogenously determined. As an indicator, I choose the investors' share of cash balances  $\theta_t \equiv \omega A_{I,t}/A_t$ . As for the choice of normalization, the equilibria do not depend on the choice of the indicator, but that choice can help in finding and characterizing the equilibria.

It is then convenient to study the following recursive (stationary) competitive equilibrium. Let  $s \equiv (q, \theta)$  be the aggregate state of the economy. Let  $\Theta \subset (0, 1)$  be an interval where  $\theta$  takes values in equilibrium, and let  $\mathbf{S} \equiv \mathbf{Q} \times \Theta$  be the set of possible values for the aggregate state s. A recursive stationary equilibrium is a set of: an interval  $\Theta \subset (0, 1)$ ; value functions  $v_i : \mathbf{R}_{++} \times \mathbf{S} \to \mathbf{R}, \ i = I, N$  for both types of households; associated policy functions  $\hat{b}_I : \mathbf{R}_{++} \times \mathbf{S} \to \mathbf{R}_+, \ \hat{c}_i : \mathbf{R}_{++} \times \mathbf{S} \to \mathbf{R}_{++}, \ \hat{a}'_i : \mathbf{R}_{++} \times \mathbf{S} \to \mathbf{R}_{++}, \ i = I, N$ ; debt function  $d : \mathbf{S} \to (0, 1)$ ; price function  $p : \mathbf{S} \to \mathbf{R}_{++}$ ; and law of motion  $\theta' : \mathbf{S} \to \Theta$  for the investors' share of nominal assets; such that the following conditions hold:

1. The investors' value function and associated policy functions solve the following Bellman equation:

$$v_I(a_I, s) = \max_{\{b_I, c_I, a_I'\}} \left\{ \log(c_I) + \beta \int_{\mathbf{Q}} v_I(a_I', q', \theta'(s)) P(q, dq') \right\}$$
  
subject to  $b_I \ge 0, \ qb_I \le a_I,$   
 $c_I > 0, \ p(s)c_I \le a_I - qb_I$ 

and  $a'_{I}[1 - d(s) + d(s)/q] \equiv b_{I} + [a_{I} - qb_{I} - p(s)c_{I}] + p(s)y_{I} > 0.$ 

The non-investors' value function and associated policy functions solve the following Bellman equation:

$$v_N(a_N, s) = \max_{\{c_N, a'_N\}} \left\{ \log(c_N) + \beta \int_{\mathbf{Q}} v_N(a'_N, q', \theta'(s)) P(q, dq') \right\}$$
  
subject to  $c_N > 0, \ p(s)c_N \le a_N$   
and  $a'_N[1 - d(s) + d(s)/q] \equiv [a_N - p(s)c_N] + p(s)y_N > 0.$ 

2. When  $a_I = \theta/\omega$  and  $a_N = (1-\theta)/(1-\omega)$ , both the bond market and the goods market are in equilibrium:

$$\omega b_I(a_I, s) = d(s)/q$$
  
and  $\omega \hat{c}_I(a_I, s) + (1 - \omega)\hat{c}_N(a_N, s) = y.$ 

Also, the law of motion  $\theta'(s)$  of the investors' share of assets  $\theta$  is compatible with individual optimization:

$$\omega \hat{a}_I'(a_I, s) = \theta'(s).$$

Notice that, when looking for an equilibrium, one must pay particular attention to the choice of the interval  $\Theta$ . As we will see, a narrow interval is crucial to obtain an existence result; but the interval cannot be too narrow, otherwise there are some  $\theta \in \Theta$  such that the next period investors' share of cash balances  $\theta'(q, \theta)$  lies outside  $\Theta$  itself.

Using the constraints for the two types of households and the equilibrium conditions of the two markets, one can easily show that, in equilibrium,

$$\omega \hat{a}'_{I}(a_{I},s) + (1-\omega)\hat{a}'_{N}(a_{N},s) = 1.$$

Also, once an equilibrium is found, one can derive the equilibrium bond supply  $b(s) \equiv d(s)/q$ , and the equilibrium money supply  $m(s) \equiv 1 - d(s)$ . Notice that the money supply, that is the cash balances available both at the beginning and at the end of the goods market session, is only a fraction 1 - d(s) of the aggregate cash balances at the beginning of the period—Part of the cash balances available at the beginning of the period are delivered to the monetary authority in exchange of nominal bonds which are only redeemed after the goods market session.

#### 3 Equilibrium behavior of the economy

Since 0 < q < 1 for all  $s \in \mathbf{S}$ , the investors invest in bonds all the money that they are not going to need in the goods market, so their cash-in-advance constraint always binds in equilibrium:

$$p(s)\hat{c}_I(a_I,s) = a_I - q\hat{b}_I(a_I,s),$$

at  $a_I = \theta/\omega$ . I focus on equilibria where the cash-in-advance constraint for non-investors is also binding. Theorem A.1 in the appendix shows that, in any equilibrium where the cash-in-advance constraint always binds also for non-investors, the goods price p(s) satisfies the relation

$$p(s)y = 1 - d(s),$$
 (1)

the law of motion  $\theta'(s)$  satisfies the relation

$$\theta'(s) = \lambda + (1 - \lambda) \frac{d(s)}{q - qd(s) + d(s)},\tag{2}$$

and the bond value d(s) solves the following functional equation, which is an equilibrium version of the investors' Euler equation:

$$\frac{d(s)}{\theta - d(s)} \equiv \int_{\mathbf{Q}} \frac{\beta}{1 - \lambda} \frac{\theta'(s) - \lambda}{\theta'(s) - d(q', \theta'(s))} P(q, dq'), \tag{3}$$

all  $s \in \mathbf{S}$ , where  $s \equiv (q, \theta)$ , and the law of motion  $\theta'(s)$  is given by 2.

The following theorem A.2 in the appendix shows that, if the functions p(s),  $\theta'(s)$  and d(s) are given by the equations 1, 2 and 3, and, in addition, p(s) satisfies the inequality

$$(1-\omega)p(s)y_N \ge \beta(1-\theta),\tag{4}$$

all  $s \in \mathbf{S}$ , where  $s \equiv (q, \theta)$ , then a binding cash-in-advance constraint is also always optimal for non-investors, and markets are in equilibrium. Notice that  $(1 - \omega)p(s)y_N$  are the noninvestors' receipts from the sale of their endowment, and that  $1-\theta$  are the non-investors' cash balances at the beginning of the period, that is, with binding cash-in-advance constraints, the non-investors' receipts from the sale of their endowment in the previous period. Hence, the inequality requires that the gross inflation rate, i.e. the inverse of the gross rate of return of investing in money, be greater than the non-investors' preferences discount factor for all  $s \in \mathbf{S}$ .

The previous theorem allows to reduce the problem of determining the equilibrium behavior of the economy to the problem of finding a space  $\Theta$  where  $\theta$  takes values in equilibrium, and a solution d(s) to the functional equation 3, such that the function  $\theta'(s)$  given by the relation 2 takes values in  $\Theta$  and the function p(s) given by the relation 1 satisfies the inequality 4, for all  $s \in \mathbf{S}$ . To this end, we need to make some assumptions constraining the process for q. First, let us define the function  $d^*(q), d^*: (0, \beta) \to (0, \beta)$ , as follows

$$d^{\star}(q) \equiv \frac{\beta - q}{1 - q}.$$

 $d^{\star}(q)$  is equal to  $\beta$  when q is equal to 0, is continuous and strictly decreasing in q, and is equal to 0 when q is equal to  $\beta$ .

 $d^{\star}(q)$  has the following economic interpretation. Consider for a moment a deterministic stationary economy where the bond price is constant and equal to q, the value of the bond issue size is constant and equal to d, all the real variables are constant, and all the nominal variables grow at the growth rate 1 - d + d/q of the aggregate cash balances. Since the investors' consumption is constant over time, from the investors' Euler equation it follows that  $q(1 - d + d/q) = \beta$ , which is simply the Fisher equation stating that the gross real interest rate is equal to the ratio of the gross nominal interest rate to the gross inflation rate. The equation can be written as  $q - qd + d = \beta$ , or  $d = (\beta - q)/(1 - q)$ , so  $d = d^{\star}(q)$ . Hence,  $d^{\star}(q)$  is the constant value of the bond issue in a deterministic stationary economy where the bond price is constant and equal to q, all the real variables are constant, and all the nominal variables grow at the growth rate of the aggregate cash balances.

Let  $\underline{q}$  and  $\overline{q}$  be respectively the smallest and greatest values of q, and let us define  $\overline{d} \equiv d^*(\underline{q})$ , and  $\underline{d} \equiv d^*(\overline{q})$ . Let  $\mathcal{M}$  be the metric space of measurable functions on  $\mathbf{S}$  taking values in  $[\underline{d}, \overline{d}]$ , with the sup norm. Theorem A.3 in the appendix proves that, if d(s) belongs to  $\mathcal{M}$ , then  $\theta'(s)$  takes values in the interval  $[\underline{\theta}, \overline{\theta}]$ , where  $\underline{\theta} \equiv \lambda + (1 - \lambda)\underline{d}/(\overline{q} - \overline{q}\underline{d} + \underline{d})$ , and  $\overline{\theta} \equiv \lambda + (1 - \lambda)\overline{d}/(\underline{q} - \overline{q}\underline{d} + \underline{d})$ . Then, if we let the interval  $\mathbf{\Theta}$  where  $\theta$  takes values in equilibrium be the interval  $[\underline{\theta}, \overline{\theta}]$ , we only need to determine a solution  $d \in \mathcal{M}$  to the functional equation 3, such that the function p(s) given by the relation 1 satisfies the inequality 4, for all  $s \in \mathbf{S}$ .

To determine such a solution, let us make the following assumption constraining the minimum value q of the process for q:

#### Assumption 3.1 $\overline{d} \leq \lambda$ .

The assumption constrains the ratio of the bond value to the sum of the bond value and the money supply to be less than the investors' share of the aggregate endowment. Empirically, the ratio of the government debt to the sum of the government debt and M2 is around .6. As I will argue in the next section, a value for the investors' share of aggregate endowment larger than 60% seems reasonable.

Let us define the operator T as follows:

$$(Td)(s) \equiv \frac{\theta R(s)}{1 + R(s)},\tag{5}$$

where R(s) is the right hand side of the functional equation 3. Theorem A.4 in the appendix proves that, under assumption 3.1,  $T : \mathcal{M} \to \mathcal{M}$ . Since a fixed point of T is a solution to the functional equation 3, we look for a fixed point of T.

It is convenient to restrict our search for a fixed point to the subset  $D \subset \mathcal{M}$  of the functions d(s) such that  $d(q,\theta)/\theta$  is weakly increasing in  $\theta$ : as the investors' share of cash balances increases, their nominal investment in bonds increases relative to their consumption expenditure. Notice that, if  $d \in D$ ,  $d(q,\theta)$  is strictly increasing in  $\theta$ : as the investors' share of cash balances increases, their nominal investment in bonds increases in absolute value. Then, theorem A.5 in the appendix proves that  $\theta'(q,\theta)$  is strictly increasing in  $\theta$ , theorem A.6 proves that  $T: D \to D$ , and theorem A.7 proves that T is monotone.

The following is the way I use the monotonicity of T to determine numerically a fixed point. First, let us construct a sequence of functions in D as follows:

$$d^0 \equiv \underline{d}, \text{ and } d^n \equiv T^n d^0, \text{ all } n \ge 1.$$
 (6)

Notice that  $T: D \to D$  implies that, if  $d^n \in D$ , then  $d^{n+1} = Td^n \in D$ , all  $n \ge 0$ . Since  $d^0 \in D$ , by induction,  $d^n \in D$ , all  $n \ge 0$ .

Then, notice that  $d^0 \equiv \underline{d}$  and  $d^1 \in D$  imply  $d^0 \leq d^1$ . Also, the monotonicity of T implies that, if  $d^n \leq d^{n+1}$ , then  $d^{n+1} = Td^n \leq Td^{n+1} = d^{n+2}$ , all  $n \geq 0$ . Hence, by induction, the sequence  $\{d^n\}_{n=0}^{\infty}$  is weakly increasing. Since  $d^n \leq \overline{d}$ , all  $n \geq 0$ , the sequence  $\{d^n\}_{n=0}^{\infty}$ converges pointwise to a function, call it  $d^{\infty}$ . Since  $d^n \in D$ , all  $n \geq 0$ , and since  $d^{\infty}$  is the pointwise limit of the sequence  $\{d^n\}_{n=0}^{\infty}$ ,  $d^{\infty}$  also belongs to D.

To compute a numerical approximation to  $d^{\infty}$ , I discretize the state space **S** with a grid of a finite number of states, I apply the operator T to the constant function  $\underline{d}$ , and I iterate until convergence is reached. Although  $d^{\infty}$  is not necessarily a fixed point of T, its numerical approximation turns out to be a fixed point of T and, therefore, a solution to the functional equation 3. Moreover, I obtain the same fixed point applying the operator T to the constant function  $\overline{d}$  and iterating until convergence is reached, Hence, abstracting from the fact that we are dealing with a numerical approximation of  $d^{\infty}$  and not with  $d^{\infty}$  itself,  $d^{\infty}$  is the only fixed point in D. The argument is the same as in the proof of the Corollary of Theorem 17.7 of Stokey and Lucas with Prescott (1989). Suppose that  $d \in D$  is a fixed point of T. Notice that  $d^0 \equiv \underline{d}$  and  $d \in D$  imply  $d^0 \leq d$ . Also, the monotonicity of T implies that, if  $d^n \leq d$ , then  $d^{n+1} = Td^n \leq Td = d$ , all  $n \geq 0$ . Hence, by induction,  $d^n \leq d$ , all  $n \geq 0$ . Since  $d^{\infty}$ is the pointwise limit of the sequence  $\{d^n\}_{n=0}^{\infty}, d^{\infty} \leq d$ . A similar argument starting with  $d^0 \equiv \overline{d}$  leads to  $d^{\infty} \geq d$ . Hence, if  $d \in D$  is a fixed point of T, then  $d = d^{\infty}$ .

Once a solution  $d \in D$  has been obtained numerically, p(s) and  $\theta'(s)$  are given by the relations 1 and 2. As explained earlier, these functions describe the equilibrium behavior

of the economy only if p(s) satisfies the additional inequality 4. I then check this condition and proceed only if it is satisfied. In this case, several aggregate variables can be determined as follows. The growth rate of the aggregate cash balances is 1 - d(s) + d(s)/q, and the ratio of the money supply to the aggregate cash balances is 1 - d(s). Since I am focusing on equilibria with binding cash-in-advance constraints, the money growth rate and the inflation rate are the same and can be determined as follows. Since non-investors spend in the current period all the receipts from the sale of their endowment in the previous period, the ratio of their real consumption to their endowment is equal to the inverse of the gross inflation rate. Since their real consumption is equal to the ratio of their initial money holdings to the goods price, the inverse of the gross inflation rate is equal to

$$\frac{1-\theta}{1-\omega}\frac{1}{p(s)}\frac{1}{y_N} = \frac{y}{(1-\omega)y_N}\frac{1-\theta}{p(s)y} = \frac{1}{(1-\lambda)}\frac{1-\theta}{1-d(s)}$$

Finally, the expected inflation rate  $\pi_e(s)$  is given by

$$1 + \pi_e(s) = \int_{\mathbf{Q}} \frac{[1 - d(s) + d(s)/q]p(q', \theta'(s))}{p(s)} P(q, dq'),$$

and the real interest rate r(s) is defined as

$$1 + r(s) \equiv \frac{1}{q} \frac{1}{1 + \pi_e(s)}$$

I prove an existence theorem under the following two additional assumptions constraining the variability of the process for q:

#### Assumption 3.2 $\overline{d}/\underline{d} \leq \beta/\overline{q}$ .

#### Assumption 3.3 $\overline{d} - \underline{d} \leq 1 - \beta$ .

Under the first additional assumption, theorem A.9 proves that  $T : D_{\theta} \to D_{\theta}$ , where  $D_{\theta}$ is the subset of D of the functions d such that  $\theta - d(q, \theta)$  is weakly increasing in  $\theta$ : as the investors' share of cash balances increases, their consumption expenditure increases. Notice that any function  $d(q, \theta)$  belonging to  $D_{\theta}$  has bounded slope (and is, therefore, continuous) with respect to its second argument  $\theta$ . As a corollary, the pointwise limit  $d^{\infty}$  of the sequence  $\{d^n\}_{n=0}^{\infty}$  defined in 6 exists and belongs to  $D_{\theta}$ . The next theorem A.10 proves that  $d^{\infty}$  solves the functional equation 3. Then, theorem A.11 proves that, under the second assumption, the function p(s) defined in 1 satisfies the inequality 4.

The result that the price level is determinate even though the monetary authority follows an interest rate rule does depend on the fact that I am focusing on a specific subset of equilibria. I restricted attention to recursive equilibria, so I am only considering stationary equilibria where the ratio of the price level to aggregate cash balances is a time-invariant function of the aggregate state of the economy. Also, I am focusing on the subset of recursive equilibria where the space  $\Theta$  depends in a specific, reasonable way on the process for q, and where the function d has specific, reasonable monotonicity properties with respect to the aggregate state variable  $\theta$ . However, although there may exist multiple equilibria, there may not exist the kind of price level indeterminacy first pointed out by Sargent and Wallace (1975), that is multiple equilibria characterized by the same real allocation but different price paths. The reason lies in the fact that the fiscal policy is non-ricardian or "passive" in the terminology of Leeper (1991). Taxes and government spending are independent of the price level (they are equal to zero!), and, given the real allocation, there is a unique price level at which the initial real value of government liabilities equals the real discounted present value of future seigniorage revenues, and the government intertemporal budget constraint is satisfied.

Empirically, the serial correlation of the process for the nominal interest rate is around .9. It is then important to characterize the equilibrium behavior of the economy under the following additional assumption constraining the process for q to be positively serially correlated:

# Assumption 3.4 The transition function P is monotone, i.e., for every non decreasing function $f: \mathbf{Q} \to \mathbf{R}$ , the function $\int_{\mathbf{Q}} f(q')P(q, dq')$ is also non decreasing.

Under this additional assumption, we look for a solution belonging to the subset  $D_q$  of Dof the functions  $d(q, \theta)$  weakly decreasing in q: as the bond price increases, the nominal investment in bonds decreases. Theorem A.12 proves that, if d belongs to  $D_q$ ,  $\theta'(q, \theta)$  is strictly decreasing in q, and theorem A.13 proves that  $T: D_q \to D_q$ , Since both constant functions  $\underline{d}$  and  $\overline{d}$  belong to  $D_q$ , the fixed point obtained applying the operator T to these functions and iterating until convergence is reached also belongs to  $D_q$ .

A number of conclusions follow from the fact that  $d(q, \theta)$  is weakly decreasing in q. Most importantly, the money supply is weakly increasing in q, so, given  $\theta$ , the money supply and the interest rate are inversely related. Since prices and money are proportional, the goods price and the interest rate are also inversely related. Consider what happens in the impact period of a monetary policy shock. An expansionary shock is defined as an unanticipated increase in the bond price q, or, equivalently, as an unanticipated decrease in the nominal interest rate. Since the second state variable  $\theta$  is given at the beginning of each period, the behavior of any variable in the impact period of an expansionary shock only depends on its response to changes in q. It increases if and only it is increasing in q. Hence, in the impact period of an expansionary shock, the nominal interest rate decreases by definition, the bond value d(s) decreases, the money supply increases and the inflation rate increases. A monetary policy shock moves, then, money and interest rates in opposite directions — A liquidity effect.

The expected behavior of the economy in the periods following the expansionary shock depends on the expected behavior of the two aggregate state variables. Under assumption 3.4, the process for q is positively serially correlated, so the expected future values of q are high. Hence, a first effect of an expansionary shock on the expected future values of any variable is to increase them if and only if that variable is increasing in q. A second effect works through the effect on the expected future values of the second state variable  $\theta$ . Theorem A.12 in the appendix proves that  $\theta'(q, \theta)$  is strictly decreasing in q, so an expansionary shock decreases the second state variable in the period immediately following the shock. In the following periods, the expected future values of the second state variable decrease for two reasons. First, because  $\theta'(q, \theta)$  is strictly decreasing in q, and the expected future values of q are high. Second, because theorem A.5 in the appendix proves that  $\theta'(q, \theta)$  is strictly increasing in  $\theta$ , and the expected future values of  $\theta$  are low. Hence, a second effect of an expansionary shock on the expected future values of any variable is to increase them if and only if that variable is decreasing in  $\theta$ . If a variable is increasing in one state variable and decreasing in the other, the two effects work in the same direction and we can determine the behavior of the expected future values of that variable.

To prove rigorously the previous intuition, let  $\mathcal{B}(\Theta)$  and  $\mathcal{B}(\mathbf{S})$  respectively denote the Borel subsets of  $\Theta$  and  $\mathbf{S}$ . Let us define the state transition functions  $\Pi^n$ : for n = 1,

$$\Pi((q,\theta), A \times B) \equiv P(q,A)\chi_B(\theta'(q,\theta)),$$

all  $(q, \theta) \in \mathbf{S}$ ,  $A \times B \in \mathcal{B}(\mathbf{S})$ , where  $\chi$  is the indicator function; and for n > 1,

$$\Pi^{n}((q,\theta), A \times B) \equiv \int_{\mathbf{Q}} \Pi^{n-1}((q',\theta'(q,\theta)), A \times B)P(q,dq'),$$

all  $(q, \theta) \in \mathbf{S}, A \times B \in \mathcal{B}(\mathbf{S})$ . The expected future values of prices and quantities are defined in terms of these state transition functions. For instance, the expectation of the n periods ahead future value of the ratio d(s) of the debt value to aggregate cash balances is  $d_{e}^{n}(s) =$  $\int_{\mathbf{S}} d(s') \Pi^n(s, ds')$ . Theorem A.14 in the appendix proves that the expected future values of any function inherit the monotonicity properties with respect to  $\theta$  of the function itself. For instance, the ratio  $d(q, \theta)$  of the debt value to aggregate cash balances is strictly increasing in  $\theta$ , so the expected future ratio  $d_e^n(q,\theta)$  is also strictly increasing in  $\theta$ . The theorem is used to prove the following theorem A.15 establishing that, if a function is increasing (decreasing) in q and decreasing (increasing) in  $\theta$ , then the expected future values of that function are increasing (decreasing) in q. The monotonicity is strict if the function is strictly monotone with respect to  $\theta$ . For instance, the ratio  $d(q, \theta)$  of the debt value to aggregate cash balances is decreasing in q and strictly increasing in  $\theta$ , so the expected future ratio  $d_e^n(q,\theta)$  is strictly decreasing in q. Hence, an expansionary shock decreases both the current value and the expected future values of the ratio of the bond value to the aggregate cash balances. Similarly, one shows that an expansionary shock increases both the current value and the expected future values of the ratio of the money supply and the goods price to the aggregate cash balances. Together with the positive serial correlation of the process for the bond price, this implies again an inverse relation between money and prices on one side and interest rates on the other.

#### 4 Numerical analysis

The effects of monetary policy shocks depend crucially on the economic weight of investors, namely their share  $\lambda$  of aggregate endowment, while neither the percentage of investors, nor the individual endowment of any households play an independent role. But what is the investor's share  $\lambda$  of the aggregate endowment in the U.S. economy is not obvious at all.

A first problem is that, in the model, households can only invest in bonds and money, so it is not clear what is the empirical counterpart of a bond. In my opinion, investment in transactions accounts (checking, savings and money market accounts) should be considered as investment in money. I also would not consider as investment in bonds the investment in assets, like retirement accounts, whose demand is inelastic with respect to the interest rate. The reason is that the crucial difference in the model between investors and non-investors is that only the investors' demand for assets is interest rate elastic, so, if the demand for an asset is not interest rate elastic, investing in that asset should not be considered as investing in bonds. A second problem in estimating the investors' share  $\lambda$  of aggregate endowment is that Tables 3 and 5 B of Kennickell, Starr-McCluer and Sunden (1995) let us only infer what is the investors' share of aggregate *income*, not their share of aggregate *endowment*. Since income is equal to labor earnings (endowment) plus return to investment, the investors' share of aggregate income is greater than their share of aggregate endowment. To clarify with an example, in the case that the investors receive an endowment of 1 and a return to investment of 1, and the non-investors only receive an endowment of 1, the investors' share of aggregate income is 2/3, while their share of aggregate endowment is 1/2.

With the previous considerations in mind, let us now try to infer what is the investor's share  $\lambda$  of the aggregate endowment in the U.S. economy. Table 5 B of Kennickell, Starr-McCluer and Sunden (1995) shows, for each income bracket, the percentage of households investing in each category of assets. For each category of assets, an upper bound for the investors' share of aggregate income is the largest percentage of investors across income brackets. That is 21.1% for certificates of deposits, 39.9% for savings bonds, 14.5% for bonds, 45.2% for stocks, 38% for mutual funds, and 54.1% for life insurance. On one hand, the percentage of households investing in any of these categories of assets is, of course, larger than any of the previous percentages. On the other hand, the previous percentages are upper bounds for the investors' share of aggregate endowment. Values between 50% and 75% seem, then, the most plausible for the investors' share  $\lambda$  of aggregate endowment in the U.S. economy.

To better understand the mechanisms at work in more realistic economies where the monetary policy process is positively serially correlated, let us first consider what happens in economies where the bond price q follows an i.i.d. process. Figure 1 plots the expected evolution of three such economies after a contractionary monetary policy shock in period 0. The dashed line refers to a full participation economy, where the investors' share  $\lambda$  of aggregate endowment is 100%, while the dotted and solid lines refer to two segmented markets economies where  $\lambda$  is respectively 75% and 50%. The bond price q is assumed to take only three values. The intermediate value, which is also the mean value, is set so that the implied nominal interest rate is 6.5%, which is approximately equal to the mean of the federal funds rate over the period 1959(1)–2001(1)<sup>11</sup>. The low and high values are set such that the implied nominal interest rates differ from 6.5% by 10 basis points. q is assumed to follow an i.i.d. process. I consider quarterly periods and set  $\beta$  equal to .99.  $\underline{d}$  and  $\overline{d}$  turn out to be respectively .3650 and .3839, so assumption 3.1 is satisfied. In period 0, I assume

<sup>&</sup>lt;sup>11</sup>All data are from the FRED database of the Federal Reserve Bank of St. Louis at the web address http://www.stls.frb.org/fred/.



Figure 1: Expected evolution after an unanticipated increase in the nominal interest rate in period 0. The nominal interest rate is i.i.d. The solid, dotted and dashed lines refer to three economies where the investors' economic weight, namely their share  $\lambda$  of aggregate endowment, is respectively 50%, 75% and 100%. Utility is logarithmic.

that a contractionary monetary policy shock hits the economy, and the bond price takes the low value. After period 0, the figure plots the evolution of the three economies expected in period 0. The evolutions of the two segmented markets economies depend on the value that the second state variable  $\theta$  takes in period 0, or, equivalently, on the past realizations of the monetary policy process. I then assume that q takes the mean value before period 0 for a sufficiently long period of time until  $\theta$  becomes stationary<sup>12</sup>. The autocorrelation and cross-correlation functions of the processes for the nominal interest rate, the money growth rate and the real interest rate are also helpful in understanding the three economies, so I simulated the three economies for 10000 periods, and plotted the autocorrelation and cross-correlation functions in figure 2.

The top-left panel of figure 1 plots the experiment. Before period 0, the nominal interest rate is constant and equal to 6.5%. In period 0, a contractionary monetary policy shock, namely a 10 basis points unanticipated increase in the nominal interest rate, hits the three economies. After period 0, the expected values of the nominal interest rate are plotted. Since the monetary policy process is i.i.d., the nominal interest rate is expected to return immediately to its mean value. The autocorrelation function of the nominal interest rate in the top-left panel of figure 2 simply confirms that the process is i.i.d.

The bottom-left panel of figure 1 plots the expected evolution of the money growth rate<sup>13</sup> Notice that the lower is the investors' economic weight, the smoother is the behavior of money, as the autocorrelation of the money growth rate in the center-left panel of figure 2 confirms. The reason is that, for a given increase in the nominal interest rate, the lower is the investors' economic weight, the smaller is the increase in aggregate bonds investment, the smaller is the decrease in aggregate money demand, and the smaller is the decrease in aggregate money supply. Also, notice the inverse relationship between the nominal interest rate and the money growth rate in the impact period of the monetary policy shock — The *liquidity* effect. The money growth rate, however, increases in the following periods. The reason lies in the fact that a one-period bond is a claim to one dollar in the next period, so an increase in the bond supply decreases the money supply in the impact period, but increases the next period aggregate cash balances. When the monetary policy process is i.i.d., all next period nominal variables increase, and the money growth rate increases. The top-right panel of figure 2 shows that the nominal interest rate is strongly negatively correlated with the contemporaneous money growth rate, but strongly positively correlated with the one-period ahead money growth rate.

The top-right panel of figure 1 plots the expected evolution of the investors' consumption. As pointed out by Alvarez and Atkeson (1996), in periods when the money growth rate and the inflation rate are low with respect to their expected future values, the non-investors' consumption is high with respect to its expected future value. The reason is that, in equilib-

<sup>&</sup>lt;sup>12</sup>The stationary value for  $\theta$  associated with the mean value of q is unique and can be obtained iterating on the law of motion  $\theta'(q, \theta)$  with q equal to the mean value, starting with any initial value of  $\theta$  in the space  $\Theta$ . The reason is that theorem A.5 in the appendix proves that, for any given q, the next period value of the second state variable  $\theta'(q, \theta)$  is strictly increasing in the current period value  $\theta$ . Also, I obtain the same stationary value starting from the smallest and the greatest values for  $\theta$ , namely  $\frac{\theta}{\theta}$  and  $\frac{\overline{\theta}}{\overline{\theta}}$ .

<sup>&</sup>lt;sup>13</sup>The inflation rate is the same as the money growth rate since the cash-in-advance constraints bind for both types of households in equilibrium.



Figure 2: Autocorrelation and cross-correlation functions of the processes for the nominal interest rate, the money growth rate and the real interest rate. The nominal interest rate is i.i.d. The solid, dotted and dashed lines refer to three economies where the investors' economic weight, namely their share  $\lambda$  of aggregate endowment, is respectively 50%, 75% and 100%. Utility is logarithmic.

rium, the non-investors spend all the cash earned in the previous period, so the lower is the inflation rate, the higher is their consumption. Since the equilibrium aggregate consumption is equal to the constant aggregate endowment, in periods when the money growth rate is low with respect to its expected future value, the equilibrium investors' consumption is low with respect to its expected future value. From the expected evolution of the investors' consumption, the expected evolution of the real interest rate can be inferred. When the investors' consumption is low with respect to its expected to its expected future value, the investors' Euler equation implies that the real interest rate is high. This mechanism creates an inverse relationship between the money growth rate and the real interest rate. The bottom-right panel of figure 1 plots the expected evolution of the real interest rate. The lower is the investors' economic weight, the stronger is the impact of the shock on the real interest rate. Of course, in the full participation economy the real interest rate is constant and equal to the preferences discount rate. The center-right and bottom-right panels of figure 2 show that the real interest rate is strongly negatively correlated with the money growth rate, and strongly positively correlated with the nominal interest rate.

Let us now see the previous mechanisms at work in more realistic economies where the monetary policy process is positively serially correlated, as in the U.S. economy. Figure 3 plots the expected evolution of three such economies after a contractionary monetary policy shock in period 0. In all three economies the first-order serial correlation of the process for q is .9, which is approximately equal to the first-order serial correlation of the federal funds rate over the period 1959(1)-2001(1)<sup>14</sup> Specifically, the transition matrix for q is

$$\begin{bmatrix} \rho & \frac{1-\rho}{2} & \frac{1-\rho}{2} \\ \frac{1-\rho}{2} & \rho & \frac{1-\rho}{2} \\ \frac{1-\rho}{2} & \frac{1-\rho}{2} & \rho \end{bmatrix},$$

with  $\rho = 1/3 + .9 * 2/3$ , so that the first-order serial correlation of the process for q is .9. Figure 4 plots the autocorrelation and cross-correlation functions of the processes for the nominal interest rate, the money growth rate and the real interest rate. For comparison, figure 5 plots the corresponding moments of the federal funds rate, the M2 growth rate and the real interest rate over the period 1959(1)-2001(1). The real interest rate is approximated with the difference between the federal funds rate and the contemporaneous GDP Price Deflator growth rate, as if expectations of inflation were equal to current inflation. Approximating it with the difference between the federal funds rate and the one-period ahead GDP Price Deflator growth rate leads to the same conclusions. The three processes resemble autoregressive processes with high serial correlation: .95 for the federal funds rate, .63 for the M2 growth rate, and .82 for the real interest rate. The federal funds rate is strongly positively correlated with the past and future M2 growth rate, but weakly positively correlated (.06) with the current M2 growth rate. The correlation between the M2 growth rate and the real interest rate is -.14. The federal funds rate is strongly positively correlated with the past, current and future real interest rate, the contemporaneous correlation being .62.

<sup>&</sup>lt;sup>14</sup>Clarida, Gali' and Gertler (1999) point out that typical estimates for the serial correlation are .8 or .9. Such a high serial correlation is what is commonly referred to as "interest rate smoothing" by the Federal Reserve.



Figure 3: Expected evolution after an unanticipated increase in the nominal interest rate in period 0. The nominal interest rate is first-order Markov with .9 serial correlation. The solid, dotted and dashed lines refer to three economies where the investors' economic weight, namely their share  $\lambda$  of aggregate endowment, is respectively 50%, 75% and 100%. Utility is logarithmic.



Figure 4: Autocorrelation and cross-correlation functions of the processes for the nominal interest rate, the money growth rate and the real interest rate. The nominal interest rate is first-order Markov with .9 serial correlation. The solid, dotted and dashed lines refer to three economies where the investors' economic weight, namely their share  $\lambda$  of aggregate endowment, is respectively 50%, 75% and 100%. Utility is logarithmic.



Figure 5: Autocorrelation and cross-correlation functions of the federal funds rate, the M2 growth rate and the real interest rate over the period 1959(1)-2001(1). The real interest rate is approximated with the difference between the federal funds rate and the contemporaneous GDP Price Deflator growth rate. Data are from the FRED database of the Federal Reserve Bank of St. Louis at the web address http://www.stls.frb.org/fred/.

The top-left panel of figure 3 plots the experiment. Before period 0, the nominal interest rate is constant and equal to 6.5%. In period 0, a contractionary monetary policy shock, namely a 10 basis points unanticipated increase in the nominal interest rate, hits the three economies. After period 0, the expected values of the nominal interest rate are plotted. Since the monetary policy process is strongly serially correlated, the nominal interest rate is expected to be high for several periods after the shock. The autocorrelation of the nominal interest rate in the top-left panel of figure 4 simply confirms that the process is first-order Markov with serial correlation equal to .9.

The bottom-left panel of figure 3 plots the expected evolution of the money growth rate. In the full participation economy, the money growth rate only decreases in the impact period, while, in the segmented markets economies, the evolution of the money growth rate is smoother. The autocorrelation of the money growth rate in the center-left panel of figure 4 confirms that the lower is the investors' economic weight, the smoother and more persistent is the endogenous process of the money growth rate. In the economy where the investors' economic weight  $\lambda$  is 50%, the autocorrelation of the money growth rate closely resembles that of a first-order autoregressive process with a serial correlation of .5. Christiano, Eichenbaum and Evans (1998) document that the empirical process for the M2 growth rate is well approximated by an AR(1) process with serial correlation of .5. The topright panel of figure 4 shows that the nominal interest rate is weakly negatively correlated with the current money growth rate, and strongly positively correlated with the future money growth rate.

The two right panels of figure 3 plot respectively the expected evolution of the investors' consumption, and the expected evolution of the real interest rate. When a contractionary shock hits the segmented markets economies, the real interest rate increases immediately, and remains high for up to one year. The lower is the investors' economic weight, the stronger and more persistent is the effect of the shock on the real interest rate. The size of the effect is important: a 10 basis points increase in the nominal interest rate makes the real interest rate increase by 12 basis points in the economy where the investors' economic weight is 75%, and by 20 basis points in the economy where the investors' economic weight is 50%. Of course, in the full participation economy, the real interest rate is constant and equal to the preferences discount rate. The autocorrelation of the real interest rate in the bottom-left panel of figure 4 shows that the lower is the investors' economic weight, the more persistent and close to data is the process for the real interest rate. The center-right and bottom-right panels of figure 4 show that the real interest rate is strongly negatively correlated with the money growth rate and strongly positively correlated with the nominal interest rate.

What drives this important and interesting dynamics is the fact that, in this segmented markets economy, perfectly anticipated monetary policy has impact effects which are similar to those that unanticipated monetary policy has in limited participation economies. In the limited participation framework, a contractionary shock causes the money growth rate to decrease, the participants' consumption to decrease, and the real interest rate to increase in the impact period. In the following periods, the shock does not have any further real effects, since anticipated changes in the money growth rate do not affect the participants' consumption. In this segmented markets economy, however, a perfectly anticipated decrease in the money growth rate does cause the investors' consumption to decrease. Hence, when the monetary policy process is positively serially correlated, a contractionary shock causes the money growth rate to remain low for several periods, the investors' consumption to remain low for several periods and to increase towards its stationary level, and the real interest rate to remain high for several periods.

#### 5 CES preferences

In this section, I show that the lower is the investors' intertemporal elasticity of substitution, the stronger and more persistent is the effect of a monetary policy shock on the real interest rate, and the higher is the serial correlation of the processes for the money growth rate and the real interest rate. Economies where the intertemporal elasticity of substitution is low exhibit, then, the same qualitative behavior as economies where the market segmentation is high.

Let us consider the same economy as in the previous sections, except that investors' preferences are

$$E\left\{\sum_{t=0}^{\infty}\beta^{t}\frac{c_{I,t}^{1-\sigma}}{1-\sigma}\right\},$$

with a constant elasticity of substitution  $1/\sigma$  less than one. To determine the equilibrium behavior of the economy, one can proceed as in the logarithmic utility case, and show that, in equilibria where the cash-in-advance constraints bind for both types of households, the ratio d(s) of government debt to aggregate cash balances solves the functional equation

$$\left(\frac{d(s)}{\theta - d(s)}\right)^{\sigma} = \int_{\mathbf{Q}} \frac{\beta}{1 - \lambda} \frac{[\theta'(s) - \lambda][1 - d(q', \theta'(s))]^{\sigma - 1}}{[\theta'(s) - d(q', \theta'(s))]^{\sigma}} \left(\frac{d(s)}{1 - d(s)}\right)^{\sigma} P(q, dq'), \tag{7}$$

all  $s \in \mathbf{S}$ , where  $s \equiv (q, \theta)$ , and the law of motion  $\theta'(s)$  is given by the relation 2. Also, with steps analogous to the logarithmic utility case, one can show that the problem of determining the equilibrium behavior of the economy can be reduced to the problem of finding a space  $\Theta$  where  $\theta$  takes values in equilibrium, and a solution d(s) to the functional equation 7, such that the function  $\theta'(s)$  given by the relation 2 takes values in  $\Theta$  and the function p(s) given by the relation 1 satisfies the inequality 4, for all  $s \in \mathbf{S}$ .

To determine such a solution, we make the following additional assumption.

#### Assumption 5.1 $[\overline{\theta} - \overline{d}] - \sigma[\overline{\theta} - \lambda] \ge 0.$

Notice that the assumption specializes to assumption 3.1 in the logarithmic utility case, and serves the same purpose — under assumption 5.1, the right hand side of the functional equation 7 is strictly increasing in  $\theta'(s)$ . We also define the operator T as follows:

$$(Td)(s) \equiv \frac{\theta R(s)^{1/\sigma}}{1 + R(s)^{1/\sigma}},\tag{8}$$

where R(s) is the right hand side of the functional equation 7. As in the logarithmic utility case, one can show that the operator T satisfies  $T: \mathcal{M} \to \mathcal{M}$ . We then let the interval  $\Theta$  where  $\theta$  takes values in equilibrium be the interval  $[\underline{\theta}, \overline{\theta}]$ , and look for a fixed point of T, which is a solution to the functional equation 7. Also, the operator T satisfies  $T: D \to D$ , and is monotone, so, with the same numerical procedure used in the logarithmic utility case, a fixed point  $d \in D$ , which turns out to be unique, is determined. The other equilibrium aggregate variable are then determined and the inequality 4 is checked.

Let us now consider how the effects of a contractionary monetary policy shock on money, prices and interest rates change, as the intertemporal elasticity of substitution changes. Figure 6 plots the expected evolution of three economies after a contractionary monetary policy shock in period 0. In all three economies, the investors' share  $\lambda$  of aggregate endowment is 75%, and the first-order serial correlation of the monetary policy process is .9. The dashed line refers to an economy where the investors' utility function is logarithmic, while the dotted and solid lines refer to two economies where the intertemporal elasticity of substitution  $1/\sigma$  is respectively 1/3 and 1/5. Though assumption 5.1 does not hold when  $\sigma = 5$ , the same numerical procedure allows to determine the equilibrium behavior even in this case. Figure 7 plots the autocorrelation and cross-correlation functions of the processes for the nominal interest rate, the money growth rate and the real interest rate.

The experiment, plotted in the top-left panel of figure 6, is the same as the one plotted in the top-left panel of figure 3. Before period 0, the nominal interest rate is constant and equal to 6.5%. In period 0, a contractionary monetary policy shock, namely a 10 basis points increase in the nominal interest rate, hits the three economies. After period 0, the expected values of the nominal interest rate are plotted. Since the monetary policy process is strongly serially correlated, the nominal interest rate is expected to be high for several periods after the shock.

The bottom-left panel of figure 6 plots the expected evolution of the money growth rate. The lower is the investors' elasticity of substitution, the smoother is the behavior of the money growth rate. The autocorrelation of the money growth rate in the center-left panel of figure 7 confirms that the lower is the investors' elasticity of substitution, the smoother and more persistent is the process for the money growth rate.

The two right panels of figure 6 plot respectively the expected evolution of the investors' consumption and the expected evolution of the real interest rate. The lower is the investors' elasticity of substitution, the stronger is the effect of a monetary policy shock on the real interest rate. For instance, in the economy where the investors' share  $\lambda$  of aggregate endowment is 75% and their elasticity of substitution is 1/3, which I consider plausible figures, a 10 basis points increase in the nominal interest rate increases the real interest rate by 20 basis points. More importantly, the real interest rate remains high for up to 4 quarters, with an increase of about 6 basis points two quarters after the shock. When their intertemporal elasticity of substitution is low, investors require a stronger change in the real interest rate to change their consumption and money demand, and to absorb the monetary policy shock. The bottom-left panel of figure 7 shows that the lower is the investors' elasticity of substitution, the more persistent and close to data is the process for the real interest rate.

Comparing the plots in figures 6 and 7 with the corresponding in figures 3 and 4, it is apparent that lowering the investors' intertemporal elasticity of substitution has the same qualitative effects as increasing the markets segmentation. Economies where the intertempo-



Figure 6: Expected evolution after an unanticipated increase in the nominal interest rate in period 0. The nominal interest rate is first-order Markov with .9 serial correlation. The investors' economic weight, namely their share of aggregate endowment  $\lambda$ , is 75%. The solid, dotted and dashed lines refer to three economies where the investors' intertemporal elasticity of substitution is respectively 1/5, 1/3 and 1 (log-utility).



Figure 7: Autocorrelation and cross-correlation functions of the processes for the nominal interest rate, the money growth rate and the real interest rate. The nominal interest rate is first-order Markov with .9 serial correlation. The investors' economic weight, namely their share of aggregate endowment  $\lambda$ , is 75%. The solid, dotted and dashed lines refer to three economies where the investors' intertemporal elasticity of substitution is respectively 1/5, 1/3 and 1 (log-utility).

ral elasticity of substitution is low exhibit the same qualitative behavior as economies where the market segmentation is high. This suggests that economies with segmented markets are promising models to address some of the asset pricing puzzles which can only be solved assuming implausibly low values for the intertemporal elasticity of substitution.

## 6 Conclusion

In this paper, I modelled the fact that a large part of the households never invests in bonds, and obtained that, holding aggregate output constant, a contractionary monetary policy shock increases persistently the real interest rate, which is central in the explanation of the monetary transmission mechanism. Among other results, I showed that in an endowment economy where the investors' economic weight is 75% and their intertemporal elasticity of substitution is 1/3, after an unanticipated 10 basis points increase in the nominal interest rate, the real interest rate increases by 20 basis points in the impact period, and remains high for up to one year. The shock has a liquidity effect, moving money and interest rates in opposite directions. The endogenous processes for the money growth rate and the real interest rate are strongly serially correlated, the money growth rate being remarkably close to an AR(1) process with a .5 serial correlation.

A natural and important direction for further research is the introduction of production and sources of uncertainty other than monetary policy. That will allow to assess the properties of different monetary policy rules in the segmented markets framework, and to characterize the optimal response of monetary authority to exogenous shocks. More importantly, it will be possible to use the segmented markets framework to explain the persistent effects of monetary policy shocks on aggregate output. Production will be introduced as in Fuerst (1992), where monetary policy shocks affect directly the financial and the firms sectors. In the segmented markets framework, the investors are the ones directly affected by monetary policy shocks, so the investors will be identified with the firms. Monetary policy shocks will, then, affect persistently the distribution of cash balances between the firms and the households sectors. Since the distribution of cash balances will affect aggregate output as in limited participation models, monetary policy shocks will have persistent effects on aggregate output. I am conducting research along these directions, and I will report the results in later work.

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## A Appendix

**Theorem A.1** Suppose that, in equilibrium, the cash-in-advance constraints bind for both types of households:  $p(s)\hat{c}_I(a_I, s) = a_I - q\hat{b}_I(a_I, s)$ , and  $p(s)\hat{c}_N(a_N, s) = a_N$ , for all  $s \in \mathbf{S}$ ,  $a_I = \theta/\omega$ ,  $a_N = (1-\theta)/(1-\omega)$ . Then, the goods price p(s) satisfies the relation 1, the law of motion  $\theta'(s)$  satisfies the relation 2, and the bond value d(s) solves the functional equation 3.

*Proof.* Relation 1 follows from

$$p(s)y = p(s)[\omega \hat{c}_{I}(a_{I}, s) + (1 - \omega)\hat{c}_{N}(a_{N}, s)]$$
  

$$= \omega p(s)\hat{c}_{I}(a_{I}, s) + (1 - \omega)p(s)\hat{c}_{N}(a_{N}, s)$$
  

$$= \omega [a_{I} - q\hat{b}_{I}(a_{I}, s)] + (1 - \omega)a_{N}$$
  

$$= 1 - \omega q\hat{b}_{I}(a_{I}, s)$$
  

$$= 1 - d(s),$$

where the first equality follows from the goods market equilibrium condition, the third from the assumption of binding cash-in-advance constraints, the fourth from  $a_I = \theta/\omega$  and  $a_N = (1 - \theta)/(1 - \omega)$ , and the last from the bond market equilibrium condition.

Relation 2 follows from

$$\begin{aligned} \theta'(s)[1 - d(s) + d(s)/q] &= \omega \hat{a}'_I(a_I, s)[1 - d(s) + d(s)/q] \\ &= \omega [\hat{b}_I(a_I, s) + a_I - q\hat{b}_I(a_I, s) - p(s)\hat{c}_I(a_I, s) + p(s)y_I] \\ &= \omega [\hat{b}_I(a_I, s) + p(s)y_I] \\ &= d(s)/q + p(s)\omega y_I \\ &= d(s)/q + \omega y_I [1 - d(s)]/y \\ &= d(s)/q + \lambda [1 - d(s)] \\ &= (1 - \lambda)d(s)/q + \lambda [1 - d(s) + d(s)/q], \end{aligned}$$

where the first equality follows from the equilibrium condition for  $\theta'(s)$ , the second from the investors' budget constraint, the third from the investors' binding cash-in-advance constraint, the fourth from the bond market equilibrium condition, the fifth from relation 1, and the sixth from the definition of  $\lambda$ .

Finally, to derive the functional equation 3, consider the investors' optimization problem. Neither constraint on  $b_I$  binds since  $c_I > 0$  and, in equilibrium,  $b_I > 0$ . Hence, from the envelope condition

$$v'_I(a_I, s) = \frac{1}{p(s)\hat{c}_I(a_I, s)},$$

and the first order condition

$$\frac{1}{p(s)\hat{c}_I(a_I,s)} = \frac{\beta}{q[1-d(s)+d(s)/q]} \int_{\mathbf{Q}} v'_I(\hat{a}'_I(a_I,s),q',\theta'(s))P(q,dq'),$$

one obtains the Euler equation<sup>15</sup>

$$\frac{1}{p(s)\hat{c}_I(a_I,s)} = \frac{\beta}{q[1-d(s)+d(s)/q]} \int_{\mathbf{Q}} \frac{1}{p(q',\theta'(s))\hat{c}_I(\hat{a}'_I(a_I,s),q',\theta'(s))} P(q,dq').$$

The investors' cash-in-advance constraint always binds in equilibrium, so

$$p(s)\hat{c}_{I}(a_{I},s) = a_{I} - qb_{I}(a_{I},s),$$
$$p(q',\theta'(s))\hat{c}_{I}(\hat{a}'_{I}(a_{I},s),q',\theta'(s)) = \hat{a}'_{I}(a_{I},s) - q'\hat{b}_{I}(\hat{a}'_{I}(a_{I},s),q',\theta'(s)),$$

for all  $s \in \mathbf{S}$ , all  $q' \in \mathbf{Q}$ , at  $a_I = \theta/\omega$ . Substituting these expressions in the previous Euler equation, one obtains

$$\frac{1}{a_I - q\hat{b}_I(a_I, s)} = \frac{\beta}{q - qd(s) + d(s)]} \int_{\mathbf{Q}} \frac{1}{\hat{a}'_I(a_I, s) - q'\hat{b}_I(\hat{a}'_I(a_I, s), q', \theta'(s))} P(q, dq').$$

In equilibrium,  $\hat{a}'_I(a_I, s) = \theta'(s)/\omega$ ,  $\hat{b}_I(a_I, s) = d(s)/q\omega$ , and  $\hat{b}_I(\hat{a}'_I(a_I, s), q', \theta'(s)) = d(q', \theta'(s))/q'\omega$ . Substituting these expressions in the previous equation, and multiplying both sides by  $d(s)/\omega$ , one obtains

$$\frac{d(s)}{\theta - d(s)} = \frac{\beta d(s)}{q - qd(s) + d(s)} \int_{\mathbf{Q}} \frac{1}{\theta'(s) - d(q', \theta'(s))} P(q, dq').$$

Finally, using relation 2, one obtains that d(s) solves the functional equation 3.

**Theorem A.2** Suppose that the goods price p(s) satisfies the relation 1, the law of motion  $\theta'(s)$  satisfies the relation 2, and the bond value d(s) solves the functional equation 3 and satisfies the inequality 4. Then, the non-investors' cash-in-advance constraint binds in equilibrium:  $p(s)\hat{c}_N(a_N, s) = a_N$ , for all  $s \in \mathbf{S}$ , and  $a_N = (1 - \theta)/(1 - \omega)$ . Also, when  $a_I = \theta/\omega$  and  $a_N = (1 - \theta)/(1 - \omega)$ ,

$$\omega b_I(a_I, s) = d(s)/q,$$
  

$$\omega \hat{c}_I(a_I, s) + (1 - \omega)\hat{c}_N(a_N, s) = y$$
  
and  $\omega \hat{a}'_I(a_I, s) = \theta'(s).$ 

*Proof.* To show that  $p(s)\hat{c}_N(a_N, s) = a_N$  is optimal for all  $s \in \mathbf{S}$ ,  $a_N = (1 - \theta)/(1 - \omega)$ , I show that deviating from this policy is not optimal. Per absurd, suppose there exists a  $s \in \mathbf{S}$  such that  $p(s)\hat{c}_N(a_N, s) < a_N$ . In this case, from the envelope condition

$$v'_N(a_N, s) = \frac{1}{p(s)\hat{c}_N(a_N, s)},$$

<sup>&</sup>lt;sup>15</sup>It is possible to derive the Euler equation even without assuming the differentiability of the value function. One simply substitutes once the value function on the right hand side of the Bellman equation with the expression on the left hand side, and then uses a variational argument.

and the first order condition

$$\frac{1}{p(s)\hat{c}_N(a_N,s)} = \beta \frac{1}{1 - d(s) + d(s)/q} \int_{\mathbf{Q}} v'_N(\hat{a}'_N(a_N,s),q',\theta'(s)) P(q,dq'),$$

one obtains the Euler equation<sup>16</sup>

$$\frac{1}{p(s)\hat{c}_N(a_N,s)} = \frac{\beta}{1 - d(s) + d(s)/q} \int_{\mathbf{Q}} \frac{1}{p(q',\theta'(s))\hat{c}_N(\hat{a}'_N(a_N,s),q',\theta'(s))} P(q,dq').$$

Since  $p(q', \theta'(s))\hat{c}_N(\hat{a}'_N(a_N, s), q', \theta'(s)) = \hat{a}'_N(a_N, s),$ 

$$\frac{1}{p(s)\hat{c}_N(a_N,s)} = \frac{\beta}{1 - d(s) + d(s)/q} \frac{1}{\hat{a}'_N(a_N,s)} = \frac{\beta}{a_N - p(s)\hat{c}_N(a_N,s) + p(s)y_N}$$

and since  $p(s)\hat{c}_N(a_N, s) < a_N$ ,

$$\frac{1}{a_N} < \frac{\beta}{p(s)y_N}$$

Since  $a_N = (1 - \theta)/(1 - \omega)$ ,

$$\frac{1-\omega}{1-\theta} < \frac{\beta}{p(s)y_N};$$
  
(1-\omega)p(s)y\_N < \beta(1-\theta),

which is a contradiction. Hence,  $p(s)\hat{c}_N(a_N,s) = a_N$ , for all  $s \in \mathbf{S}$ , and  $a_N = (1-\theta)/(1-\omega)$ .

Taking into account the previous result, I now need to show that the following equilibrium conditions hold:

$$\omega b_I(a_I, s) = d(s)/q,$$
  

$$\omega \hat{c}_I(a_I, s) + (1 - \theta)/p(s) = y$$
  
and  $\omega \hat{a}'_I(a_I, s) = \theta'(s),$ 

for all  $s \in \mathbf{S}$ , and  $a_I = \theta/\omega$ . First, notice that the function  $\hat{c}_I(a_I, s)$  defined by

$$\omega \hat{c}_I(a_I, s) + (1 - \theta)/p(s) = y$$

for all  $s \in \mathbf{S}$ , and  $a_I = \theta/\omega$ , satisfies the investors' Euler equation derived in the previous lemma. Then, it is easy to derive the functions  $\hat{b}_I(a_I, s)$  and  $\hat{a}'_I(a_I, s)$  from the investors' binding cash-in-advance constraint and the investors' budget constraint, and to verify that the equilibrium conditions hold.

**Theorem A.3** For any  $d \in \mathcal{M}$ , the function  $\theta'(q, \theta)$  defined in 2 takes values in the interval  $[\underline{\theta}, \overline{\theta}]$ .

 $<sup>^{16}</sup>$ Again, it is possible to derive the Euler equation without assuming the differentiability of the value function.

*Proof.* For any  $d \in \mathcal{M}$  and any  $s \in \mathbf{S}$ ,

$$\begin{aligned} \theta'(s) &= \lambda + (1-\lambda) \frac{d(s)}{q - qd(s) + d(s)} \\ &\leq \lambda + (1-\lambda) \frac{d(s)}{q - q\overline{d} + d(s)} \\ &\leq \lambda + (1-\lambda) \frac{\overline{d}}{q - q\overline{d} + \overline{d}} \\ &\leq \lambda + (1-\lambda) \frac{\overline{d}}{\underline{q} - \underline{q}\overline{d} + \overline{d}} = \overline{\theta}; \end{aligned}$$

where the first inequality follows from  $d(s) \leq \overline{d}$  for any  $s \in \mathbf{S}$ ; the second from  $\overline{d} < 1$  and  $d(s) \leq \overline{d}$  for any  $s \in \mathbf{S}$ ; the third from  $\overline{d} < 1$  and  $q \geq \underline{q}$  for any  $q \in \mathbf{Q}$ ; and the last equality from the definition of  $\overline{\theta}$ .

Similarly, one can show that  $\theta'(s) \ge \underline{\theta}$ .

**Theorem A.4** Under assumption 3.1, the operator T defined in 5 satisfies  $T : \mathcal{M} \to \mathcal{M}$ .

*Proof.* For any  $d \in \mathcal{M}$  and any  $s \in \mathbf{S}$ , the right hand side of the functional equation 3 is

$$R(s) \equiv \int_{\mathbf{Q}} \frac{\beta}{1-\lambda} \frac{\theta'(s)-\lambda}{\theta'(s)-d(q',\theta'(s))} P(q,dq')$$
$$\leq \frac{\beta}{1-\lambda} \frac{\theta'(s)-\lambda}{\theta'(s)-\overline{d}}$$
$$\leq \frac{\beta}{1-\lambda} \frac{\overline{\theta}-\lambda}{\overline{\theta}-\overline{d}}$$
$$= \frac{\beta \overline{d}}{\underline{q}-\underline{q}\overline{d}+\overline{d}} \frac{1}{\overline{\theta}-\overline{d}}$$
$$= \frac{\overline{d}}{\overline{\theta}-\overline{d}} \equiv \overline{R};$$

where the first inequality follows from  $d(s) \leq \overline{d}$  for any  $s \in \mathbf{S}$ ; the second from assumption 3.1 and theorem A.3; the next equality from the definition of  $\overline{\theta}$ ; and the following from the definition of  $\overline{d}$ . Hence,

$$(Td)(s) = \frac{\theta R(s)}{1 + R(s)} \le \frac{\overline{\theta R}}{1 + \overline{R}} = \frac{\overline{\theta d}}{\overline{\theta} - \overline{d}} \frac{\overline{\theta} - \overline{d}}{\overline{\theta}} = \overline{d};$$

where the first equality follows from the definition of T; the inequality from  $\theta \leq \overline{\theta} < 1$  for any  $\theta \in \Theta$ , and from the inequality previously derived; and the next equality from the definition of  $\overline{R}$ .

Similarly, one can show that

$$R(s) \ge \frac{\underline{d}}{\underline{\theta} - \underline{d}} \equiv \underline{R},$$

and that

$$(Td)(s) = \frac{\theta R(s)}{1 + R(s)} \ge \underline{d}.$$

The measurability of Td follows from the fact that P(q, A) is a measurable function of q for all  $A \in \mathcal{B}(\mathbf{Q})$ .

**Theorem A.5** For any  $d \in D$ , the function  $\theta'(q, \theta)$  defined in 2 is strictly increasing in  $\theta$ .

*Proof.* For any  $d \in D$ , any  $q \in \mathbf{Q}$  and any  $\theta^1, \theta^2 \in \Theta, \ \theta^1 < \theta^2$ ,

$$\begin{aligned} \theta'(q,\theta^1) &= \lambda + (1-\lambda) \frac{d(q,\theta^1)}{q - qd(q,\theta^1) + d(q,\theta^1)} \\ &< \lambda + (1-\lambda) \frac{d(q,\theta^1)}{q - qd(q,\theta^2) + d(q,\theta^1)} \\ &< \lambda + (1-\lambda) \frac{d(q,\theta^2)}{q - qd(q,\theta^2) + d(q,\theta^2)} = \theta'(q,\theta^2) \end{aligned}$$

where the first inequality follows from the fact that  $d(q, \theta)$  is strictly increasing in  $\theta$ ; and the second from  $d(s) \leq \overline{d} < 1$  for any  $s \in \mathbf{S}$ , and the fact that  $d(q, \theta)$  is strictly increasing in  $\theta$ .

**Theorem A.6** Under assumption 3.1, the operator T defined in 5 satisfies  $T: D \to D$ .

*Proof.* In light of theorem A.4, I only need to show that  $(Td)(q,\theta)/\theta$  is weakly increasing in  $\theta$ . Indeed, I will show that the monotonicity is strict. For any  $d \in D$ , any  $q \in \mathbf{Q}$  and any  $\theta^1, \theta^2 \in \mathbf{\Theta}, \ \theta^1 < \theta^2$ , the right hand side of the functional equation 3 is

$$\begin{split} R(q,\theta^{1}) &\equiv \int_{\mathbf{Q}} \frac{\beta}{(1-\lambda)} \frac{\theta'(q,\theta^{1}) - \lambda}{\theta'(q,\theta^{1}) - d(q',\theta'(q,\theta^{1}))} P(q,dq') \\ &< \int_{\mathbf{Q}} \frac{\beta}{(1-\lambda)} \frac{\theta'(q,\theta^{1}) - \lambda}{\theta'(q,\theta^{1}) - d(q',\theta'(q,\theta^{2}))} P(q,dq') \\ &\leq \frac{\beta}{(1-\lambda)} \int_{\mathbf{Q}} \frac{\theta'(q,\theta^{2}) - \lambda}{\theta'(q,\theta^{2}) - d(q',\theta'(q,\theta^{2}))} P(q,dq') = R(q,\theta^{2}); \end{split}$$

where the strict inequality follows from the fact that  $d(q, \theta)$  is strictly increasing in  $\theta$  and from theorem A.5; and the weak inequality from  $d(s) \leq \overline{d}$  for any  $s \in \mathbf{S}$ , from assumption 3.1, and from theorem A.5. It follows that

$$\frac{(Td)(q,\theta^1)}{\theta^1} = \frac{R(q,\theta^1)}{1+R(q,\theta^1)} < \frac{R(q,\theta^2)}{1+R(q,\theta^2)} = \frac{(Td)(q,\theta^2)}{\theta^2},$$

which concludes the proof that  $(Td)(q,\theta)/\theta$  is strictly increasing in  $\theta$ .

#### **Theorem A.7** Under assumption 3.1, the operator $T: D \rightarrow D$ defined in 5 is monotone.

*Proof.* Consider any  $d^1, d^2 \in D$  such that  $d^1(s) \leq d^2(s)$  for any  $s \in \mathbf{S}$ . Let us define  $\theta'^1(s)$  and  $\theta'^2(s)$  the function  $\theta'(s)$  respectively when  $d = d^1$  and  $d = d^2$ . Then, the following steps show that  $\theta'^1(s) \leq \theta'^2(s)$  for any  $s \in \mathbf{S}$ :

$$\begin{aligned} \theta'^{1}(s) &= \lambda + (1-\lambda) \frac{d^{1}(s)}{q - qd^{1}(s) + d^{1}(s)} \\ &\leq \lambda + (1-\lambda) \frac{d^{1}(s)}{q - qd^{2}(s) + d^{1}(s)} \\ &\leq \lambda + (1-\lambda) \frac{d^{2}(s)}{q - qd^{2}(s) + d^{2}(s)} = \theta'^{2}(s); \end{aligned}$$

where the first inequality follows from  $d^1(s) \leq d^2(s)$  for any  $s \in \mathbf{S}$ ; and the second from  $d^2(s) \leq \overline{d} < 1$  for any  $s \in \mathbf{S}$ , and  $d^1(s) \leq d^2(s)$  for any  $s \in \mathbf{S}$ .

Let us define  $R^1(s)$  and  $R^2(s)$  the right hand side of the functional equation 3 respectively when  $d = d^1$  and  $d = d^2$ . Then, the following steps show that  $R^1(s) \leq R^2(s)$  for any  $s \in \mathbf{S}$ :

$$R^{1}(s) = \int_{\mathbf{Q}} \frac{\beta}{1-\lambda} \frac{\theta^{\prime 1}(s) - \lambda}{\theta^{\prime 1}(s) - d^{1}(q^{\prime}, \theta^{\prime 1}(s))} P(q, dq^{\prime})$$

$$\leq \int_{\mathbf{Q}} \frac{\beta}{1-\lambda} \frac{\theta^{\prime 1}(s) - \lambda}{\theta^{\prime 1}(s) - d^{1}(q^{\prime}, \theta^{\prime 2}(s))} P(q, dq^{\prime})$$

$$\leq \int_{\mathbf{Q}} \frac{\beta}{1-\lambda} \frac{\theta^{\prime 2}(s) - \lambda}{\theta^{\prime 2}(s) - d^{2}(q^{\prime}, \theta^{\prime 2}(s))} P(q, dq^{\prime})$$

$$\leq \int_{\mathbf{Q}} \frac{\beta}{1-\lambda} \frac{\theta^{\prime 2}(s) - \lambda}{\theta^{\prime 2}(s) - d^{2}(q^{\prime}, \theta^{\prime 2}(s))} P(q, dq^{\prime}) = R^{2}(s);$$

where the first inequality follows from  $\theta'^1(s) \leq \theta'^2(s)$  for any  $s \in \mathbf{S}$ , and from the fact that  $d^2(q,\theta)$  is strictly increasing in  $\theta$ ; the second from  $d^1(s) \leq d^2(s)$  for any  $s \in \mathbf{S}$ ; and the third from  $d^2(s) \leq \overline{d}$  for any  $s \in \mathbf{S}$ , from assumption 3.1, and from  $\theta'^1(s) \leq \theta'^2(s)$  for any  $s \in \mathbf{S}$ .

It follows that, for any  $s \in \mathbf{S}$ ,

$$(Td^{1})(s) = \frac{\theta R^{1}(s)}{1 + R^{1}(s)} \le \frac{\theta R^{2}(s)}{1 + R^{2}(s)} = (Td^{2})(s),$$

which concludes the proof that T is monotone.

**Theorem A.8** For any  $d \in D_{\theta}$ , any  $q \in \mathbf{Q}$ , and any  $\theta^1, \theta^2 \in \Theta$ ,  $\theta^1 < \theta^2$ , the law of motion  $\theta'(q, \theta)$  defined in 2 satisfies

$$\theta'(q,\theta^2) - \theta'(q,\theta^1) \le \frac{\theta'(q,\theta^1) - \lambda}{d(q,\theta^1)} \frac{\overline{q}}{\beta} [\theta^2 - \theta^1].$$

**Proof.** Consider any  $d \in D_{\theta}$ , any  $q \in \mathbf{Q}$ , and any  $\theta^1, \theta^2 \in \Theta, \ \theta^1 < \theta^2$ . Then,

$$\begin{split} \theta'(q,\theta^2) - \theta'(q,\theta^1) &= (1-\lambda) \left[ \frac{d(q,\theta^2)}{q - qd(q,\theta^2) + d(q,\theta^2)} - \frac{d(q,\theta^1)}{q - qd(q,\theta^1) + d(q,\theta^1)} \right] \\ &= (1-\lambda) \left[ \frac{q[d(q,\theta^2) - d(q,\theta^1)]}{[q - qd(q,\theta^2) + d(q,\theta^2)][q - qd(q,\theta^1) + d(q,\theta^1)]} \right] \\ &= \frac{\theta'(q,\theta^1) - \lambda}{d(q,\theta^1)} \frac{q[d(q,\theta^2) - d(q,\theta^1)]}{q - qd(q,\theta^2) + d(q,\theta^2)} \\ &\leq \frac{\theta'(q,\theta^1) - \lambda}{d(q,\theta^1)} \frac{q[\theta^2 - \theta^1]}{q - qd(q,\theta^2) + d(q,\theta^2)} \\ &\leq \frac{\theta'(q,\theta^1) - \lambda}{d(q,\theta^1)} \frac{q}{q - q\underline{d} + \underline{d}} [\theta^2 - \theta^1] \\ &\leq \frac{\theta'(q,\theta^1) - \lambda}{d(q,\theta^1)} \frac{\overline{q}}{\overline{q} - \overline{q}\underline{d} + \underline{d}} [\theta^2 - \theta^1] \\ &= \frac{\theta'(q,\theta^1) - \lambda}{d(q,\theta^1)} \frac{\overline{q}}{\overline{q} - \overline{q}\underline{d} + \underline{d}} [\theta^2 - \theta^1] \\ &= \frac{\theta'(q,\theta^1) - \lambda}{d(q,\theta^1)} \frac{\overline{q}}{\overline{q} - \overline{q}\underline{d} + \underline{d}} [\theta^2 - \theta^1] \end{split}$$

where the first inequality follows from the fact that  $\theta - d(q, \theta)$  is weakly increasing in  $\theta$ ; the second from q < 1 for any  $q \in \mathbf{Q}$  and from  $d(s) \ge \underline{d}$  for any  $s \in \mathbf{S}$ ; the third from  $q \le \overline{q}$  for any  $q \in \mathbf{Q}$ ; and the following equality from the definition of  $\underline{d}$ .

**Theorem A.9** Under assumptions 3.1 and 3.2, the operator T defined in 5 satisfies  $T : D_{\theta} \to D_{\theta}$ .

*Proof.* In light of the previous theorems, I only need to show that, if  $d \in D_{\theta}$ , then  $\theta - (Td)(q, \theta)$  is weakly increasing in  $\theta$ . Consider any  $d \in D_{\theta}$ , any  $q \in \mathbf{Q}$ , and any  $\theta^{1}, \theta^{2} \in \Theta$ ,  $\theta^{1} < \theta^{2}$ . Then, the difference of the right hand side of the functional equation 3 evaluated respectively in  $\theta = \theta^{2}$  and  $\theta = \theta^{1}$  is

$$\begin{split} R(q,\theta^2) - R(q,\theta^1) \\ \equiv \int_{\mathbf{Q}} \frac{\beta}{1-\lambda} \frac{\theta'(q,\theta^2) - \lambda}{\theta'(q,\theta^2) - d(q',\theta'(q,\theta^2))} P(q,dq') \\ - \int_{\mathbf{Q}} \frac{\beta}{1-\lambda} \frac{\theta'(q,\theta^1) - \lambda}{\theta'(q,\theta^1) - d(q',\theta'(q,\theta^1))} P(q,dq') \\ \leq \int_{\mathbf{Q}} \frac{\beta}{1-\lambda} \frac{\theta'(q,\theta^2) - \lambda}{\theta'(q,\theta^1) - d(q',\theta'(q,\theta^1))} P(q,dq') \\ - \int_{\mathbf{Q}} \frac{\beta}{1-\lambda} \frac{\theta'(q,\theta^1) - \lambda}{\theta'(q,\theta^1) - d(q',\theta'(q,\theta^1))} P(q,dq') \end{split}$$

$$= \int_{\mathbf{Q}} \frac{\beta}{1-\lambda} \frac{\theta'(q,\theta^2) - \theta'(q,\theta^1)}{\theta'(q,\theta^1) - d(q',\theta'(q,\theta^1))} P(q,dq')$$
  
$$\leq \int_{\mathbf{Q}} \frac{\beta}{1-\lambda} \frac{\theta'(q,\theta^1) - \lambda}{\theta'(q,\theta^1) - d(q',\theta'(q,\theta^1))} P(q,dq') \frac{1}{d(q,\theta^1)} \frac{\overline{q}}{\beta} [\theta^2 - \theta^1]$$
  
$$= R(q,\theta^1) \frac{1}{d(q,\theta^1)} \frac{\overline{q}}{\beta} [\theta^2 - \theta^1];$$

where the first inequality follows from theorem A.5 and the fact that  $\theta - d(q, \theta)$  is weakly increasing in  $\theta$ ; the second inequality from theorem A.8; and the last equality from the definition of R(s).

Now, the following steps show that, as  $\theta$  increases, the percentage increase in the function  $1 + R(q, \theta)$  is less or equal than the percentage increase in  $\theta$  itself:

$$\begin{split} \frac{(1+R(q,\theta^2))-(1+R(q,\theta^1))}{1+R(q,\theta^1)} \\ &\leq \frac{1}{1+R(q,\theta^1)}R(q,\theta^1)\frac{1}{d(q,\theta^1)}\frac{\overline{q}}{\beta}[\theta^2-\theta^1]; \\ &= \frac{\theta^1R(q,\theta^1)}{1+R(q,\theta^1)}\frac{1}{d(q,\theta^1)}\frac{\overline{q}}{\beta}\frac{\theta^2-\theta^1}{\theta^1} \\ &= (Td)(q,\theta^1)\frac{1}{d(q,\theta^1)}\frac{\overline{q}}{\beta}\frac{\theta^2-\theta^1}{\theta^1} \\ &\leq \frac{\overline{d}}{\underline{d}}\frac{\overline{q}}{\beta}\frac{\theta^2-\theta^1}{\theta^1} \\ &\leq \frac{\theta^2-\theta^1}{\theta^1}; \end{split}$$

where the first inequality follows from the inequality previously obtained; the second equality from the definition of T; the following inequality from  $d(s) \ge \underline{d}$  for all  $s \in \mathbf{S}$  and theorem A.4; and the last inequality from assumption 3.2.

The previous inequality implies that

$$\frac{1+R(q,\theta^2)}{1+R(q,\theta^1)} \le \frac{\theta^2}{\theta^1},$$
$$\frac{\theta^1}{1+R(q,\theta^1)} \le \frac{\theta^2}{1+R(q,\theta^2)},$$

so  $\theta/(1+R(q,\theta))$  is weakly increasing in  $\theta$ . Since the definition of T implies that

$$\theta - (Td)(s) = \theta - \frac{\theta R(s)}{1 + R(s)} = \frac{\theta}{1 + R(s)},$$

it follows that  $\theta - (Td)(q, \theta)$  is also weakly increasing in  $\theta$ .

**Theorem A.10** Under assumptions 3.1 and 3.2, the pointwise limit  $d^{\infty} \in D_{\theta}$  of the sequence  $\{d^n\}_{n=0}^{\infty}$  defined in 6 solves the functional equation 3.

*Proof.* Let us define  $\theta'^n(s)$  and  $\theta'^{\infty}(s)$  the function  $\theta'(s)$  respectively when  $d = d^n$  and  $d = d^{\infty}$ . Since the sequence  $\{d^n\}$  converges pointwise to  $d^{\infty}$ , the sequence  $\{\theta'^n\}$  converges pointwise to  $\theta'^{\infty}$ .

Now, for any  $s \in \mathbf{S}$ , any  $q' \in \mathbf{Q}$ , and any  $n \ge 0$ ,

$$\begin{aligned} |d^{\infty}(q',\theta'^{\infty}(s)) - d^{n}(q',\theta'^{n}(s))| \\ &\leq |d^{\infty}(q',\theta'^{\infty}(s)) - d^{n}(q',\theta'^{\infty}(s))| + |d^{n}(q',\theta'^{\infty}(s)) - d^{n}(q',\theta'^{n}(s))| \\ &\leq |d^{\infty}(q',\theta'^{\infty}(s)) - d^{n}(q',\theta'^{\infty}(s))| + |\theta'^{\infty}(s) - \theta'^{n}(s)|, \end{aligned}$$

where the last inequality follows from the facts that  $d^n(q,\theta)$  is strictly increasing in  $\theta$ , and  $\theta - d^n(q,\theta)$  is weakly increasing in  $\theta$ , so the absolute value of the slope of  $d^n(q,\theta)$  with respect to  $\theta$  is less than one. As  $n \to \infty$ , the first absolute value converges to zero since the sequence  $\{d^n\}$  converges pointwise to  $d^\infty$ , and the second absolute value converges to zero since the sequence that the slopes of the functions belonging to  $D_{\theta}$  with respect to their second argument are uniformly bounded.

Let us define  $f^n(s,q')$ ,  $f^{\infty}(s,q')$  and  $\overline{f}(s,q')$  the argument of the integral on the right hand side of the functional equation 3 respectively when  $d = d^n$ ,  $d = d^{\infty}$  and when dis constant and equal to  $\overline{d}$ . From the results obtained so far, it follows that the sequence  $\{f^n\}_{n=0}^{\infty}$  converges pointwise to  $f^{\infty}$ . Also,  $f^n \leq \overline{f}$ , all  $n \geq 0$ ,  $f^n$  are integrable, all  $n \geq 0$ , and  $\overline{f}$  is also integrable. By the Lebesgue Dominated Convergence Theorem,  $f^{\infty}$  is integrable, and its integral is equal to the limit of the integrals of  $f^n$ .

Let us define  $R^n(s)$  and  $R^{\infty}(s)$  the right hand side of the functional equation 3 respectively when  $d = d^n$  and  $d = d^{\infty}$ . From the results obtained so far, it follows that the sequence  $\{R^n\}_{n=0}^{\infty}$  converges pointwise to  $R^{\infty}$ . Hence, for any  $s \in \mathbf{S}$ ,

$$d^{\infty}(s) = \lim_{n \to \infty} d^{n+1}(s) = \lim_{n \to \infty} (Td^n)(s)$$
$$= \lim_{n \to \infty} \frac{\theta R^n(s)}{1 + R^n(s)} = \frac{\theta R^{\infty}(s)}{1 + R^{\infty}(s)} = (Td^{\infty})(s),$$

that is  $d^{\infty}$  solves the functional equation 3. Notice, here, that the fourth equality follows from the results previously obtained, and not from the continuity of T. The reason is that T is uniformly continuous in the sup norm, while the sequence  $\{d^n\}$  converges to  $d^{\infty}$  only pointwise.

**Theorem A.11** Under assumption 3.3, for any  $d \in \mathcal{M}$ , the function p(s) defined in 1 satisfies the inequality 4.

*Proof.* For any  $d \in \mathcal{M}$ ,

$$(1-\omega)p(s)y_N = \frac{(1-\omega)y_N}{y}[1-d(s)] = (1-\lambda)[1-d(s)] \ge (1-\lambda)(1-\overline{d}),$$

where the inequality follows from  $d(s) \leq \overline{d}$  for all  $s \in \mathbf{S}$ . Also,

=

$$\beta(1-\theta) \le \beta(1-\underline{\theta}) = \beta[1-\lambda-(1-\lambda)\underline{d}/(\overline{q}-\overline{q}\underline{d}+\underline{d})]$$
$$= \beta[1-\lambda-(1-\lambda)\underline{d}/\beta] = \beta-\beta\lambda-(1-\lambda)\underline{d} = (1-\lambda)(\beta-\underline{d});$$

where the first inequality follows from  $\theta \geq \underline{\theta}$  for all  $\theta \in \Theta$ ; the first equality from the definition of  $\underline{\theta}$ ; and the second from the definition of  $\underline{d}$ . Since assumption 3.3 requires that  $\beta - \underline{d} \leq 1 - \overline{d}$ , the two previous inequalities imply that the inequality 4 is satisfied.

**Theorem A.12** For any  $d \in D_q$ , the law of motion  $\theta'(q, \theta)$  defined in 2 is strictly decreasing in q.

*Proof.* For any  $q^1, q^2 \in \mathbf{Q}, q^1 < q^2$ , and any  $\theta \in \mathbf{\Theta}$ ,

$$\begin{split} \theta'(q^1,\theta) &= \lambda + (1-\lambda) \frac{d(q^1,\theta)}{q^1 - q^1 d(q^1,\theta) + d(q^1,\theta)} \\ &\geq \lambda + (1-\lambda) \frac{d(q^2,\theta)}{q^1 - q^1 d(q^1,\theta) + d(q^2,\theta)} \\ &\geq \lambda + (1-\lambda) \frac{d(q^2,\theta)}{q^1 - q^1 d(q^2,\theta) + d(q^2,\theta)} \\ &> \lambda + (1-\lambda) \frac{d(q^2,\theta)}{q^2 - q^2 d(q^2,\theta) + d(q^2,\theta)} = \theta'(q^2,\theta); \end{split}$$

where the first inequality follows from the fact that  $d(q, \theta)$  is weakly decreasing in q and from  $d(s) \leq \overline{d} < 1$  for any  $s \in \mathbf{S}$ ; the second inequality from the fact that  $d(q, \theta)$  is weakly decreasing in q; and the strict inequality from  $d(s) \leq \overline{d} < 1$  for any  $s \in \mathbf{S}$ .

**Theorem A.13** Under assumptions 3.1 and 3.4, the operator T defined in 5 satisfies  $T : D_q \to D_q$ .

*Proof.* In light of theorem A.6, I only need to show that, if  $d \in D_q$ , then  $(Td)(q, \theta)$  is weakly decreasing in q. Indeed, I will show that the monotonicity is strict.

Consider any  $d \in D_q$ , any  $q^1, q^2 \in \mathbf{Q}$ ,  $q^1 < q^2$ , and any  $\theta \in \Theta$ . Then, the right hand side of the functional equation 3 evaluated at  $q = q^1$  is

$$R(q^{1},\theta) \equiv \int_{\mathbf{Q}} \frac{\beta}{1-\lambda} \frac{\theta'(q^{1},\theta) - \lambda}{\theta'(q^{1},\theta) - d(q',\theta'(q^{1},\theta))} P(q^{1},dq')$$

$$\geq \int_{\mathbf{Q}} \frac{\beta}{1-\lambda} \frac{\theta'(q^{2},\theta) - \lambda}{\theta'(q^{2},\theta) - d(q',\theta'(q^{1},\theta))} P(q^{1},dq')$$

$$> \int_{\mathbf{Q}} \frac{\beta}{1-\lambda} \frac{\theta'(q^{2},\theta) - \lambda}{\theta'(q^{2},\theta) - d(q',\theta'(q^{2},\theta))} P(q^{1},dq')$$

$$\geq \int_{\mathbf{Q}} \frac{\beta}{(1-\lambda)} \frac{\theta'(q^2,\theta) - \lambda}{\theta'(q^2,\theta) - d(q',\theta'(q^2,\theta))} P(q^2,dq') = R(q^2,\theta);$$

where the first inequality follows from  $d(s) \leq \overline{d}$  for any  $s \in \mathbf{S}$ , from assumption 3.1, and from theorem A.12; the strict inequality from theorem A.12 and the fact that  $d(q,\theta)$  is strictly increasing in  $\theta$ ; and the last inequality from the fact that  $d(q,\theta)$  is weakly decreasing in q and from assumption 3.4.

Hence,

$$(Td)(q^{1},\theta) = \frac{\theta R(q^{1},\theta)}{1 + R(q^{1},\theta)} > \frac{\theta R(q^{2},\theta)}{1 + R(q^{2},\theta)} = (Td)(q^{2},\theta),$$

where the equalities follow from the definition of T and the inequality from the inequality previously derived.

**Theorem A.14** Under assumption 3.1, for any function  $f(q, \theta)$  increasing (decreasing) in  $\theta$ , the expected future values

$$f_e^n(q,\theta) \equiv \int_{\mathbf{S}} f(q',\theta') \Pi^n((q,\theta), d(q',\theta'))$$

are also increasing (decreasing) in  $\theta$ , for all  $n \ge 1$ . The monotonicity is strict if  $f(q, \theta)$  is strictly monotone.

*Proof.* Consider any  $q \in \mathbf{Q}$ , any  $\theta^1, \theta^2 \in \Theta$ ,  $\theta^1 < \theta^2$ , and any  $f(q, \theta)$  strictly increasing in  $\theta$ . Then, after defining  $f_e^n \equiv f$  when n = 0,  $f_e^n(q, \theta)$  is strictly increasing in  $\theta$  for n = 0.

For  $n \ge 1$ , suppose that  $f_e^{n-1}(q,\theta)$  is strictly increasing in  $\theta$ . Then,

$$f_{e}^{n}(q,\theta^{1}) = \int_{\mathbf{Q}} f_{e}^{n-1}(q',\theta'(q,\theta^{1}))P(q,dq')$$
$$< \int_{\mathbf{Q}} f_{e}^{n-1}(q',\theta'(q,\theta^{2}))P(q,dq') = f_{e}^{n}(q,\theta^{2});$$

where the strict inequality follows from theorem A.5, and  $f_e^{n-1}(q,\theta)$  is strictly increasing in  $\theta$ .

By induction,  $f_e^n(q,\theta)$  is strictly increasing in  $\theta$  for all  $n \ge 1$ .

The other claims of the theorem can be shown similarly.

**Theorem A.15** Under assumptions 3.1 and 3.4, for any function  $f(q, \theta)$  increasing (decreasing) in q and decreasing (increasing) in  $\theta$ , the expected future values

$$f_e^n(q,\theta) \equiv \int_{\mathbf{S}} f(q',\theta') \Pi^n((q,\theta), d(q',\theta'))$$

are increasing (decreasing) in q, for all  $n \ge 1$ . The monotonicity is strict if  $f(q, \theta)$  is strictly monotone in  $\theta$ .

*Proof.* Consider any  $q^1, q^2 \in \mathbf{Q}$ ,  $q^1 < q^2$ , any  $\theta \in \Theta$ , and any  $f(q, \theta)$  weakly decreasing in q and strictly increasing in  $\theta$ . Then, after defining  $f_e^n \equiv f$  when n = 0,  $f_e^n(q, \theta)$  is weakly decreasing in q and strictly increasing in  $\theta$  for n = 0.

For  $n \ge 1$ , suppose that  $f_e^{n-1}(q, \theta)$  is weakly decreasing in q and strictly increasing in  $\theta$ . Then,

$$f_{e}^{n}(q^{1},\theta) = \int_{\mathbf{Q}} f_{e}^{n-1}(q',\theta'(q^{1},\theta))P(q^{1},dq')$$
$$> \int_{\mathbf{Q}} f_{e}^{n-1}(q',\theta'(q^{2},\theta))P(q^{1},dq')$$
$$\ge \int_{\mathbf{Q}} f_{e}^{n-1}(q',\theta'(q^{2},\theta))P(q^{2},dq') = f_{e}^{n}(q^{2},\theta)$$

where the strict inequality follows from theorem A.12, and the fact that  $f_e^{n-1}(q,\theta)$  is strictly increasing in  $\theta$ ; and the weak inequality from assumption 3.4 and the fact that  $f_e^{n-1}(q,\theta)$  is weakly decreasing in q. Also, theorem A.14 implies that  $f_e^n(q,\theta)$  is strictly increasing in  $\theta$ .

By induction,  $f_e^n(q,\theta)$  is weakly decreasing in q and strictly increasing in  $\theta$  for all  $n \ge 1$ . The other claims of the theorem can be shown similarly.