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The Pricing of Derivatives on Assets with Quadratic Volatility

by

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THE PRICING OF DERIVATIVES ON ASSETS WITH QUADRATIC VOLATILITY

CHRISTIAN ZÜHLSDORFF

ABSTRACT. The basic model of financial economics is the Samuelson model of geometric Brownian motion because of the celebrated Black-Scholes formula for pricing the call option. The asset's volatility is a linear function of the asset value and the model guarantees positive asset prices. In this paper it is shown that the pricing partial differential equation can be solved for level-dependent volatility which is a quadratic polynomial. If zero is attainable, both absorption and negative asset values are possible. Explicit formulae are derived for the call option: a generalization of the Black-Scholes formula for an asset whose volatility is affine, the formula for the Bachelier model with constant volatility, and new formulae in the case of quadratic volatility. The implied Black-Scholes volatilities of the Bachelier and the affine model are frowns, the quadratic specifications imply smiles.

JEL Classification G12, G13

Keywords strong solutions, stochastic differential equation, option pricing, quadratic volatility, implied volatility, smiles, frowns

INTRODUCTION

In their seminal article Black and Scholes (1973) derive a formula for the value of a call option if the underlying asset follows geometric or economic Brownian motion, a model introduced by Nobel laureate Paul Samuelson. For an account of the development of the geometric Brownian motion asset price model and pre-Black-Scholes attempts to price options in this model see Samuelson (1964).

Despite the overwhelming success of the Black-Scholes formula there were and are other models for asset prices. At the beginning of the century Louis Bachelier (1900) already tried to evaluate derivatives when changes in asset prices are normally distributed. This can be implemented as a model where volatility is a constant independent of the asset value. Cox and Ross (1976) develop the *constant elasticity of variance* (CEV) model where volatility is a power function. Ingersoll (1997) uses a quadratic volatility function with two real roots for an exchange rate futures.

In the following we first show that the conditions which are usually assumed to ensure the existence of strong solution to a stochastic differential equation (SDE) can be considerably

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relaxed: we only need local Lipschitz continuity of the coefficients and a condition on the drift, *boundedness on the diagonal*.

Then it is shown that the pricing partial differential equation (PDE) can be solved for level dependent volatility which is the product of a time dependent function and a quadratic polynomial. This adds the *affine*, the *quadratic with one real root*, and the *quadratic with no real root* volatility specification to the range of models with closed-form pricing formulae.

For most of the specifications presented in this paper the asset's price process can go negative. In this case, the two arbitrage-free behaviours in zero are absorption or to allow negative values. We will show that for quadratic volatility the pricing PDE is explicitly solvable in both cases. This invalidates a critique often put against the Bachelier model, that it would allow asset prices to become negative. In subsection 3.1 we derive the pricing formula for a call option in the positive Bachelier model in closed form.

For real option markets, the choice between a normal or a lognormal dynamics is not at all an academic one:

In addition, Bacheliers work seemed perfectly credible as the market (euroyen) started paying for 100 calls (puts struck at zero) based on the belief that the possibility of the market going negative could not be ruled out.

Taleb (1997, Perhaps Bachelier was right)

The structure of the paper is as follows: section 1 specifies the diffusion model and discusses the existence of solutions and pricing in the general model. In section 2, the solution for the pricing function of a European contingent claim in the quadratic volatility model is given. In section 3, the formulae for the value of a call option for the three possible specifications are presented: constant, affine, and quadratic. In section 4, the Black-Scholes implied volatilities of these specifications are compared. It turns out that with models of this type a wide variety of implied volatility *smiles* and *frowns* is possible. Section 5 concludes.

1. THE GENERAL DIFFUSION MODEL

Take a frictionless financial market where traders can costlessly store money and trade in an asset whose discounted (or futures) price X is modeled as the solution to the stochastic differential equation (SDE)

$$dX = \mu(t, X) dt + \sigma(t, X) dW \quad X(0) = x_0.$$

We assume that W is a one-dimensional Brownian motion with respect to a stochastic base $(\Omega, (\mathcal{F}_t), P)$, the filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. Notice that this is a model for the discounted price: for example, in the classic Samuelson model with a bond B as numeraire

$$\frac{dS}{S} = \mu dt + \sigma dW \quad dB = r dt$$

it is the forward price $X = S/B$ with dynamics

$$\frac{dX}{X} = (\mu - r) dt + \sigma dW.$$

Assumption 1.1. *The drift function μ and the volatility function σ are jointly continuous in (t, x) and locally Lipschitz-continuous globally in t , ie, there are constants $K_n \geq 0$ such that*

$$|\mu(t, x) - \mu(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq K_n |x - y| \quad \text{for all } |x|, |y| \leq n.$$

It is well known that under this assumption the SDE has a unique strong solution which may explode, see Kunita (1984, Theorem II.5.2), explosion of the process X means

$$P\{\sup_t |X(t)| = \infty\} > 0.$$

It is a commonly held belief in the literature that one needs a linear growth condition on the drift and the volatility function to ensure non-explosion. For example, Dumas, Fleming and Whaley (1998) ensure linear growth by capping their polynomial volatility at 50%

$$\sigma(t, x) = x \min(a_0 + a_1 x + a_2 x^2 + a_3 T + a_4 T^2 + a_5 x T, 0.5),$$

where T is the maturity of the claim. Andersen and Andreasen (2000) specify the limited CEV Libor market model with a truncated volatility

$$\sigma(t, x) = x \min(\varepsilon^{\alpha-1}, x^{\alpha-1}) \quad \alpha > 0.$$

Choosing a strictly positive ε ensures linear growth. The following proposition 1.3 shows that only the drift needs a restriction stronger than local Lipschitz-continuity:

Assumption 1.2. *μ is bounded on the diagonal globally in t , ie*

$$x \mu(t, x) \leq B(1 + x^2)$$

for a constant $B \geq 0$.

Proposition 1.3. *Under assumptions 1.1 and 1.2, the SDE has a unique non-exploding solution up to any finite time horizon $T < \infty$.*

The proof is in appendix A.1. Condition 1.2 is

- easier to check than other assumptions which are commonly used to ensure non-explosion, for example Feller conditions, see Karatzas and Shreve (1991, 5.5.29).
- less restrictive than linear growth, take e.g. the drift

$$\mu(t, x) = -\text{sign}(x) |x|^3$$

which is bounded-on-the-diagonal

$$x \mu(t, x) = -|x|^4 \leq 0$$

but not linearly bounded. This drift pulls X strongly towards zero.

Denote the state space of the price process X by

$$\mathcal{D} = [l, u], \quad -\infty \leq l < u \leq \infty.$$

It is the smallest interval such that

$$P\{X(t) \notin \mathcal{D}\} = 0 \quad \text{for all } t \geq 0.$$

A self-financing portfolio is given by its initial value $\pi(0)$ and a \mathcal{F} -previsible X -integrable process δ (the portfolio strategy), its value process

$$\pi(t) = \pi(0) + \int_0^t \delta(s) dX(s)$$

fulfilling the self-financing condition

$$d\pi = \delta dX.$$

A strategy is called *admissible* if its associated value process is a.s. bounded from below. In the following, we will deal with specifications that allow for $l = -\infty$, in this case the asset value is unbounded from below. This implies that the bite-and-hold portfolio, which is defined as lending x_0 for buying the asset in $t = 0$ and holding it till maturity, is not admissible:

$$\pi(0) = 0 \quad \delta(t) = 1 \quad \pi(t) = X(t) - X(0) = X(t) - x_0$$

To insure that the model is arbitrage-free, we make the following

Assumption 1.4.

1) The volatility function is strictly positive on the interior of the domain \mathcal{D} , ie

$$\sigma(t, x) > 0 \quad \text{for all } (t, x) \in [0, \infty) \times (l, u),$$

and

$$\frac{\mu(t, x)^2}{\sigma(t, x)^2} < \infty \quad \text{for all } (t, x) \in [0, \infty) \times \mathcal{D}.$$

2)

$$\int_0^t \frac{\mu(s, X(s))^2}{\sigma(s, X(s))^2} ds < \infty \quad P - a.s. \quad \text{for all } t < \infty$$

3) The exponential local martingale

$$\mathcal{E}(t, X) = \exp \left(- \int_0^t \frac{\mu(s, X(s))}{\sigma(s, X(s))} dW(s) - \frac{1}{2} \int_0^t \frac{\mu(s, X(s))^2}{\sigma(s, X(s))^2} ds \right)$$

is a true martingale.

From Girsanov's theorem, see Karatzas and Shreve (1991, 5.3.1), we know that in this case the equivalent measure P_T^* defined by

$$P_T^*(A) = E [1_A \mathcal{E}(T, X)]$$

is a (local) martingale measure for X on \mathcal{F}_T . The associated Brownian motion is

$$dW^* = dW + \frac{\mu(t, X)}{\sigma(t, X)} dt$$

and the dynamics of X under the measure P^* is

$$dX = \sigma(t, X) dW^* \quad X(0) = x_0.$$

A sufficient condition to ensure that assumption 1.4.3 holds is the Novikov criterion, see Karatzas and Shreve (1991, 3.5.13):

$$E \exp \left(\frac{1}{2} \int_0^T X(t)^2 dt \right) < \infty$$

A simple example for which condition 1.4 holds is to take $\mu = \sigma$.

If the boundary $\{l, u\}$ is attainable the only arbitrage-free behaviour of the process is to be absorbed in the boundary: if the process is reflected either instantly or after a stopping time, for example in the left boundary l , an arbitrageur just has to buy the asset and sell it later for a price which is surely higher.

For any arbitrage-free price process X with state space $[l, u]$ it is possible to construct another one with a smaller state space

$$[\tilde{l}, \tilde{u}] \subset [l, u].$$

This new arbitrage-free price process \tilde{X} is obtained by just stopping the process X at the new boundary. We can take for example the Wiener process W^* itself and obtain an arbitrage-free model with positive asset values by stopping in zero. This procedure gives the positive Bachelier model which is a standard Brownian motion absorbed in zero, cf. subsection 3.1.

The usual choice for the boundary is $l = 0$ and $u = \infty$ to model an asset, attainability of zero could be interpreted as bankruptcy. But also other choices of the boundary are reasonable, e.g. $l = 0$ and $u = 1$ for a zero-coupon bond.

A European option on the asset is given by its payoff, a non-negative measurable function g , and its maturity T . The fair value of the option under the pricing measure P^* is the expectation

$$E^* [g(X(T)) \mid X(t) = x].$$

Assumption 1.5. *There is a function $V \in C^{1,2}$ which satisfies the PDE*

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} = 0$$

with terminal value

$$V(T, x) = g(x).$$

If any of the bounds l or u is finite and attainable, the boundary condition for the PDE is

$$V(t, l) = g(l) \quad \text{or} \quad V(t, u) = g(u) \quad \text{for all } t.$$

Proposition 1.6. *Under assumptions 1.1, 1.2, and 1.5, the fair price of the contingent claim with payoff g and maturity T is equal to the solution V to the PDE:*

$$V(t, x) = E^* [g(X(T)) \mid X(t) = x]$$

Notably, this solution V is unique.

The **proof** is in appendix A.2. By applying Itô's formula to the value function

$$dV(t, X) = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} \right) dt + \frac{\partial V}{\partial x} dX = \frac{\partial V}{\partial x} dX$$

we see that the Δ -hedge $\delta(t) = \frac{\partial V}{\partial x}(t, X(t))$ gives the hedge strategy for the claim $g(X(T))$.

2. PRICING WITH QUADRATIC VOLATILITY

Assumption 2.1. *The volatility function σ can be split up in the form*

$$\sigma(t, x) = \gamma(t) p(x)$$

for a strictly positive bounded function γ and a two times differentiable function p .

Proposition 2.2. *Under the conditions of section 1 and assumption 2.1, the function p is a quadratic polynomial if and only if the pricing PDE has a solution of the following form*

$$V(t, x) = \lambda(\theta(t)) \xi(Z(x)) h(\theta(t), Z(x))$$

such that the time and space changes θ and Z are defined by

$$\theta(t) = \int_t^T \gamma^2(s) ds \quad \text{and} \quad Z(x) = \int_c^x \frac{dy}{p(y)} \quad \text{for any } c \in (l, u).$$

The function $h \in \mathcal{C}^{1 \times 2}$ satisfies the heat equation

$$\frac{\partial h}{\partial \theta} = \frac{1}{2} \frac{\partial^2 h}{\partial z^2}$$

in the transformed time and space domain $[0, \theta(T)) \times \mathcal{R}$ with $\mathcal{R} = [L, U]$ for $L = Z(l)$ and $U = Z(u)$. The transformed initial condition is

$$h(0, z) = \frac{g(Z^{-1}(z))}{\xi(z)}$$

and the boundary condition is

$$h(\theta, L) = \frac{h(0, L)}{\lambda(\theta)} \quad \text{and} \quad h(\theta, U) = \frac{h(0, U)}{\lambda(\theta)}.$$

In the case that the proposition holds, p is a quadratic polynomial

$$p(x) = a + bx + cx^2$$

and the correction functions λ and ξ are given by

$$\lambda(\theta) = \exp(-\zeta \theta / 8) \quad \text{for } \zeta = b^2 - 4ac$$

and

$$\xi(z) = \sqrt{p(Z^{-1}(z))},$$

ie

$$V(t, x) = \exp(\zeta\theta/8) \sqrt{p(x)} h(\theta(t), Z(x)).$$

The **proof** is in appendix B.1. The choice of the constant c in the definition of Z is not important for the formula. It just shifts the bounds L, U and the initial condition for the solution to the heat equation h on this domain. This does not affect the values of h in $z = Z(x)$.

Tian (2000) showed that for volatility functions which satisfy assumption 2.1 the pricing PDE can be transformed to the heat equation if and only if

$$\frac{\partial^2}{\partial^2 x} \left(p \frac{\partial^3 p}{\partial^3 x} \right) = 0.$$

If p is not a quadratic polynomial, the time and spatial transformations θ and Z may depend themselves jointly on t and x . Unfortunately, we were unable to find an example for a function which fulfils the condition except quadratic polynomials. Tian (2000)'s result is an application of the general method by Carr, Lipton and Madan (April 2000) for reducing PDEs to the heat equation. Define

$$\tau(t) = \sqrt{\theta(t)}$$

which is more convenient later for the closed-form call price formulae.

Corollary 2.3. *Under the assumption of proposition 2.2, the transition density of the price process X with dynamics $dX = \gamma(t)p(X)dW^*$ under the pricing measure P^* is given by*

$$\begin{aligned} \varrho^X(t, x, T, y) &= \frac{P^*(X(T) \in dy \mid X(t) = x)}{dy} \\ &= \exp(\zeta\tau^2(t)/8) \frac{\sqrt{p(x)}}{\sqrt{p(y)}} \frac{\varrho(\tau(t), Z(x), Z(y))}{p(y)} \end{aligned}$$

where $\varrho(\tau, z, v)$ is the transition density of absorbed Brownian motion in the interval $[L, U]$.

The **proof** is in appendix B.2. Using the explicit representation of the risk-neutral transition density, we can price any European option by numerical integration.

3. PRICING FORMULAE FOR THE CALL OPTION

In this section, we give closed form solutions for the call option

$$g(x) = (x - k)^+ \quad k \geq 0$$

for three possible specifications of the volatility function: constant, affine and quadratic. The density and the distribution function of the one-dimensional normal distribution are denoted by

$$\phi(x) = \frac{\exp(-x^2/2)}{\sqrt{2\pi}} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^x \phi(y) dy.$$

All proofs are in appendix C.

3.1. Constant Volatility.

$$p = 1$$

The first model of asset prices and as well the first description of Brownian motion, the thesis of Louis Bachelier (1900), english translation Bachelier (1964), is also the first attempt to price an option given the dynamics of the asset. The Bachelier model is generally assumed to imply that the price process can get negative, ie $\mathcal{D} = (-\infty, \infty)$. Take

$$dX = \sigma dW^* \quad \text{for a constant } \sigma > 0$$

as specification under the pricing measure. The pricing formula is

$$\text{Bac}(k, x, \tau) = \tau [d \Phi(d) + \phi(d)], \quad d = (x - k)/\tau.$$

If we model the asset as positive, ie with absorption in zero, the pricing formula is

$$\text{Bac}^+(k, x, \tau) = \text{Bac}(k, x, \tau) - \text{Bac}(k, -x, \tau).$$

In section 4 we will plot an example of the implied volatilities of the standard and the positive Bachelier model.

3.2. Affine Volatility.

$$p(x) = x - l \quad l \in \mathbb{R}$$

Volatility is an affine function with root in l , the asset is log-normally distributed on (l, ∞) . The famous model by Samuelson (1964) with a linear volatility function corresponds to the choice $l = 0$. In this case, the call price is given by the Black-Scholes formula:

$$\text{BS}(k, x, \tau) = x\Phi(d + \tau/2) - k\Phi(d - \tau/2) \quad d = \log(x/k)/\tau$$

The general affine pricing formula is

$$\text{AV}(k, x, \tau, l) = \text{BS}(k - l, x - l, \tau).$$

For $l < 0$ the asset price reaches zero with positive probability. If we choose to model with absorption in zero the price is given by the positive AV formula:

$$\text{AV}^+(k, x, \tau, l) = \text{BS}(k - l, x - l, \tau) - \text{BS}((k - l)(l - x)/l, -l, \tau)$$

In section 4 we will discuss the influence of the absorption term on implied volatilities.

Sondermann (1988) analyses a lognormal model where the asset price is bounded by some $u < \infty$ using a binomial tree. The no-arbitrage argument of section 1 tells us that the asset has to be absorbed in u . The value of the European call option is given by the Black-Scholes price minus a correction term:

$$\text{So}(k, x, \tau) = \text{BS}(k, x, \tau) - \text{BS}\left(u, \frac{xk}{u}, \tau\right)$$

The price of the call is lower than in the Black-Scholes case. This gives the market participant a price and a hedge if he attaches zero probability to the event

$$\{X > u\}.$$

3.3. Quadratic Volatility with two real roots and bounded domain. The price process is specified on the open interval $\mathcal{D} = (l, u)$ by the volatility:

$$p(x) = \frac{(x-l)(u-x)}{u-l} \quad l < u$$

The first one¹ to use this quadratic function with two real roots is Ingersoll (1997) who proposed a volatility of this form for exchange rates futures. Rady and Sandmann (1994) use it for forward bond prices, and Miltersen, Sandmann and Sondermann (1997) with $l = 0$ and $u = 1$ for (simple) forward rates. Amazingly, like the affine formula this quadratic pricing formula can be represented as a modified Black-Scholes formula

$$\text{Q2}(k, x, \tau, l, u) = \text{BS}(\tilde{k}, \tilde{x}, \tau)$$

for

$$\tilde{x} = \frac{(x-l)(u-k)}{u-l} \quad \text{and} \quad \tilde{k} = \frac{(k-l)(u-x)}{u-l}.$$

Ingersoll (1997) proves this formula using the PDE approach, Rady (1997) and Goldys (1997) give probabilistic proofs. Obviously

$$\lim_{u \rightarrow \infty} \text{Q2}(k, x, \tau, l, u) = \text{AV}(k, x, \tau, l).$$

The pricing formulae for the following three quadratic specifications seem to be new.

3.4. Quadratic Volatility with two real roots and unbounded domain. The value process with volatility

$$p(x) = (x-l)(x-l+d)/d \quad d > 0$$

has state space (l, ∞) . The minimum $-d^2/4 < 0$ of the parabola is in $l - d/2 < l$. Given the solution to the pricing PDE for the case of a bounded domain in the previous section, we are able to guess the solution in the unbounded case $x \in (l, \infty)$:

$$\text{Q2}_u(k, x, \tau, l, d) = \text{BS}(\tilde{k}, \tilde{x}, \tau)$$

¹to the author's knowledge

for

$$\tilde{x} = \frac{(x-l)(k-l+d)}{d} \quad \text{and} \quad \tilde{k} = \frac{(k-l)(x-l+d)}{d}.$$

It holds

$$\lim_{d \rightarrow \infty} Q2_u(k, x, \tau, l, d) = AV(k, x, \tau, l).$$

3.5. Quadratic Volatility with one real root.

$$p(x) = (x-l)^2$$

In this case, the pricing formula can be expressed as a modified positive Bachelier formula:

$$Q1(k, x, \tau, l) = (x-l)(k-l) \text{Bac}^+ \left(\frac{1}{x-l}, \frac{1}{k-l}, \tau \right)$$

The correction term for absorption in zero is given by

$$(x-l)(k-l) \text{Bac}^+ \left(\frac{1}{x-l}, \frac{l-2x}{(x-l)(k-l)}, \tau \right).$$

As we are going to see in the next section, this pricing rule is perfectly suited to model smiles of implied volatility. As it offers an analytical pricing formula, it is more convenient than the quadratic with no real root specification presented in the next subsection. In its pure form

$$dX = X^2 dW^* \quad X(0) = x_0$$

this specification is an important example: Delbaen and Schachermayer (1994) showed that this process is a local martingale for which there is no equivalent martingale measure. By Itô's formula the inverse process $R = 1/X$ fulfils

$$dR = X dt - dW^* = \frac{1}{R} dt - dW^* \quad \text{and} \quad R(0) = \frac{1}{x_0}.$$

So R is a Bessel process of dimension 3 which implies

$$\lim_{t \rightarrow \infty} X(t) = 0 \quad \text{a.s.}$$

as R reaches ∞ a.s., see Karatzas and Shreve (1991, 3.3.24). Contrary to intuition, the process X does not explode but converges to zero for almost all paths.

The process R is not arbitrage-free with respect to admissible integrands as was shown in Delbaen and Schachermayer (1995). So, if we take X as a model for an exchange rate, in the model's domestic country there is no arbitrage as X is a local martingale, but in the foreign country arbitrageurs can make riskless profits if they are allowed to use general admissible portfolios. Both processes are arbitrage-free with respect to simple predictable integrands. This remarkable fact shows the dependence of the concept of no-arbitrage on the notion of admissibility: an admissible portfolio in one country may not be admissible after a numeraire change.

3.6. Quadratic Volatility with no real roots.

$$p(x) = d \left[1 + \left(\frac{x - m}{d} \right)^2 \right] \quad d > 0$$

This specification is the most flexible one. It allows to model extreme forms of implied volatility, cf. the examples in section 4. The parameter m defines the minimum of the parabola and d its slope. For any state space $[l, u]$ the space transformation is

$$Z(x) = \arctan \left(\frac{x - m}{d} \right) - a \quad \text{for} \quad a = \arctan \left(\frac{l - m}{d} \right).$$

Define

$$A = U + a = Z(u) + a = \arctan \left(\frac{u - m}{d} \right).$$

The pricing PDE is solvable in semi-closed form as a sine expansion

$$Q0(k, x, \tau, m, d) = \frac{2e^{\tau^2/2}\sqrt{d}}{U \cos(a + Z(x))} \sum_{n>0} c_n e_n(\tau^2) \sin_n(Z(x))$$

for

$$e_n(\tau^2) = \exp \left(-n^2 \pi^2 \tau^2 / 2U^2 \right) \quad \text{and} \quad \sin_n(z) = \sin(n\pi z/U)$$

and coefficients c_n given in appendix C.5. For the implementation, we recommend to use the fast recursive algorithms of Goertzel and Reintsch, cf. appendix C.6.

4. EXAMPLES OF IMPLIED VOLATILITIES

In this section we give examples of the behaviour of the (Black-Scholes) implied volatilities for different models. We will compute prices according to different specifications and then calculate the implied volatilities from these prices.² Recall the risk-neutral dynamics of X :

$$dX = p(X)\gamma dW^* \quad X(0) = x_0$$

The initial value of the price process is $x_0 = 100$. The specifications are chosen such that the at-the-money volatilities are equal to an absolute volatility of 20% = 1/5, ie $p(x_0)\gamma = 20$. The reference model is

Black-Scholes: a lognormal model $p(x) = x$ with $\gamma = 1/5$, its implied volatility is by definition constant.

As a first example, figure 1 plots two affine volatility function, the Bachelier constant, and a quadratic with two roots volatility function.

AV (I=20): is an affine model $p(x) = x - l$ with $l = -20$ and $\gamma = 1/6$.

AV (I=200): is an affine model with $l = -200$ and $\gamma = 1/15$.

Bac: is a Bachelier model $p(x) = 1$ with constant volatility $\gamma = 20$.

²To obtain accurate values far out-of- and in-the-money, the values were computed using the C++ library CLN by Bruno Haible which provides arbitrary precision arithmetic, see www.gnu.org/software/ or www.ginac.de/cln/.

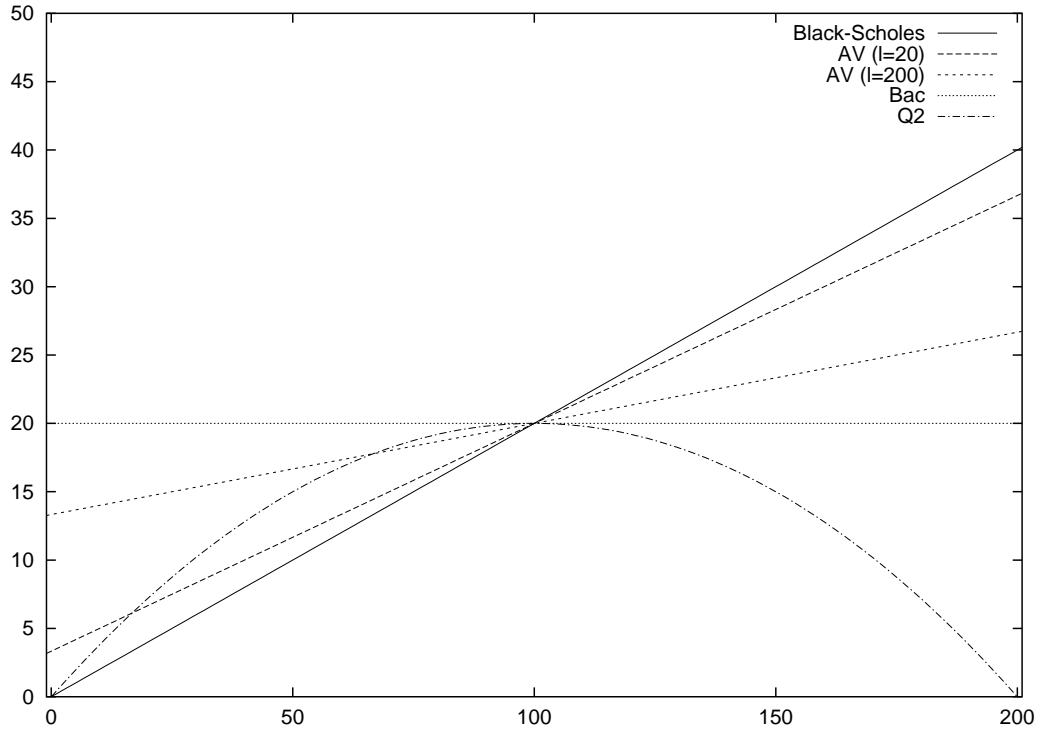


FIGURE 1. Affine and Q2 Volatility Functions

Q2: is a quadratic model $p(x) = (x - l)(u - x)/(u - l)$ with two real roots and bounded domain on $l = 0$ and $u = 200$, $\gamma = 2/5$.

Figure 2 plots the implied volatilities for the two affine and the Bachelier specifications with an option maturity $T = 1$. The implied volatilities of the affine models vary between the Bachelier, which is the limit for $l \rightarrow -\infty$, and the Black-Scholes ($l = 0$) implied volatility. Implied Black-Scholes volatilities of affine specifications are all *frowning*, ie, out-of-the-money implied volatility is higher and in-the-money implied volatility is lower than the constant Black-Scholes implied volatility.

The second example are two models with an upper bound.

So: is a lognormal Sondermann model with $\gamma = 1/5$ and $u = 200$ as absorbing upper bound.

Q2: is the quadratic model specified above, aslo with upper boundary $u = 200$.

Both models imply a stronger frown effect than the affine models. This is because their implied volatilities tend to zero at the upper boundary u .

Intuitively, the transformation from volatility function to implied volatility function is a clock-wise 45 degrees turn around the at-the-money point, in our examples the point $(100, 20)$. The previous examples did only show *frowns* of implied Black-Scholes volatilities. To make the implied volatility *smile* we have to use an input volatility function which is everywhere above

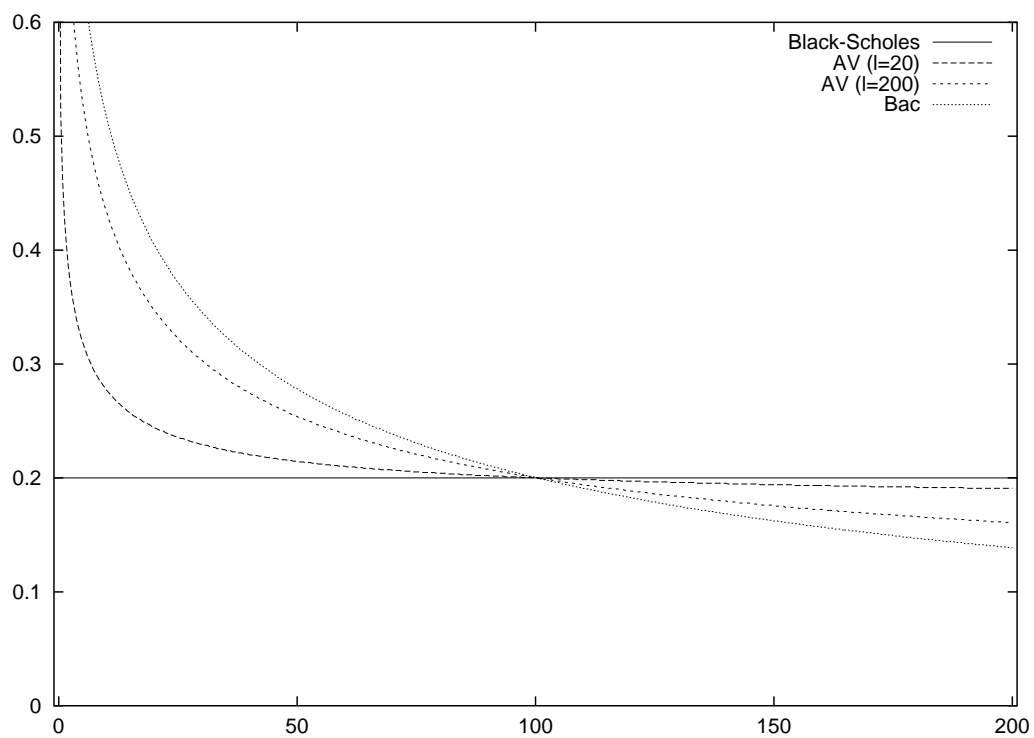


FIGURE 2. Implied Black-Scholes Volatilities of the Affine Specifications

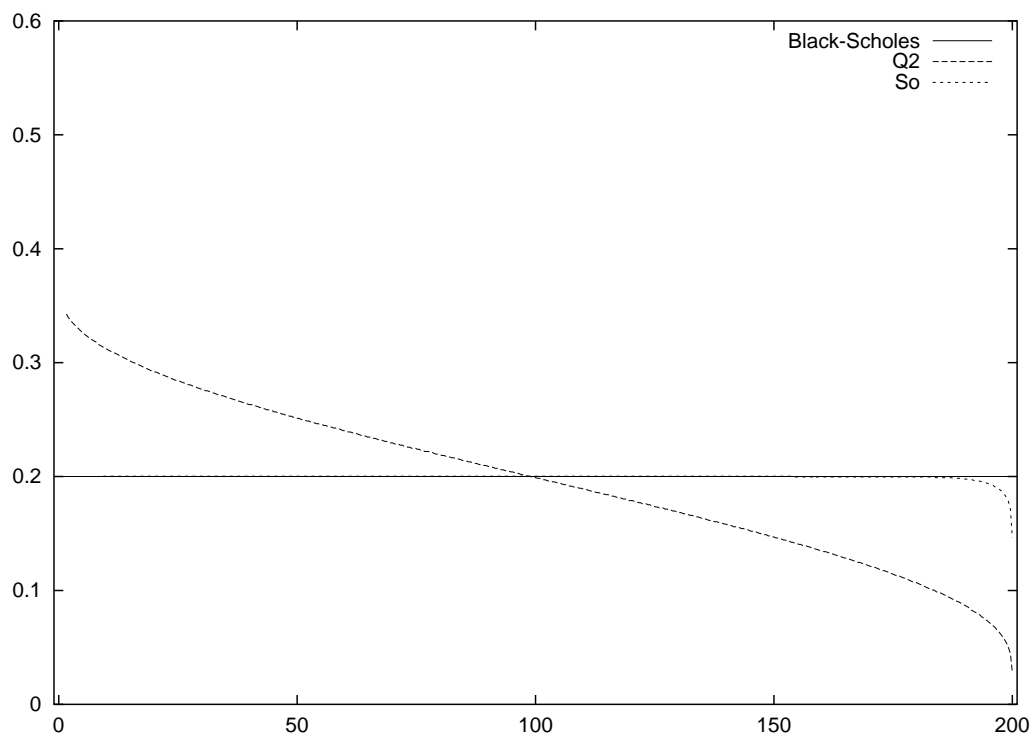


FIGURE 3. Implied Volatilities for Specifications with an Absorbing Upper Bound

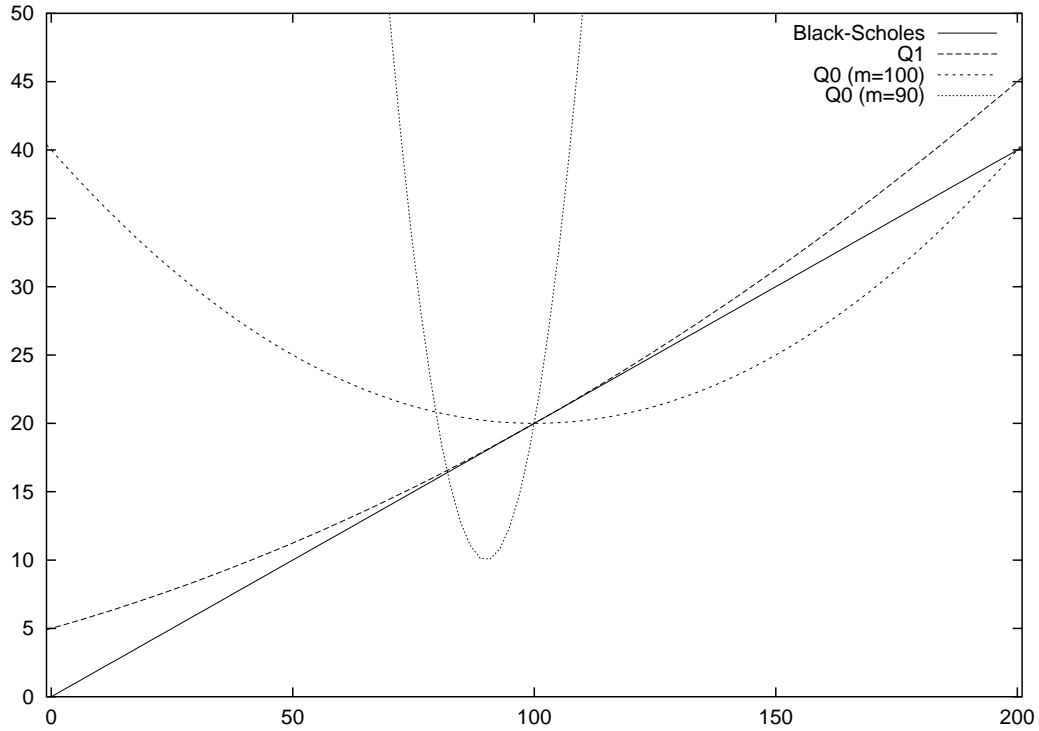


FIGURE 4. Quadratic Volatility Functions

the linear one. As third example, we plot in figure 4 the following specifications of the new quadratic models.

Q1: is a quadratic model $p(x) = (x-l)^2$ with one real root in $l = -100$, and $\gamma = 1/2000$.

Q0 (m=100): is a quadratic model with no real root, $m = 100$, $d = 100$, and $\gamma = 1/5$.

Q0 (m=90): is a quadratic model with no real root, $m = 90$, $d = 10$, and $\gamma = 1$.

Figure 4 plots their implied volatilities for a maturity of one year. Using these strongly non-linear volatility functions it is possible to obtain smiles, ie upward sloping implied volatilities. With extreme parameters like the second $Q0(m = 90)$, specification we can even have hockey-stick like shapes.

Finally, to show the difference in implied volatilities between an absorbed and an unrestricted model, figure 6 plots the implied volatilities for three different models in both cases.

Bac: is a Bachelier model with constant volatility $\gamma = 20$.

AV: is an affine model with $l = 100$ and $\gamma = 1/10$.

Q0: is a quadratic model with no real root, $m = 100$, $d = 100$, and $\gamma = 1/5$.

To see the effect more clearly we used an option expiry of ten years. The affine implied volatilities differ only for strikes far out-of-the-money and only if the volatility in zero is close to the constant (Bachelier) volatility. For practical purposes the difference is only of importance if the asset price is close to zero. For the unabsorbed specifications the implied volatilities diverge to infinity if asset values tend to zero.

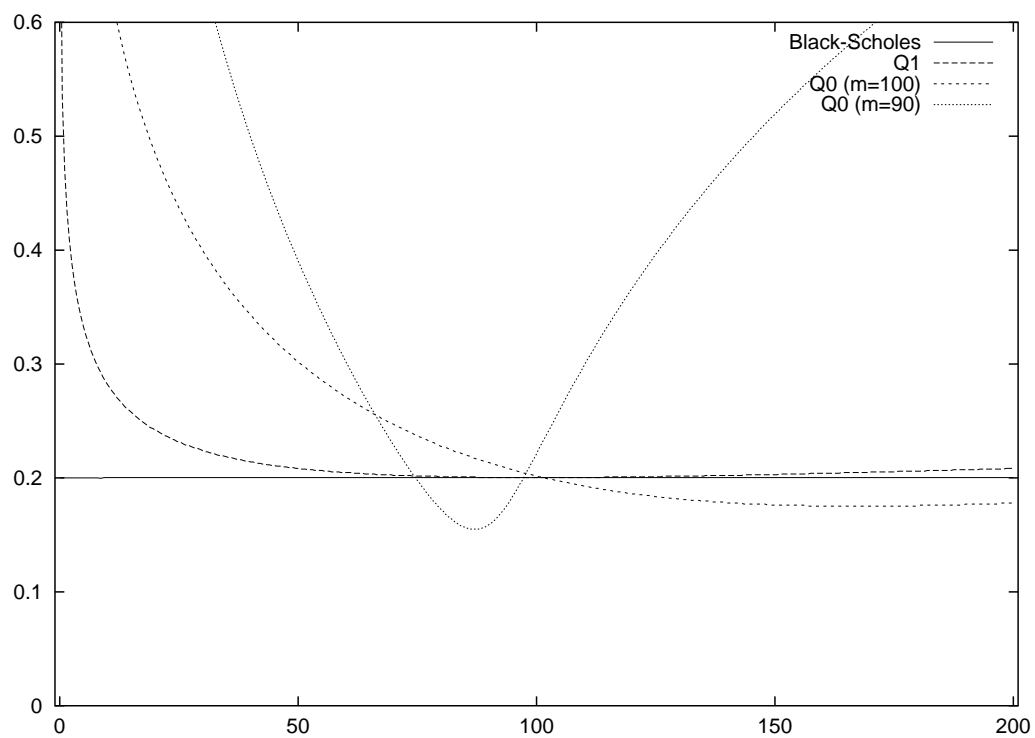


FIGURE 5. Implied Black-Scholes Volatilities for Quadratic Models

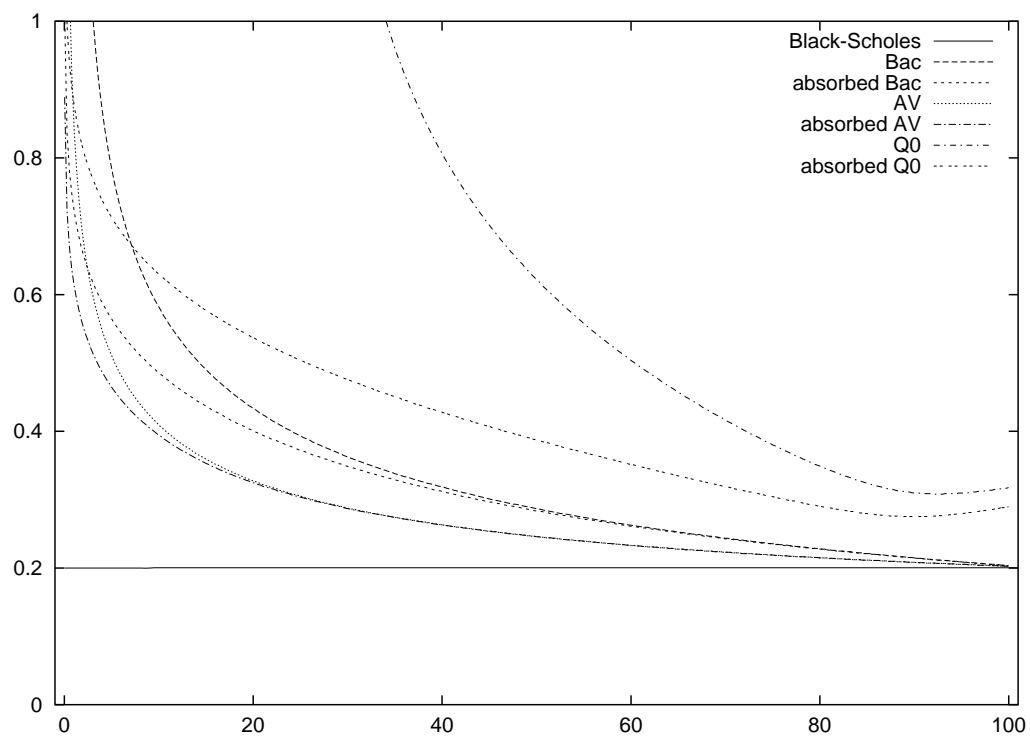


FIGURE 6. Implied Volatilities in 10 years

For models where the volatility in zero is larger than the constant one the effect is stronger. For the extreme Q0 in our example, the at-the-money implied volatility is pushed up for more than 30%. This happens because the quadratic volatility is higher than the linear Black-Scholes volatility below and above the at-the-money point. The affine volatility function is symmetrically higher below and lower above at-the-money, so higher volatility below balances lower volatility above.

5. CONCLUSION

In this paper it was shown that for an SDE to have a strong solution it is sufficient for the drift to be bounded on the diagonal. If we model with absorption in a left boundary $l > -\infty$ the condition is equivalent to linear growth of the drift on the positive axis. The volatility function is unrestricted except local Lipschitz-continuity.

Then it was shown that the pricing PDE can be solved for any quadratic volatility function, also in the presence of an absorbing boundary. Explicit formulae for the call option were given. They allow to model skews in implied Black-Scholes volatilities, both *frowns* and *smiles*. The formulae for quadratic volatility with one real root, no real root, and with two real roots on the unbounded domain seem to be new and allow notably to model smiles.

The discussion of the absorbing boundary showed that criticising the Bachelier specification for allowing the price to get negative is not valid: specifying absorption in zero is arbitrage-free and does give closed-form call option prices. This model seems to be of practical importance, for example in Japan in the late 1990s when interest rates were virtually zero. An application of these results to the market model of interest rates is given in Zühlendorff (2000).

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APPENDIX A. PROOFS FOR SECTION 1

A.1. Proof of Proposition 1.3. Suppose that X is a weak solution to the SDE

$$dX = \mu(t, X) dt + \sigma(t, X) dW.$$

By Theorem 10.2.1 in Stroock and Varadhan (1979), the following condition is sufficient to ensure non-explosion up to time T : there is a non-negative function F such that

$$\lim_{|x| \rightarrow \infty} F(x) = \infty$$

and

$$\mu(t, x)F'(x) + \frac{1}{2}\sigma(t, x)^2F''(x) \leq \lambda F(x)$$

for a constant $\lambda > 0$. To prove proposition 1.3 we construct a function fulfilling this condition. By continuity of σ , there is a constant C_T such that

$$|\sigma(t, 0)| \leq C_T \quad \text{for } t < T$$

and by the local Lipschitz condition

$$|\sigma(t, x)| \leq |\sigma(t, 0)| + |\sigma(t, x) - \sigma(t, 0)| \leq C_T + K_1|x|.$$

Let B be the constant from μ 's bounded-on-the-diagonal condition 1.2. The function

$$F(x) = B + \frac{1}{2} [(C_T + K_1)^2 + \log(1 + x^2)]$$

is strictly positive and it holds

$$F'(x) = \frac{x}{1+x^2} \quad \text{and} \quad F''(x) = \frac{1-x^2}{(1+x^2)^2} \leq \begin{cases} 0 & |x| \geq 1 \\ 1 & |x| \leq 1 \end{cases}.$$

Now we have

$$\begin{aligned} \mu(t, x)F'(x) + \frac{1}{2}\sigma(t, x)^2F''(x) &= \frac{x\mu(t, x)}{1+x^2} + \frac{1}{2} \frac{(1-x^2)\sigma(t, x)^2}{(1+x^2)^2} \\ &\leq B + \frac{1}{2}(C_T + K_1)^2 \\ &\leq F(x), \end{aligned}$$

so F satisfies the condition of the theorem with $\lambda = 1$. □

A.2. Proof of Proposition 1.6. The setup is that of Heath and Schweizer (2000), but we consider a closed domain so we have to specify a boundary condition for absorption. Proposition 1.6 is the convers to their theorem 1. Where they provide a condition (A.3) which is sufficient for the equivalence of the PDE and the martingale approach to option pricing, we show that the condition is necessary.

Take a sequence of bounded domains

$$[l_n, u_n] \uparrow [l, u].$$

If l or u is finite we can use $l_n = l$ or $u_n = u$ for all n . For any $t \leq T$ and any finite $x \in [l, u]$ we can find an n with $x \in [l_n, u_n]$. Let $X^{t,x}$ denote the solution to the SDE

$$dX^{t,x}(s) = \sigma(s, X^{t,x}(s)) dW^*(s), \quad X(t) = x$$

with absorbing boundary $\{l_n, u_n\}$. Define the stopping time τ_n as the hitting time by $X^{t,x}$ of the boundary before T , ie

$$\tau_n = T \wedge \inf\{s \leq T \mid X^{t,x}(s) = l_n \text{ or } X^{t,x}(s) = u_n\}.$$

By continuity of $X^{t,x}$ and V we have that $V(\tau_n, X^{t,x}(\tau_n)) < \infty$. As σ is locally Lipschitz-continuous, it is Lipschitz-continuous on $[l_n, u_n]$. Using theorem 6.5.2 of Friedman (1975) we know that the function

$$w_n(t, x) = E^* [V(\tau_n, X^{t,x}(\tau_n))]$$

is the unique solution to the PDE

$$\frac{\partial w_n}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 w_n}{\partial x^2} = 0$$

with boundary condition

$$w_n(t, l_n) = V(t, l_n) \quad w_n(t, u_n) = V(t, u_n) \quad \text{for } t \leq T$$

and terminal condition

$$w_n(T, x) = V(T, x) = g(x) \quad \text{for } x \in [l_n, u_n].$$

As V does also solve the equation with the same boundary conditions it holds

$$V(t, x) = w_n(t, x) \quad \text{for } (t, x) \in [0, T) \times [l_n, u_n].$$

On the other hand, by Karatzas and Shreve (1991, 5.4.D) or the arguments in the proof of theorem 1 of Heath and Schweizer (2000) we know that $X^{t,x}$ is a strong Markov process, thus

$$E^* [g(X^{t,x}(T)) \mid \mathcal{F}_{\tau_n}] = E^* [V(T, X^{t,x}(T)) \mid \mathcal{F}_{\tau_n}] = V(\tau_n, X^{t,x}(\tau_n)).$$

Taking expectations we obtain

$$V(t, x) = w_n(t, x) = E^* [g(X^{t,x}(T))].$$

If l or u is finite, starting the process in each bound immediately gives the boundary condition

$$V(t, l) = g(l) \quad \text{and} \quad V(t, u) = g(u) \quad \text{for } t \leq T,$$

see also Cox and Miller (1965). □

APPENDIX B. PROOFS FOR SECTION 2

B.1. Proof of Proposition 2.2. The following proof is a generalisation of the one given by Rady and Sandmann (1994) for a model of discounted bond prices with roots in 0 and 1. In this section, function arguments are omitted and subscripts denote partial differentials. Note that for the time change τ and the spatial change Z it holds

$$\theta_t = \frac{\partial \theta}{\partial t} = -\gamma^2$$

and

$$Z_x = \frac{\partial Z}{\partial x} = \frac{1}{p} \quad Z_{xx} = \frac{\partial^2 Z}{\partial^2 x} = -\frac{p_x}{p^2} \quad Z^{-1}_z = \frac{\partial Z^{-1}}{\partial z} = p(Z^{-1})$$

First, we will show that the existence of a solution to the pricing PDE of the form given in assumption 2.1 is a sufficient condition for p to be a quadratic polynomial. Suppose that the solution of the pricing PDE is of the form

$$V(t, x) = \lambda(\theta(t)) \xi(Z(x)) h(\theta(t), Z(x))$$

where h is a solution to the heat equation $h_\theta = h_{zz}/2$. Then it holds

$$V_\theta = \xi(\lambda_\theta h + \lambda h_\theta), \quad V_{xx} = \frac{\lambda}{p^2} [\xi_{zz} h + 2\xi_z h_z + \xi h_{zz} - p_x(\xi_z h + \xi h_z)],$$

and the PDE becomes

$$\begin{aligned} 0 &= V_\theta - \frac{1}{2}p^2V_{xx} \\ &= \lambda\xi \underbrace{\left(h_\theta - \frac{1}{2}h_{zz}\right)}_{=0} - \lambda h_z \underbrace{\left(\xi_z - \frac{1}{2}p_x\xi\right)}_{E_1} - h \underbrace{\left(\frac{1}{2}\lambda(\xi_{zz} - p_x\xi_z) - \lambda_\theta\xi\right)}_{E_2}. \end{aligned}$$

To solve the PDE it must hold $E_1 = E_2 = 0$. As the correction function ξ does only depend on z the condition $E_1 = 0$ implies that

$$\xi_z = \frac{1}{2}p_x\xi.$$

Using $x = Z^{-1}(z)$, the solution to the differential equation is

$$\xi(z) = \exp \left[\frac{1}{2} \int_0^z p_x(Z^{-1}(y)) dy \right] = \sqrt{p(Z^{-1}(z))}$$

as

$$\int_0^z p_x(Z^{-1}(y)) dy = \int_c^{Z^{-1}(z)} \frac{p_x(v)}{p(v)} dZ^{-1}(z) = \log [p(Z^{-1}(z))].$$

For ξ it holds

$$\xi_{zz} = \frac{1}{4}\xi(2pp_{xx} + p_x^2),$$

so $E_2 = 0$ is equivalent to

$$\lambda_\theta = \frac{1}{4}\lambda(pp_{xx} - p_x^2/2).$$

As by assumption the function λ does only depend on θ the term in braces has to be constant. Differentiating $p_x^2 - 2pp_{xx} = \zeta = \text{const}$ once we obtain the equation

$$pp_{xxx} = 0$$

which implies that p is a quadratic polynomial as $p > 0$ on the domain by assumption. So we can solve

$$\lambda_\theta = -\lambda\theta/8 \quad \text{for} \quad \lambda(\theta) = \exp(-\zeta\theta/8).$$

The initial condition is

$$g(x) = V(0, x) = \xi(Z(x)) h(0, Z(x))$$

which gives

$$h(0, z) = g(Z^{-1}(z))/\xi(z),$$

the transformed initial condition.

Second, we have to proof that a quadratic polynomial

$$p(x) = a + bx + cx^2$$

implies a solution of the form stated in assumption 2.1. This is trivial as we can easily build the solution using to construct Z and λ . \square

B.2. Proof of Corollary 2.3. We have to verify that

$$\int_{\mathbb{R}} g(y) \varrho^X(t, x, T, y) dy = E^* [g(X(T)) \mid X(t) = x]$$

for any continuous function g with bounded support, see Karatzas and Shreve (1991, 5.7.9ff). The transition density $\varrho(\tau, v, z)$ of absorbed Brownian motion on the interval $[L, U]$ is the fundamental solution of the heat equation on this interval, see Karatzas and Shreve (1991, 2.8.C and 4.3.D). Combining this with the representation of theorem 2.2 we get

$$\begin{aligned} E^* [g(X(T)) \mid X(t) = x] &= \lambda(\tau^2(t)) \xi(Z(x)) h(\tau^2(t), Z(x)) \\ &= \lambda(\tau^2(t)) \xi(Z(x)) \int_{\mathbb{R}} \frac{g(Z^{-1}(v))}{\xi(v)} \varrho(\tau(t), v, Z(x)) dv \\ &= \lambda(\tau^2(t)) \xi(Z(x)) \int_{\mathbb{R}} g(y) \frac{\varrho(\tau(t), Z(x), Z(y))}{\xi(Z(y)) p(y)} dy. \end{aligned}$$

For the first equality we used the representation of the solution of the heat equation as an integral of its initial condition with respect to the transition density of absorbed Brownian motion. For the second we used the substitution $v = Z(y)$ with

$$dv = Z'(y) dy = \frac{dy}{p(y)}.$$

Now we can identify the fundamental solution of the pricing PDE as

$$\varrho^X(t, x, T, y) = \lambda(\tau^2(t)) \frac{\xi(Z(x))}{\xi(Z(y))} \frac{\varrho(\tau(t), Z(x), Z(y))}{p(y)}$$

which is also the risk-neutral transition density. \square

APPENDIX C. PROOFS FOR SECTION 3

Notice the following symmetry which will ease the computation of the formulae in the case of an absorbing boundary in zero: the boundary condition for h in L will always be of Dirichlet type because

$$h(\tau^2, L) = (Z^{-1}(L) - k)^+ = (l - k)^+ = 0.$$

Suppose that we have solved the pricing PDE on the domain $[l, u]$ for an $l < 0$. If we want to find a solution on the domain $[0, u]$, the only characteristic of the PDE that changes is the boundary condition which is for the call to be 0 in 0. The obvious solution is to take

$$h_0(\tau^2, z) = h(\tau^2, z) - h(\tau^2, 2Z(0) - z)$$

as the solution to the heat equation on $\mathcal{R} = [Z(0), U]$.

The transition density of the one-dimensional Wiener process

$$\varrho(\tau, x, y) = \frac{P\{W(\tau^2) \in dy \mid W(0) = x\}}{dy} = \frac{1}{\tau} \phi\left(\frac{x - y}{\tau}\right).$$

is the fundamental solution of the heat equation on the unbounded domain $(-\infty, \infty)$. Throughout the proofs we will use the abbreviations

$$z = Z(x) \quad \text{and} \quad K = Z(k).$$

C.1. Constant Volatility.

$$p = 1$$

It holds

$$\zeta = 0 \quad \lambda = 1 \quad Z(x) = x \quad Z^{-1}(z) = z \quad \xi(Z^{-1}(z)) = 1$$

The state space for this specification is the whole real line $\mathcal{D} = (-\infty, \infty)$, there is no boundary condition as $\mathcal{R} = (-\infty, \infty)$. With the substitution $y = x - \tau\tilde{y}$ we easily solve

$$\begin{aligned} \text{Bac}(k, x, \tau) = h(\tau^2, x) &= \int_{-\infty}^{\infty} h(0, y) \varrho(\tau, x, y) dy \\ &= \int_{-\infty}^{\infty} (y - k)^+ \varrho(\tau, x, y) dy \\ &= \int_{-\infty}^d (x - \tau\tilde{y} - k) \phi(\tilde{y}) d\tilde{y} \\ &= (x - k)\Phi(d) + \tau\phi(d) \quad \text{for} \quad d = \frac{x - k}{\tau}. \end{aligned}$$

Using

$$Z(0) = 0 \quad \text{and} \quad Z(x) = x$$

the price for the case of an absorbing boundary in 0 is

$$\text{Bac}^+(k, x, \tau) = \text{Bac}(k, x, \tau) - \text{Bac}(k, -x, \tau).$$

C.2. Affine Volatility.

$$p(x) = x - l$$

We have

$$\begin{aligned} \zeta &= 1 \quad \lambda(\tau^2) = \exp(-\tau^2/8) \\ Z(x) &= \log(x - l) \quad Z^{-1}(z) = e^z + l \quad \xi(z) = e^{z/2} \end{aligned}$$

The initial value is

$$h(0, z) = (e^z + l - k)^+ e^{-z/2} = e^{z/2} - (k - l)e^{-z/2}.$$

For the state space $\mathcal{D} = (l, \infty)$ with $\mathcal{R} = \mathbb{R}$ we get

$$\begin{aligned} h(\tau^2, z) &= \int_{-\infty}^{\infty} h(0, y) \varrho(\tau, z, y) dy \\ &= \int_{-\infty}^d (e^{z-\tau y} + l - k) e^{-\frac{z-\tau y}{2}} \phi(y) dy \\ &= e^{\tau^2/8} [e^{z/2} \Phi(d + \tau/2) - (k - l)e^{-z/2} \Phi(d - \tau/2)]. \end{aligned}$$

with

$$d = \frac{z - K}{\tau} = \frac{1}{\tau} \log \left(\frac{x - l}{k - l} \right), \quad K = Z(k).$$

Combining the factors we get

$$\begin{aligned} \text{AV}(k, x, \tau, l) &= \lambda(\tau^2) e^{z/2} h(\tau^2, z) \\ &= (x - l) \Phi(d + \tau/2) - (k - l) \Phi(d - \tau/2) \\ &= \text{AV}(k - l, x - l, \tau, 0) \\ &= \text{BS}(k - l, x - l, \tau). \end{aligned}$$

For $l < 0$ it holds $Z(0) = \log(-l)$ and the correction for absorption in zero is

$$\exp(\tau^2/8) e^{z/2} h(\tau^2, 2Z(0) - z) = \text{BS} \left(\frac{(k - l)(l - x)}{l}, -l, \tau \right).$$

The easiest way to proof the Sondermann formula for absorption in $u < \infty$ is to proceede like in the case for absorption in zero: try to find a correction function of the form

$$h_U(\tau, z) = h(\tau, B \pm z)$$

with B to be determined. The new function fulfils the heat equation, so we have to choose a B such that the new function satisfies the boundary constraint

$$h(\tau, U) - h_U(\tau, U) = h(0, U).$$

The candidat

$$h_U(\tau, z) = h(\tau, z + 2(K - U))$$

satisfies this constraint and also the initial condition as

$$h_U(0, z) = 0 \quad \text{for } z \in (-\infty, U].$$

C.3. Quadratic Volatility with two real roots.

$$p(x) = \frac{(x - l)(u - x)}{u - l} \quad l < x < u$$

The relevant parameters and functions are

$$\begin{aligned} \zeta &= 1 & \lambda(\tau^2) &= \exp(-\tau^2/8) \\ Z(x) &= \log \left(\frac{x - l}{u - x} \right) & Z^{-1}(z) &= \frac{u - l}{1 + e^{-z}} - l & \xi(z) &= \frac{\sqrt{u + l}}{e^{z/2} + e^{-z/2}} \end{aligned}$$

The initial value is

$$h(0, z) = \frac{1}{\sqrt{u + l}} [(u - k)e^{z/2} - (k + l)e^{-z/2}]^+$$

so for $\mathcal{D} = \mathcal{R} = \mathbb{R}$:

$$\begin{aligned}
\sqrt{u+l} \lambda(\tau^2) h(\tau^2, z) &= \lambda(\tau^2) \int_K^\infty [(u-k)y^{1/2} - (k+l)e^{-y/2}] \varrho(\tau, z, y) dy \\
&= \lambda(\tau^2) \int_{-\infty}^{\frac{z-K}{\tau}} [(u-k)e^{(z-\tau y)/2} - (k+l)e^{-(z-\tau y)/2}] \phi(y) dy \\
&= (u-k)e^{z/2} \Phi\left(\frac{z-K}{\tau} + \frac{\tau}{2}\right) - (k+l)e^{-z/2} \Phi\left(\frac{z-K}{\tau} - \frac{\tau}{2}\right) \\
\xi(z) \lambda(\tau^2) h(\tau^2, z) &= \frac{(x+l)(u-k)}{u+l} \Phi\left(d + \frac{\tau}{2}\right) - \frac{(k+l)(u-x)}{u+l} \Phi\left(d - \frac{\tau}{2}\right) \\
d &= \frac{1}{\tau} \log\left(\frac{(x+l)(u-k)}{(u-x)(k+l)}\right)
\end{aligned}$$

C.4. Quadratic Volatility with one real root.

$$\begin{aligned}
p(x) &= (x-l)^2 & \zeta &= 0 & \mathcal{D} &= (l, \infty) \\
z = Z(x) &= \frac{-1}{x-l} & Z^{-1}(z) &= l - \frac{1}{z} & \mathcal{R} &= (-\infty, 0) \\
\xi(z) &= |-1/z| = -1/z & \xi(Z(x)) &= x-l \\
h(0, z) &= -z \left(l - k - \frac{1}{z}\right)^+ = (z(k-l) + 1)^+
\end{aligned}$$

The solution in the transformed state space is

$$\begin{aligned}
h(\tau^2, z) &= \int_{-\infty}^0 h(0, y) [\varrho(\tau, z, y) - \varrho(\tau, z, -y) + \delta_0 P(y=0 | z, \tau)] dy \\
&= \int_{-\frac{1}{k-l}}^0 [y(k-l) + 1] [\varrho(\tau, z, y) - \varrho(\tau, z, -y)] dy + 2\Phi\left(\frac{z}{\tau}\right) \\
&= (k-l) \int_{-\frac{1}{k-l}}^{\frac{1}{k-l}} y \varrho(\tau, z, y) dy + \int_{-\frac{1}{k-l}}^0 \varrho(\tau, z, y) dy - \int_0^{\frac{1}{k-l}} \varrho(\tau, z, y) dy + 2\Phi\left(\frac{z}{\tau}\right) \\
&= (k-l) [d_+ \Phi_+ - d_- \Phi_- + \tau(\phi_+ - \phi_-)]
\end{aligned}$$

using the abbreviations

$$d_{\pm} = z \pm \frac{1}{k-l} \quad \phi_{\pm} = \phi\left(\frac{d_{\pm}}{\tau}\right) \quad \Phi_{\pm} = \Phi\left(\frac{d_{\pm}}{\tau}\right)$$

and

$$d_+(z) = \frac{x-k}{(x-l)(k-l)} \quad d_-(z) = \frac{2l-k-x}{(x-l)(k-l)}$$

It holds

$$Q1(k, x, \tau, l) = (x-l)(k-l) \text{Bac}^+\left(\frac{1}{x-l}, \frac{1}{k-l}, \tau\right).$$

To compute the correction term for absorption in zero we need

$$d_{\pm}^0 = d_{\pm}(2Z(0) - z) = d_{\pm}\left(\frac{2x-l}{l(x-l)}\right),$$

then

$$d_+^0(z) = \frac{3x - 2l}{(x-l)(k-l)}, \quad d_-^0(z) = \frac{x}{(x-l)(k-l)},$$

so the correction is

$$(x-l)(k-l) \text{Bac}^+ \left(\frac{1}{x-l}, \frac{l-2x}{(x-l)(k-l)}, \tau \right).$$

C.5. Quadratic with no real root.

$$p(x) = d \left[1 + \left(\frac{x-m}{d} \right)^2 \right] \quad d > 0$$

This implies

$$\begin{aligned} \zeta = -4 \quad \lambda(\tau^2) = e^{\tau^2/2} \quad Z(x) = \arctan \left(\frac{x-m}{d} \right) - a \\ L = 0 \quad U = A - a \leq \pi \end{aligned}$$

for

$$a = \arctan \left(\frac{l-m}{d} \right) \geq -\frac{\pi}{2} \quad \text{and} \quad A = \arctan \left(\frac{u-m}{d} \right) \leq \frac{\pi}{2},$$

then

$$Z^{-1}(z) = m + d \tan(z + a) \quad \text{and} \quad \xi(z) = \frac{\sqrt{d}}{\cos(z + a)}.$$

The initial value is

$$\begin{aligned} h(0, z) &= \frac{\cos(z+a)}{\sqrt{d}} (m + d \tan(z+a) - k)^+ \\ &= \frac{1}{\sqrt{d}} [(m-k) \cos(z+a) + d \sin(z+a)]^+. \end{aligned}$$

Define

$$H = h(0, U) = \frac{1}{\sqrt{d}} [(m-k) \cos(A) + d \sin(A)].$$

The general solution of the heat equation on the finite domain $(0, U)$ has a sine expansion of the form

$$\begin{aligned} h(\tau^2, z) &= \frac{2}{U} \sum_{n>0} a_n e_n(\tau^2) \sin_n(z) \\ e_n(\tau^2) &= e^{-n^2 \pi^2 \tau^2 / 2U^2} \quad \sin_n(z) = \sin(n\pi z/U) \end{aligned}$$

with coefficients

$$c_n = \int_0^U h(0, z) \sin_n(z) dz,$$

see Carslaw and Jaeger (1959, 3.3). The general integrals we need are

$$S(n, z) = \int \sin(z+a) \sin_n(z) dz = -\frac{\nu_n^+ \sin(a - \nu_n^- z) + \nu_n^- \sin(a + \nu_n^+ z)}{2q_n}$$

and

$$C(n, z) = \int \cos(z+a) \sin_n(z) dz = -\frac{\nu_n^+ \cos(a - \nu_n^- z) + \nu_n^- \cos(a + \nu_n^+ z)}{2q_n}$$

for

$$\nu_n^\pm = \frac{n\pi}{U} \pm 1, \quad \text{and} \quad q_n = \frac{n^2\pi^2}{U^2} - 1 = \nu_n^+ \nu_n^-.$$

So, the coefficients for the solution are

$$c_n = \frac{1}{\sqrt{d}} [(m - k) (C(n, U) - C(n, K)) + d (S(n, U) - S(n, K))].$$

As

$$a \pm \nu_n^\pm U = A \pm n\pi,$$

it holds

$$\begin{aligned} C(n, U) &= -\frac{\nu_n^+ \cos(A - n\pi) + \nu_n^- \cos(A + n\pi)}{2q_n} \\ &= -\frac{(\nu_n^+ + \nu_n^-) \cos(A + n\pi)}{2q_n} \\ &= (-1)^{n+1} \frac{n\pi \cos(A)}{U q_n} \end{aligned}$$

and

$$\begin{aligned} S(n, U) &= -\frac{\nu_n^+ \sin(A - n\pi) + \nu_n^- \sin(A + n\pi)}{2q_n} \\ &= -\frac{(\nu_n^+ + \nu_n^-) \sin(A + n\pi)}{2q_n} \\ &= (-1)^{n+1} \frac{n\pi \sin(A)}{U q_n}. \end{aligned}$$

Combining all this, we get

$$c_n = (-1)^{n+1} \frac{n\pi}{U} H - \frac{1}{\sqrt{d}} [(m - k)C(n, K) + dS(n, K)].$$

By the general pricing formula from proposition 1.6 the boundary condition in U is

$$h(\tau^2, U) = \frac{h(0, U)}{\lambda(\tau^2)} = H e^{-\tau^2/2}.$$

Now we have to compute the solution for the time dependent boundary. By Carslaw and Jaeger (1959, 3.5) it is given by a sine expansion with coefficients

$$\begin{aligned} b_n &= (-1)^{n+1} \frac{n\pi}{2U} \int_0^{\tau^2} \frac{h(\theta, U)}{e_n(\theta)} d\theta \\ &= H (-1)^{n+1} \frac{n\pi}{2U} \int_0^{\tau^2} \exp(q_n \theta/2) d\theta \\ &= (-1)^{n+1} \frac{n\pi \exp(q_n \tau^2/2) - 1}{U q_n} H. \end{aligned}$$

Combining all factors we get the coefficients a_n for the sine expansion of the solution:

$$a_n = b_n + c_n = (-1)^{n+1} \frac{n\pi \exp(q_n \tau^2/2)}{U q_n} H - \frac{1}{\sqrt{d}} [(m - k)C(n, K) + dS(n, K)]$$

which finally gives

$$\begin{aligned}
& \lambda(\tau^2)\xi(z)h(\tau^2, z) \\
&= \frac{\sqrt{d}}{\cos(z+a)} \frac{2}{U} \sum_{n>0} a_n e^{-q_n \tau^2/2} \sin_n(z) \\
&= \frac{1}{\cos(z+a)} \left[\sqrt{d} H \frac{\sin(z)}{\sin(U)} - \frac{2}{U} \sum_{n>0} [(m-k)C(n, K) + dS(n, K)] e^{-q_n \tau^2/2} \sin_n(z) \right]
\end{aligned}$$

as

$$\int_0^U \sin(z) \sin_n(z) dz = (-1)^{n+1} \frac{n\pi}{U} \frac{\sin(U)}{q_n}.$$

If $l = m$, which implies $a = 0$, the first term simplifies

$$\frac{\sqrt{d}H}{\sin(U)} \frac{\sin(z)}{\cos(z)} = \frac{\sqrt{d}H}{\sin(U)} \tan(z) = \frac{H}{\sin(U)} \frac{x-m}{\sqrt{d}}.$$

If further $u = \infty$, which implies $U = \frac{\pi}{2}$ and $H = \sqrt{d}$, the first term simplifies further

$$\frac{H}{\sin(U)} \frac{x-m}{\sqrt{d}} = x-m.$$

C.6. Implementing the Q0 formula. Because of the fast decreasing factors e_n we need only 200–300 terms in the expansion. To compute the $\sin_n(z)$ We can use two simple recursive algorithms by Goertzel and Reintsch, see e.g. Stoer and Bulirsch (1993). Define $\xi = \pi z/U \in [0, \pi]$ then it holds

$$\sin_0(z) = 0 \quad \sin_1(z) = \sin(\xi).$$

The Goertzel algorithm is given by

$$\sin_n(z) = c \sin_{n-1}(z) - \sin_{n-2}(z) \quad \text{with } c = 2 \cos \xi.$$

The algorithm is numerically unstable for small values of ξ . In this case, we can alternatively use the slightly more complicated Reintsch algorithm: if $0 \leq \xi \leq \frac{\pi}{2}$ use

$$\delta_n(z) = \lambda \sin_{n-1}(z) + \delta_{n-1}(z) \quad \delta_1(z) = \sin_1(z)$$

with

$$\begin{aligned}
\sin_n(z) &= \delta_n(z) + \sin_{n-1}(z) \\
\lambda &= 2(\cos \xi - 1) = -4 \sin^2(\xi/2).
\end{aligned}$$

For $\frac{\pi}{2} < \xi \leq \pi$ use

$$\begin{aligned}
\delta_n(z) &= \lambda \sin_{n-1}(z) - \delta_{n-1}(z) \quad \delta_1(z) = \sin_1(z) \\
\sin_n(z) &= \delta_n(z) - \sin_{n-1}(z)
\end{aligned}$$

with

$$\lambda = 2(\cos \xi + 1) = 4 \cos^2(\xi/2).$$