Topper, Jürgen

Working Paper

A Finite Element Implementation of Passport Options

Diskussionspapiere der Wirtschaftswissenschaftlichen Fakultät, Universität Hannover, No. 224e

Provided in Cooperation with:
School of Economics and Management, University of Hannover

Suggested Citation: Topper, Jürgen (2001) : A Finite Element Implementation of Passport Options, Diskussionspapiere der Wirtschaftswissenschaftlichen Fakultät, Universität Hannover, No. 224e

This Version is available at:
http://hdl.handle.net/10419/78300

Terms of use:
Documents in EconStor may be saved and copied for your personal and scholarly purposes.
You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.
If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.
A Finite Element Implementation of Passport Options

Discussion Paper No. 224
ISSN 0949-9962
MSC: 65M60, 65M70, 90C39, 49L25, 49L20, 91B30
JEL: C61, C63, G13

Version October 27, 2001

Jürgen Topper: Andersen
Financial and Commodity Risk Consulting (FCRC)
Mergenthalerallee 55
65760 Eschborn/Frankfurt
Germany
Juergen.Topper@De.ArthurAndersen.com

Department of Economics (Mathematical Economics)
University of Hannover
Königsworther Platz
30167 Hannover
Germany

I would like to thank Leif Andersen, Hans-Peter Deutsch, Vicky Henderson, Alexander Lipton-Lifschitz, Marian Maas, Vijay Parmar, Riccardo Rebonato, Granville Sewell, Jan Večer and Paul Wilmott with respect to the discussions leading to the present article.
Abstract

Except for special cases, passport options do not have closed-form solutions. Here we show how to derive approximate solutions using finite element methods. We also show that finite elements offer advantages in computing the hedge parameters.

Contents

1 Introduction 3

2 The Pricing Model for Passport Options 4
   2.1 Different Approaches ........................................ 4
   2.2 European Passport Options .................................. 4
   2.3 Non-Convex Payoffs .......................................... 6
   2.4 A Correction .................................................. 6
   2.5 General Payoffs ............................................... 7
   2.6 Integrating Early Exercise .................................. 7
   2.7 Extending Passport Options to General Options on Trading Accounts ................. 7

3 A Numerical Solution with Finite Elements 9
   3.1 One Spatial Variable .......................................... 9
   3.2 Two Spatial Variables ......................................... 10

4 Applications 13
   4.1 Plain Vanilla Passport Options .............................. 13
      4.1.1 European Passport Options ............................... 13
      4.1.2 American Passport Options .............................. 14
   4.2 Relative Exotics ............................................... 14
   4.3 Asian Options .................................................. 15
   4.4 Absolute Exotics ............................................... 16
      4.4.1 The Algorithm .............................................. 16
      4.4.2 Knock-Out Barriers ....................................... 17

Bibliography 18
1 Introduction

Passport options are a new kind of financial instrument introduced by Bankers Trust in 1997. They are used to protect trading accounts. The basic passport option allows the holder to take the profit from a trading account while any losses are covered by the writer of the option. The maximal amount a trader can go either long or short is limited to the same pre-specified amount. To make passport options cheaper, or to reduce the risk to the writer, certain exotic features such as caps, floors and barriers have been employed. The concept of passport options has been extended to general options on trading accounts where the limits for going short or long do not necessarily have to be equal anymore. This concept of an option on a trading account contains many special cases such as plain vanilla European and American options, passport options, and Asian options.

Passport options can be used to:

- protect the trading account of inexperienced traders;
- to price life insurance claims contingent on the performance of a reference fund [4];
- to develop new commodity hedging strategies [14].
2 The Pricing Model for Passport Options

2.1 Different Approaches

Currently, there are three approaches to pricing passport options which differ in the assumption on interest rates:

- Ahn/Penault/Wilmott ([1], [2], [30]): There is only one interest rate for borrowing and lending. The underlying does not pay any dividends.
- Andersen/Andreasen/Brotherton-Ratliffe [3]: There is only one interest rate \( r \) for borrowing and lending. The underlying pays a continuous dividend yield \( \gamma \).
- Hyer/Lipton-Lifschitz/Pugachevsky [16]: There are different rates for borrowing and lending.

We have chosen the approach by Andersen/Andreasen/Brotherton-Ratliffe because it can be extended easily by different interest rates for borrowing and lending ([3], S. 16). Also, we have adapted the notation of that paper. Different interest rates for borrowing and lending are a rather unusual assumption; even the authors of this approach discuss the special case of equal rates (the so-called symmetric case) in much more detail than the general case ([16], S. 129f). The approach by Ahn/Penault/Wilmott is, therefore, obviously a special case of the model by Andersen/Andreasen/Brotherton-Ratliffe with \( \gamma = 0 \).

2.2 European Passport Options

The starting point is the Black-Scholes framework, in which the underlying \( S \) follows the following stochastic differential equation; see [3]:

\[
\frac{dS(t)}{S(t)} = (r - \gamma) \, dt + \sigma dW(t)
\]

Consider an investor in \( t_i \) holding \( u(t_i) \in [-1, 1] \) in this underlying. From \( t_i \) to \( t_{i+1} \) the investor gains \( u(t_i) [S(t_{i+1}) - S(t_i)] \). Summing up over all periods, the total investor's total gain \( w \) is

\[
w = \sum_{i=0}^{H-1} u(t_i) [S(t_{i+1}) - S(t_i)]
\]

Assuming continuous trading, i.e. \( \lim_{i \to 0} (t_i - t_{i+1}) = 0 \) the gain can be expressed as:

\[
w(t) = \int_0^t u(s) \, dS(s)
\]

\[
\iff \quad dw(t) = u(t) \, dS(t) \text{ with } w(0) = 0
\]

The European passport option gives the holder the right but not the obligation to receive \( w \) in \( T \). In case \( w < 0 \) the rational investor is not interested in delivery so that the payoff equals

\[
[w(T)]^+ \equiv \max[0, w(T)]
\]

For deriving the pricing equation we will use a similar argument as Black and Scholes: An instantaneously riskless portfolio \( \Pi \) consists of one passport option and \( -k \) units of the underlying

\[
\Pi = V - kS
\]

Within the time interval \( (t, t + dt) \) the value of this portfolio changes by

\[
d\Pi = dV - k(dS + \gamma S dt)
\]

We assume the existence of an optimal strategy \( u^* \) and the derivatives \( V_{SS}, V_{Sw}, \) and \( V_{S^2} \). We also presuppose that the holder of the option maximizes his revenues without being hindered from taking \( u^* \) by hedging necessities or other superimposed circumstances. Then the following holds:

\[
dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial w} dw + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS)^2 + \frac{\partial^2 V}{\partial S \partial w} (dS \, dw) + \frac{1}{2} \frac{\partial^2 V}{\partial w^2} (dw)^2
\]
To simplify this equation, two more results are needed. Squaring eq. (1) gives:

\[(dS)^2 = \sigma^2 S^2 dt\] (9)

The profit-maximizing behavior of the holder turns eq. (4) into:

\[dw = u^* dS\] (10)

Plugging these results into eq. (8) leads to:

\[dV = \frac{\partial V}{\partial t} dt + \left( \frac{\partial V}{\partial S} + u^* \frac{\partial V}{\partial w} \right) dS + \frac{1}{2} \left( \frac{\partial^2 V}{\partial S^2} + 2u^* \frac{\partial^2 V}{\partial S \partial w} + \frac{\partial^2 V}{\partial w^2} \right) \sigma^2 S^2 dt\] (11)

The parameter \(k\) has to be chosen for the portfolio II to become instantaneously riskless.

\[k = \frac{\partial V}{\partial S} + u^* \frac{\partial V}{\partial w}\] (12)

Because of the absence of arbitrage the riskless portfolio II has to grow by the same rate as a money market account \(r\).

\[dII = r II dt\] (13)

Combining above results:

\[\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \left( \frac{\partial^2 V}{\partial S^2} + 2u^* \frac{\partial^2 V}{\partial S \partial w} + \frac{\partial^2 V}{\partial w^2} \right) + (r - \gamma) S \left( \frac{\partial V}{\partial S} + u^* \frac{\partial V}{\partial w} \right) = r V\] (14)

With the following final condition:

\[V(T, S, w) = w^+\] (15)

the substitution of variables \(x \equiv w/S\), reduces the dimensions of the problem by one. Using this substitution generates the following PDE:

\[\frac{\partial v}{\partial t} + (u^* - x)(r - \gamma) \frac{\partial v}{\partial x} + \frac{1}{2} (u^* - x)^2 \sigma^2 \frac{\partial^2 v}{\partial x^2} = \gamma v\] (16)

with

\[u^* = \text{sign} \left( (r - \gamma) \frac{\partial v}{\partial x} - x \sigma^2 \frac{\partial^2 v}{\partial x^2} \right)\] (17)

where \(v(T)\) has to be monotonously increasing and convex in \(x\).\(^1\) Equivalent formulations of eq. 17 are:

\[\frac{\partial v}{\partial t} - x(r - \gamma) \frac{\partial v}{\partial x} + \frac{1}{2} (1 + x^2) \sigma^2 \frac{\partial^2 v}{\partial x^2} + u^* \left( (r - \gamma) \frac{\partial v}{\partial x} - x \sigma^2 \frac{\partial^2 v}{\partial x^2} \right) = \gamma v\] (18)

and:

\[\frac{\partial v}{\partial t} - x(r - \gamma) \frac{\partial v}{\partial x} + \frac{1}{2} (1 + x^2) \sigma^2 \frac{\partial^2 v}{\partial x^2} + \left| (r - \gamma) \frac{\partial v}{\partial x} - x \sigma^2 \frac{\partial^2 v}{\partial x^2} \right| = \gamma v\] (19)

with:

\[v(T, x) = v_T(x)\] (20)

as a final condition. Only for \(r = \gamma\) a closed-form solution is known (compare sec. 4.1.1). For \(r \neq \gamma\) we have to resort to numerical techniques.

Eq. (19) is the PDE for the passport option value, given a strategy \(u^*\). It is also possible to view this problem from a different perspective by looking for an equation that defines the optimal strategy \(\mu^*\). Via the principles of dynamic programming, a PDE, called the Hamilton-Jacobi-Bellmann equation (HJB), can be derived which defines this optimal strategy [18]:

\[(r - \gamma) S \frac{\partial V}{\partial S} + rw \frac{\partial V}{\partial w} - r V - \frac{\sigma^2 S^2}{2} \max_{-1 \leq \mu^* \leq 1} \left[ V_{SS} + 2\mu^* V_{Sw} + (\mu^*)^2 V_{ww} \right] = V_t\]

\(^1\)Non-convex Payoffs are discussed below in sec. 2.3.
This equation has to be used as a basis for numerical computations when \( u^* \) and \( \mu^* \) are not known. Such problems arise when caps or barriers on either \( w \) and/or \( S \) are introduced. A transformation as \( x \equiv w/S \) is then not possible anymore.

The **Hedge Ratio** \( k \) is slightly different to the Black-Scholes framework. The basic idea is that a portfolio \( \Pi \) consisting of a long call \( C \) and a short position in \( k \) shares \( S \)

\[
\Pi = C - k S \tag{22}
\]

is riskfree for an infinitesimal amount of time.\(^2\) The hedge parameter in the Black-Scholes model is

\[
k = \frac{\partial C}{\partial S} = \Delta \tag{23}
\]

Portfolios containing passport options can be immunized against infinitesimal changes in the share via eq. (22). For \( k \) we have (compare ([3], S. 33f)):

\[
k = \nu + (u^* - x) \frac{\partial v}{\partial x} \tag{24}
\]

This implies that numerical difficulties arising in computing \( \Delta \) are also present in computing \( k \). Finite elements provide approximate solutions to the entire domain consisting of simple algebraic functions. Whenever \( u^*(S) \) changes its sign, \( k(x) \) shows a jump.

### 2.3 Non-Convex Payoffs

#### 2.4 A Correction

According to ([3], **Proposition 5**) the result of the previous section can be generalized to non-convex payoffs by changing the control to \( u^* \in [-1, 1] \):

\[
u^* = \begin{cases} 
\psi & \text{if } \psi(x, t) \in [-1, 1] \text{ and } \frac{\partial v}{\partial x} < 0 \\
\text{sign}(\psi(x, t)) & \text{otherwise}
\end{cases} \tag{25}
\]

with

\[
\psi(x, t) = x - \frac{r - \gamma}{\sigma^2} \frac{\partial v}{\partial x} \tag{26}
\]

With the help of a simple counterexample it can be shown that this proposition is wrong. Convex payoffs are supposed to be a special case of eq. (25). This is not the case. We consider the special case of \( r = \gamma \) which can easily be extended to \( r \neq \gamma \). The unique optimal control \( u^* \) for convex payoffs is according to eq. (17) (see also [3], **Proposition 2**):

\[
u^*(x, t) = \text{sign} \left( (r - \gamma) \frac{\partial v}{\partial x} - x \sigma^2 \frac{\partial^2 v}{\partial x^2} \right) \tag{27}
\]

Inserting \( r = \gamma \) simplifies the expression:

\[
u^*(x, t) = \text{sign} \left( -x \sigma^2 \frac{\partial^2 v}{\partial x^2} \right) \tag{28}
\]

Convexity of \( v \) in \( x \) implies \( \frac{\partial^2 v}{\partial x^2} \geq 0 \). For \( t < T \) even strong convexity holds: \( \frac{\partial^2 v}{\partial x^2} > 0 \). Together with \( \sigma > 0 \) it has to hold in \( t \in [0, T] \)

\[
\text{sign} \left( -x \sigma^2 \frac{\partial^2 v}{\partial x^2} \right) = \text{sign} (-x) \tag{29}
\]

Inserting \( r = \gamma \) into the general payoff eq. (25) gives:

\[
u^* = \text{sign}(\psi) = \text{sign}(x) \tag{30}
\]

Therefore, the convex payoff is not a special case of the general payoff. That shows that ([3], **Proposition 5**) is wrong.

---

\(^2\) To be exact: Black and Scholes [5] consider a portfolio short in the call and long in the share. In order to keep conformity to [3] we have changed the positions.
2.5 General Payoffs

We will present the general control $u^*$ for arbitrary payoffs in this section first. Then we will deduce the special controls for convex and concave payoffs.

The general control is:

$$ u^* = \begin{cases} 
\psi & \text{if } \psi(x, t) \in [-1, 1] \text{ and } \frac{\partial^2 v}{\partial x^2} < 0 \\
\text{sign}(-\psi \frac{\partial^2 v}{\partial x^2}) & \text{otherwise}
\end{cases} $$. \hspace{1cm} (31)

For strictly concave payoff functions ($\frac{\partial^2 v}{\partial x^2} < 0$) eq. (31) simplifies to:

$$ u^* = \begin{cases} 
\psi & \text{if } \psi(x, t) \in [-1, 1] \\
\text{sign}(\psi) & \text{otherwise}
\end{cases} $$. \hspace{1cm} (32)

For (strictly and simply) convex payoff functions the following control function holds:

$$ u^* = \text{sign} \left(-\psi \frac{\partial^2 v}{\partial x^2} \right) $$. \hspace{1cm} (33)

$$ = \text{sign} \left( x - r - \gamma \frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x^2} \right) $$. \hspace{1cm} (34)

$$ = \text{sign} \left( \frac{1}{\sigma^2} \left( (r - \gamma) \frac{\partial v}{\partial x} + \sigma^2 x \frac{\partial^2 v}{\partial x^2} \right) \right) $$. \hspace{1cm} (35)

$$ = \text{sign} \left( (r - \gamma) \frac{\partial v}{\partial x} + \sigma^2 x \frac{\partial^2 v}{\partial x^2} \right) $$. \hspace{1cm} (36)

since $\sigma^2 \geq 0$. This is eq. (18) from [3].

2.6 Integrating Early Exercise

Early exercise of the option can be integrated with a penalty function $p$. This function $p$ ensures that in areas of the $(t, S)$ space where early exercise is optimal, the pricing eq. (19) is forced to take on the intrinsic value of the option while it vanishes on the rest of the domain. For details of this technique and various specifications of $p$ see [8] and [19].

$$ \gamma v = \frac{\partial v}{\partial t} - x(r - \gamma) \frac{\partial v}{\partial x} + \frac{1}{2} \left( 1 + x^2 \right) \sigma^2 \frac{\partial^2 v}{\partial x^2} + \left( r - \gamma \right) \frac{\partial v}{\partial x} - x \sigma^2 \frac{\partial^2 v}{\partial x^2} + p $$

$$ p = e_{\text{penalty}} \min \left( \max(S - E, 0), 0 \right) $$. \hspace{1cm} (37)

$$ (38) $$

According to [8], $e_{\text{penalty}}$ depends on the type of the element; according to our experience, it suffices to choose $e_{\text{penalty}}$ sufficiently large such as $e_{\text{penalty}} = 10^8$. Computing the value of American passport options can be greatly simplified when all interest rates in the model are identical and there are no dividends. Since you can always emulate early exercise with an European option by entering a zero position in the stock, the American option is worth the same.

2.7 Extending Passport Options to General Options on Trading Accounts

The defining property of an option on a trading account is that the control is not restricted to the closed interval $[-1, 1]$ anymore ([23], [24]). Besides, we generalize eq. (4) to

$$ dw(t) = u^*(t)dS(t) + \tilde{r} \left[ w - u^*(t)S(t) \right] dt $$

$$ w(0) = w_0 $$

(39) $w(0) = w_0$.

The initial wealth is represented by $w_0$, and $\tilde{r}$ is the interest rate corresponding to reinvesting the cash position $w - u^*(t)S(t)$ (possibly different from the risk-neutral interest rate $r$).

$$ rSV_S - rV - \max_{a \leq u \leq b} \left( [\tilde{r} w + u^*(r - \tilde{r})] + \frac{\sigma^2 S^2}{2} [V_S S + 2u^*V_S w + (u^*)^2 V_{ww}] \right) = V_t $$

$$ (41) $$

---

For the results of this section I want to thank Leif Anderson.
The plain vanilla passport contract is defined by the payoff function

\[ V(T, S, w) = w^+ \] (42)

The optimal control is known to be

\[ u^* = \alpha - \text{sign}\left(\frac{w}{S} - \alpha\right)\beta \] (43)
\[ \alpha = \frac{b - a}{2} \] (44)
\[ \beta = \frac{b + a}{2} \] (45)

Using the substitution \( x \equiv w/S \) this HJB equation becomes [28]:

\[ 0 = v_t + \max_{u \leq u^* \leq b} \left[ (r - \rho)(u^* - x) + \frac{\sigma^2}{2}(u^* - z)^2 u_{xx} \right] \] (47)
\[ u(T, x) = x^+ \] (48)

Again the change of the variable has reduced the dimensionality of the problem by one.

It can be shown in several ways that Asian options are a special case of the general option on a trading account. Two approaches leading to simple PDEs have been explored by [28] and [29].
3 A Numerical Solution with Finite Elements

3.1 One Spatial Variable

Most passport option models have to be solved numerically since they are non-linear parabolic PDEs. For this reason, we have chosen a collocation finite element method. First, we consider the stationary problem

\[ L(u) = f, \quad u(x_{\text{min}}) = u_{\text{min}}, \quad u(x_{\text{max}}) = u_{\text{max}} \]  

(49)

An approximate solution \( \tilde{u} \) is of the following form:

\[ \tilde{u} = \sum_{k=0}^{N+1} a_k \phi_k(x) \]  

(50)

The task is to find values for \( a_k \) which make \( \tilde{u} \) the "best" approximation. While the Galerkin finite element methods normally used in derivative pricing (cf. [17], [9], [25]) determine the \( a_k \) by solving

\[ \int_{x_{\text{min}}}^{x_{\text{max}}} (L(\tilde{u}) - f) \phi_k \ dx = 0, \quad k = 1, \ldots, N \]  

(51)

collocation finite element methods take the following approach:

\[ (L(\tilde{u}) - f)(\xi_k) = 0, \quad k = 1, \ldots, N \]  

(52)

The collocation points for each element \([x_i; x_{i+1}]\) in the method used here are

\[ \xi_{2i+1, 2i+2} = x_i + \left( \frac{1}{2} \pm \frac{1}{2\sqrt{3}} \right) (x_{i+1} - x_i) \]  

(53)

The approximate function is defined by

\[ \tilde{u} = \sum_{k=0}^{N+1} a_k \phi_k \]  

(54)

\[ = a_0 H_0 + a_1 S_0 + a_2 H_1 + a_3 S_1 + \ldots + a_N H_N + a_{N+1} S_{N+1} \]  

(55)

for

\[ H_k(x) = 3 \left( \frac{x - x_{k-1}}{x_k - x_{k-1}} \right)^2 - 2 \left( \frac{x - x_{k-1}}{x_k - x_{k-1}} \right)^3 \] \text{for } x_{k-1} \leq x \leq x_k \]

(56)

\[ = 0 \quad \text{elsewhere} \]

\[ S_k(x) = \frac{(x - x_{k-1})^2}{(x_k - x_{k-1})^2} + \frac{(x - x_{k-1})^3}{(x_k - x_{k-1})^3} \] \text{for } x_{k-1} \leq x \leq x_k \]

(57)

\[ = \frac{(x_{k+1} - x)^2}{(x_k - x_k)^2} - \frac{(x_{k+1} - x)^3}{(x_k - x_k)^3} \] \text{for } x_k \leq x \leq x_{k+1} \]

\[ = 0 \quad \text{elsewhere} \]

The integration of the boundary conditions is achieved by

\[ a_0 = u(x_{\text{min}}) \]  

(58)

\[ a_N = u(x_{\text{max}}) \]  

(59)

Finding the \( N \) parameters \( a_1, \ldots, a_{N-1}, a_{N+1} \) results in a system of non-linear equations:

\[ f_1(a_1, \ldots, a_{N-1}, a_{N+1}) = 0 \]

(60)

\[ \vdots \]

\[ f_N(a_1, \ldots, a_{N-1}, a_{N+1}) = 0 \]
This system of non-linear equations is solved via Newton’s method. The integration of time is similar to
the Galerkin finite element methods. Spatial variables are discretized with finite elements while time is treated
with finite differences. This can be visualized as the non-linear elliptic operator \( L(u) \) evolving through time.
Each equation corresponds to a collocation point. The dynamic counterpart to eq. (60) is given by

\[
\hat{a} = \begin{pmatrix}
\hat{a}_1 \\
\vdots \\
\hat{a}_{N-1} \\
\hat{a}_{N+1}
\end{pmatrix} = \begin{pmatrix}
\gamma_1 (a_1, \ldots, a_{N-1}, a_{N+1}) \\
\vdots \\
\gamma_N (a_1, \ldots, a_{N-1}, a_{N+1})
\end{pmatrix}
\] (61)

This stiff system of non-linear ordinary differential equations can be solved with various time-stepping
procedures. We have chosen a second order Adams-Moulton and a first order backward difference
implementation. The initial conditions to eq. (61) are given by a discretization of the final condition belonging to
the PDEs. The final conditions as well as the boundary conditions depend on the type of contract.

Approximating the solution of a PDE using finite element methods (Galerkin or collocation), using piece-
wise polynomials of degree \( n \), the approximation to the solution itself should be of the order \( O(h^{n+1}) \); the
approximation to the \( m \)-th derivative should be \( O(h^{n+1-m}) \), i.e. one is losing one order of precision for each
derivative. Why is this? Let us assume a smooth function \( f(x) \) in the interval \( 0 \leq x \leq h \). We want to
approximate this function using a Taylor approximant on degree \( n \):

\[
T(x) = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 + \ldots + f^{(n)}(0)\frac{x^n}{n!}
\] (62)

By the Taylor series remainder theorem, the error is

\[
f(x) - T(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}
\] (63)

\[
0 \leq c \leq x
\] (64)

which is \( O(h^{n+1}) \) when \( 0 < x < h \). If the first derivative of \( T(x) \) is taken, it can be noted that it is exactly
the Taylor polynomial approximant of degree \( n-1 \) to \( f'(x) \), and thus \( f'(x) - T'(x) \) is of \( O(h^n) \). Similarly,
\( f''(x) - T''(x) \) is of \( O(h^{n-1}) \), etc. Though it requires a little more work to prove other cases, similar error
bounds on \( f^n(x) - T^n(x) \), \( f^n(x) - T^n(x) \), etc. can be derived when \( T(x) \) is the Lagrange polynomial interpolant,
Hermite polynomial interpolant or spline interpolant of degree \( n \).

### 3.2 Two Spatial Variables

We again consider the stationary problem \( L(u(x, y)) = f \), \( L \) being a non-linear differential operator. We
search for an approximate solution \( \tilde{u}(x, y) \) for the following problem:

\[
L(u) = f, \quad \tilde{u}(x_{min}) = \tilde{u}(x_{max}) = u_{max}, \quad \tilde{u}(y_{min}) = \tilde{u}(y_{max}) = u_{max},
\] (65)

That is, we consider a rectangular domain with Dirichlet conditions. This method can easily be generalized
for non-rectangular domains and Neumann and mixed boundary conditions [21]. An approximate solution \( \tilde{u} \)
can take the following form:

\[
\tilde{u} = \sum_{k=1}^{N} a_k \phi_k(x, y),
\] (66)

Again, the task is to find values for \( a_k \) which make \( \tilde{u} \) the “best” approximation. While the Galerkin finite
element methods sometimes used in two-asset derivative pricing (cf. [9], [25]) determine the \( a_k \)'s by solving

\[
\int_{x_{min}}^{x_{max}} \int_{y_{min}}^{y_{max}} (L(\tilde{u}) - f) \phi_k \, dy \, dx = 0, \quad k = 1, \ldots, N,
\] (67)

collocation finite element methods take the following approach. They enforce that at certain points in the
domain, the so-called collocation points, the PDE is exactly satisfied.

\[
(L(\tilde{u}) - f)(\xi_k) = 0, \quad k = 1, \ldots, N
\] (68)
The domain is divided into disjoint elements, the finite elements. These finite elements are usually either rectangular or triangular. The latter type of element can form either a structured or an unstructured grid. Here we employ rectangular elements which are not necessarily of the same size.

The four collocation points for each element \([x_i; x_{i+1}] \times [y_i; y_{i+1}]\) in the method used here are

\[
\begin{align*}
(x_i + \frac{1}{2} \pm \frac{1}{2\sqrt{3}}) \ (x_{i+1} - x_i); \ y_i + \frac{1}{2} \pm \frac{1}{2\sqrt{3}}) \ (y_{i+1} - y_i)
\end{align*}
\] (71)

Let \(s_i\) denote the number of gridlines in direction \(i\). The approximate solution is defined by a linear combination of \(4s_x s_y\) basis functions

\[
H_i(x)H_j(y), \ H_i(x)S_j(y), \ S_i(x)H_j(y), \ S_i(x)S_j(y)
\] (72)

with

\[
H_k(x) = \begin{cases} 
3 \left( \frac{x - x_{k-1}}{x_k - x_{k-1}} \right)^2 - 2 \left( \frac{x - x_{k-1}}{x_k - x_{k-1}} \right)^3 & \text{for } x_{k-1} \leq x \leq x_k \\
3 \left( \frac{x_{k+1} - x}{x_{k+1} - x_k} \right)^2 - 2 \left( \frac{x_{k+1} - x}{x_{k+1} - x_k} \right)^3 & \text{for } x \leq x \leq x_{k+1} \\
0 & \text{elsewhere}
\end{cases}
\] (73)

\[
S_k(x) = \begin{cases} 
-\frac{(x - x_{k-1})^2}{(x_k - x_{k-1})} + \left( \frac{x - x_{k-1}}{x_k - x_{k-1}} \right)^3 & \text{for } x_{k-1} \leq x \leq x_k \\
\left( x_{k+1} - x \right)^2 \left( x_{k+1} - x_k \right) - \left( x_{k+1} - x \right)^2 \left( x_{k+1} - x_k \right)^2 & \text{for } x \leq x \leq x_{k+1} \\
0 & \text{elsewhere}
\end{cases}
\] (74)

\[
H_k(y) = \begin{cases} 
3 \left( \frac{y - y_{k-1}}{y_k - y_{k-1}} \right)^2 - 2 \left( \frac{y - y_{k-1}}{y_k - y_{k-1}} \right)^3 & \text{for } y_{k-1} \leq y \leq y_k \\
3 \left( \frac{y_{k+1} - y}{y_{k+1} - y_k} \right)^2 - 2 \left( \frac{y_{k+1} - y}{y_{k+1} - y_k} \right)^3 & \text{for } y \leq y \leq y_{k+1} \\
0 & \text{elsewhere}
\end{cases}
\] (75)

\[
S_k(y) = \begin{cases} 
-\frac{(y - y_{k-1})^2}{(y_k - y_{k-1})} + \left( \frac{y - y_{k-1}}{y_k - y_{k-1}} \right)^3 & \text{for } y_{k-1} \leq y \leq y_k \\
0 & \text{elsewhere}
\end{cases}
\]
\[
\frac{\left(y_{k+1} - y\right)^2}{(y_{k+1} - y_{k})} - \frac{(y_{k+1} - y)^3}{(y_{k+1} - y_k)^2} \quad \text{for} \quad y_k \leq y \leq y_{k+1}
\]

The approximate solution \( \tilde{u}(x, y) \) has the form

\[
\tilde{u}(x, y) = \sum_{i=1}^{s_x} \sum_{j=1}^{s_y} \left( A_{ij} H_i(x) H_j(y) + B_{ij} H_i(x) S_j(y) + C_{ij} S_i(x) H_j(y) + D_{ij} S_i(x) S_j(y) \right)
\]

so the \( N = 4s_x s_y \) unknowns are \( A_{11}, B_{11}, C_{11}, D_{11}, A_{12}, \ldots \), which are relabeled to \( \alpha_1, \ldots, \alpha_N \) for notational convenience. The approximating function \( \tilde{u}(x, y) \) has a continuous mixed derivative \( \tilde{u}_{x,y} \) since the mixed derivative of each of the basis functions is continuous. To be successful in this setting, the key feature of continuous mixed derivatives has to hold. This feature is not common to most FE methods in use today. The basis functions normally used by Galerkin methods do not even have continuous \( \tilde{u}_x \) or \( \tilde{u}_y \).

The approximate solution is required to satisfy the PDE exactly at the four collocation points in each of the \( (s_x - 1)(s_y - 1) \) subrectangles, and to satisfy the boundary conditions at certain points. The number of boundary collocation points plus the number of interior collocations points \( 4(s_x - 1)(s_y - 1) \) is equal to the number of basis functions \( 4s_x s_y \), which is equal to the number of unknowns \( N \). Finding the \( N \) parameters \( \alpha_1, \ldots, \alpha_N \) results in a system of non-linear equations:

\[
\begin{align*}
\mathbf{f}_1(\alpha_1, \ldots, \alpha_N) &= 0 \\
& \vdots \\
\mathbf{f}_N(\alpha_1, \ldots, \alpha_N) &= 0
\end{align*}
\]

This system of non-linear equations is solved via Newton's method. The integration of time is similar to the Galerkin finite element method. Spatial variables are discretized with finite elements while time is treated with finite differences. This can be visualised as the non-linear elliptic operator \( L(u) \) evolving through time. Each equation corresponds to a collocation point. The dynamic counterpart to eq. (78) is given by

\[
\dot{\mathbf{\alpha}} = \begin{pmatrix} \dot{\alpha}_1 \\ \vdots \\ \dot{\alpha}_N \end{pmatrix} = \begin{pmatrix} g_1(\alpha_1, \ldots, \alpha_N) \\ \vdots \\ g_N(\alpha_1, \ldots, \alpha_N) \end{pmatrix}
\]

This stiff system of non-linear ordinary differential equations can be solved with various time-stepping procedures. We have chosen the same routines as for the problem with only one spatial variable. The initial conditions to eq. (79) are given by a discretization of the final condition of the PDEs of ch. 2. All computations have been performed with PDE2D, a general purpose finite element solver described in [22].
4 Applications

4.1 Plain Vanilla Passport Options

4.1.1 European Passport Options

A Linear Model For $r = \gamma$ eq. (16) reduces to:

$$\frac{\partial v}{\partial t} + \frac{1}{2}(1 + |x|)^2 \sigma^2 \frac{\partial^2 v}{\partial x^2} = \gamma v$$

(80)

For the payoff function $v(T) = \max(0, X) \equiv X^+$ an analytical solution has been found [3]:

$$v(t, x) = e^{-\gamma(T-t)} \left[ x^+ + N(d) - (1 + |x|) N \left( d - \sigma \sqrt{T-t} \right) + \Omega \right]$$

(81)

with

$$d = \frac{-\ln(1 + |x|) + \frac{1}{2} \sigma^2 (T-t)}{\sigma \sqrt{T-t}}$$

(82)

$$\Omega = \frac{1}{2} \left[ \frac{\sigma \sqrt{T-t} - 1}{N(d)} + \frac{1}{2} (1 + |x|) N \left( d - \sigma \sqrt{T-t} \right) + \frac{1}{2} \sigma \sqrt{T-t} N'(d) \right]$$

(83)

$N$ and $N'$ denote the cumulative function and the density function of the normal distribution respectively. This analytical solution is to be used as a benchmark. We will repeat the example from ([3], Table 1). Some of our results do not agree with the results reported there because some of the results in [3] are erroneous.\footnote{Andersen et al. compare a Crank-Nicolson FD Schema with the analytical solution in ([3], Tab. 3). Seemingly, the numerical values do not converge to the analytical solution except for high values of $\omega$. However, the numerical algorithm does converge, but not to the wrong analytical solution.}

Eq. (81) is sensitive against the accuracy of the approximation of the normal distribution. We show the results using approximations which are accurate to four and seven digits, respectively; compare ([15], S. 243f).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot $S$</td>
<td>100.0</td>
</tr>
<tr>
<td>Dividend yield $\gamma$</td>
<td>0.0</td>
</tr>
<tr>
<td>Interest rate $r$</td>
<td>0.0</td>
</tr>
<tr>
<td>Volatility $\sigma$</td>
<td>0.3</td>
</tr>
<tr>
<td>Time to maturity</td>
<td>1 Year</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>Result from [3]</th>
<th>Correct Solution: $N(\cdot)$ accurate to ...</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\times 10^3$</td>
<td>... 4 digits</td>
</tr>
<tr>
<td>100</td>
<td>100.1560</td>
<td>100.1560</td>
</tr>
<tr>
<td>50</td>
<td>51.6456</td>
<td>51.5818</td>
</tr>
<tr>
<td>20</td>
<td>22.9063</td>
<td>22.8877</td>
</tr>
<tr>
<td>10</td>
<td>18.8846</td>
<td>18.8794</td>
</tr>
<tr>
<td>0</td>
<td>13.1381</td>
<td>13.1399</td>
</tr>
<tr>
<td>-10</td>
<td>8.8808</td>
<td>8.8794</td>
</tr>
<tr>
<td>-20</td>
<td>5.8876</td>
<td>5.8876</td>
</tr>
<tr>
<td>-50</td>
<td>1.5893</td>
<td>1.5818</td>
</tr>
<tr>
<td>-100</td>
<td>0.1566</td>
<td>0.1597</td>
</tr>
</tbody>
</table>

For the numerical solution of the PDE (80), besides the initial condition given by eq. (15), two boundary conditions are necessary which turn the unbounded domain of the PDE $-\infty < x < \infty$ into a compact one. Andersen et al. suggest ([3], S. 23f)

$$v(t, -e^{\omega \sqrt{T-t}}) = 0$$

(84)

$$\frac{\partial v(t, e^{\omega \sqrt{T-t}})}{\partial x} = e^{-r(T-t)}$$

(85)

$$h = 4$$

(86)
We solve this problem several times using different numbers of elements and time steps to observe the convergence behavior.

<table>
<thead>
<tr>
<th>Spatial Steps</th>
<th>25</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>400</th>
<th>800</th>
<th>1600</th>
<th>3200</th>
</tr>
</thead>
</table>

For the following computations we employ 400 spatial steps (equaling 399 elements) and 800 time steps. For the time integration a first order backward difference method is used:

<table>
<thead>
<tr>
<th>w</th>
<th>Analytical Solution</th>
<th>Numerical Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>100.15600</td>
<td>100.1575</td>
</tr>
<tr>
<td>50</td>
<td>51.58181</td>
<td>51.5833</td>
</tr>
<tr>
<td>20</td>
<td>25.88757</td>
<td>25.8878</td>
</tr>
<tr>
<td>10</td>
<td>18.88084</td>
<td>18.8881</td>
</tr>
<tr>
<td>0</td>
<td>13.13810</td>
<td>13.1378</td>
</tr>
<tr>
<td>-10</td>
<td>8.88084</td>
<td>8.8806</td>
</tr>
<tr>
<td>-20</td>
<td>5.88757</td>
<td>5.8878</td>
</tr>
<tr>
<td>-50</td>
<td>1.58181</td>
<td>1.5833</td>
</tr>
<tr>
<td>-100</td>
<td>0.15600</td>
<td>0.1575</td>
</tr>
</tbody>
</table>

A Non-Linear Model Here we solve eq. (18) with eq. (20). The data are taken from [3].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot S</td>
<td>100.0</td>
</tr>
<tr>
<td>Dividend yield γ</td>
<td>5.0</td>
</tr>
<tr>
<td>Interest rate r</td>
<td>4.5</td>
</tr>
<tr>
<td>Volatility σ</td>
<td>0.3</td>
</tr>
<tr>
<td>Time to maturity</td>
<td>2 Years</td>
</tr>
</tbody>
</table>

This problem has been solved already with Finite Differences by [3] and a Galerkin Finite Element method by [26]. For w = 0 no result can be given because of the jump there.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>28.2277</td>
<td>28.2239</td>
<td>28.2260</td>
<td>-0.4074</td>
<td>-0.4079</td>
</tr>
<tr>
<td>10</td>
<td>22.3144</td>
<td>22.3015</td>
<td>22.3008</td>
<td>-0.4024</td>
<td>-0.4029</td>
</tr>
<tr>
<td>0</td>
<td>18.1253</td>
<td>18.1123</td>
<td>18.1123</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>-10</td>
<td>13.5180</td>
<td>13.5143</td>
<td>13.5135</td>
<td>0.5189</td>
<td>0.5176</td>
</tr>
<tr>
<td>-20</td>
<td>10.4261</td>
<td>10.4293</td>
<td>10.4329</td>
<td>0.4392</td>
<td>0.4390</td>
</tr>
</tbody>
</table>

4.1.2 American Passport Options

Here we introduce early exercise to the non-symmetric example from above. Again, this problem has already been solved with Finite Differences by [3] and a Galerkin Finite Element method by [26].

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>28.1264</td>
<td>28.2110</td>
<td>28.2139</td>
<td>-0.5672</td>
<td>-0.5641</td>
</tr>
<tr>
<td>10</td>
<td>23.0204</td>
<td>23.0227</td>
<td>23.0228</td>
<td>-0.3974</td>
<td>-0.3971</td>
</tr>
<tr>
<td>0</td>
<td>17.8418</td>
<td>17.8048</td>
<td>17.8068</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>-10</td>
<td>13.7776</td>
<td>13.7873</td>
<td>13.7902</td>
<td>0.5330</td>
<td>0.5331</td>
</tr>
<tr>
<td>-20</td>
<td>10.6031</td>
<td>10.6124</td>
<td>10.6150</td>
<td>0.4406</td>
<td>0.4406</td>
</tr>
</tbody>
</table>

4.2 Relative Exotics

So-called relative exotics in the world of passport options have the exotic feature on $x$; i.e. the cap, floor and/or barrier(s) is/are applied to the quotient of wealth and stock $x = \frac{S}{w}$. Absolute exotics bear the exotic feature(s) on $w$ and/or $S$ individually. These contracts are discussed further below.

As a numerical example we add a knock-out barrier at $x = 20$ to the non-symmetric contract from above. Note that by introducing a rebate of $R = 20$ the payoff function does not lose its convexity.
4 Applications

<table>
<thead>
<tr>
<th>$w$</th>
<th>FE</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>10</td>
<td>16.5356</td>
</tr>
<tr>
<td>0</td>
<td>13.4722</td>
</tr>
<tr>
<td>-10</td>
<td>10.8090</td>
</tr>
<tr>
<td>-20</td>
<td>8.3884</td>
</tr>
</tbody>
</table>

4.3 Asian Options

It can be shown that Asian options can be interpreted as options on a trading account as well. For a derivation see [28].

$$v_t + r(q - x)v_z + \frac{1}{2}(q - x)^2\sigma^2v_{zz} = 0$$  \hspace{1cm} (87)

with the final condition

$$v(T, x) = x^+$$  \hspace{1cm} (88)

The function $q$ depends on the contract at hand. This approach can be simplified leading to an even simpler PDE [29].

Continuous Sampling  For this type of sampling $q$ is a continuous function:

$$q = 1 - \frac{t}{T}$$  \hspace{1cm} (89)

In order to be able to tackle the final-value problem eq. (87) and (88) numerically, it has to be converted into a final-boundary-value problem, i.e., the infinite domain has to be cut off by replacing $-\infty$ and $\infty$ by taking finite values for $z$. Also, these points have to be prescribed with values for $u$ or its derivative. Following [28] we choose

$$u(z_{min} = -1) = 0$$  \hspace{1cm} (90)

$$\frac{\partial u(z_{max} = 1)}{\partial z} = 1$$  \hspace{1cm} (91)

Additionally, to be comparable to [28] and [10] we take the same number of space and time points. The spatial variable is discretized with 200 elements of equal length with cubic Hermite basis functions. Time is integrated by a Crank-Nicholson scheme with 400 steps.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest rate</td>
<td>$r$</td>
<td>0.15</td>
</tr>
<tr>
<td>Price of Underlying in $t = 0$</td>
<td>$S_0$</td>
<td>100</td>
</tr>
<tr>
<td>Time to Maturity</td>
<td>$T$</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.05</td>
<td>95</td>
<td>11,094</td>
<td>11,094</td>
<td>11,094</td>
<td>11,094</td>
<td>11,094</td>
<td>11,114</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td>6,794</td>
<td>6,794</td>
<td>6,794</td>
<td>6,794</td>
<td>6,794</td>
<td>6,810</td>
</tr>
<tr>
<td></td>
<td></td>
<td>105</td>
<td>2,746</td>
<td>2,744</td>
<td>2,744</td>
<td>2,744</td>
<td>2,744</td>
<td>2,761</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>90</td>
<td>15,399</td>
<td>15,399</td>
<td>15,399</td>
<td>15,399</td>
<td>15,399</td>
<td>15,445</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td>7,028</td>
<td>7,029</td>
<td>7,030</td>
<td>7,028</td>
<td>7,028</td>
<td>7,066</td>
</tr>
<tr>
<td></td>
<td></td>
<td>110</td>
<td>1,415</td>
<td>1,415</td>
<td>1,415</td>
<td>1,415</td>
<td>1,415</td>
<td>1,461</td>
</tr>
<tr>
<td></td>
<td>0.20</td>
<td>90</td>
<td>15,642</td>
<td>15,643</td>
<td>15,643</td>
<td>15,642</td>
<td>15,641</td>
<td>15,748</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td>8,409</td>
<td>8,412</td>
<td>8,409</td>
<td>8,409</td>
<td>8,408</td>
<td>8,515</td>
</tr>
<tr>
<td></td>
<td></td>
<td>110</td>
<td>3,556</td>
<td>3,560</td>
<td>3,554</td>
<td>3,556</td>
<td>3,554</td>
<td>3,661</td>
</tr>
<tr>
<td></td>
<td>0.30</td>
<td>90</td>
<td>16,513</td>
<td>16,516</td>
<td>16,516</td>
<td>16,516</td>
<td>16,512</td>
<td>16,732</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td>10,210</td>
<td>10,215</td>
<td>10,210</td>
<td>10,210</td>
<td>10,208</td>
<td>10,429</td>
</tr>
<tr>
<td></td>
<td></td>
<td>110</td>
<td>5,731</td>
<td>5,736</td>
<td>5,729</td>
<td>5,731</td>
<td>5,728</td>
<td>5,948</td>
</tr>
</tbody>
</table>

Discrete Sampling  This kind of sampling calls for a different $q$:

$$q = 1 - \frac{1}{n} \left\lfloor \frac{n}{T} \right\rfloor$$  \hspace{1cm} (92)

with \( \lfloor \cdot \rfloor \) denoting the integer part function. Note that this formula allows for non-equidistant time-steps between the fixings.
4 APPLICATIONS

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>K</td>
</tr>
<tr>
<td></td>
<td>FE</td>
</tr>
<tr>
<td></td>
<td>3 Fixings</td>
</tr>
<tr>
<td>0.05</td>
<td>95</td>
</tr>
<tr>
<td></td>
<td>4.943</td>
</tr>
<tr>
<td>0.10</td>
<td>90</td>
</tr>
<tr>
<td></td>
<td>4.026</td>
</tr>
<tr>
<td>0.20</td>
<td>90</td>
</tr>
<tr>
<td></td>
<td>5.772</td>
</tr>
<tr>
<td>0.30</td>
<td>90</td>
</tr>
<tr>
<td>100</td>
<td>13.257</td>
</tr>
<tr>
<td></td>
<td>8.912</td>
</tr>
</tbody>
</table>

The details of the numerical implementation are the same as in the previous section. The Monte Carlo results are achieved in a most simple fashion as described in [13]. We use 100,000 sample paths for all examples so that these results are rather crude estimates. Obviously, the computational burden of solving the PDE is much smaller than generating the Monte Carlo paths.

**Sensitivity Analysis: Delta** With another example from the literature [27] we will show that finite elements can be used to compute $\Delta$ directly from the numerical solution for the option premium.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest rate</td>
<td>$r$</td>
<td>0.09</td>
</tr>
<tr>
<td>Price of Underlying in $t=0$</td>
<td>$S_0$</td>
<td>100</td>
</tr>
<tr>
<td>Time to Maturity</td>
<td>$T$</td>
<td>16 days</td>
</tr>
<tr>
<td>Equal distance samplings</td>
<td></td>
<td>14</td>
</tr>
</tbody>
</table>

For the only run necessary for the finite element computation we use 199 elements of equal length; time integration is performed with an adaptive second order Adams-Moulton scheme using 1000 time steps. For details of this time integration technique, see [21], [6].

<table>
<thead>
<tr>
<th>Premium</th>
<th>Delta</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strike Approximation</td>
<td>MC</td>
</tr>
<tr>
<td>99</td>
<td>5.292</td>
</tr>
<tr>
<td>100</td>
<td>1.139</td>
</tr>
<tr>
<td>105</td>
<td>0.038</td>
</tr>
</tbody>
</table>

### 4.4 Absolute Exotics

#### 4.4.1 The Algorithm

Absolute exotics refer to passport options with a cap, floor or barrier put on either the current value of the trading account $w(t)$ or the underlying $S$. Putting a barrier or cap on $w(t)$ seems natural because beyond some limit the interest in further hedging should be small. Besides, it makes the option cheaper [2]. For this kind of option, the HJB equation has to be solved directly since no control is known a priori. In this section, we outline the idea of the algorithm which involves some basic idea from (static) optimization. Then we apply this algorithm to a problem which has already been solved above.

Starting point is the HJB equation as in eq. (21):

$$(r - \gamma)SV_S + rwV_w - rV - \frac{\sigma^2 S^2}{2} \max_{-1 \leq \mu^* \leq 1} [V_{SS} + 2\mu^*V_{Sw} + (\mu^*)^2 V_{ww}] = V_t$$

(93)

We slightly generalize this problem by allowing the control to vary within $[a, b]$. Then, one has to solve the following optimization problem for each node in each time step:

$$\max_{a \leq \mu^* \leq b} (V_{SS} + 2\mu^*V_{Sw} + (\mu^*)^2 V_{ww})$$

(94)

This is just a quadratic polynomial in $\mu^*$, so that the existence of a maximum is guaranteed. The maximum has to occur at either $\mu^* = a$, $\mu^* = b$ or $\mu^* = -\frac{2V_{Sw}}{V_{ww}}$ (if $a < \mu^* < b$), the latter point being the vertex of the

---

5For the fundamental idea of this algorithm I want to thank Granville Sewell.
quadratic polynomial. All three values are calculated and the maximum is taken. Obviously, in case we do not have a bang-bang solution,\(^6\) accurate approximations of the second derivatives are needed.

As a first test, we recompute the symmetric problem from sec. 4.1.1. This problem cannot test all features of the proposed algorithm since it is known that its optimal control is of the bang-bang type. The routine should, therefore, never be forced to compute second derivatives. Since this problem has two spatial dimensions, we also provide a numerical solution based on eq. (14) so that we have two numerical solutions based on two spatial variables. For both problems the following boundary conditions have been applied:

\[
\begin{align*}
V(0, w, t) &= \max(0, w) \\
\frac{\partial^2 V(200, w, t)}{\partial S^2} &= 0 \\
V(S, -200, t) &= 0 \\
V(S, 200, t) &= w
\end{align*}
\]

With a rectangular mesh of 19 elements in \(S\) and 42 in \(w\) and an adaptive second order Adams-Moulton method for the integration of time, the following results are produced:

<table>
<thead>
<tr>
<th>(w)</th>
<th>Analytical Solution</th>
<th>Numerical Solution</th>
<th>(x_{141})</th>
<th>(x_{121})</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>100.1566</td>
<td>100.1566</td>
<td>100.1561</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>51.5818</td>
<td>51.5804</td>
<td>51.5835</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>25.8878</td>
<td>25.8878</td>
<td>25.8904</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>18.8808</td>
<td>18.8820</td>
<td>18.8840</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>13.1381</td>
<td>13.1397</td>
<td>13.1414</td>
<td></td>
</tr>
<tr>
<td>-10</td>
<td>8.8808</td>
<td>8.8820</td>
<td>8.8840</td>
<td></td>
</tr>
<tr>
<td>-20</td>
<td>5.8876</td>
<td>5.8878</td>
<td>5.8904</td>
<td></td>
</tr>
<tr>
<td>-50</td>
<td>1.5818</td>
<td>1.5804</td>
<td>1.5836</td>
<td></td>
</tr>
<tr>
<td>-100</td>
<td>0.1566</td>
<td>0.1566</td>
<td>0.1581</td>
<td></td>
</tr>
<tr>
<td>Time Steps</td>
<td>-</td>
<td>31</td>
<td>99</td>
<td></td>
</tr>
</tbody>
</table>

The algorithm which computes the optimal control implicitly gives slightly less accurate results for the same mesh while using about three times more time steps.

### 4.4.2 Knock-Out Barriers

Knock-out barriers can be applied to both stock and/or the account. In the second case the rationale is that when the loss becomes too large the coverage is off; in the first case, the usual rationale for cheapening the option is applied. As an example, we take the problem from above and introduce a knock-out barrier on \(w\). With a rectangular mesh of 19 elements in \(S\) and 42 in \(w\) and an adaptive second order Adams Moulton method for the integration of time, the following results were produced:

<table>
<thead>
<tr>
<th>(w)</th>
<th>Location of barrier</th>
</tr>
</thead>
<tbody>
<tr>
<td>(w = -20)</td>
<td>(w = -30)</td>
</tr>
<tr>
<td>100</td>
<td>100.0000</td>
</tr>
<tr>
<td>50</td>
<td>51.4286</td>
</tr>
<tr>
<td>20</td>
<td>25.3535</td>
</tr>
<tr>
<td>10</td>
<td>18.7589</td>
</tr>
<tr>
<td>0</td>
<td>11.7785</td>
</tr>
<tr>
<td>-10</td>
<td>6.0061</td>
</tr>
<tr>
<td>-20</td>
<td>0.0000</td>
</tr>
<tr>
<td>-50</td>
<td>0.0000</td>
</tr>
<tr>
<td>-100</td>
<td>0.0000</td>
</tr>
<tr>
<td>Time Steps</td>
<td>64</td>
</tr>
</tbody>
</table>

\(^6\)Controls switching back and forth between their possible extremes are called bang-bang controls; compare [7].
References

REFERENCES


