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A NOTE ON RATIONING, QUANTITY CONSTRAINTS,  
AND CONSUMPTION THEORY

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## A Note on Rationing, Quantity Constraints, and Consumption Theory

### 1. Introduction

In their classic paper, Tobin and Houthakker [9] nearly exhausted what can be said about changes in the behaviour of a consumer who is confronted with straight quantity rationing at exactly the previously chosen consumption level. Pollak [7] added some further insights for a rather similar setup. More recently, Howard [2] tried to generalize the Tobin-Houthakker theory to the more realistic situation where rationing forces the consumer away from his voluntarily chosen bundle. In this attempt, Howard relied on what he labels a reasonable definition of a disequilibrium substitution effect and on two inconspicuous looking assumptions. It is the purpose of this short article to show that his definition is unworkable and that both of his assumptions are rather problematic in the context he uses. While developing the latter, a noteworthy feature of quantity rationing will emerge.

### 2. The preliminaries

At issue is the comparative static behaviour of a consumer who maximizes a well behaved utility function  $u(x)$  under different sets of constraints. In order to save on notation and to prevent confusion, the problems investigated by Howard will be stripped to their bare bones and both the problems as well as their respective solutions will be numbered and marked with a corresponding superscript. Let  $x = (x_1, \dots, x_n)'$  and  $p = (p_1, \dots, p_n)'$  represent quantities and prices, respectively, of  $n$  commodities, let  $y$  denote fixed income and write

$b$  for the allocation of commodity  $n$  in case of rationing. Now, consider the following standard problems

- (1)  $\max_x u(x)$  s.t.  $y - p'x = 0$   
with the solution  $x^1 = x^1(p, y)$ ,  $\lambda^1 = \lambda^1(p, y)$ ,
- (2)  $\max_x u(x)$  s.t.  $y - p'x = 0$  and  $b - x_n = 0$  with  $b = x_n^1$   
with the solution  $x^2 = x^2(p, y, b)$ ,  $\lambda^2 = \lambda^2(p, y, b)$ ,  $\mu^2 = \mu^2(p, y, b)$ ,
- (3)  $\max_x u(x)$  s.t.  $y - p'x = 0$  and  $b - x_n = 0$  with  $b \neq x_n^1$   
with the solution  $x^3 = x^3(p, y, b)$ ,  $\lambda^3 = \lambda^3(p, y, b)$ ,  $\mu^3 = \mu^3(p, y, b)$

as well as the seemingly unrelated and odd problem

- (4)  $\max_x v(x) = u(x) - \alpha x_n$  s.t.  $y - p'x = 0$  with  $\alpha = \mu^3$   
with the solution  $x^4 = x^4(p, y)$ ,  $\lambda^4 = \lambda^4(p, y)$ ,

where  $\lambda$  and  $\mu$  are the Lagrange multipliers of the budget constraint and the rationing constraint<sup>1</sup>, respectively.

When comparing problem (2) with problem (1) it will be obvious that the additional constraint  $b - x_n = 0$  is 'just binding' in the sense of Silberberg [8] as commodity  $n$  is rationed at precisely the level  $x_n^1$  the consumer would buy anyway. It can be shown that in this case  $\mu^2$  will vanish while the remainder of the solution coincides, i.e.  $x^1 = x^2$ ,  $\lambda^1 = \lambda^2$  and  $\mu^2 = 0$ . This is the setup investigated by Tobin and Houthakker.

Problem (3) in contrast characterizes the situation where rationing of commodity  $n$  is binding in the sense that it forces the consumer to choose a bundle  $x^3$  which is generally different from the bundle  $x^1$  that would be chosen without rationing. In this certainly more representative case of rationing one will find that  $\mu^3 > 0$  as  $b < x_n^1$ . What is perhaps not that obvious but will be immediately intelligible when comparing the respective first order conditions is that problem (4) has been

constructed in such a way that its solution coincides with that of problem (3). Thus, we have

$$(4) \quad x^1 = x^2 \neq x^3 = x^4, \quad \lambda^1 = \lambda^2, \quad \lambda^3 = \lambda^4, \quad 0 = \mu^2 \neq \mu^3$$

The complete comparative static systems associated with problems (1) through (4) are given by<sup>2</sup>

$$(6) \quad \begin{bmatrix} x_p^k & x_Y^k \\ \lambda_p^k & \lambda_Y^k \end{bmatrix} = [H^k]^{-1} \begin{bmatrix} \lambda^k I_n & 0 \\ x^k & -1 \end{bmatrix} \quad k=1,4$$

and

$$(7) \quad \begin{bmatrix} x_p^k & x_Y^k & x_b^k \\ \lambda_p^k & \lambda_Y^k & \lambda_b^k \\ \mu_p^k & \mu_Y^k & \mu_b^k \end{bmatrix} = [H^k]^{-1} \begin{bmatrix} \lambda^k I_n & 0 & 0 \\ x^k & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad k=2,3$$

where  $I_n$  is the identity matrix of order  $n$  and  $H^k$  denote the relevant bordered Hessian matrices which are assumed here to be regular. These bordered Hessian matrices look rather similar as can be seen from

$$(8) \quad H^k = \begin{bmatrix} u_{xx}(x^k) & -p \\ -p' & 0 \end{bmatrix} \quad k=1,4$$

and

$$(9) \quad H^k = \begin{bmatrix} u_{xx}(x^k) & -p & z \\ -p' & 0 & 0 \\ z' & 0 & 0 \end{bmatrix} \quad k=2,3, \quad ,$$

where the  $n$ -vector  $z$  has the extremely simple form  $z=(0, \dots, 0, -1)'$ .

Due to the simplicity of the last row and column of the bordered

Hessian matrices corresponding to problems (2) and (3) it is possible to develop their inverses entirely in terms of elements of the inverse bordered Hessian matrices corresponding to the simple problems (1) and (4), respectively. This is at the heart of both Tobin and Houthakker's original theory of quantity rationing and Howard's generalization. Writing  $H_{ij}$  for the cofactor of the element  $h_{ij}$  in  $H$  and  $H_{ij.kl}$  for the cofactor of the element  $h_{kl}$  in  $H_{ij}$  one finds by expanding the determinant and the cofactors of  $H^k$  in (9) by its last row and column

$$\begin{aligned}
 \det H^k &= -H_{nn}^h, \\
 H_{ij}^k &= -H_{ij.nn}^h \quad \text{all } i, j \neq n, n+2, \\
 H_{in}^k &= 0 \quad \text{all } i \neq n+2, \\
 H_{i(n+2)}^k &= H_{in}^h \quad \text{all } i \neq n+2, \\
 H_{(n+2)(n+2)}^k &= \det H^h
 \end{aligned}
 \tag{10}$$

for  $k=2$  if  $h=1$  or  $k=3$  if  $h=4$ .

Note that the matrix  $H$  and thus also the matrix of its cofactors are symmetric. Assuming that the second order sufficient conditions hold for problems (1) through (4), we have furthermore

$$\begin{aligned}
 \text{sgn } \det H^k &= (-1)^n \quad \text{all } k \\
 \text{sgn } H_{ii}^k &= (-1)^{n-1} \text{ or } 0 \quad \text{all } i \leq n \text{ and all } k
 \end{aligned}
 \tag{11}$$

Finally, it will be useful to remember Jacobi's celebrated ratio theorem (see Murata [6], p. 7) which permits to write

$$H_{ij}^k = -H_{ij.nn}^h = (H_{in}^h H_{nj}^h - H_{ij}^h H_{nn}^h) / \det H^h$$

for all  $i, j \neq n, n+2$  and for  $k=2$  if  $h=1$  or  $k=3$  if  $h=4$

and thus provides the key for a wide range of LeChâtelier-type analysis.

For the sake of completeness note that (6) and (7) yield in conjunction with Cramer's rule the standard results

$$(13) \quad \frac{\partial x_i^k}{\partial p_j} = \lambda^k \frac{H_{ji}^k}{\det H^k} + x_j^k \frac{H_{(n+1)i}^k}{\det H^k} \quad \text{all } i, j \text{ and } k$$

$$(14) \quad \frac{\partial x_i^k}{\partial y} = - \frac{H_{(n+1)i}^k}{\det H^k} \quad \text{all } i \text{ and } k$$

$$(15) \quad \frac{\partial x_i^k}{\partial b} = - \frac{H_{(n+2)i}^k}{\det H^k} \quad \text{all } i \text{ and } k=2,3$$

Conventionally, commodities  $i$  and  $j$  are called Hicks-Allen substitutes or complements depending on whether the first term in the Slutsky equations (13), i.e. the so-called pure or compensated substitution term

$$(16) \quad \left. \frac{\partial x_i^k}{\partial p_j} \right|_{\bar{u}} := \lambda^k \frac{H_{ij}^k}{\det H^k}$$

is positive or negative. Also, of course, commodity  $i$  is said to be superior or inferior depending on whether  $\partial x_i^k / \partial y$  is positive or negative, i.e. whether the income effect in the Slutsky equations (13) is negative or positive.

### 3. Some basic results in the theory of quantity rationing

With the help of (15), (10), (11) and (16) it is now easy to check Tobin and Houthakker's well known result



$$(17) \quad \operatorname{sgn} \frac{\partial x_i^k}{\partial b} = \operatorname{sgn} \frac{-H^{(n+2)i}}{\det H^k} = \operatorname{sgn} \frac{-H_{in}^h}{\det H^h} = - \operatorname{sgn} \frac{\partial x_i^h}{\partial p_n} \Big|_{\bar{u}}$$

all  $i \neq n$  and  $k=2$  if  $h=1$  or  $k=3$  if  $h=4$  ,

where use has been made of the fact that  $\lambda^k$ , the marginal utility of income, is positive. (17) says for the classic Tobin-Houthakker setup ( $k=2$ ,  $h=1$ ) that the rationed consumer will buy more (less) of good  $i$  whenever the allocation of good  $n$  is increased if good  $i$  used to be a Hicks-Allen complement (substitute) to good  $n$  before rationing was introduced. We note in passing that this finding, which sounds entirely plausible at first glance, is in fact rather hard to understand because with rationing all goods  $i$  become Hicks-Allen neutral vis-à-vis the rationed commodity  $n$  ( $H_{in}^k = 0$  for  $k=2,3$ ) and it is not clear why their previous characteristics should still be of importance.

Now, (17) holds also for the setup  $k=3$  and  $h=4$ , and this is what Howard relied on in his first generalization. By defining

$$(i) \quad \frac{\partial x_i^4}{\partial p_j} \Big|_{\bar{u}} := \lambda^4 \frac{H_{ji}^4}{\det H^4}$$

to be a disequilibrium substitution effect and continuing to speak of a Hicks-Allen substitute or complement in complete analogy to (16) if the RHS of (i) is positive or negative, Howard ([2], pp. 404-5) states that the basic Tobin-Houthakker finding (17) applies also in the general case where rationing of commodity  $n$  forces the consumer away from his original choice  $x^1$ . But there is a fundamental difference: whereas Tobin and Houthakker's result may be evaluated at least in principle whenever the behavior of the consumer with utility function  $u(x)$  before rationing is completely known, Howard's generalization

rests on non-existent information. A quick glance at problem (4) shows that his disequilibrium substitution effect (i) is nothing but a conventional substitution effect, though, unfortunately, one belonging to the imaginary consumer with utility  $v(x) = u(x) - \mu^3 x_n$  where  $v(x)$  is being held constant at the level  $\bar{v} = u(x^4) - \mu^3 x_n^4$ . To identify this artefactual consumer is obviously impossible as long as  $\mu^3$ , which is part of the solution to problem (3), is not known. With this construct, then, Howard is perhaps unknowingly making a full circle in his approach: in order for his generalization to be applicable, the solution to problem (3) and thus the answer to the question which was to be answered must be known beforehand. The requirement of finding the sign of his disequilibrium substitution effect is therefore merely a different way of conceding that the theory of rationing has nothing to say about the sign of  $\partial x_1^3 / \partial b$ , i.e. the reaction to a change in the allocation by an effectively rationed consumer.

As is well known, the indeterminate sign of the income effect in the Slutsky equations (13) prevents us from signing any of the comparative static reactions with regard to price changes even if the sign of the substitution term is known. The only and trivial exception in case of rationing is, of course, the 'demand' for the rationed good itself, because

$$(18) \quad \frac{\partial x_n^k}{\partial p_j} = 0 \quad \text{all } j \text{ and } k=2,3$$

by (10) and (13). Assuming along with Howard ([2], p. 406), however, that

(ii) all goods are superior, i.e.  $\partial x_i^k / \partial Y > 0$

and that

(iii) the introduction of a quantity constraint does not affect the substitution relationships between unrationed goods

because '.. there does not seem to be any reason why ..' it should do so, a somewhat larger but still minute fraction of the  $n^2$  conceivable price responses under rationing can be signed. On account of the normality assumption (ii) follows

$$(19) \quad \frac{\partial x_i^k}{\partial p_n} < 0 \quad \text{all } i \neq n \text{ and } k=2,3$$

and in conjunction with (11) which assures the non-positivity of the own substitution terms also

$$(20) \quad \frac{\partial x_i^k}{\partial p_i} < 0 \quad \text{all } i \neq n \text{ and } k=2,3$$

Furthermore, if two goods  $i$  and  $j$  were Hicks-Allen complements before rationing, assumption (ii) together with (iii) would immediately yield Howard's second main finding

$$(21) \quad \frac{\partial x_i^k}{\partial p_j} < 0 \quad \text{for Hicks-Allen complements and } k=2,3$$

There is nothing which can be argued against the latter result, it is the two assumptions (ii) and (iii) which deserve criticism. Rationing does definitely influence the characteristics of and the relationships among the various commodities in a systematic though not easily predictable way and this appears to forbid generalizing

assumptions like (ii) and (iii). This can be most easily seen by comparing the comparative static reactions for the Tobin-Houthakker setup and will of course apply *a fortiori* to Howard's analysis where rationing forces the consumer away from his original choice. Looking first at the substitution term and the income effect separately, one finds with the help of (10), (12) and (16) or (14) and (16)

$$(22) \quad \left. \frac{\partial x_i^2}{\partial p_j} \right|_{\bar{u}} - \left. \frac{\partial x_i^1}{\partial p_j} \right|_{\bar{u}} = - \left. \frac{\partial x_n^1}{\partial p_j} \right|_{\bar{u}} \left. \frac{\partial x_i^1}{\partial p_n} \right|_{\bar{u}} / \left. \frac{\partial x_n^1}{\partial p_n} \right|_{\bar{u}} \quad \text{all } i, j$$

and

$$(23) \quad \frac{\partial x_i^2}{\partial y} - \frac{\partial x_i^1}{\partial y} = - \frac{\partial x_n^1}{\partial y} \left. \frac{\partial x_i^1}{\partial p_n} \right|_{\bar{u}} / \left. \frac{\partial x_n^1}{\partial p_n} \right|_{\bar{u}} \quad \text{all } i, j$$

respectively. For the entire price effect this combines then in conjunction with (13) to<sup>3</sup>

$$(24) \quad \frac{\partial x_i^2}{\partial p_j} - \frac{\partial x_i^1}{\partial p_j} = - \frac{\partial x_n^1}{\partial p_j} \left. \frac{\partial x_i^1}{\partial p_n} \right|_{\bar{u}} / \left. \frac{\partial x_n^1}{\partial p_n} \right|_{\bar{u}} \quad \text{all } i, j$$

Note that (24) is entirely compatible with (18) for  $i=n$ . Note also that  $\left. \frac{\partial x_n^1}{\partial p_n} \right|_{\bar{u}}$  must be non-positive by (11) and in fact can be shown to be strictly negative (see Debreu [1]). Now, (22) through (24) reveal some interesting insights into the effects of quantity rationing on the relationships among commodities.

First, (22) shows that if both commodities  $i$  and  $j$  were either Hicks-Allen complements or substitutes to commodity  $n$  before rationing, then quantity rationing has a clear tendency to strengthen Hicks-Allen substitutability or to eradicate Hicks-Allen complementarity among the unrationed commodities  $i$  and  $j$ <sup>4</sup>. In the latter case,

Howard's assumption (iii) is obviously inappropriate, and even though there appears to be no compelling intuitive reason why two Hicks-Allen complements  $i$  and  $j$  should either be both Hicks-Allen complements or substitutes to a third commodity like commodity  $n$  before rationing is being introduced, it is perhaps nevertheless correct to identify quantity rationing with a strong tendency towards eradicating Hicks-Allen complementarities among unrationed goods since, as Kamien [4] has shown, the maximum permissible number of complementarity relationships shrinks dramatically with the dimension of the commodity space and rationing is technically nothing but a reduction in this dimension. (23) in turn has another interesting implication. If commodity  $i$  was a Hicks-Allen complement to commodity  $n$  before rationing and if the latter was superior, then rationing has a clear tendency to make commodity  $i$  inferior. This plausible result conflicts with Howard's normality assumption (ii) and again proves that rationing definitely affects the characteristics of unrationed goods. Finally, (24) says similarly to (22) that the way in which rationing affects the gross substitutability among two unrationed commodities  $i$  and  $j$  depends entirely on the relationships between these two commodities and the rationed good  $n$  before rationing.

#### 4. Conclusion

There is little the theory has to say about the effects of quantity rationing on consumer behaviour in the Tobin-Houthakker case where rationing leaves the original choice of the consumer unaffected unless all relationships between the good to be rationed and the remaining commodities are known. But there is absolutely nothing

the theory can say on the basis of the latter information if rationing forces the consumer to deviate from his original choice. The Tobin-Houthakker theory of rationing should nevertheless be considered as quite useful because quantity rationing schemes do in reality rather often start with allocation levels which are closely related or even identical to the previously voluntarily chosen demand levels.

## Footnotes

<sup>1</sup> The reader may question the appropriateness of identifying the equality constraints  $b - x_n = 0$  with straight quantity rationing. But even though inequality constraints would be obviously more commensurate, we will disregard this case since it would unnecessarily complicate the following argument. Thus, 'rationing' may in effect mean forced consumption.

<sup>2</sup> In the following matrix derivatives we use the convention set out very clearly in Intriligator [3], App. B 9.

<sup>3</sup> For a completely different derivation of the following result see Pollak [7].

<sup>4</sup> In a somewhat different context Kusumoto [5] has termed this tendency of certain types of constraints to eliminate complementarity relationships a generalized Le Chatelier-Samuelson principle and given necessary and sufficient conditions under which it obtains.

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