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## Optimal stabilization policy with delayed controls and imperfect state measurements

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OPTIMAL STABILIZATION POLICY  
WITH DELAYED CONTROLS AND  
IMPERFECT STATE MEASUREMENTS

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Serie A - Nr. 133

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## Abstract

The standard optimal control solutions of the macroeconomic stabilization problem - i.e. essentially: the open- and closed-loop solution - are not necessarily implementable or optimal in real-life situations. This is because they do not take into account the time necessary to measure the economy's state and to realize the policy measures physically. In this paper, Dynamic Programming is used to derive the "best implementable" solution to the optimisation of a quadratic welfare loss-functional subject to a linear econometric model when there are such delays. Two cases are considered:

- a) Perfect, but delayed state measurements are possible;
- b) Only imperfect, delayed measurements are available.

In both cases, the analytical characterization of the solution immediately suggests practical schemes for the numerical computation of the optimal policy sequence.

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## I. Introduction

One of the foremost tasks in quantitative macro-economic policy-making is the determination of the values of the monetary and fiscal policy-variables (such as tax rates, government expenditures, discount rates ...), which ensures that some bundle of macroeconomic goals (like low inflation and unemployment rates, equilibrated trade balance, etc) is optimised, not only in single periods, but dynamically over time. In order to provide a methodological basis to help answer these and similar questions, several authors have addressed themselves in recent years to the problem of minimising a quadratic welfare loss-functional

$$(1) \quad W = E \left[ \sum_{t=1}^N \frac{1}{2} (x_t - \bar{x}_t)^T K_t (x_t - \bar{x}_t) \right] \rightarrow \min_{u_1, \dots, u_N}$$

subject to a linear or linearised econometric model in state-variable form

$$(2a) \quad x_t = A_t x_{t-1} + B_t u_t + C_t z_t + \varepsilon_t, \quad t = 1, \dots, N,$$

$$(2b) \quad x_0 = \hat{x},$$

see e.g. PINDYCK [1973], CHOW [1975]. This problem is known as the linear-quadratic-Gaussian tracking problem (short: LQG).

In the above equations as in the remainder of this article  $E[\cdot]$  is the expectation operator and  $\cdot^T$  denotes transposition.  $E[\cdot/\cdot\cdot]$  is the expectation of  $\cdot$  conditional on  $\cdot\cdot$ . The  $(n \times 1)$  vector  $x_t$  describes the state of the economy at period  $t$ ,  $\bar{x}_t$  being an exogenously specified target vector of desired

values for  $x_t$ , and  $\hat{x}$  being the given initial state. (This approach can easily be extended to the case when  $x_0$  is known only probabilistically, see section IV.). The matrix  $K_t$ , which weights the squared and cross-termed deviations of  $x_t$  from its desired value  $\bar{x}_t$ , is non-negative definite.  $u_t$  is a  $(m \times 1)$  vector of policy or control variables whose value is freely set by the policy-maker.  $z_t$  is a  $(p \times 1)$  vector of exogeneous variables.  $\varepsilon_t$  is a  $(n \times 1)$  vector of stochastic disturbance which are normal-distributed with mean 0 and known covariance matrix and which are serially uncorrelated. The parameter matrices  $A_t$ ,  $B_t$  and  $C_t$  are assumed to be known exactly.

In general it is appropriate to define (2a) so that  $x_t$  includes  $u_t$  as a subvector, thus making  $u_t$  an argument of  $W$  without complicating the notation unnecessarily, see CHOW [1975] (The weighting of  $u_t$  in  $W$  can express the technical or political costs of using macroeconomic instruments. It may also be used to keep the fluctuation of  $u_t$  within reasonable bounds without explicitly introducing restrictions on these fluctuations).

For every initial state  $\hat{x}$ , (2a) describes the possible behaviour of the economy over time as a function of the policy variables  $u_t$ , of the uncontrollable events  $z_t$  and of the random phenomena  $\varepsilon_t$ . Because of the disturbances  $\varepsilon_t$ , this behaviour is only probabilistically known. The optimal policy problem is to determine the control sequence  $u_1^*, \dots, u_N^*$  which minimizes the expected welfare loss (1) subject to the economic constraints (2).

In the formulation (1)-(2) of the macroeconomic policy-making problem, neither the time needed to measure the economy's state  $x_t$  (i.e., the measurement lag) nor the time needed to decide upon and physically realise a control  $u_t$  (i.e., the realisation lag) are taken into consideration. However, it will be shown in section II that when the control lag as sum of measurement and realisation lag is longer than one period, failure to take it into account explicitly in the problem constraints results in a solution which cannot be implemented. Thus the so-called closed-loop solution of LQG, which traditionally has been used in the optimisation of econometric models, may be meaningless in numerous real-life situations.

In this paper we derive the "best implementable" solution to the macroeconomic policy-making problem when there is a control lag. First, in section II, we present the "classical" solutions of LQG, i.e., the closed-loop and the open-loop solutions. As previously stated, we show that these solutions are not implementable or optimal when there is a non-trivial control lag. In the following sections, two cases of macroeconomic policy-making with control lag are considered:

- a) The state of the economy is measured exactly (section III);
- b) Only a linear transformation of the state is measured, corrupted by additive stochastic disturbances (section IV).

In both cases analytical expressions for the solution of the delayed control problem, for the state variance/covariance and for the minimum expected welfare loss are given, which define easily realisable numerical solution schemes. Finally, section V is devoted to some conclusions.

## II. Closed-loop and open-loop solutions

In optimal stochastic control there exists a well-defined correspondence between some characteristics of the optimal solution and the problem's assumptions on which information is used for determining the optimal control in any period. It is therefore usual to speak of the different solutions of a problem - meaning the solutions of the different problems derived one from the other by varying only the constraints on information availability. Two different solutions of LQG are generally considered in the literature:

- The closed-loop solution defines the optimal control in any period  $t$  as a function of the current state and of time, i.e. in the form,

$$(3) \quad u_t^* = u^*(x_{t-1}, t) \quad , \quad t = 1, \dots, N.$$

- The open-loop solution defines the optimal control as a function of the initial state and of time alone,

$$(4) \quad u_t^{**} = u^{**}(x_0, t) \quad , \quad t = 1, \dots, N.$$

The other major types of solution discussed in the literature for the general stochastic control problem, see e.g. BAR-SHALOM and TSE [1976], reduce to these two types in the special case of LQG.

Whatever assumptions are made on information availability the corresponding solution of LQG can be derived using Dynamic Programming, see CHOW [1975]. According to Bellman's Principle

of Optimality (BELLMAN [1957]), the optimal control for the period  $t$ ,  $u_t^*$ ,  $t = 1, \dots, N$ , can be obtained as the solution of the one-period optimisation problem:

$$(5) \quad V_t = E \left[ \frac{1}{2} (x_t - \bar{x}_t)^T K_t (x_t - \bar{x}_t) / \sigma_t \right] + V_{t+1}^* \rightarrow \min_{u_t},$$

where  $V_{t+1}^* = \min V_{t+1}$  and  $\sigma_t$  describes the information available on the past process behaviour when determining  $u_t^*$ . Thus, loosely speaking,  $\sigma_t = \{x_0, u_1, \dots, u_{t-1}\}$  in the open-loop case and  $\sigma_t = \{x_0, \dots, x_{t-1}, u_1, \dots, u_{t-1}\}$  in the closed-loop case. It can easily be shown that  $V_\tau^* = V^*(x_{\tau-1}, \tau)$  is the minimum expected welfare loss over  $t = \tau, \dots, N$ .

Since the  $\varepsilon_t$  are serially uncorrelated and independent of  $x_\tau$  and  $u_\tau$ ,  $\tau = 1, \dots, N$ ,

(5) can be rewritten in the closed-loop case in the form

$$(6) \quad V_t = \frac{1}{2} x_t^T K_t x_t - x_t^T K_t \bar{x}_t + \frac{1}{2} \bar{x}_t^T K_t \bar{x}_t + E \left[ \frac{1}{2} \varepsilon_t^T K_t \varepsilon_t \right] + V_{t+1}^*.$$

This last equation is the starting point for deriving the closed-loop solution of LQG and will be referred to in later parts of this paper. The derivation of the optimal solution of LQG starting with (6) can be found in CHOW [1975] and will not be presented here.

The closed-loop solution of LQG always exists. It is unique and given by the deterministic, linear feedback rule

$$(7) \quad u_t^* = G_t x_{t-1} + g_t, \quad t = 1, \dots, N,$$

with

$$(8a) \quad G_t = -(B_t^T H_t B_t)^{-1} B_t^T H_t A_t ,$$

$$(8b) \quad g_t = -(B_t^T H_t B_t)^{-1} B_t^T (H_t C_t z_t - h_t) ,$$

$$(9a) \quad H_{t-1} = K_{t-1} + (A_t + B_t G_t)^T H_t (A_t + B_t G_t) , \quad H_N = K_N ,$$

$$(9b) \quad h_{t-1} = K_{t-1} \bar{x}_{t-1} + (A_t + B_t G_t)^T (h_t - H_t C_t z_t) , \quad h_N = K_N \bar{x}_N ,$$

provided  $B_t^T H_t B_t$  is not singular. When this matrix is singular, the solution is no longer unique. However, a well defined optimal feedback rule (out of several possible) is easily obtained by replacing  $(B_t^T H_t B_t)^{-1}$  in (8) with the corresponding Moore-Penrose generalised inverse, which is always uniquely defined. See GARBADE [1976], PRESTON [1977] for details.

The system (7)-(9) not only analytically characterises the optimal solution of (1)-(2), but offers computational schemes for numerical solutions as well. Using alternatively (8) and (9) for  $t = N, N-1, \dots, 1$  together with the initial condition  $H_N = K_N$ ,  $h_N = K_N \bar{x}_N$ , the matrices and vectors  $G_N, g_N$ ;  $H_{N-1}, h_{N-1}$ ;  $G_{N-1}, g_{N-1}$ ;  $\dots$ ,  $G_1, g_1$ , can be determined backward in time. As previously noted, the feedback matrices  $G_t$  and forcing vectors  $g_t$  are independent of the values taken by  $x$  and  $u$  over  $t = 0, \dots, \bar{N}$ ; therefore, they can be predetermined before even  $x_0$  is known. Given  $G_t$  and  $g_t$ , the optimal control  $u_t^*$  is determined by (7) as a linear, deterministic function of the current state of the economy,  $x_t$ .

The closed-loop solution (7)-(9) is the optimal solution of (1)-(2) in the absence of any restriction on information

availability. Accordingly, it makes "maximal" use of information. (In fact, (7)-(9) shows that this "maximal use" of information consists in using the actual value of  $x_{t-1}$  - and this value alone - to determine  $u_t^*$ . This reflects the independence and Markovian properties of the different sequences defined by LQG). No other solution to LQG yields a smaller expected minimum welfare loss. For these and other reasons, systematic use of the closed-loop solution has been advocated for use within the macroeconomic policy-making framework, see e.g. CHOW [1972].

However, practical implementation of the closed-loop solution puts stringent requirements on the length of the measurement and realisation lags. Although the optimal decision-rule (7) can be predetermined, the corresponding optimal controls  $u_t^*$  cannot be computed before the state  $x_{t-1}$  is known.

By (2a)  $x_{t-1}$  is a random variable. Its value can therefore only be determined by measuring it after it has been realised, i.e., at the earliest in  $t-1$ . Thus the closed-loop solution (7)-(9) can only be implemented if  $x_{t-1}$  is measured without delay ("on line" or "real time" measurement). Furthermore, this solution requires that the control  $u_t^*$  can be physically realised immediately after its value has been computed. To put it more precisely: implementation of the closed-loop solution is impossible if the control lag as the sum of measurement lag and of realisation lag is longer than one time period.

In actual macroeconomic policy-making, measurement lag and control lag are indeed far from trivial. (The measurement lag corresponds largely to the recognition lag of the theory of economic policy. The realisation lag covers, among

others, the so-called legislative and administrative lags. For a detailed discussion of the different kinds of delay in macroeconomic policy-making and of their relative and absolute importance, see e.g. FRIEDMAN [1948].) These lags will often amount to more than one period, particularly if the econometric model used has short sampling intervals (quarterly, monthly, or weekly models). If so, the solution (7)-(9) has no practical significance and cannot be used as a decision-help.

The open-loop solution, on the other hand, is the solution of the problem (1)-(2) augmented with the constraint (4) which excludes any use of state measurements for  $t > 0$ . It is given by (7)-(9) with  $x_{t-1}$  replaced in (7) with its minimum variance prediction conditional on  $x_0$ . (This result can be obtained by trivially modifying the analysis of section III). Although in attenuated form, real-life implementations of this solution are still restricted by eventual measurement and realisation lags: the initial state  $x_0$  (or, in the case of a random  $x_0$ , its distribution) must be known at  $t = 0$ ; the realisation lag for  $u_t^{**}$  cannot be longer than  $t$  periods. Within the context of macroeconomic policy-making, however, it is another characteristic of the open-loop solution which appears to be its major shortcoming. This solution disregards any measurement of the economy's state in periods  $t > 0$  which might (and as a rule will) become available between  $t = 0$  and  $t = N$ . This results in an unnecessarily large uncertainty about the real economy's state in  $t > 0$  and in an accordingly unnecessarily large

minimum expected welfare loss. In other words, the open-loop solution as a rule is not the best implementable solution to the concrete macroeconomic problem. Note, however, that for the deterministic linear-quadratic problem, knowledge of the deterministic equivalent of (2) is sufficient to make a perfect, costless prediction of  $x_t$ ,  $t = 1, \dots, N$ . Therefore, there is no advantage in using direct state measurements and both the open-loop and closed-loop solution determine the same optimal control sequence and the same minimum loss.

Summarising it can be said that both closed-loop and open-loop solutions are attractively clear-cut. They permit adequate treatment of the kind of problems most often encountered in engineering, see BRYSON and HO [1975]. However, they do not appear to be generally appropriate as solutions of the optimisation of econometric models for macroeconomic policy-making. The closed-loop solution will often not be implementable. The open-loop solution, too, may not be implementable; in any case, it unnecessarily restricts the use of the delayed measurements of the economy's state which are continuously available in policy-making. The purpose of section III is thus to derive the solution to LQG which will make the best possible use of available information while remaining implementable despite the given control delays.

It is important to note that we are exclusively concerned with the delays in determining and realising the optimal economic measures  $u_t^*$ . These delays take place out-

side of the economic structure described by (2a). They are not to be confused with the "transmission lags" within the economy. These express the fact that economic measures possibly do not influence the economy at once after they are realised and/or exert an influence over several periods. Such transmission lags will normally be explicitly taken into account in the econometric model - although their existence may be "hidden" in the state variable form (2a), which expresses the original model as a system of first-order difference equations.

### III. Delayed controls

For simplicity's sake we assume in this section that measurement lag and realisation lag are constant and equal to  $\alpha$  and  $\beta$  periods respectively, with  $\alpha + \beta = \gamma$ . At this point it is crucial to recognize that the conceptual differentiation between measurement lag and realisation lag is irrelevant for the solution of the delayed optimal control problem. According to our previous discussion of the delayed problem, the controls  $u_t$  we are concerned with are to be interpreted as "effective in period  $t$ , decided upon in period  $t - \beta$ , when the most recent state measurement is  $x_{t - \gamma - 1}$ ". The information that can be used in the determination of the optimal  $u_t$  is given by a) the deterministic knowledge of  $x_{t - \gamma - 1}$ ; b) the deterministic knowledge of the previous controls  $u_\tau$ ,  $\tau < t$ ; and c), as we will show in detail, a probabilistic knowledge (a prediction) of the states  $x_{t - \gamma}, \dots, x_{t - 1}$ ; this prediction is based on the knowledge of the a priori system (2a), of  $x_{t - \gamma - 1}$ , and of  $u_{t - \gamma}, \dots, u_{t - 1}$ . The information a)-c) does not depend on when exactly within  $[t - \gamma - 1, t]$  the value of  $u_t$  is decided upon. It is already physically available in  $t - \gamma$ . Thus the absolute and relative values of  $\alpha$  and  $\beta$  are unimportant. Only the value of  $\gamma$  is relevant, i.e. the length of the control lag.

Summarising, the control problem can be expressed as:

$$(10) \quad W = E \left[ \sum_{t=\gamma}^N \frac{1}{2} (x_t - \bar{x}_t)^T K_t (x_t - \bar{x}_t) \right] \rightarrow u_\gamma, \dots, u_N \min$$

subject to

$$(11a) \quad x_t = A_t x_{t-1} + B_t u_t + C_t z_t + \varepsilon_t, \quad t = 1, \dots, N,$$

$$(11b) \quad x_0 = \hat{x},$$

$$(12a) \quad u_t^* = u^*(u_\mu, u_\nu, t), \quad \mu < t - \gamma, \quad \nu < t,$$

$$(12b) \quad u_1 = \hat{u}_1, \dots, u_{\gamma-1} = \hat{u}_{\gamma-1},$$

where  $\hat{u}_t$ ,  $0 < t < \gamma$ , are the known historical values of  $u_t$ . The constraints (12) express exactly the minimum requirements necessary to ensure implementability of the solution when there is a control lag of  $\gamma$  periods. Since by (12b) the values of  $u_1, \dots, u_{\gamma-1}$  are fixed, the delayed problem is concerned only with the determination of the optimal controls over  $t = \gamma, \gamma+1, \dots, N$ , see (10). (Alternatively one can, for example, define the problem over  $t = 1, 2, \dots, N$ . In this case (11a) must be extended to cover the periods  $t = -\gamma, -\gamma+1, \dots, N$ ; the initial conditions (11b) and (12b) must define  $x_{(-\gamma)}$  instead of  $x_0$  and  $u_{(-\gamma+1)}, \dots, u_0$  instead of  $u_1, \dots, u_{\gamma-1}$ .) The constraints (11) are identical to the constraints (2) in the standard LQG problem.

As before, it follows from Bellman's Principle of Optimality that the optimal control  $u_t^*$  can be obtained by solving the problem

$$(13) \quad V_t = E \left[ \frac{1}{2} (x_t - \bar{x}_t)^T K_t (x_t - \bar{x}_t) / x_\mu, u_\nu, \mu < t - \gamma, \nu < t \right] + \\ + V_{t+1}^* \rightarrow \min_{u_t}$$

We will show that the optimal solution of (13) can be expressed in the form of a linear-feedback rule acting on a minimum variance estimate of  $x_t$ . We first analyse the prediction process

$$(14) \quad \tilde{x}_t = E[x_t/x_\mu, u_\nu, \mu < t - \gamma, \nu < t] .$$

Let  $k = t - \gamma - 1$ . By successively eliminating  $x_t$  from (2a) for  $t = t-1, \dots, k+1$ , one sees that  $x_t$  can be decomposed into a deterministic part  $x_t^d$  and in a stochastic part  $\eta_t$ ,

$$(15) \quad x_t = (S_{t/\gamma} x_k + s_{t/\gamma}) + (\eta_t) := (x_t^d) + (\eta_t)$$

with

$$(16a) \quad S_{t/\gamma} = \prod_{\tau=0}^{\gamma} A_{t-\tau} ,$$

$$(16b) \quad s_{t/\gamma} = B_t u_t + C_t z_t + \sum_{\theta=0}^{\gamma-1} \left( \prod_{\tau=0}^{\theta} A_{t-\tau} \right) \cdot (B_{t-\theta-1} u_{t-\theta-1} + C_{t-\theta-1} z_{t-\theta-1}) ,$$

$$(17) \quad \eta_t = \varepsilon_t + \sum_{\theta=0}^{\gamma-1} \left( \prod_{\tau=0}^{\theta} A_{t-\tau} \right) \varepsilon_{t-\theta-1} .$$

By (17)  $\eta_t$  is a linear-combination of normal-distributed variables with mean 0. It is also independent of  $x_t$  and  $u_t$  for all  $t$ . Therefore

$$(18) \quad \tilde{x}_t = E[x_t^d + \eta_t] = E[x_t^d] + E[\eta_t] = x_t^d ,$$

i.e.,  $x_t^d$  is the least mean-squared prediction of  $x_t$ , with prediction error  $\eta_t = x_t - \tilde{x}_t = x_t - x_t^d$ .

Let us now turn back to the optimisation problem (13).

We have

$$\begin{aligned}
 (19) \quad V_t &= E\left[\frac{1}{2}(x_t - \bar{x}_t)^T K_t (x_t - \bar{x}_t) / x_\mu, u_\nu, \mu < t - \gamma, \nu < t\right] + V_{t+1}^* = \\
 &= E\left[\frac{1}{2}(x_t^d + \eta_t - \bar{x}_t)^T K_t (x_t^d + \eta_t - \bar{x}_t) / x_\mu, u_\nu, \mu < t - \gamma, \nu < t\right] + V_{t+1}^* = \\
 &= \frac{1}{2} x_t^{dT} K_t x_t^d - x_t^{dT} K_t \bar{x}_t + \frac{1}{2} \bar{x}_t^T K_t \bar{x}_t + E\left[\frac{1}{2} \eta_t^T K_t \eta_t\right] + V_{t+1}^* .
 \end{aligned}$$

This last expression is of the same type as (6). Since the derivation of (7)-(9) from (6) together with (2a) is independent of the particular values taken by  $x_t$ , it follows immediately that the optimal solution of (13) is given by (7)-(9) with  $x_{t-1}$  replaced in (7) by the least mean-squared prediction  $x_{t-1}^d$  according to (15)-(17).

Thus, the optimal feedback rule remains the same in the case of arbitrarily long control lags as in the case of no lags. However, in the former case the feedback rule in  $t$  is acting on the best prediction  $x_{t-1}^d$  of  $x_{t-1}$  available in the period when  $u_t^*$  is to be decided upon, and not, as in the latter case, on actual deterministic knowledge of  $x_{t-1}$ . Furthermore, the minimum expected welfare loss increases with the length of the control lags. To recognize this, consider first the problem without delays, (1)-(2). From (6) in conjunction with (7) one can see that the direct contribution of every period  $t$

to the expected minimum total welfare loss,  $V_t^* - V_{t+1}^*$ , can be interpreted as the sum of the following two components:

a) the minimum period loss in  $t$  for the deterministic problem obtained by neglecting the stochastic disturbances  $\varepsilon_t$  in (2); and b) a "stochastic part"  $E\left[\frac{1}{2}\varepsilon_t^T K_t \varepsilon_t\right] = \frac{1}{2} \text{trace } \Sigma K_t$ . The stochastic part expresses the expected welfare loss which is caused by the uncertainty over the economy's dynamic behaviour expressed in the  $\varepsilon_t$ . This part is independent of  $x_0$  and of the target values  $\bar{x}_t$ .

The same distinction can immediately be made for the problem with delayed controls (10)-(12). By (19) together with (15)-(16) it is evident that the deterministic part of the period loss is the same in the delayed case as in the case when there is no control lag. However, the stochastic part now takes the value  $E\left[\frac{1}{2}\eta_t^T K_t \eta_t\right]$ , i.e. by (17),

$$\begin{aligned}
 (20) \quad E\left[\frac{1}{2}\eta_t^T K_t \eta_t\right] &= E\left[\frac{1}{2}\varepsilon_t^T K_t \varepsilon_t\right] + E\left[\frac{1}{2}\left[\sum_{\theta=0}^{\gamma-1} \left(\prod_{\tau=0}^{\theta} A_{t-\tau}\right)\varepsilon_{\theta-1}\right]^T K_t\right. \\
 &\quad \left.\cdot \left[\sum_{\theta=0}^{\gamma-1} \left(\prod_{\tau=0}^{\theta} A_{t-\tau}\right)\varepsilon_{\theta-1}\right]\right] = \\
 &= \frac{1}{2} \text{trace } \phi_t K_t,
 \end{aligned}$$

with  $\phi_t$ , the covariance matrix of  $\eta_t$ , given by

$$(21) \quad \phi_t = \Sigma + \left[\sum_{\theta=0}^{\gamma-1} \left(\prod_{\tau=0}^{\theta} A_{t-\tau}\right)\right] \Sigma \left[\sum_{\theta=0}^{\gamma-1} \left(\prod_{\tau=0}^{\theta} A_{t-\tau}\right)\right]^T.$$

(Of course  $\phi_t$  is also the covariance matrix of the state  $x_t$ .)

Since  $K_t$  is non-negative definite a)  $E\left[n_t^T K_t n_t\right] \geq E\left[\varepsilon_t^T K_t \varepsilon_t\right]$  for all  $t$ ; strict inequality will normally hold;

b)  $E\left[n_t^T K_t n_t\right]$  is a non-decreasing, in general increasing, function of the control lag  $\gamma$ . In other words, the larger the control lag, the lower is the expected level of goal achievement. An increase in the control lag is exactly equivalent to an increase in the system's uncertainty. As previously noted, in the deterministic case ( $\varepsilon_t = 0$  for all  $t$ ) future states can be exactly predicated; control lags have no influence on the minimum loss.

#### IV. Optimal delayed controls with imperfect state information

In the real world, perfect measurements of the economy's state in any given period are not available. The first ex post estimate of a state is a mixture of imperfect measurements, complemented and modified by judgemental opinions, statistical inferences and forecasts. As more or better data become available, this first estimate is revised until eventually some "final" estimate is obtained. Even the final estimate, however, represents only a very imperfect measurement of the state considered.

In order to take this into account, we reformulate the problem (10)-(12) in the form

$$(22) \quad W = E \left[ \sum_{t=\gamma}^N \frac{1}{2} (x_t - \bar{x}_t)^T K_t (x_t - \bar{x}_t) \right] \rightarrow \min_{u_\gamma, \dots, u_N}$$

subject to

$$(23a) \quad x_t = A_t x_{t-1} + B_t u_t + C_t z_t + \varepsilon_t, \quad t = 1, \dots, N,$$

$$(23b) \quad x_0 = \hat{x} + \varepsilon_0, \quad \varepsilon_0 \text{ is } N(0, \Sigma),$$

$$(24a) \quad u_t^* = u^*(y_\mu, u_\nu, t), \quad \mu < t - \gamma, \nu < t,$$

$$(24b) \quad u_1 = \hat{u}_1, \dots, u_{\gamma-1} = \hat{u}_{\gamma-1},$$

$$(25a) \quad y_t = M_t x_t + e_t, \quad t = 0, \dots, N,$$

$$(25b) \quad e_t \text{ is } N(0, \theta), \quad E \left[ (e_t) (e_\tau)^T \right] = 0, \quad \tau \neq t,$$

$$(25c) \quad E \left[ (e_t) (\varepsilon_\tau)^T \right] = 0, \quad t, \tau = 0, \dots, N,$$

where  $N(\cdot, \cdot)$  is the normal distribution with mean  $\cdot$  and covariance matrix  $\cdot \cdot \cdot$ . That is, we assume that  $x_t$  can be measured only indirectly through the stochastic vector  $y_t$  governed by (25). This way of modelling imperfect measurements is standard in engineering literature. For the macro-economic policy-making problem it should offer a reasonable compromise between mathematical tractability and realism. The assumption (23b), that  $x_0$  is not known exactly and is only described by a Gaussian random variable, follows naturally from assumption (25a-b). (Note, however, that (23b) includes a deterministic  $x_0 = \hat{x}$  as special case; alternatively, as previously noted, the results of the earlier section can be extended trivially to cover a random, normal-distributed  $x_0$ .) Assumption (25c) can easily be relaxed to allow for correlated  $\{\varepsilon_t\}$  and  $\{e_t\}$  processes. The measurement lag is now to be understood as the time needed to obtain the measurement  $y_t$  after  $x_t$  is realized. As before, we assume a total control lag (measurement + realization lag) of  $\gamma$  periods.

The functional equation of Dynamic Programming for this problem is given by

$$(26) \quad V_t = E \left[ \frac{1}{2} (x_t - \bar{x}_t)^T K_t (x_t - \bar{x}_t) / y_\mu, u_\nu, \mu < t - \gamma, \nu < t \right] + V_{t+1}^* .$$

Again let  $k = t - \gamma - 1$ . From section III we know that  $x_t$  can be expressed as a linear transformation of  $x_k$  and the stochastic disturbance  $\eta_t$ , which is independent of all control and states:

$$(27) \quad x_t = S_{t/\gamma} x_k + s_{t/\gamma} + \eta_t ,$$

with  $S_{t/\gamma}$ ,  $s_{t/\gamma}$  and  $\eta_t$  given by (16)-(17). In the present case  $x_k$  is not known. However, Optimal Filtering Theory tells us that the minimum variance estimate of  $x_k$ , conditional on the whole process history,

$$(28) \quad x_k^f = E\left[x_k / \sigma_\mu, \mu \leq k\right], \quad k = 0, \dots, N,$$

can be computed recursively as a linear-combination of  
 a) the current measurement  $y_k$ ; and b) the minimum variance estimate of the previous state  $x_{k-1}^f$ . The weights of this linear-combination express the relative importance of the uncertainty in the measurement  $y_k$  and of the uncertainty in a minimum-variance prediction of  $x_k$ , conditional on  $x_{k-1}$ . The recursive rule leading to the estimates  $x_k$  allows also computation of the corresponding error covariance matrices  $\Gamma_k = \text{cov}(\omega_k)$ , with  $\omega_k := x_k - x_k^f$ ,  $k = 0, \dots, N$ . This rule is called a Kalman-Filter, the estimates  $x_k^f$  are known as filtered estimates. For details see, for example, ANDERSON and MOORE [1979].

The important result for our purpose is that the estimate error  $\omega_k$  associated with the estimate  $x_k$  is normal-distributed with mean 0 and known covariance  $\Gamma_k$ . Setting  $(x_k^f + \omega_k)$  for  $x_k$  in (27), one sees immediately that in the present case as in the case of section III  $x_t$  can be expressed as the sum of a deterministic part  $x_t^g$  and of a zero-mean normal distributed disturbance  $\delta_t$ :

$$(29) \quad x_t = x_t^g + \delta_t$$

with

$$(30a) \quad x_t^g = S_{t/\gamma} x_k + s_{t/\gamma} ,$$

$$(30b) \quad \delta_t = S_{t/\gamma} \omega_k + \eta_t ,$$

the covariance matrix of  $\delta_t$  (i.e., of  $x_t$ ) being given by

$$(31) \quad \Delta_t = S_{t/\gamma} \Gamma_k S_{t/\gamma}^T + \Sigma + \left[ \sum_{\theta=0}^{\gamma-1} \left( \prod_{\tau=0}^{\theta} A_{t-\tau} \right) \right] \Sigma \left[ \sum_{\tau=0}^{\gamma-1} \left( \prod_{\tau=0}^{\theta} A_{t-\tau} \right) \right]^T = \\ = S_{t/\gamma} \Gamma_k S_{t/\gamma}^T + \phi_t .$$

By an argument similar to the one of section III it follows immediately that in the case of delayed controls and imperfect state measurements:

1) The optimal solution of (26) is given by (7)-(9) with  $x_{t-1}$  replaced in (7) with its minimum variance prediction conditional on the optimal (Kalman) filtering of  $x_{t-\gamma-1}$ . In other words, here again the optimal feedback rule is unchanged compared with the basic case (1)-(2), except that it is now acting on the best available state prediction and not, as previously, on the actual value of the state.

2) The "stochastic part" of period  $t$ 's contribution,  $t = \gamma, \dots, N$ , to the minimum expected welfare loss is increased by trace  $S_{t/\gamma} \Gamma_k S_{t/\gamma}^T K_t$  compared with the case with delays but exact measurements. This stochastic part is now the sum of the three elements

$$\text{trace } \Sigma K_t, \text{ trace } \left[ \sum_{\theta=0}^{\gamma-1} \left( \prod_{\tau=0}^{\theta} A_{t-\tau} \right) \right] \Sigma \left[ \sum_{\tau=0}^{\gamma-1} \left( \prod_{\tau=0}^{\theta} A_{t-\tau} \right) \right]^T K_t, \text{ and trace } S_{t/\gamma} \Gamma_k S_{t/\gamma}^T K_t$$

corresponding respectively to the irreducible uncertainty in the economy's behaviour, to the increased uncertainty caused by the control lag, and to the uncertainty caused by the measurement's inexactitude. The deterministic part of the loss is unchanged by the assumption of imperfect measurements.

It should be noted here that the problem with imperfect measurements and no control lag and its solution have long been extensively discussed in the literature, see, e.g. CHOW [1975]. The solution of this problem is a special case of the solution presented in this section and is, of course, given by (7)-(9) with  $x_t$  replaced in (7) by the corresponding Kalman-Filtering of  $x_t$  given the whole process history over  $0, \dots, t$ .

The difference between imperfect measurements and perfect, lagged measurements is that in the former case it is never possible to determine the exact values of the states while in the latter this information eventually becomes available, but too late to be of use in determining the optimal control.

As previously pointed out, estimates of the economy's state in any given period are in general revised several times. The policy-maker is therefore likely to have at his disposal several imperfect measurements  $y_t, y_t', \dots$ , of the past state  $x_t$ , corresponding to the different revised estimates. If the statisticians do their job properly, however,  $(y_t, y_t', \dots)$  should not yield any more information about the true state  $x_t$  than the most recent estimate,  $y_t$ . Thus the  $y_\mu$  taken into consideration in the formulation of  $V_t$ , see (26), can be taken

without loss of generality as the "newest" estimates available at the moment when  $u_t$  is to be decided upon.

Finally, let us remark that the problems of sections III and IV can easily be extended to cover the case when different control variables are associated with different control lags. This extension allows, for example, the explicit modelling of the fact that monetary policy measures can in general be realized faster than fiscal ones, etc.

## V. Conclusions

The preceding sections have revealed that in every case studied the optimal control in  $t$ ,  $u_t^*$ , is to be determined as a function of the minimum variance estimate of  $x_{t-1}$ , conditional on whatever information is available when  $u_t$  is to be decided upon. Control lags and imperfect measurements do reduce the quality of this estimate and, accordingly, do decrease the expected level of goal achievement. However, they have no effect on the linear transformation which expresses  $u_t^*$  as a function of the available estimate for  $x_{t-1}$ , i.e. on the optimal feedback equation in  $t$ . This result, while hardly surprising considering the given linear-quadratic framework, is both satisfactory and disappointing. On the one hand, it shows that numerical optimisations of linear econometric models remain valid in their most important aspects even if they neglect considerable control lags. In particular, these optimisations use the correct feedback decision rules. On the other hand the extreme simplicity and handiness of our results demonstrate once again the extreme poverty of the stochastic structure underlying LQG. In fact, even after the introduction of delays, the control problem can barely be called stochastic. (However, stochastic aspects play a considerable role in the filtering problem, which was not explicitly developed here.)

Thus, in spite of extensive presentation and discussion of this problem, we only find that it can be easily solved with

very elementary mathematics, and that the simplest, most intuitive ideas for dealing with control lags are indeed optimal. Even more disturbing may be the fact that this paper's results depend crucially on the linearity assumption. However, since a proved method to optimise non-linear econometric models proceeds through successive local linearisation (see CHOW [1975]), the results retain some operational validity even in the non-linear case. The Gaussian assumption is less crucial and can be significantly relaxed.

Finally, our results provide an immediate way of determining numerically the influence of control lags on the minimum expected welfare loss. We believe that a study of this influence using existing econometric models, although necessarily subject to serious qualifications, could lead to instructive insights as to the extent of potential economic costs due to delays in data gathering, decision-making and policy-implementation.

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