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to the Multiple-Goal Problem

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Nr. 94

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MAN-MACHINE INTERACTION AS AN ANSWER TO  
THE MULTIPLE-GOAL PROBLEM

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SURVEY

In many real problems the evaluation of alternative actions must take multiple heterogenous and complex indices or criteria into account. In this case, the standard mathematical optimization models cannot be applied to determine the "best" decision. What methods can then be used to help the decision-maker in his choice? What supplementary information is needed and how can it be obtained?

This article attempts to answer these questions by giving a critical overview of several man-machine interactive methods which were recently devised to solve the multiple-goal problem. Three types of methods are outlined and compared. The mathematical structure and the implementation of one of the methods, which appear to be operational for a wide range of problems, are presented in greater detail and illustrated by means of an example.

INTRODUCTION

In this article we are considering complex decision situations characterized by a multiplicity of partially conflicting goals, all of which are desirable to some extent. For simplicity's sake these goals are assumed to be of the form <sup>1)</sup>

$$x_i = f^i(z_1, \dots, z_m) \rightarrow \sup_z, \quad x_i \in R, \quad i = 1, \dots, n.$$

We assume the functions  $f^i$  to be point-to-point mappings so that exactly one  $x$  corresponds to each  $z$  and vice-versa. The functions  $f^i$ , the set  $D \subseteq R^m$  of feasible problem variables  $z$  and the corresponding set  $T \subseteq R^n$  of feasible goal variables  $x$  are supposed to be clearly defined and explicitly known. The decision-maker is assumed to have an implicit overall objective function  $U$  as a function of the  $n$  single variables  $x_i$ . The function  $U$ , however, is not explicitly known. Formally, this decision-making situation can be formalized as a constrained maximization problem,

$$\begin{aligned} &U(x_1, \dots, x_n) \rightarrow \sup, && U \text{ unknown}, \\ &\text{subject to} \\ &x_i = f^i(z_1, \dots, z_m) && i = 1, \dots, n; \\ &x \in T \iff z \in D \iff G(z_1, \dots, z_m) = 0; \end{aligned} \tag{1}$$

in which  $G$  describes a set of direct and functional constraints on the variables  $z_j$ .

Multiple-goal situations of the kind described above are most common in practical economic decision-making processes and have been recognized for a long time in economic theory<sup>2)</sup>. Two examples of these kinds of problems are:

a) In macroeconomics the "stabilization policy problem": unemployment and maintenance of full-capacity output, control of inflation, reasonable rate of economic growth, maintenance of a high level of investment, and redistribution of income through taxes and transfers;

b) In microeconomics the problem of maximizing profit, cash flow, profitability, stabilizing prices and beneficiary margins, assuring future investment possibilities and growth, augmenting the market share, etc.

Note that multiple-goal problems typically appear in organizations too, in which there is not a single decision-maker but a board, each member of which is more crucially interested in certain specific goals.

However, the assumption of an explicitly known, single-valued, overall objective function is still the rule in economic theory as well as in the decision-oriented models of Operations Research. The main reason for the apparent neglect of multiple-goal situations and their specific problems is probably the difficulty in handling multiple goals in a satisfactory manner. Knowledge of  $n$  single goals is insufficient in itself in enabling the formulation of an optimization problem capable of being used for actual policy recommendations to the decision-maker. When only the  $n$  goals are known, this can lead only to the vector-maximization problem

$$\begin{aligned} (x_1, \dots, x_n) &\rightarrow \sup, \\ \text{subject to} & \\ x_i &= f^i(z_1, \dots, z_m), \quad i = 1, \dots, n; \\ G(z_1, \dots, z_m) &= 0. \end{aligned} \tag{2}$$

The solution of (2), however, is not uniquely defined but consists of a whole set of points, i. e. the efficient set. Each element  $\bar{x}$  of this set, i. e. each efficient point, is characterized by the fact that no feasible  $x$  exists such that  $x_i \geq \bar{x}_i$ ,  $i = 1, \dots, n$ , with the strong inequality  $>$  holding for at least one  $i$ . That means there is no feasible point  $x$  which fulfills one single goal better than  $\bar{x}$  without simultaneously fulfilling some other goal less well than  $\bar{x}$  (Pareto optimality). Therefore, the solution of the decision problem (1) given a specific objective function  $U$  is necessarily an efficient point. Nevertheless, even when the computation of the efficient set is possible, the decision-maker is left with the problem of determining the "best" point in the efficient set. Since the efficient set is, as a rule, a complicated,  $n$ -dimensional structure, this reduced problem is not significantly simpler than

the original problem of finding the best point in the feasible set.

The purpose of this paper is to present some interactive man-machine approaches which can be effectively applied in actual multiple-goal decision situations to help the decision-maker make his choice.

There is no general agreement concerning the solution of the multiple-goal problem. The solution is straight-forward in two special classes: (a) No real difficulty arises when the preference ordering of the decision-maker is lexicographic; (b) The best solution can easily be found by directly comparing alternatives when the feasible or the efficient set contains only a few points or when they can be represented in an easily understandable way, for example graphically. However, a graphical representation is, in general, only possible for  $n = 2$ . For  $n > 2$  the feasible or the efficient set can, as a rule, only be represented analytically in the form of an equation, which in most practical situations will be too intuitively meaningless to serve as the basis for a decision.

For the general case, several interactive approaches have been studied to date. (For an excellent review, see Roy (1971).) Some of these approaches aim at the explicit specification of the overall objective function of the decision-maker or of some weaker surrogate of it, in order to permit the expression of the decision problem as a standard mathematical program. The highly interesting questions arising therefrom are central to welfare theory. However, these approaches are often too ambitious for practical application. In particular, the information which the decision-maker has to supply within the interactive process is often too complex. In any case, these approaches have mostly been studied on a theoretical level and there has been little or no attempt to apply them in actual problem solving.

We will therefore restrict our attention to the second main kind of approach, which has already reached the operational stage. The aim here is to determine a "satisfactory" solution through sequential exploration of the feasible set without having recourse to overall information about the preference ordering of the decision-maker. This exploration, however, is guided to some extent by local information furnished by the decision-maker and leads, among other things, to a progressive definition of his preferences. Procedures of this kind are typically based on an interactive procedure in which the decision-maker is faced at the first iteration with a feasible first solution. He is then asked by the algorithm to answer a few simple, intuitively clear questions, directed at revealing his evaluation of the solution (decision phase). The answers are subsequently used in a calculation phase to determine a new, "improved" solution, which forms the basis for a new iteration. The process is assumed to converge within a reasonable number of iterations to a "satisfactory" or "suboptimal" solution.

Let us already state at this point one very important characteristic of the methods we will present. The whole computation, which takes place in the problem variable space  $\{z\}$ , is relegated to the computer. The decision-

maker can regard the entire procedure as taking place in the goal variable space  $\{x\}$ ; he can disregard the values taken by the variables  $z_j$ , the function  $f^i$  and the constraints  $G$ . This has two advantages: a) The decision-maker can concentrate on his choice with no extraneous details to distract him; b) Practically, the only limiting factor in the use of the method is the maximal number of goal variables the decision-maker can comprehend at one time. Depending on the method, this number lies between 4 and 10 (see section III, B). The number of problem variables  $z_i$  can be much higher. The dimension of  $\{z\}$  and the complexity of the functions  $f^i$  and the constraints  $G$  is limited only by the computer's capacity to solve in a reasonable amount of time the classical optimization problem corresponding to (1) when the objective function  $U$  is explicitly known.

This article is organized as follows: In part I we discuss some general characteristics of interactive methods for solving the multiple-goal problem and their consequences, and present the logic underlying the three main classes of methods. The implementation of the GEOFFRION approach, which we consider to be the most promising of the three, is discussed in part II. A numerical example is given. Part III is devoted to comparative evaluation of the different methods presented. Some conclusions are drawn relative to their practical application and to further research.

## I - BASIC METHODS

Three main classes of approaches for solving the multiple-goal problem through sequential exploration of the feasible set can be distinguished:

- direct search or unstructured approaches;
- best compromise approaches (prototype: STEM method);
- interactive large-step procedures of the GEOFFRION-type.

The corresponding methods all aim at determining the best solution to a constrained optimization problem. However, the fact that they are based on a man-machine dialogue and are designed especially for assistance in managerial decision-making, implies the existence of important general properties that distinguish them in the family of optimization methods. These properties and their implications are discussed briefly in section A. The basic ideas underlying the unstructured approaches, the STEM and the GEOFFRION methods are presented in sections B - D.

### A - General properties

Normally, the cost of building and optimizing a multiple-goal model do not differ significantly from the cost of building and optimizing the corresponding classical optimization model, given a known overall objective function. To be sure, the interactive procedure may occasion increased programming and computing expenditures, although this may be offset by increased willingness

to tolerate an inexact solution (see below). The supplementary expenditures, however, are likely to constitute only a small fraction of the total analysis cost and will not play a decisive role in the choice of a given method.

Nevertheless, one specific cost factor places narrow constraints on the interactive methods: the high cost, for example in time elapsed, of the decision-maker's participation in the interactive process. This cost factor will be of decisive importance if the decision-maker participating in the interactive procedure is - as should ideally be the case - the "policy-maker" himself, i. e. usually a (board of) top-level manager(s) or high government official(s).

Another specific aspect of interactive procedures, which distinguishes them from regular iterative optimization problems, is the fact that "inexactitude" is introduced during the iterative process by the human element. Interactive procedures are based on an information exchange which takes place at each iteration between decision-maker and computer. It is likely that the decision-maker will not be able to exactly express the information he wants to communicate to the computer. Moreover, he will probably undergo a learning process during the interactive procedure, so that the information he furnishes at a later iteration may be "inconsistent" with the information he furnished at earlier iterations. On the other hand, the decision-maker will not be able to interpret exactly the information provided by the computer. In particular, he will consider small changes in the values of the variables insignificant, although even such seemingly insignificant changes may be of great importance in a standard optimization routine.

A third aspect is the fact that the decision-maker will not consider the "best solution" attained in the interactive procedure to be the decisive answer to his decision problem. Rather, he will rely on the procedure to provide him with supplementary information on the problem and to help him in his appreciation of relevant alternatives. In other words, confidence in the method is an essential factor in its practical success, and the key to the successful conduct of multiple-goal methods is the joint exercise of good judgement by executives and professional operations researchers.

From the various arguments discussed above, it is obvious that only approximate solutions can realistically be expected from interactive methods for solving the multiple criteria problem. The important issue, however, is not whether the proposed solution is optimal, but whether the method permits an improvement in the solution significant enough to justify its use. Therefore, a good approximation of the optimal solution is satisfactory, too. Furthermore, since the model to be optimized is itself an approximation, solutions will be acceptable which do not exactly satisfy the constraints (i. e., pseudo solutions). The consequence of this standpoint is that the formal problem of the existence of a solution becomes uninteresting except if the solution is unbounded. The problem is then meaningless. In the following, we will assume that all solutions are bounded, unless stated otherwise.



Since the inexactitude of the solution and the limited number of iterations are accepted, the infinite convergence of the method is irrelevant. The initial convergence, however, is crucial. A few iterations should lead to a reasonably small neighborhood of the optimum. The convergence must be robust against possible inconsistencies in the information provided by the decision-maker. The interactive procedures altogether should be compatible with the decision-maker's eventually changing his mind; he should be helped, but not restricted, in his search.

All the methods presented in this paper satisfy the above requirements. They are not bound to a special mathematical structure of the multiple-goal problem either, but can be applied to a wide variety of models. Only the STEM method is more restrictive in its use. However, it should be clear that quick convergence to a global optimum presupposes "reasonable" formal properties of the problem. The existence of several local optima in itself is not necessarily very disturbing in an interactive setting. In most cases, experience and intuition will allow easy recognition of a given local solution as globally unsatisfactory.

The implementation of a multiple-goal procedure in a concrete case should be carefully planned, if possible in close collaboration with the executives and the staff of the firm concerned. Before it is practically used as decision aid for a given problem, the procedure should be extensively tested. The corresponding trial runs should, if possible, be conducted with middle managers. Particular regard should be given to the instructions describing the model, the method, and the way they can contribute to the solution of the concrete multiple-goal problem. It is also extremely important to organize the information exchange between decision-maker and computer so that inexactitudes and/or misunderstandings as well as time required are reduced to a minimum. In general, the information exchange should be made with the help of a time-sharing program directly on a computer console whereby the calculation results are displayed on the screen.

Before the decision-maker is allowed to use the interactive procedure for solving his concrete problem, he should be thoroughly trained in the use of the method. In his training he should receive both theoretical explanations and practice in solving typical examples. Experience has shown that this training is not only important as an instrument in increasing the decision-maker's confidence and his acceptance of the method. It is also crucial in increasing the reliability of his answers to the computer. The solution of the real problem should be repeated at least once in order to increase confidence in the results and to check for eventual learning processes and local optima. The results must be filed and analyzed ex post in collaboration with the decision-maker in order to interpret his choices and their consequences. For the description of a practical case of implementation, see section III, A.

B - The unstructured approaches

Unstructured approaches refer to the direct application to multiple-goal problems of methods known in mathematical programming under the generic name of "Sequential Optimum Seeking by Experimentation". The designation "unstructured" is common but unfortunate, since many of these methods have a non-trivial structure. Their common characteristic is that they do not require elaborate information from the decision-maker. In general, a pairwise ranking of alternatives is all that is necessary. They neither seriously restrict the decision-maker in his search nor help him to any great extent. The rate at which they proceed toward the optimum depends on the decision-maker's luck and cannot be predicted in advance. Nevertheless, their importance should not be underestimated. These methods are generally easy to understand, easy to implement, and give very satisfactory results in situations which are not too complex.

In the simplest case the decision-maker is invited at each iteration to enter a solution x. Then the computer tells him whether x is feasible or not. (See printout and flow chart of the GEOFFRION method). Supplementary information can eventually be supplied; for example, the computer can indicate if x is efficient or not, which constraints are binding, etc. The decision-maker is completely on his own in choosing which new solution he will enter at the next iteration. Contrary to the STEM and the GEOFFRION methods, this approach can also be applied to discrete problems.

More complicated methods present the decision-maker at each iteration with a whole set of feasible solutions. He is then asked for a subjective ranking of the solutions. This ranking is used to determine the set of solutions to be presented at the next iteration, using, for example, a combined gradient and golden search method. A detailed exposition of these methods cannot be given here. The interested reader, however, may refer to the corresponding chapter in Simmons (1975) (in a classical optimization context) for an overview.

C - The STEM Method

1. General remarks. The STEM method devised by Benayoun and Tergny (1969) is applicable to linearly constrained problems in which the goal variables  $x_i$  are linear functions of the problem variables  $z_i$ . That is, it is applicable to multiple objective linear programs of the type:

$$\begin{aligned}
U(x) \rightarrow \max; \quad & x_i = c^i z, \quad i = 1, \dots, n; \quad U \text{ unknown;} \\
\text{subject to} & \\
z \in D \iff & (Az \leq b, \quad z \geq 0). \\
\text{with } x = (x_1, \dots, x_n), \quad & z = (z_1, \dots, z_m)', \\
c^i = (c_1^i, \dots, c_m^i), \quad & b = (b_1, \dots, b_\ell), \\
A \text{ a } (l \times m) \text{ matrix.} &
\end{aligned}
\tag{3}$$

The concept behind the STEM method is to sequentially explore feasible modifications (i. e., compromise solutions) of an unfeasible "ideal" solution, whereby the exploration is guided to some extent by the decision-maker's answers to simple questions posed by the algorithm. The decision-maker is not asked for information about the overall objective function  $U$  at any stage of the iterative process.

Prior to the interactive process with alternating calculation and decision phases, a preparation phase is necessary in order to calculate an "ideal" solution and the first compromise solution.

2. Preparation phase. This phase begins with the determination of the  $n$  extreme solutions which correspond to the maximization of each variable  $x_i$  considered as the unique goal. In other words, these extreme solutions are solutions to the  $n$  LPs

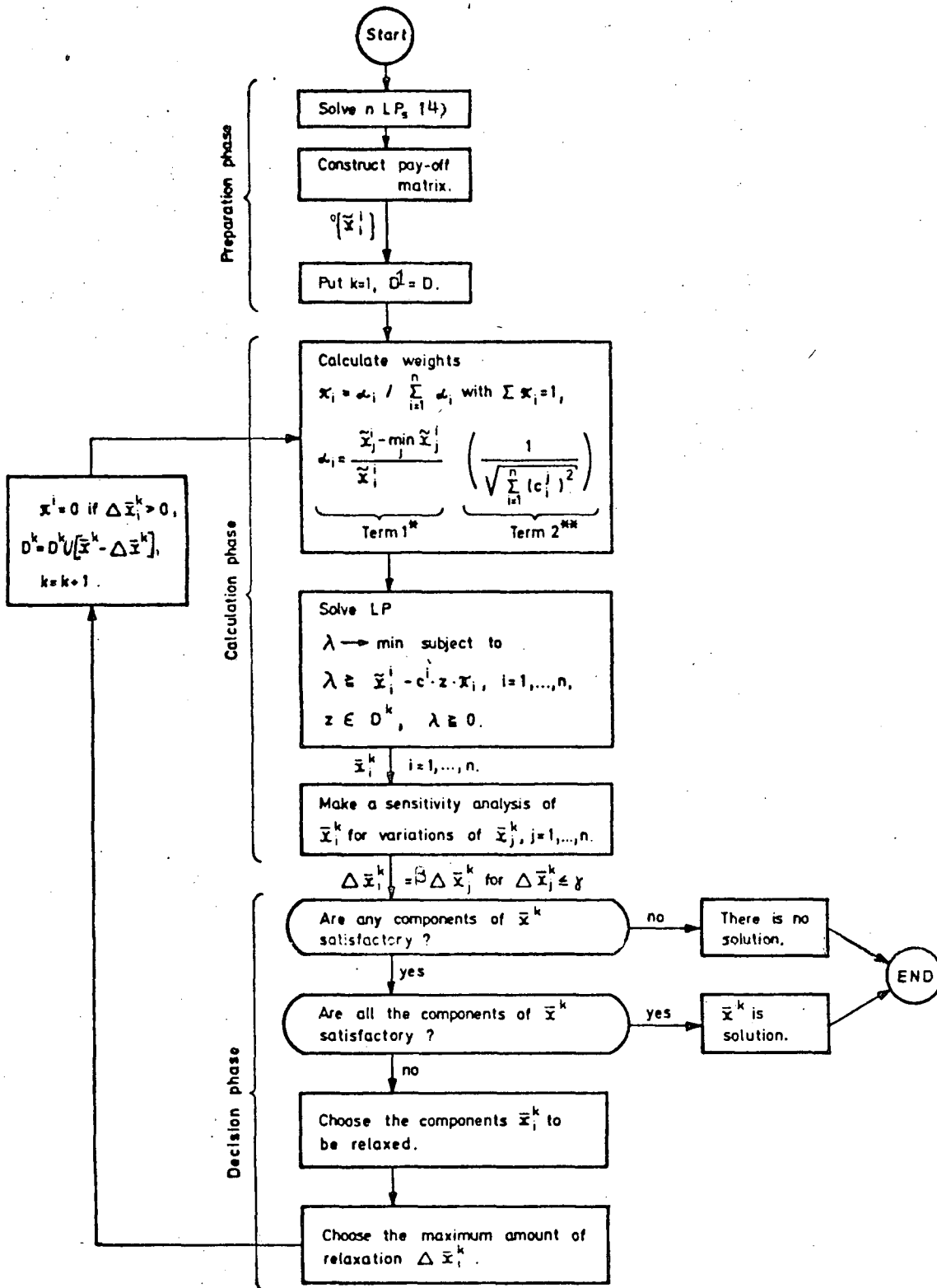
$$\begin{aligned} x_i &= c^i \rightarrow \max \\ \text{subject to} & \\ z &\in D, \end{aligned} \tag{4}$$

$i = 1, \dots, n$ . Let  $\tilde{x}^i = (\tilde{x}_1^i, \dots, \tilde{x}_n^i)$  be the solution of the  $i$ -th problem (4) and  $\tilde{z}^i = (\tilde{z}_1^i, \dots, \tilde{z}_m^i)$  the corresponding value of  $z$ . The value  $\tilde{x}_i^i$  represents the maximum value that the goal variable  $x_i$  can attain under the constraint  $z \in D$ , irregardless of the "goodness" of the values of the other goal variables.

The vectors  $\tilde{x}^i$  are used to build a pay-off matrix  $\{\tilde{x}_i^i\}$ , the  $i$ -th row of this matrix being given by the extreme solution  $\tilde{x}^i$ . The main diagonal of the pay-off matrix,  $\{\tilde{x}_i^i\}$ , which consists of the "maxima maximorum" of the goal variables  $x_i$ , is considered to be the reference or "ideal" solution. Because of the way they were obtained as solutions to the  $n$ -problems (4), the values  $\tilde{x}_i^i$ ,  $i = 1, \dots, n$ , cannot be reached simultaneously, so that  $\{\tilde{x}_i^i\}$  is unfeasible.

3. Calculation phase. The preparation phase is followed by a calculation phase in which the feasible  $\bar{x}^k = (\bar{x}_1^k, \dots, \bar{x}_n^k)$ , which is "closest" in the minimax sense to the ideal solution  $\{\tilde{x}_i^i\}$ , is calculated as the solution of a LP (see flow chart, Fig. 1).

In this calculation, weights  $\pi_i$  are introduced to express the relative importance of the distance of each goal variable  $x_i$  from its ideal value  $\tilde{x}_i^i$ . These weights are chosen in such a way that they will be small when the value of  $x_i$  is not much improved by a change in  $z$  from its ideal value  $\tilde{z}^i$  (see "Term 1" in flow chart). On the contrary,  $\pi_i$  will be large if  $x_i$  is very sensitive to a change in the value of  $z$ . The weights are normalized to one in order to enable easy comparison of different solutions obtained from different weighting strategies. Fig. 2 presents a printout example of the calculation phase.



\* "Minimax" term. This term becomes  $(\min \bar{x}_j - x_i) / \min \bar{x}_j$  when  $\bar{x}_i \leq 0$ .  
 \*\* "Normalizing" term.

FIG.1: Flow chart of STEM

Fig. 2 - STEM method: Calculation phase printout example  
(data input previously)

IDEAL AND COMPROMISE SOLUTIONS ARE

	COST	STOCKOUT	SALES
IDEAL	800000	6.4	1000000
COMP	1200000	9.3	700000

] =  $\bar{x}^k$

ARE ALL COMPONENTS OF COMP UNSATISFACTORY? Y or N.

□: N

ARE ALL COMPONENTS OF COMP SATISFACTORY? Y or N:

□: N

IF VARIABLES ARE NOT RELAXED BY MORE THAN  
200000 1.3 -3000

THEN ONE UNIT RELAXATION PERMITS

	DELTACOST	DELTASTOCK	DELTASALES
COST	1	-0.00000043	1.1
STOCK	-60000	1	27400
SALES	-2.5	-0.00000018	-1

] Sensivity  
Analysis

ENTER ABSOLUTE AMOUNTS OF MAXIMUM RELAXATION FOR ALL VARIABLES

□: 100000. 0. 0.

] =  $\Delta \bar{x}_i^k, i=1, \dots, 3$

IDEAL AND COMPROMISE SOLUTIONS ARE

	COST	STOCKOUT	SALES
IDEAL	800000	6.4	1000000
COMP	1300000	8.87	810000

] =  $\bar{x}^{k+1}$

4. Decision phase. Following the calculation phase, the ideal solution and the compromise solution are presented to the decision-maker, who compares them and decides if the compromise solution is satisfactory or not.

Three different cases can be distinguished:

- If no component of  $\bar{x}^k$  is satisfactory, the problem is declared unsolvable and computation is terminated.
- Likewise computation is terminated if all components of  $\bar{x}^k$  are satisfactory. The current compromise solution  $\bar{x}^k$  is then chosen as the "best compromise" or the definitive solution.
- If some of the components of  $\bar{x}^k$  are satisfactory and others not, the decision-maker must accept a certain worsening or relaxation of the satisfactory components in order to permit an improvement of the unsatisfactory ones. Therefore, he is asked which satisfactory component  $\bar{x}_i^k$  can be relaxed and how much relaxation  $\Delta\bar{x}_i^k$  is acceptable at the most. To help him with his decision, he is presented with the results of a sensitivity analysis, which indicate the behaviour of the different goal variables in the neighbourhood of the present compromise solution  $\bar{x}^k$ . Then the program returns to the calculation phase.

For the next iteration, the feasible set  $D^k$  is modified according to the decision made. It is requested for the next iteration that the new value for each element of the new compromise solution,  $\bar{x}_i^{k+1}$ , not be less than the old value minus the accepted maximal relaxation, if any:

$$\bar{x}_i^{k+1} \geq \bar{x}_i^k - \Delta\bar{x}_i^k, \Delta\bar{x}_i^k \geq 0, i = 1, \dots, n. \quad (5)$$

That is, the new compromise solution must be in the domain defined by the decision-maker. The weight  $\pi_i$  is set equal to zero in the  $k+1$  iteration whenever  $\Delta\bar{x}_i^k > 0$ . This expresses the fact that the decision-maker doesn't consider relaxation  $\Delta\bar{x}_i^k$  to be very costly.

At the end of the procedure it may be interesting to calculate a posteriori weights such that  $\pi_i / (\bar{x}_i^i - \bar{x}_i^k) = \text{constant}$ . In this way local approximations can be determined for the marginal rates of substitution between goals at the optimum.

5. Characteristics. The calculation phase of STEM can be easily programmed with the help of standard LP programs. The convergence of STEM to the best compromise solution is fast. The solution can be obtained in less than  $n$  iterations when one of the constraints of the type (5) has been relaxed at each iteration to such an extent that further relaxation is out of question. Fewer iterations are needed when during a single iteration more than one constraint is relaxed "to the limits of the acceptable". It should be emphasized here that in the case of STEM, in contrast to the unstructured of GEOFFRION approach, the stopping rule cannot be chosen according to the specific problem, but is stringently imposed by the algorithm.

In general, the compromise solutions are efficient. If not, they can always be made efficient by means of minor alterations.

STEM has one unpleasant characteristic. In later iterations it is impossible to "undo" partially or completely a relaxation  $\Delta \bar{x}_i^k$  made at an earlier stage. In this way, the decision-maker is impeded in his exploration of the feasible set. He cannot change his mind during the decision-making process or correct incorrect decisions. The only possibility of taking learning processes into consideration, which eventually may take place during the interactive procedure, is, therefore, to begin the process again from the beginning.

STEM can be considered to be the prototype of several interactive procedures, all of which:

- are designed within the framework of linear programming;
- replace the concept "best solution" with the concept "best compromise";
- use the vector  $\{\bar{x}_i^i\}$  as an "ideal solution".

These methods differ in the way in which they work toward the "best compromise". We mention here the method devised by Saska (1968) at the same time as, but independently of, STEM; the more highly developed Progressive Orientation Procedure (POP) of Benayoun and Tergny (1969); the algorithm of Belenson and Kapur (1973); the interactive procedures based on the Goal Programming Method of Charnes and Cooper (1961) (see for example Dyer (1972a). In some important aspects, however, these differ from STEM to a greater extent.) See also Aubin and Näslund (1972) and Contini and Zionts (1968).

#### D - The GEOFFRION Method

1. General remarks. Some of the most effective methods for the numerical solution of non-linear, constrained optimization problems are known in optimization theory under the generic name of large-step algorithms. A large-step procedure is typically defined by the following iterative steps:

Step 0. Select an (arbitrary) initial feasible point

$$z^0 = (z_1^0, \dots, z_m^0). \text{ Let } k = 0.$$

Step 1. At iteration  $k$ , determine the "best" direction  $y_j^k$ ,  $j = 1, \dots, m$ , to improve the current point  $z^k$ . The choice of  $y^k$  is to be based on the gradient in direction of  $z$  of the objective function  $U$  at the current point  $z^k$ ,  $\nabla_z U(z^k)$ .<sup>3)</sup>

Step 2. Find a step-size  $t^k$  such that  $U(z^k + t^k y^k)$  is maximized over all the feasible points  $z^k + t^k y^k$ .

Step 3. Compute the new current point  $z^{k+1} = z_j^k + t^k y_j^k$ ,  $j = 1, \dots, m$ . Return to step 1, where  $z^{k+1}$  replaces  $z^k$ .

The iterative process 1-3 produces a sequence  $z^0, z^1, \dots$  of improving feasible points  $z$ . It ends when  $z^k$  is a local maximum or a "satisfying" point, i.e. when it is impossible to find a better point than the current one by means of local considerations.

Large-step methods have been designed for the most general types of non-linear optimization problems (see, for example, Blum and Oettli (1975)). Under appropriate regularity conditions, they converge very fast to a neighborhood of the optimum. 4)

In his (1970) paper, Geoffrion conjectured that large-step methods could offer a very effective way of solving the multiple-goal problem (1) if the decision-maker is able to give the local information about his implicit objective function  $U$  necessary in carrying out the optimization calculation. This information is: (a) Some expression for the gradient of  $U$  at the current point and (b) the optimal step-size. In other words, the logic underlying Geoffrion's proposal is to adopt a mathematical programming technique of known efficiency, but to implement it interactively in order to do away with the need for knowledge of the overall objective function.

Since Geoffrion's original paper, this approach has been studied in detail and applied by numerous theoreticians (see, for example, Deissenberg (1976), Dyer (1973a), (1973b), (1974), Feinberg (1972), Geoffrion and Hogan (1972), Geoffrion and Dier (1972), Geoffrion et al. (1972), Wallenius (1975)). Contrary to the STEM method in which implementation is determined within narrow limits by the STEM algorithm itself, the GEOFFRION approach permits a large number of ways of obtaining the needed information from the decision-maker, and can be applied to the whole range of large-step algorithms. Since it is impossible to cover all possible versions of the method, we will present in the following the logic underlying steps 1 and 2 in an interactive large-step algorithm within the specific framework of the FRANK-WOLFE algorithm<sup>5)</sup>, but in terms as general as possible. Except for the special form of the optimization problem serving to determine  $t^k$ , our presentation is valid for all interactive large-step algorithms if the objective function  $U$  has continuous derivations on  $\{z\}$  and the feasible set  $D$  is compact and convex. Application of an interactive large-step algorithm will be discussed in more detail in part II.

2. Step 1. In a large-step procedure, the determination of the best direction  $y^k$  uses the property of the gradient  $\nabla_z U(z^k)$  as the "direction of steepest ascent" of  $U$  in the vicinity of  $z^k$ . In the FRANK-WOLFE algorithm,  $y^k$  is given by  $y^k = \bar{z}^k - z^k$ , where  $\bar{z}^k$  is the solution of a standard LP approximating the problem

$$\begin{aligned} &\nabla_z U(z^k) \cdot z \rightarrow \sup_z \\ &\text{subject to} \\ &z \in D. \end{aligned} \tag{6}$$



In other words, in this particular algorithm the direction  $y^k$  is chosen such that the product of the maximum rate of improvement multiplied by the feasible rate of improvement in this direction is maximized.

In a multiple-goal problem,  $U$  is unknown and, therefore,  $\nabla_z U$  is unknown, too. Before solving the LP corresponding to (6), it is necessary to specify  $\nabla_z U(z^k)$  in a decision phase on the basis of information given by the decision-maker.

In general, however, the decision-maker evaluates the current situation in terms of  $x_i^k = f^i(z^k)$ ,  $i = 1, \dots, n$ , and not in terms of  $z^k$ . Hence, he will be unable to give direct information on the partial derivatives  $\partial U(z^k) / \partial z_j$ . Fortunately the problem of estimating  $\nabla_z U$  can easily be expressed in terms of  $x^k$ .  $\nabla_z U(z^k)$  can be written

$$\nabla_z U(z^k) = \sum_{i=1}^n (\partial U / \partial x_i^k) \nabla_z f^i(z^k).$$

In this expression,  $\nabla_z f^i(z^k)$  can be directly calculated from the known  $f^i$ . Note further that the solution of (6) is not affected by a positive scaling of  $\nabla_z U(z^k)$ .  $\nabla_z U(z^k)$  can, for example, be divided by  $(\partial U / \partial x_1^k)$ , which, according to our assumptions, is positive. Hence (6) can be expressed equivalently as

$$\sum_{i=1}^n w_i^k \nabla_z f^i(z^k) \cdot z \rightarrow \sup_z$$

subject to (6')

$$z \in D,$$

with  $w_i^k = (\partial U / \partial x_i^k) / (\partial U / \partial x_1^k)$ ,  $i = 1, \dots, n$ .

Since the objective function has been scaled by  $(\partial U / \partial x_1^k)$ , we will call the goal variable  $x_1$  the reference variable. Note that the choice of  $x_1$  as the reference variable is arbitrary and that any other goal variable  $x_i$  could be chosen to play this distinguished role.

Thus, in order to permit the construction of  $\nabla_z U(z^k)$ , the decision-maker needs only to give a subjective appreciation of the current point  $x^k$  (and not  $z^k$ ) in a form suitable for approximating the  $n-1$  weights  $w_i^k$ ,  $i=2, \dots, n$ . (Since  $x_1^k$  was chosen as the reference variable,  $w_1^k$  is identically equal to 1). This subjective appreciation can be made from two different standpoints:

- a) On the one hand  $(1/w_i^k)$  is the slope of the projection of  $\nabla_z U(z^k)$  on the  $(x_1, x_i)$  plane. That is,  $w_i^k$  gives the marginally "ideal proportion of change" for  $x_1$  and  $x_i$ . Let  $w_i^k = (\delta x_i / \delta x_1)$ ; then  $U$  increases most rapidly if  $x_i^k$  is marginally augmented by  $\delta x_i$  units for each  $\delta x_1$  augmentation of  $x_1^k$ , all other variables being constant. Therefore, the weight  $w_i^k$  approximately corresponds to the decision-maker's answer to the question: "How much should  $x_i$  ideally increase from its current level  $x_i^k$  if  $x_1$  is increased by  $\delta x_1$  from its current level  $x_1^k$ ?"

- b) On the other hand,  $-w_i^k$  is the slope of a tangent to the indifference curve through  $x^k$  in the  $(x_1, x_i)$  plane. In other words,  $w_i^k$  expresses the marginal rate of transformation or trade-off between  $x_1$  and  $x_i$  at  $x^k$ . If the decision-maker is indifferent to a change in the values of  $x_1^k$  and  $x_i^k$  in the marginal amounts  $\Delta x_1$  and  $\Delta x_i$  respectively, all other variables being constant, then  $w_i^k = -(\Delta x_1 / \Delta x_i)$ . The weight  $w_i^k$  can therefore be approximately determined by letting the decision-maker answer the question: "What loss in  $x_i$  from its current level  $x_i^k$  would compensate exactly for a given increase  $\Delta x_1$  in  $x_1$  from its present level  $x_1^k$ ?"

The relationship between the two standpoints is expressed graphically in Fig. 3. The interactive evaluation of the weights  $w_i^k$  is followed by a calculation phase. The weights are used to build the objective function of the LP corresponding to (6). The computer solves the LP determining  $\bar{z}^k$  and  $y^k$ . The program then proceeds to step 2.

3. Step 2. In all likelihood the point  $\bar{z}^k$  is not in the immediate neighborhood of  $z^k$  and the behavior of  $U$  can change significantly between  $z^k$  and  $\bar{z}^k$ . This implies that the "best" direction  $y^k$ , which was myopically determined as a function of  $U$ 's behavior in the immediate neighborhood of  $z^k$ , may become a "bad" direction as one advances farther from  $z^k$  in the direction of  $\bar{z}^k$ . However, because of the continuity of  $U$  and its partial derivatives, there is a feasible combination  $z^{k+1} = (1-t^k)z^k + t^k \bar{z}^k$ ,  $j = 1, \dots, m$ ,  $0 \leq t^k \leq 1$ , of  $z^k$  and  $\bar{z}^k$  that yields  $U(z^{k+1}) > U(z^k)$ . The problem in step 2 is to determine the optimal step-size  $t^k$ , i.e. the value of  $t^k$  which maximizes  $U(z^{k+1})$ . 6)

In the interactive procedure, the optimal value of  $t^k$  is directly determined by the decision-maker. He is guided in his choice by plots of the values  $x_i^{k+1} = f^i(z^k + t^k y^k)$ ,  $i = 1, \dots, m$ , as a function of  $t$  between 0 and 1 (Fig. 4a). These plots permit a comprehensive overview of the feasible bundles  $x_i^{k+1}$ ,  $i = 1, \dots, m$  as a function of the single variable  $t^k$ . Alternatively, the computer can tabulate the values of the goal variables at selected values of  $t$ . Given the important case that the goal variables are linear functions of the variables  $z_j$ , the  $x_i$ 's are also linear in  $t$  (Fig. 4b).

4. Summary of the method. As a whole the application of an interactive large-step algorithm to problem (1) proceeds in the following manner:

- Step 0 (Initialization): The decision-maker chooses or is presented with an initial feasible solution  $x_i^0 = f^i(z_i^0)$ ,  $i = 1, \dots, n$ .
- Step 1, a) (Decision phase): At iteration  $k$ , the decision-maker assesses his trade-off weights  $w_i^k$  by subjectively analyzing the current solution  $x^k$ . This analysis can follow either approach 2a) or 2b) above.
- b) (Calculation phase): Compute the best direction  $y^k$  from the solution of a LP formulated on the basis of both  $w_i^k$ ,  $i = 1, \dots, n$ , and the constraints defining the feasible set  $D$ . Plot the function  $f^i(z^k + t^k y^k)$  for  $0 \leq t \leq 1$ ,  $i = 1, \dots, n$ .

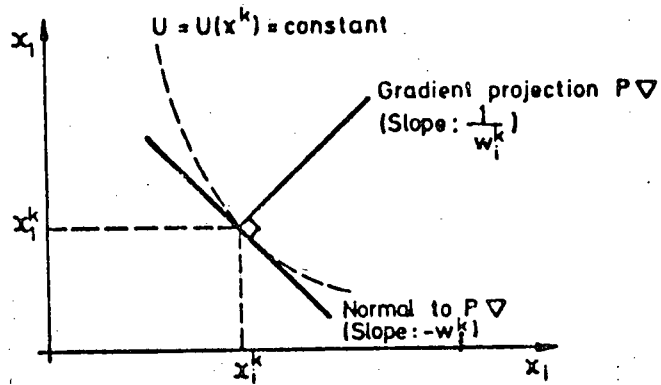


FIG. 3

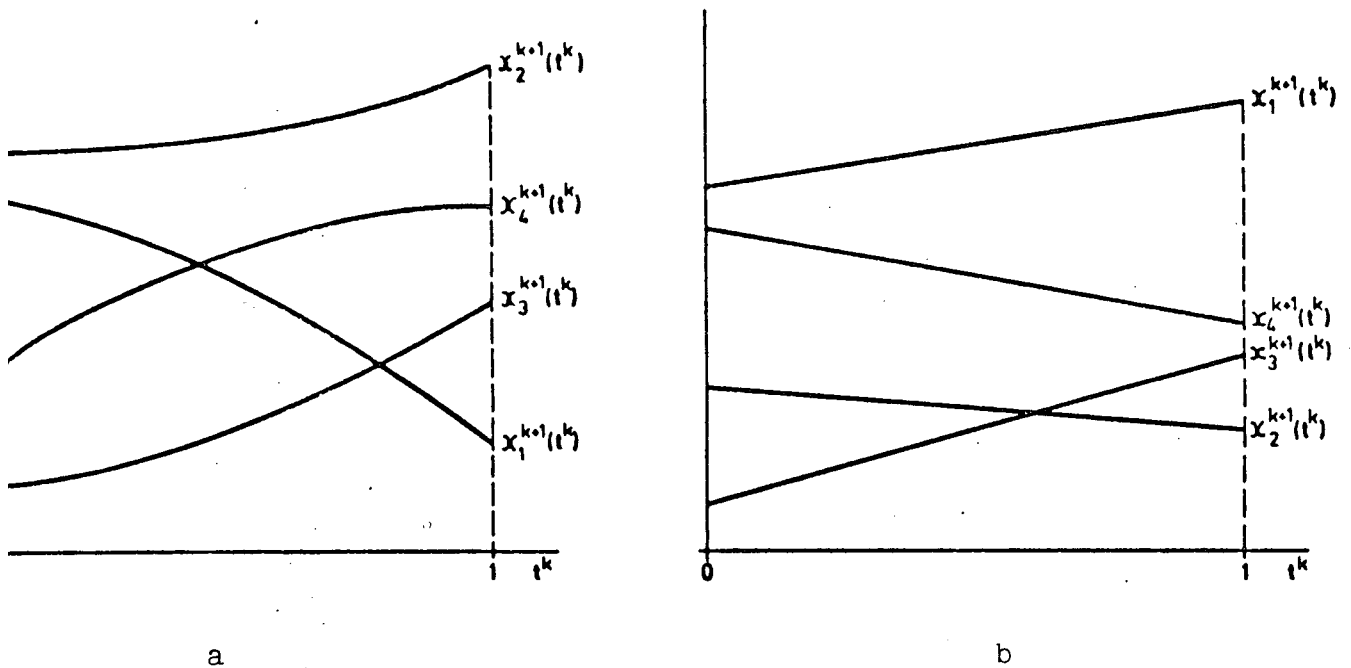


FIG. 4

- Step 2 (Decision phase): Let the decision-maker choose the "best"  $t^k$  by subjective analysis of the plots of  $f^i(z^k + t^k y^k)$ .
- Step 3 (Calculation phase): Use the "best"  $t^k$  from step 2 to compute the new current solution  $z^{k+1} = z^k + t^k y^k$ . Let  $x_i^{k+1} = f^i(z^{k+1})$ ,  $i = 1, \dots, n$ .  $k = k + 1$ , and return to step 1.

5. Characteristics. The calculation phase of the FRANK-WOLFE algorithm and of most other interactive large-step procedures are easy to program with the help of standard LP programs. The initial convergence of the large-step algorithms is very fast in general. In the case of the FRANK-WOLFE algorithm, Wolfe (1970) has shown that under certain regularity conditions the difference between the maximal value of the objective function,  $U(x^*)$ , and the current value,  $U(x^k)$ , is at least halved at each of the first  $K$  iterations ( $K$  unknown). That is,

$$\frac{U(x^*) - U(x^{k+1})}{U(x^*) - U(x^k)} \leq \frac{1}{2}, \quad k \leq K.$$

The improvement  $U(x^{k+1}) - U(x^k)$  is in general greater, the farther  $x^k$  is from the solution  $x^*$ . Accordingly, the algorithm leads very quickly to a neighborhood of  $x^*$ .

The convergence is robust against errors in the specification of the weights  $w_i^k$ . The convergence toward  $x^*$  is assured if, generally speaking, the error on the  $w_i^k$  decreases as  $x^k$  comes closer to  $x^*$  (see Dyer (1974)). The decision-maker can modify his preference at any time. He can express his present preference through his choice of the  $w_i^k$  and  $t^k$  at the current iteration without being restricted in any way by the algorithm. The flexibility of the algorithm also gives the decision-maker the possibility to experiment at any stage of the interactive process, i. e. to use arbitrary weights  $w_i^k$  in order to see the results or to explore extreme areas of the feasible set. To determine whether or not the preferences of the decision-maker have stabilized themselves during the interactive process, it may be useful to begin the procedure again by starting from another initial point  $z^0$  and to see if both corresponding sequences  $x^0, x^1, \dots$  and  $x^{0'}, x^{1'}, \dots$  converge to the same area in  $x$ .

A significant characteristic of GEOFFRION-type methods is their high informational content. In particular it is possible to construct a family of indifference curves in the vicinity of the path  $x^0, x^1, \dots$ , which is uniquely determined up to a monotonous transformation. In this way, in case of stable preferences one gets an a posteriori approximation of the behavior of the overall objective function  $U$  along with the path  $x^0, x^1, \dots$ . When different paths, which all converge to the same area in  $x$ , are known, it may be possible to integrate the local information along each path to approximate the overall objective function of the decision-maker (Fig. 5).

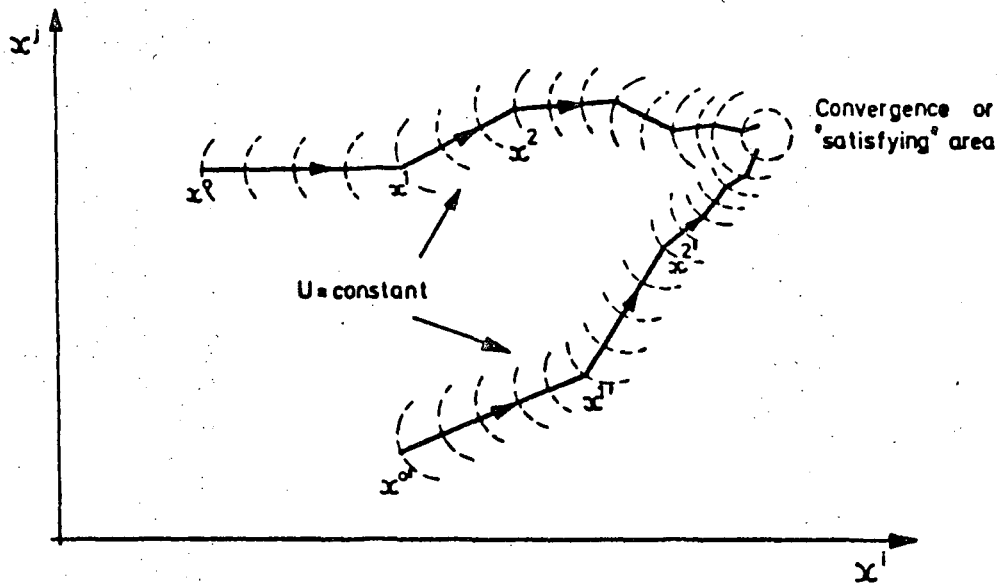


FIG. 5

Two paths  $x^0, x^1, \dots$  and  $x^{0'}, x^{1'}, \dots, x^0 \neq x^{0'}$ , will not converge to the same area in two cases: a) If the decision-maker has changed his mind to a significant degree between the last iteration of the first experiment and the last iteration of the second experiment; b) If there are several local optima. It may be impossible to recognize which case is true in a given situation. In any case, however, an effort should be made to analyze the reasons for the divergence. The decision-maker's opinion will, of course, play a central role in such an analysis.

## II - APPLICATION OF INTERACTIVE LARGE-STEP ALGORITHMS

Having presented the logic underlying GEOFFRION's approach, we now turn to its practical application. Some aspects of implementation are discussed in Section A; Section B is devoted to an example. A flow chart is given in Fig. 6.

### A - Implementation

1. Preparation. Any practical application of an interactive large-step procedure must be carefully planned according to the "General Remarks" made in Section I, A.

2. Step 0 (Initialization): Any feasible point  $z^0$  can be chosen as the initial solution. However, the number of iterations necessary to reach a satisfying solution may be decreased to a great extent by choosing a "good" if possible efficient, starting point  $z^0$ .

A simple but effective procedure to this effect consists of letting the decision-maker choose a good  $x^0$  by himself with the help of an unstructured approach and then compute the corresponding  $z^0$ . In most practical applications another possibility is to use actual or historical values of the problem variables for  $z^0$ , possibly modified to satisfy the model constraints. Other more sophisticated methods for determining a favorable initial solution have been devised in optimization theory. Let us mention here the possibility of projecting an "ideal", feasible point given by the decision-maker on the efficient surface. (Note the analogy with the determination of the first compromise solution in STEM). For details see, for example, Dikin (1967).

The determination of a feasible  $z^0$  may not be a trivial task if the feasible set is defined by complicated functional constraints  $G(z) = 0$ .

3. Step 1 (Decision phase): In practical applications it may be difficult for the decision-maker to directly answer the questions corresponding to the "ideal proportional change" or the "trade-off" approaches presented in subsection I D 2. Therefore, indirect methods have been devised which permit the determination of the weights  $w_k^k$  by letting the decision-maker answer a sequence of yes-no questions generated by the computer. We present here such an indirect method devised by Dyer (1973) and extensively tested in numerous applications (see also flow chart Fig. 7 and printout example Fig. 8).

Dyer's interactive routine is based on the "Trade-off" approach. In order to determine  $w_1^k$ , it aims at finding a modified vector  $x^k \bar{\Delta} := (x_1^k + \bar{\Delta} x_1, x_2^k, \dots, x_{i-1}^k, x_i^k - \bar{\Delta} x_i, x_{i+1}^k, \dots, x_n^k)$ ,  $\bar{\Delta} x_1, \bar{\Delta} x_i \geq 0$ , which is considered by the decision-maker to be equivalent to the current solution  $x^k = (x_1^k, \dots, x_n^k)$ . The interactive procedure first determines upper and lower bounds for  $x^k \bar{\Delta}$ . With the help of a bisection algorithm, these bounds are gradually brought closer together until a close approximation for  $x^k \bar{\Delta}$  is reached.

a) Initialization. At the beginning of the procedure the decision-maker is confronted with the current solution  $x^k$  and with the first, arbitrarily modified vector  $x^k \Delta^1 = (x_1^k + \bar{\Delta} x_1, x_2^k, \dots, x_{i-1}^k, x_i^k - \Delta^1 x_i, x_{i+1}^k, \dots, x_n^k)$ . Let  $\ell = 1$ . Since we are interested in the marginal rates of substitution  $w_1^k$ , the perturbations  $\bar{\Delta} x_1$  and  $\Delta^1 x_i$  must be small relative to the current values  $x_1^k$  and  $x_i^k$ , in order to keep the errors of higher order in an acceptable range. However, they must be large enough to be considered significant by the decision-maker. To ensure this last characteristic, it is convenient to let the decision-maker himself choose  $\bar{\Delta} x_1$  and  $\Delta^1 x_i$ ,  $i = 2, \dots, n$ . ("ENTER DESIRED PERTURBATIONS" in example printout.)

b) Determination of the bounds. At iteration  $\ell$ , the decision-maker is asked if he prefers  $x^k$  to  $x^k \Delta^\ell$ ,  $x^k \Delta^\ell$  to  $x^k$ , or if he is indifferent between the two. If the decision-maker prefers  $x^k \Delta^\ell$  to  $x^k$ , i. e. if in his opinion the decrease  $-\Delta^\ell x_i$  in the value of  $x_i^k$  is more than compensated for by the increase  $\bar{\Delta} x_1$  in the value of  $x_1^k$ , then  $\Delta^\ell x_i$  is doubled in the next iteration  $\ell+1$  and the question is repeated until  $x^k$  is preferred to  $x^k \Delta^\ell$ . One iteration before, however,  $x^k \Delta^{\ell-1}$  was preferred to  $x^k$ . Thus we know that the perturbation  $-\bar{\Delta} x_i$ , which exactly compensates for the perturbation  $\bar{\Delta} x_1$ , has as upper and lower bounds  $-\Delta^{\ell-1} x_i$  and  $-\Delta^\ell x_i$ , respectively.

c) Reduction of the interval. A bisection algorithm using the general formula

$$\Delta^{\ell+1} x_i = (\text{upper bound} + \text{lower bound at iteration } \ell) / 2 \quad (7)$$

is then applied to explore and narrow the interval  $(\Delta^\ell x_i, \Delta^{\ell-1} x_i)$ .  $\Delta^{\ell+1} x_i$  is computed according to (7). It is then used as the new lower bound ( $x^k \Delta^{\ell+1}$  preferred to  $x^k$ ) or as the new upper bound ( $x^k$  preferred to  $x^k \Delta^{\ell+1}$ ) in a new iteration using (7). The process is repeated until the decision-maker is indifferent between  $x^k$  and  $x^k \Delta^{\ell+1}$ , i. e.  $\Delta^{\ell+1} x_i = \bar{\Delta} x_i$ . The calculation of  $w_1^k$  follows immediately from  $w_1^k \simeq -(\bar{\Delta} x_1 / \bar{\Delta} x_i)$ . To assure the working of the algorithm in case  $x^k$  is preferred to  $x^k \Delta^1$  at the first iteration, it is practical to use the convention  $x^k = (x_1^k + \Delta^0 x_1, x_2^k, \dots, x_{i-1}^k, x_i^k - \Delta^0 x_i, x_{i+1}^k, \dots, x_n^k)$  with  $\Delta^0 x_1 = \Delta^0 x_i = 0$ .

The above procedure is repeated until all  $n-1$  weights  $w_i^k$ ,  $i = 2, \dots, n$ , have been calculated.  $w_1^k$  is identically equal to 1. Remark that  $\bar{\Delta} x_1$  remains constant during the whole procedure. In general,  $w_1^k$  is obtained within a few iterations because of the limited ability of the decision-maker to discriminate between two similar vectors.

In the above presentation of DYER's method, changes  $\Delta x_i$  in the values of  $x_i^k$ ,  $i = 2, \dots, n$ , were systematically modified until they exactly compensated for a given change  $\bar{\Delta} x_1$  in the value of  $x_1^k$ . In other words,  $x_1$  was used as the reference variable. As stated before, any other goal variable  $x_i$  can assume the role of reference variable. As a matter of fact, it may be interesting to let the decision-maker calculate a second set of weights,  $w_i^{k'}$ , using another goal variable  $x_j$  as the reference variable. If the decision-maker is consistent in his appreciation, we should have

$$w_j^k w_1^{k'} = (\bar{\Delta} x_1^k / \bar{\Delta} x_j^k) / (\bar{\Delta} x_j^k / \bar{\Delta} x_1^k) \simeq 1 \quad (8)$$

$$w_i^k / w_i^{k'} = (\bar{\Delta} x_1^k / \bar{\Delta} x_i^k) / (\bar{\Delta} x_i^k / \bar{\Delta} x_j^k) \simeq w_j^k, \quad i, j = 1, \dots, n, \text{ etc.} \quad (9)$$

Thus, the calculation of a second set of weights  $w_i^{k'}$  offers a possibility of computing a measure for the "inconsistency" or "inexactitude" of the information provided by the decision-maker. If the inconsistency of the information exceeds a given tolerance limit, one may ask the decision-maker to think over his statements. A corresponding routine can easily be built into the computer program so as to include such an automatic consistence check at every iteration.

3. Step 1 (Calculation phase): As mentioned previously, the calculation phase in step 1 consists of little more than the solution of a standard LP. The LP corresponding to the FRANK-WOLFE algorithm in the case of linear restrictions is given in the flow chart. For additional cases see Simmons (1975).

Two particular outcomes of the step 1 calculation phase are of interest: a) If the solution of the LP is unbounded, the solution of (1) is unbounded and the interactive problem has no meaningful solution; b) If the solution  $\bar{z}^k$  of the LP is the current point  $z^k$  itself, there is no "improving" feasible direction in the neighborhood of  $z^k$ . In other words,  $z^k$  is a (local) optimum and, from a purely mathematical point of view, the problem is solved. However, the decision-maker may consider  $z^k$  an implausible solution or may want to resume the search for supplementary information about the feasible set. It is therefore useful to give him the possibility at this point of entering directly a modified feasible point  $z^{k+1}$ . From a formal point of view, this is, of course, equivalent to starting a new search from the beginning. From a psychological point of view, however, this can make a great deal of difference.

4. Step 2: The fundamental aspects of the implementation of the step-size problem have already been discussed in part I. If the computer tabulates the values of  $(1-t^k)z^k + t^k \bar{z}^k$  for  $\ell$  equidistant values of  $t^k$ , i. e. for  $t^k = 1/\ell, 2/\ell, \dots, \ell/\ell$ , the decision-maker should be allowed to choose the number  $\ell$  according to his subjective information needs, and/or given the possibility of modifying a previously chosen number.

If  $\bar{z}^k$  is in the neighborhood of  $z^k$ , the solution of the step-size problem can produce only a small modification of the current solution. Thus, many iterations may be needed to leave the neighborhood of  $z^k$ , a feature which can be



time-consuming as well as nerve-racking for the decision-maker. In this case, too, it may be advantageous to present the option: "Enter directly a modified current solution  $z^{k+1}$ ". For simplicity's sake and in order to give the decision-maker the most complete freedom in his search, we propose to systematically offer such an option in step 2, regardless of the characteristics of  $\bar{z}^k$ .

5. Step 3: This step is trivial.

6. Stopping rule: As stated already, the calculation ends when the solution of the LP in step 1 is unbounded (meaningless problem) or when  $z^k = \bar{z}^k$ . However,  $z^k = \bar{z}^k$  is an exact mathematical requirement which may be fulfilled only after a very high or even infinite number of iterations. Hence, it is necessary to define some other, more practical rule for terminating the iterative process.

Of course, it is possible to weaken the requirement  $z^k = \bar{z}^k$  to  $z^k \approx \bar{z}^k$ , i. e. to let the computation end whenever  $\bar{z}^k$  is in a neighborhood of  $z^k$ . One can also impose a ceiling on the number of iterations. However, any formal stopping rule impairs the freedom of the decision-maker in his search. Therefore it seems preferable to simply let the decision-maker decide by himself when he wants to end the computation. Experience has shown that, as a rule, the path  $x^k, x^{k+1}, \dots$  tends to exhibit at the end of the search some kind of cyclical behavior within the "satisfactory" area (Fig. 9).

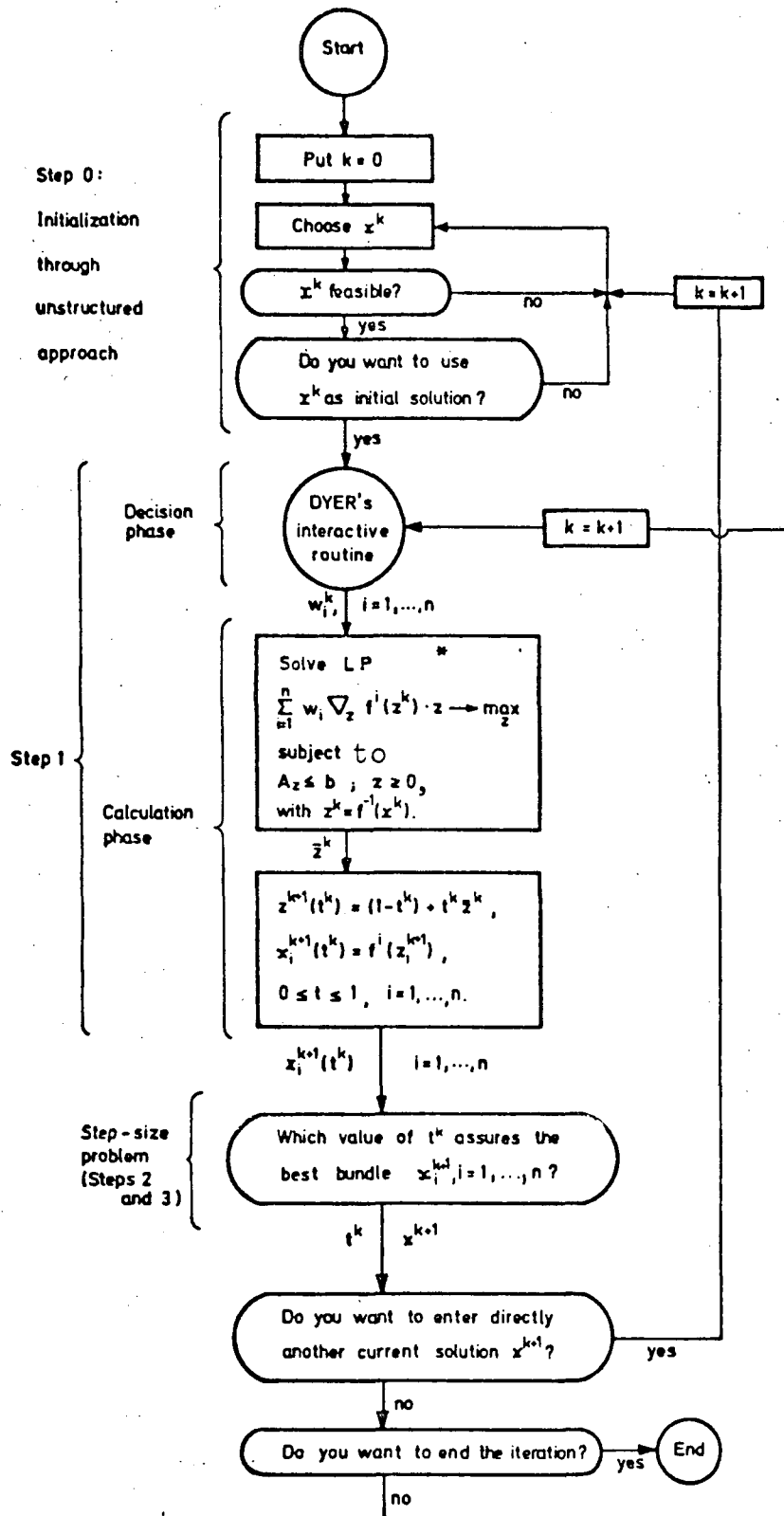
7. Additional remarks. Let us state again that repetition of the iterative procedure, starting from different initial points, may provide useful information. The data and, in particular, the decision and the paths for  $x^k$ , should be filed and analyzed ex post with the help of the decision-maker.

8. Printout. The following printout corresponds largely to Dyer's original printout. However, Dyer's program contains an option "ENTER TRADE-OFFS DIRECTLY". Step 0 uses an unstructured interactive approach for choosing  $z^0$ . Note that the perturbations  $\Delta x_1$  and  $\Delta x_2$ , corresponding to "costs" and "stockout" respectively, are negative. This is due to the fact that these variables are to be minimized, i. e. the program maximizes the variables "-costs" and "-stockout".

## B - Example

The following numerical example was chosen solely on the basis of its simplicity, which makes it possible to easily understand the structure of the problem. Although the problem is presented somewhat like a case study, it does not at all pretend to be realistic. For case studies and more realistic examples, see the literature listed in part III. See also the example described in section III, A.

Consider a firm that produces three products 1, 2 and 3 in the respective quantities  $z_1, z_2, z_3$ . The production capacity is limited by the constraint  $z_1 + 2z_2$



\* This LP corresponds to the solution of the problem  $U(x) \rightarrow \sup$  subject to  $x_i = f^i(z)$ ,  $i=1, \dots, n$ ,  $Az \leq b$ ,  $z \geq 0$ , with the FRANK-WOLFE algorithm.

FIG. 6 : GEOFFRION Method: Flow Chart

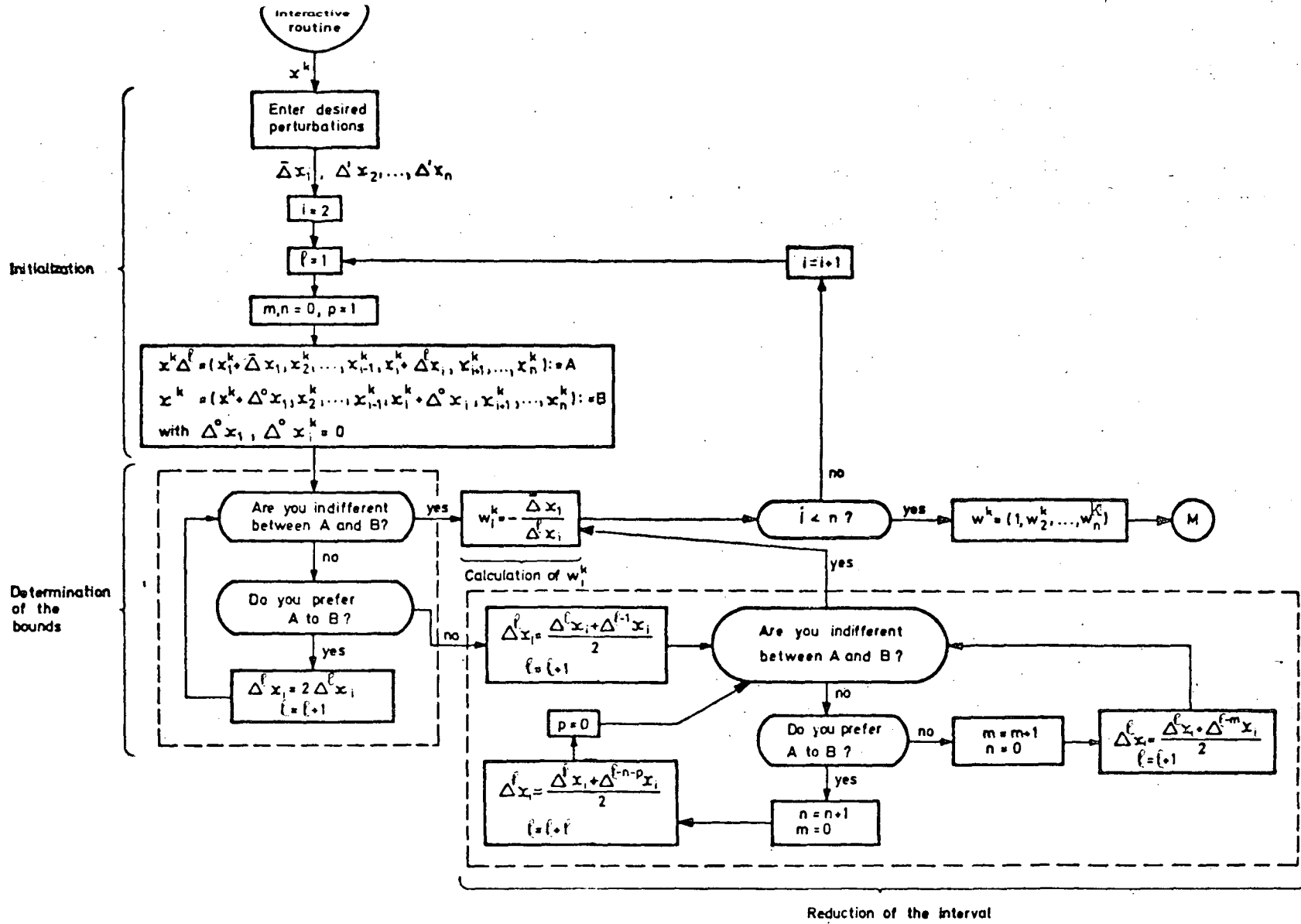


FIG. 7: DYER's interactive procedure: Flow Chart

Fig. 8 - Geoffrion's approach with Dyer's trade-off estimation routine: printout example (data input previously).

ENTER BEST INITIAL SOLUTION COST, STOCKOUT, SALES.

□: 800000. 6.4 1000000.

THIS SOLUTION IS NOT FEASIBLE. CORRECT YOUR CHOICE

□: 1300000. 9. 800000.

THIS SOLUTION IS FEASIBLE. IF YOU WANT TO USE IT AS THE INITIAL SOLUTION, TYPE "Y". IF NO, ENTER NEW SOLUTION.

□:Y

ENTER DESIRED PERTURBATIONS

□: -10000. -0.5 5000.

	A	B
COST	1300000	1290000
STOCK	9	9
SALES	800000	795000

IF YOU PREFER A (B) TYPE "A" ("B"). IF YOU ARE INDIFFERENT, TYPE "I".

□:A

	A	B
COST	1300000	1290000
STOCK	9	9
SALES	800000	797500

IF YOU PREFER A (B) TYPE "A" ("B"). IF YOU ARE INDIFFERENT, TYPE "I".

□:B

	A	B
COST	1300000	1290000
STOCK	9	9
SALES	800000	798750

IF YOU PREFER A (B) TYPE "A" ("B"). IF YOU ARE INDIFFERENT, TYPE "I".

□:I

Step 0:  
Initialization

$\bar{\Delta} x_1, \Delta^1 x_2, \Delta^1 x_3$

Determination  
of  $w_3^k$

Fig. 8 cont'd

THE TRADE-OFFS ARE

COST -1  
STOCK -3  
SALES 8

The vector  $w^k$

ENTER NO. OF POINTS TO SEE IN STEP-SIZE PROBLEM

□:10

$\epsilon$

COST	STOCK	SALES
1300000	9.	800000
1310000	8.9	802000
1320000	8.8	804000
1330000	8.7	806000
1340000	8.6	808000
1350000	8.5	810000
1360000	8.4	812000
1370000	8.3	814000
1380000	8.2	816000
1390000	8.1	818000
1400000	8.	820000

Step-size  
problem

DO YOU WANT A DIFFERENT NUMBER OF POINTS? IF YES,  
TYPE TOTAL NUMBER. IF NO, TYPE "N".

□:N

CHOOSE A PREFERRED SOLUTION FROM THE ABOVE VECTORS.

□: 137000. 8.3 81400.

$x^{k+1}, t^k = 0, 7.$

IF YOU WISH TO END ITERATIONS, TYPE "E". OTHERWISE  
TYPE "C".

□:C

CURRENT SOLUTION IS

137000 8.3 81400

DO YOU WANT TO ENTER ANOTHER CURRENT SOLUTION DIRECTLY? Y OR N.

□:N

ENTER DESIRED PERTURBATIONS

□;

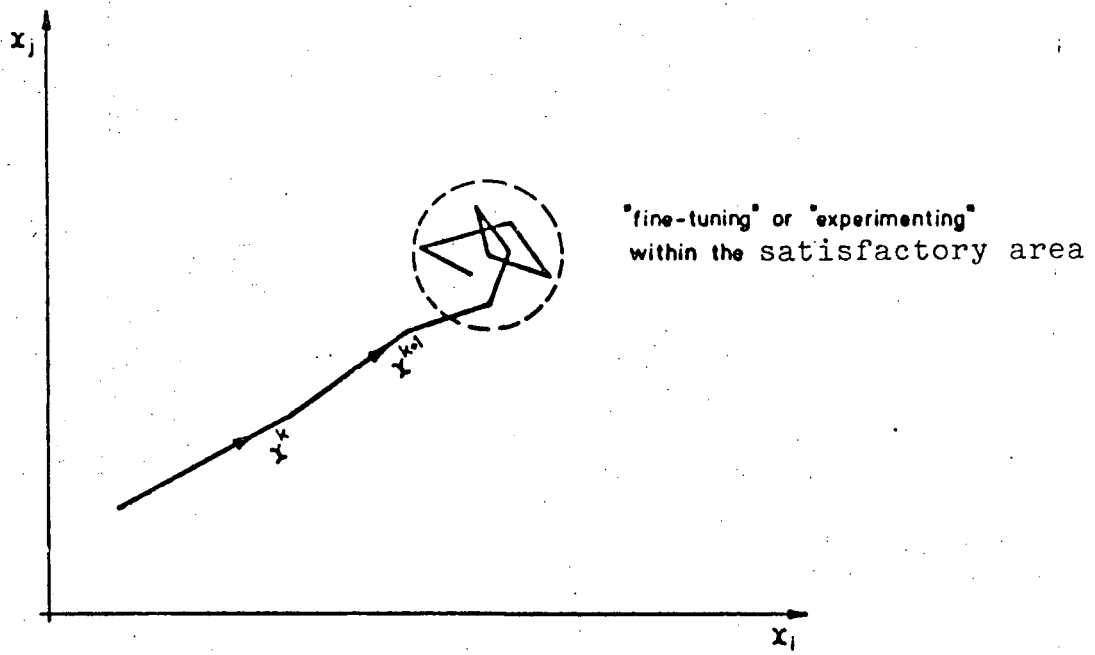


FIG. 9

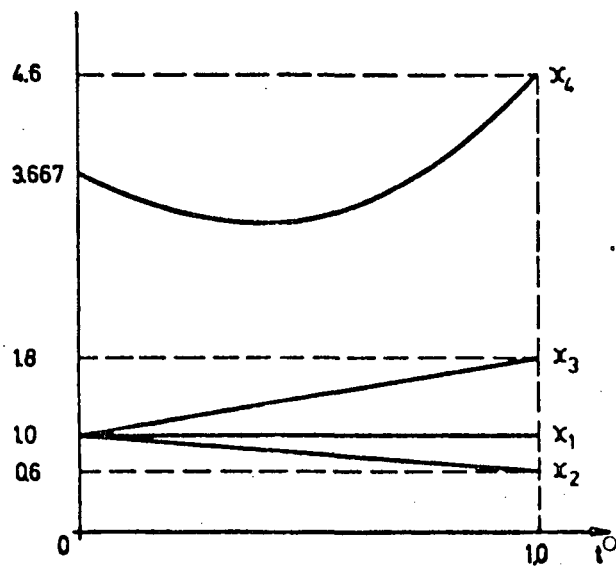


FIG. 10

+  $z_3 \leq 4$ . The OR department of the firm has determined that the cost of the production of 1, 2 and 3 is  $3z_1^2$ ,  $2z_2^2$  and  $3z_3^2$ , respectively. The market price for 1, 2 and 3 is 4, 3 and 2, respectively. Thus the profit function of the firm is

$$\pi(z) = (4z_1 - 3z_1^2) + (3z_2 - 2z_2^2) + (2z_3 - (1/3)z_3^2).$$

However, profit maximization is not the only goal pursued by the management of the firm. In order to be well-represented on all markets 1, 2 and 3 and to thereby reduce the firm's sensitivity to business cycles, the management is also interested in high production levels for each product. For this reason, the firm is currently producing the same quantity of each product:  $z_1 = z_2 = z_3 = 1$ . Nevertheless, the management is unsatisfied with this solution. Being unable to find a satisfying solution immediately or to define an overall objective function over the four simultaneous, conflicting goals

$$z_1 \rightarrow \max, \quad z_2 \rightarrow \max, \quad z_3 \rightarrow \max, \quad \pi \rightarrow \max,$$

the management turns to its OR staff for help.

The OR men realize they are faced with the following multiple-goal problem:<sup>7)</sup>

$$\begin{aligned} &\max U(x), \quad U \text{ unknown} \\ &\text{subject to} \\ &x_1 = z_1, \quad x_2 = z_2, \quad x_3 = z_3, \\ &x_4 = \pi = (4z_1 - 3z_1^2) + (3z_2 - 2z_2^2) + (2z_3 - (1/3)z_3^2), \\ &z_1 + z_2 + z_3 \leq 4, \\ &z_1, z_2, z_3 \geq 0. \end{aligned} \tag{10}$$

OR staff and management decide to solve this problem with the help of an interactive large-step algorithm (modified interactive FRANK-WOLFE algorithm).

After an introduction to the method, to the problem and to the use of the computer, an experiment is performed using the current operating point  $z = (1, 1, 1)$ , i.e.  $x = (1, 1, 1, 3.667)$ , as the initial solution.

Presented with  $x^0$ , the managerial board is asked to determine a trade-off weight  $w_2^0$  for  $x_2$ , the production level of product 2, versus the reference variable  $x_1$ . The weight  $w_2^0$  is estimated by using the "ideal proportional change" approach. The ratio of  $x_2$  to  $x_1$  in the current solution is one to one. The management, however, feels that more emphasis should be put on the production of product 1. It therefore comes to the conclusion that an ideal proportional change from the current values of these two goal variables would be 0.02 units of product 1 for every 0.01 unit<sup>8)</sup> of product 2, or

$$w_2^0 = \delta x_2 / \delta x_1 = 0.01 / 0.02 = 0.5.$$

Similar reasoning relating  $x_3$  to  $x_1$ , leads to  $w_3^0 = 0$ . That is, the management is not interested in changing the production of product 3 following a change in the production of product 1 from its present value.

The weight  $w_4^0$  is estimated by using the "trade-off" approach. The management considers the loss of 0.01 profit units exactly compensated for, in terms of its preferences, by an increase of 0.005 units in the production of product 1, so that

$$w_4 = (\Delta x_1 / \Delta x_4) = -(0.005 / -0.01) = 0.5.$$

The trade-off weights  $w_i^0$  are used to compute the best direction  $y^0$  and the management is confronted with the plots of  $x^0 + t^0 y^0$  presented in Fig. 10. Since  $x_1$ ,  $x_2$  and  $x_3$  are linear functions of  $z$ , the corresponding plots are line segments.

The management decides that the best combination of values for  $x_1 - x_4$  corresponds to  $t^0 = 1$ . This value is used by the computer to calculate the new solution,  $x^1 = (1, 0.6, 1.8, 4.6)$ .  $x^1$  is used as the initial solution for a new iteration. After 6 iterations, which are summarized in Table 1, the management considers the current solution  $x_6$  satisfactory and decides to terminate the procedure.

Iteration k	$x_1^k$	$x_2^k$	$x_3^k$	$x_4^k$	Optimal step-size $t^k$	$U(x^k)^*$	$s^k$
1	1	1	1	3.667	1.0	6.667	0
2	1	.6	1.8	4.6	.4	7.2	0
3	.88	.6	1.92	4.888	.12	7.248	0
4	.88	.624	1.872	4.866	.025	7.2499	0
5	.8752	.624	1.8768	4.875	.005	7.2499	0
6	.8752	.6250	1.8745	4.875	.0010	7.2499	0.0003
Optimal values*	.875	.625	1.875	4.875	X	7.25	0

\* for  $U = 2x_1 + x_2 + x_4$

Table 1



In fact, Table 1 reproduces the search process of a decision-maker a) whose preferences are represented by the overall objective function  $U(x) = 2x_1 + x_2 + x_4 \rightarrow \max$ , and b) who gives the exact value of  $x^k$  and  $t^k$  corresponding to his preference. In addition, Table 1 contains the values of  $x_i$ ,  $i = 1, \dots, 4$ , corresponding to the optimal solution of problem (10) given the above objective function  $U$ , and the values assumed by  $U$  at the different points  $x^k$ . The last column,  $s^k$ , expresses the idle capacity corresponding to each solution  $z^k$ , i. e.  $s^k = 4 - (z_1^k + 2z_2^k + z_3^k)$ . The value of  $s^k$  is a measure of the distance of  $z^k$  from the efficient surface. In case of an efficient solution,  $s^k = 0$ .

It is clear from Table 1 that the convergence, measured in terms of the improvements  $U(x^{k+1}) - U(x^k)$ , is very fast at the beginning but decreases very quickly as  $x^k$  approaches  $x^*$ . (See also Table 3.) Nevertheless, a nearly optimal solution is reached after just 4 iterations.

Note that even in such a simple problem the relationship between objective function  $U$  and optimal solution is far from trivial. One cannot intuitively relate the optimal solution to the constrained problem. Hence, as a rule, an objective function does not have a meaning intuitively. Decision-makers do not think in terms of overall objective functions, but in terms of the desirability of a solution or of a change. The interactive procedure exactly interprets such desirability information to help the decision-maker recognize the best feasible alternative. Note, for example, that the overall objective function does not contain  $x_3$  as an argument although we stated  $x_3 \rightarrow \max$  as a goal. Such a case can take place when, because of special structure of the problem, a goal variable (here  $x_3$ ) consistently takes values that the decision-maker considers to be optimal relative to the values of the other variables (degeneracy).

Table 2 illustrates what can happen when the decision-maker is unable to express exactly the weights  $w_i^k$  corresponding to his preferences. The values of  $x^k$  in this table were obtained by stochastic simulation using the value  $w_i^k = (1 + \epsilon_i^k) (\partial U / \partial x_i^k) / (\partial U / \partial x_1^k)$  for  $w_i^k$ , with  $\epsilon_i^k$  an equally distributed random variable on  $[-0.10, +0.10]$  and  $U(x) = 2x_1 + x_2 + x_4$  as before. In other words, we simulated a decision-maker who expresses the  $w_i^k$ 's with an error factor up to 10% from their true values. The values presented in Table 2 are, of course, the result of one particular simulation; they are not, however, atypical. As can be seen, the introduction of uncertainty does not seriously impair the quality of the results.

Iteration k	$x_1^k$	$x_2^k$	$x_3^k$	$x_4^k$	Optimal step-size $t^k$	$U(x^k)$	$s^k$
1	1	1	1	3.667	.82	6.667	0
2	.854	.595	1.755	4.788	.38	7.091	.201
3	.877	.619	1.779	4.795	.54	7.168	.106
4	.872	.606	1.770	4.786	.92	7.136	.146
5	.881	.628	1.786	4.799	.19	7.189	.077
6	.890	.650	1.801	4.809	.47	7.239	.009
7	.883	.634	1.790	4.803	.77	7.203	.059
Optimal values	.875	.625	1.875	4.875	X	7.25	0

Table 2

Finally, Table 3 illustrates the importance of the choice of  $x^0$  by showing the working of the algorithm for  $U$  as above, but  $x^0 = (0, 0, 0, 0)$ . The convergence becomes very slow after the 5th iteration, and the current solution is still quite distant from the optimal solution at the 10th iteration. In fact, over 30 iterations are needed in order for the current solution to agree up to two decimals with the optimal one.

Iteration k	$x_1^k$	$x_2^k$	$x_3^k$	$x_4^k$	Optimal step-size $t^k$	$U(x^k)$	$s^k$
0	0	0	0	0	.25	0	4
1	1	0	0	1	.48	3	3
2	.52	0	1.92	3.88	.12	4.92	1.56
3	.920	0	1.699	3.677	.26	5.417	1.381
4	.677	.529	1.250	4.248	.06	6.222	1.015
5	.871	.498	1.177	4.098	.16	6.338	1.056
6	.730	.418	1.633	4.602	.04	6.480	.901
7	.811	.501	1.843	4.461	.07	6.584	.394
8	.753	.524	1.645	4.602	.02	6.632	.545
9	.792	.463	1.772	4.616	.05	6.668	.500
10	.763	.484	1.670	4.697	.03	6.717	.594
Optimal values	.875	.625	1.875	4.875	X	7.25	0

Table 3

The choice of  $x^0$  is important for a second reason, which is independent of the formal working of the algorithm. It may be very difficult for the decision-maker to determine the weights  $w_i^k$  in a meaningful way if he considers the current point  $x^k$  a completely unreasonable solution. In this example, for instance, the question "What loss in  $x_i$  compensates exactly for a one-unit increase in  $x_1$ ?" is bound to be considered by the decision-maker as meaningless when  $x_1 = x_i = 0$ .

### III - COMPARATIVE EVALUATION OF THE METHODS

#### A - Experimental Results

Knowledge of the theoretical properties of the different man-machine procedures is insufficient for purposes of practical application or further development of the methods. It is at least as important to know how well they perform in actual situations. However, only a small number of experimental results have been published to date. (See, for example, Agarwal (1973), Dyer (1973b), Feinberg (1972), Geoffrion et al. (1972), Wallenius and Zionts (1975), Zionts (1967). They are all almost exclusively concerned with the interactive FRANK-WOLFE algorithm. Since the decision-making situation, the experimental framework and the interactive routing vary considerably from study to study, the partially contradictory findings are difficult to interpret in general terms.

Nevertheless, a most interesting experiment was conducted by Wallenius with the purpose of comparing from a decision-maker's point of view the relative performance of (a) an unstructured approach, (b) the STEM method and (c) the interactive FRANK-WOLFE algorithm. The ensuing results presented in Wallenius (1975) are summarized in the following sub-sections.<sup>9)</sup> The experiment itself, important for the interpretation of the results and interesting as an application example, is presented in some detail.

1. Decision problem. The framework for the decision-making situation consisted of a fictitious, but realistic, dynamic model of a company manufacturing a seasonal product. The model contained 19 linear constraints and 25 decision variables defining inventory balances, production capacity, labor and overtime for each of the seven periods of the planning horizon. Three conflicting goals related to production, inventory and work-force planning were to be taken into consideration.

1st goal: minimize cumulated cost due to regular payroll and overtime and cost of holding inventory over the seven periods of the planning horizon,

2nd goal: minimize average stockouts over the planning horizon;

3rd goal: minimize cumulated number of employees temporarily laid off over the planning horizon.

2. Methods. The unstructured method used was of the simplest kind. The decision-maker was instructed at each iteration to enter a vector of values for the goal variables into the computer. The only information provided by the computer program was whether or not the choice was feasible. The subjects did not receive information about the values of the non-goal variables corresponding to their choice.

The STEM method was applied according to the schema described above with the exception that the decision-maker was not shown the results of parametric variations of the goal variables (sensitivity analysis).

Within the interactive FRANK-WOLFE algorithm, the determination of the trade-offs was carried out with the help of DYER's indirect procedure.

3. Experimental design. Several pilot studies were made first to assure a satisfactory experimental design. The main experiment was then conducted with 18 undergraduate students of business administration and 18 top managers with formal training in engineering or business administration. All of the students, but only a few of the managers, had basic knowledge of mathematical programming.

The subjects were introduced one by one into the decision-making framework with the help of training sessions and a 9-page booklet describing the problem, the plant and its environment. Each subject was subsequently asked to play the role of plant manager. He was informed of the mechanical aspects of operating the terminal before being given instructions for using the first method. The instructions were accompanied by an introduction to the logic of the method to be tested and an example. The subjects were allowed to iterate as long as they wanted, except, of course, in the case of the STEM method. After they had finished, they were presented with instructions for the next method and the procedure was repeated. (The average length of the session was 2 hours and 40 minutes for the students and 3 hours for the managers.) The order in which the methods were presented was changed from subject to subject in order to compensate for eventual systematic learning effects. For the same reason, the parameters of the decision problem were changed in a random manner with each method. The feasible points found during the use of any one method were not feasible for any other method. Afterwards each subject was interviewed about the method and the experimental design. The responses were statistically tested.

4. Measures of performance. Seven measures of performance were used to evaluate the methods:

- (a) Decision-maker's confidence in the solution obtained (ordinal ranking);
- (b) Ease of use of the method (ordinal ranking);
- (c) Ease of understanding the logic of the method on the basis of the instructions (ordinal ranking);
- (d) Usefulness of information provided by the decision-maker (ordinal ranking);

- (e) Rapidity of convergence, measured by the number of iterations and the total time for solving the problem;
- (f) CPU-time on UNIVAC 1108;
- (g) Distance of the solution from the efficient surface.

5. Results. Both managers and students considered the importance of the information obtained during the interactive procedure to be the factor playing the main role in their evaluation of the methods. This factor was followed closely by confidence in the solution obtained. In third and fourth place came the ease of use and understanding of the method, respectively. Decision-making time (time spent on the computer terminal and for preparation) was ranked last.

No statistically significant ranking of confidence in the results obtained could be determined for either group. This reflects the fact that the evaluation of the goodness of a solution in a fictitious setting, i. e. without recourse to past experience or to "intuition", is a very precarious task. An analysis of the subjects' answers leads, moreover, to the conclusion that their confidence in the results of the GEOFFRION method was weakened by the difficulties they experienced in determining the trade-off weights. On the other hand, they thought the STEM method left them too little freedom in their search for a desirable solution ("built-in" stopping criterion; impossibility of "undoing" a previous relaxation).

Both groups considered the GEOFFRION method the most difficult to use, followed by the STEM method. The main difficulty in the use of the GEOFFRION method lay in the determination of the trade-offs. The step-size problem was considered relatively easy to solve. In the case of the STEM method, the subjects had some difficulty in specifying a value for the relaxation  $\Delta \bar{x}_i^k$ . Finally, Wallenius came to the conclusion that with the help of the unstructured approach, it was relatively easy to find feasible solutions, but difficult to find good feasible solutions.

Managers and students varied in the way they judged ease of understanding. Both groups considered the GEOFFRION method more difficult to understand than the other approaches, but the managers more so. The students considered the unstructured approach easier to understand than the STEM method. The managers, however, considered both methods equally difficult.

Divergence between the two groups was also evident in their evaluation of the usefulness of the information provided. Both groups considered the unstructured approach the least informative. However, the managers ranked the GEOFFRION method below the STEM method in terms of information provided, whereas the students considered both methods equally informative.

The rapidity of convergence is shown in the following Table 4, which is for the most part self-explanatory.

	Students		Managers		Students and Managers		
	Iter-ations	Total time (min)	Iter-ations	Total time (min)	CPU-time pro problem (sec)	CPU-time pro iteration (sec)	CPU-time pro min. of total time(sec)
GEOFFRION Approach	3	31.5	3	38	4.6	1.53	0.13
STEM Method	4.5	14.5	4	13.5	5.8	1.39	0.42
Unstructured Approach	13.5	19.5	10	19.5	8.1	0.69	0.42

Table 4 - Median number of iterations, median total time (decision-making time + time for operating the terminal) and median CPU-time for solving the problem on a UNIVAC 1108 time-sharing system.

On the average, the distance of the solutions from the efficient surface was small in both the GEOFFRION and the unstructured approaches. (As stated earlier, the STEM method always leads to an efficient point.) However, single results showed large deviations from the efficient surface. It seems that the students were able to find slightly more efficient solutions than the managers.

Finally, managers and students were asked which method they preferred on the whole in terms of implementation willingness. The students' answers were randomly distributed. The managers considered the GEOFFRION and STEM methods the same but clearly preferred the unstructured method.

## B - Conclusions

At first glance, Wallenius's findings may be interpreted as a clear rejection of sophisticated methods. Both the GEOFFRION and STEM methods performed less well than the unstructured approach in nearly every respect, with the exception of the information they provided.

Yet, a more thorough analysis shows that the validity of the rankings obtained by Wallenius is restricted to the particular experiment he conducted. On one hand, this experiment was characterized by a very simple structure of the feasible set and by a small number of goal variables (three). There is strong evidence, however, that, due to the unsystematic way in which the search is conducted, the efficiency of the unstructured method decreases rapidly with increasing complexity of the efficient set and, above all, with increasing number of variables. In particular, the number of iterations needed becomes unacceptably large. Therefore, 4 to 5 goal variables may be the upper limit for practical use of the unstructured approach.

In comparison, the experiments mentioned at the beginning of section IV, A, have shown that the STEM and GEOFFRION methods are practically applicable even with as many as 7 to 10 goal variables. Even in particularly complicated problems the number of iterations needed remains within reasonable limits. In the case of STEM it is seldom larger than twice the number of goals. With the GEOFFRION method, the number of iterations rarely exceeds ten, including "fine tuning" in the final iterations.

On the other hand, the relatively poor performance of the STEM and GEOFFRION methods in the Wallenius experiment is partly due to the fact that students and managers did not have enough training in the use of the methods. Success with these two methods crucially depends on the decision-maker's ability at correctly interpreting the information provided by the computer and at stating exactly the information input corresponding to his preferences. A priori, his ability will be insufficient because the information exchange is based on data that the decision-maker usually does not quantify. However, the quality of the information exchange can be greatly improved by practice. <sup>10)</sup>

The main contribution of Wallenius's study is that it highlights the weak points in the use of the STEM and GEOFFRION methods. In both cases, the critical factor is the man-machine collaboration - in the STEM method the determination of the relaxations, in the GEOFFRION method the determination of the trade-off weights. The logical direction for further research would, therefore, seem to be the attempt to improve this collaboration, both by adjusting the information exchange in order to more adequately correspond to the characteristics of a human decision-maker, and by devising training programs to improve the information processing by the decision-maker.

We note that the STEM and GEOFFRION methods are not applicable in the very important case of discrete decision problems. There is a need here for more powerful methods. In this context, an apparently promising research field could be the investigation of interactive branch and bound algorithms.

In summarizing, it can be said that the STEM and GEOFFRION methods

- are expensive methods in the sense that they place high demands on the decision-maker, both in terms of time and intellectual effort. Their programming is also rather complicated;
- are restricted to continuous problems;
- permit a more efficient search of the "best solution" to complex problems. Their relative advantage as compared to the unstructured method increases rapidly with the complexity of the problem;
- provide the decision-maker with significantly more decision-relevant information than the unstructured method.

As for the GEOFFRION method, it offers important advantages as compared to the STEM method.

- It is not restricted to the linear programming framework;

- It is more efficient than the STEM method, both in terms of number of iterations and of quality of the solution, when the number of goal variables is large;
- It does not restrict the decision-maker in his search;
- It provides the decision-maker with more information than the STEM method;
- Generally, it is a more powerful and versatile decision-making tool.

However, it places many more demands on the decision-maker in terms of training required and of intellectual effort than STEM. In other words, no one method is superior to the others in all respects. In a practical situation, the choice of the method will depend both on technical factors (structure of the problem) and on the preferences and needs of the decision-maker relative to information, costs, etc. The above presentation should offer preliminary guide-lines for making such a choice.



FOOTNOTES

- 1) When a goal variable is supposed to be minimized, it is assumed to be transformed to maximization by an appropriate sign change. We use in the following the notation  $x := (x_1, \dots, x_n)$ ,  $z^k := (z_1^k, \dots, z_m^k)$ , etc.  $A' (x')$  designates the transposition of the matrix  $A$  (the vector  $x$ ).
- 2) See, for example, Baumol (1959), Johnsen (1968).
- 3)  $\nabla_z(z^k) := (\partial U / \partial z_1, \dots, \partial U / \partial z_m) |_{z = z^k}$  is the vector of partial derivatives  $\partial U / \partial z_i$ ,  $i = 1, \dots, m$ , evaluated at  $z = z^k$ . We use the notation  $\partial U / \partial z_i^k := \partial U / \partial z_i |_{z_i = z_i^k}$ .  $\nabla_z(z^k)$  represents a local linear approximation for the "direction of greatest increase" of  $U$  at  $z = z^k$ . The main difference between the different large-step algorithms lies in the definition of the "best direction"  $y^k$ . As a rule, however,  $y^k$  will be derived from the solution of a standard LP formulated on the basis of both  $\nabla_z U$  and (a linear approximation of) the constraints defining the feasible set  $D$ . (See, for example, Simmons (1975).)
- 4) The term "large-step method" expresses the fact that the improved point  $z^{k+1}$  is not restricted to a neighborhood of the current point  $z^k$ , but may lie at some distance from it.
- 5) The FRANK-WOLFE algorithm was used as the basis for most of the studies of GEOFFRION's approach. It is more specifically designed for problems in which the objective function  $U$  is concave on  $\{z\}$ . However, it converges even when  $U$  is not concave.
- 6) The problem  $U [(1-t^k) z^k + t^k \bar{z}^k] \rightarrow \max$  is formally equivalent to the problem  $U [z^k + t_1^k y^k] \rightarrow \max$ ,  $z^{k+1}$  feasible, stated under "General Remarks".
- 7) This problem is atypical in so far as  $x$  is of a higher order than  $z$ .
- 8) Note that the  $w_i$ 's are only marginally valid, i. e. express the preference of the decision-maker only with regard to a relatively small change from the current solution. In order to grasp the intuitive contents of this requirement, each  $z$  unit can be thought of in terms of thousands of physical units.
- 9) A new comparative study, Tell (1976), was not available in time to be considered in this paper.
- 10) In an experiment with 7 goals conducted by the author, graduate students needed for an example 6 to 7 hours practice with the GEOFFRION method before considering themselves able to correctly solve trade-off and step-size problems. In comparison, Wallenius's students had on the average less than 1 hour console time to familiarize themselves with GEOFFRION's method.

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