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Existence, uniqueness and  
continuity of portfolio  
choice

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# Existence, uniqueness and continuity of portfolio choice.

Laurie Davies and Gerd Ronning.

## 1 Introduction and notation

An investor with initial wealth  $Y > 0$  buys assets  $(A_j)_j^n$  in period 1 whose values in period 2 (per unit of wealth invested) are given by the non-negative random variables  $(X_j)_1^n$ . The total wealth of the investor in period 2 is therefore  $\sum_{j=1}^n y_j X_j$  where  $y_j$  is the amount of wealth allotted to the asset  $A_j$ . We assume that the investor has a utility function  $u(x)$  and that he selects the  $(y_j)_1^n$  (subject to his budget constraint  $\sum_{j=1}^n y_j = Y$ ) so as to maximize his expected utility in the second period (see [1]).

If  $F(x_1, \dots, x_n)$  is the joint distribution function of the random variables  $(X_j)_1^n$  the problem may be formulated as follows:

Select  $(y_j)_1^n$  subject to

$$y_j \geq 0, \quad j = 1, \dots, n, \quad \sum_{j=1}^n y_j = Y$$

so as to maximize

$$\int_0^\infty \int_0^\infty u\left(\sum_{j=1}^n y_j x_j\right) F(dx_1, \dots, dx_n) .$$

We introduce the following notation which we shall use in the remainder of the paper.

A row vector will be denoted by a bold-face letter y and its components by  $y_j$  ( $j = 1, \dots, n$ ) i.e.

$$\underline{y} = (y_1, \dots, y_n) .$$

The inner-product  $\underline{y}_1 \cdot \underline{y}_2$  of two vectors is defined by

$$\underline{y}_1 \cdot \underline{y}_2 = \sum_{j=1}^n y_{1j} y_{2j} .$$

The non-negative orthant  $R_+^n$  of  $R^n$  is given by

$$R_+^n = \{ \underline{x} : x_j \geq 0 \quad j = 1, \dots, n \}$$

and for each  $K > 0$  we write

$$R_+^n(K) = \{ \underline{x} : 0 \leq x_j \leq K \quad j = 1, \dots, n \} .$$

The simplex  $S_n = S_n(Y)$  is defined by

$$S_n = \{ \underline{y} : \underline{y} \in R_+^n, \sum_{j=1}^n y_j = Y \}$$

and for the integral

$$\int_0^\infty \dots \int_0^\infty v(x_1, \dots, x_n) F(dx_1, \dots, dx_n)$$

we write

$$\int v(\underline{x}) F(d\underline{x}) .$$

All integrals will be over  $R_+^n$  unless otherwise stated.

The portfolio selection problem may now be stated as follows:

Select  $\underline{y}^*$  in  $S_n$  such that

$$(1) \quad \int u(\underline{y}^* \cdot \underline{x}) F(d\underline{x}) = \sup_{\underline{y} \in S_n} \int u(\underline{y} \cdot \underline{x}) F(d\underline{x}) .$$

The set of solutions of (1) will be called the solution set for the distribution function  $F$  and will be denoted by  $\textcircled{Y}$ .

In future a star will indicate that a vector belongs to the solution set. If  $\textcircled{Y}$  is non-empty and  $\underline{y}^* \in \textcircled{Y}$  we write

$$M = \int u(\underline{y}^* \cdot \underline{x}) F(d\underline{x}) .$$

$M$  will be called the maximal expected utility associated with  $F$ .

We shall be concerned with the following problems.

(a) Under what conditions is  $\textcircled{Y}$  non-empty ?

(b) Under what conditions does  $\textcircled{Y}$  contain exactly one point ?

(c) Under what conditions is  $\underline{y}^*$  a continuous function of  $F$  ?

(a) will be dealt with in 2, (b) in 3 and (c) in 4-5.

The results we give are quite general although in some places we sacrifice generality for simplicity. In particular we shall always assume that

(2)  $u(x)$  is continuous on  $[0, \infty)$

and

(3)  $u(x)$  is non-negative.

From the point of view of portfolio analysis such assumptions are innocuous. Readers requiring full generality should have no difficulty in adopting the proofs given here.

## 2 Existence

We prove the following theorem.

Theorem 1. *If  $u(x)$  is non-decreasing then*

(4)  $\textcircled{Y}$  is non-empty

and

(5)  $M$  is finite or infinite according as

$$\max_{1 \leq j \leq n} \left( \int u(Yx_j) F(d\underline{x}) \right)$$

is finite or infinite.

Proof. For  $\underline{y} \in S_n$  and  $\underline{x} \in R_+^n$

$$\underline{y} \cdot \underline{x} \leq Y(\max_{1 \leq j \leq n} x_j)$$

and hence, as  $u(x)$  is positive and increasing

$$(6) \quad 0 \leq u(\underline{y} \cdot \underline{x}) \leq u(Y(\max_{1 \leq j \leq n} x_j))$$

which implies

$$S = \sup_{\underline{y} \in S_n} \left( \int u(\underline{y} \cdot \underline{x}) F(d\underline{x}) \right) \leq \int u(Y(\max_{1 \leq j \leq n} x_j)) F(d\underline{x}) .$$

Suppose now

$$\max_{1 \leq j \leq n} \left( \int u(Y x_j) F(d\underline{x}) \right) < \infty .$$

Then as  $u(x)$  is increasing

$$\int u(Y(\max_{1 \leq j \leq n} x_j)) F(d\underline{x}) \leq n \left( \max_{1 \leq j \leq n} \left( \int u(Y x_j) F(d\underline{x}) \right) \right)$$

and thus

$$(7) \quad \int u(Y(\max_{1 \leq j \leq n} x_j)) F(d\underline{x}) < \infty .$$

The simplex  $S_n$  is compact and therefore there exists a sequence  $(\underline{y}_i)_{i=1}^{\infty}$  tending to a point  $\underline{y}_0$  in  $S_n$  such that

$$(8) \quad \lim_{i \rightarrow \infty} \int u(\underline{y}_i \cdot \underline{x}) F(d\underline{x}) = S .$$

Because of (6) and (7) we may apply Lebesgue's dominated convergence theorem to obtain

$$\int \lim_{i \rightarrow \infty} u(\underline{y}_i \cdot \underline{x}) F(d\underline{x}) = S .$$

By (2)  $u(x)$  is continuous and hence

$$\int u(\underline{y}_0 \cdot \underline{x}) F(d\underline{x}) = S$$

which implies  $\underline{y}_0 \in \textcircled{Y}$ .

Suppose

$$\max_{1 \leq j \leq n} \left[ \int u(Y \ x_j) F(d\underline{x}) \right] = \infty.$$

Then there exists a  $j_0$  such that

$$\int u(Y \ x_{j_0}) F(d\underline{x}) = \infty$$

and thus

$$\underline{y}_0 = (0, \dots, 0, \underbrace{Y}_{j_0}, 0 \dots 0)$$

belongs to  $\textcircled{Y}$ . This completes the proof of the theorem.

### 3 Uniqueness

We prove the following theorem.

Theorem 2.      If

(9)       $u(x)$  is strictly concave,

$$(10) \quad \max_{1 \leq j \leq n} \left[ \int u(Y \ x_j) F(d\underline{x}) \right] < \infty$$

and

(11)      the random variables  $(X_j)_1^n$  are linearly



independent <sup>†</sup> then  $\textcircled{Y}$  contains exactly one point.

Proof. As by (3)  $u(x)$  is always non-negative (9) implies that  $u(x)$  is increasing and hence the conditions of Theorem 1 are satisfied. The solution set  $\textcircled{Y}$  is therefore non-empty and  $M$  is finite.

Suppose now that  $\textcircled{Y}$  contains two distinct points  $\underline{y}_1^*$  and  $\underline{y}_2^*$ . Let  $\theta$ ,  $0 < \theta < 1$ , be a real number and we set

$$\underline{y}_0 = \theta \underline{y}_1^* + (1-\theta) \underline{y}_2^* .$$

As  $u(x)$  is strictly concave

$$u(\underline{y}_0 \cdot \underline{x}) \geq \theta u(\underline{y}_1^* \cdot \underline{x}) + (1-\theta)u(\underline{y}_2^* \cdot \underline{x})$$

with strict inequality holding if  $\underline{y}_1^* \cdot \underline{x} \neq \underline{y}_2^* \cdot \underline{x}$ . The linear independence of the random variables  $(X_j)_1^n$  therefore implies that strict inequality holds for the set of  $\underline{x}$  of positive F-measure. This gives

$$\int u(\underline{y}_0 \cdot \underline{x})F(d\underline{x}) > \theta \int u(\underline{y}_1^* \cdot \underline{x})F(d\underline{x}) + (1-\theta) \int u(\underline{y}_2^* \cdot \underline{x})F(d\underline{x}) = 1$$

which is an obvious contradiction. The solution set  $\textcircled{Y}$  therefore contains at most one point and the theorem is proved.

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<sup>†</sup> The random variables  $(X_j)_1^n$  are said to be linearly independent if

$$P\left(\sum_{j=1}^n c_j X_j = 0\right) = 1 \text{ implies } c_1 = \dots = c_n = 0.$$

That the conditions of the theorem are not necessary is shown by the following example.

Example 1. We set  $Y = 1$ ,  $u(x) = e^x$  and suppose that  $X_1$  is exponentially distributed with mean 1. If  $X_2$  is equal to  $1/2 X_1$  with probability one then none of the conditions of the theorem are satisfied. However, it is easily verified that  $\underline{y} = (1, 0)$  is the only choice which gives an infinite expected utility and is therefore the unique optimal choice.

The conditions of the theorem are, however, necessary in the restricted sense that if any of them are dropped counter examples may be found as the reader may easily verify.

#### 4 Continuity: $u(x)$ bounded.

4.1 In this section we prove that if the sequence of distributions  $(F_v)_1^\infty$  converges weakly to  $F_0$  (written

$$w\text{-}\lim_{v \rightarrow \infty} F_v = F_0 : \quad \text{see [2] p. 40)}$$

then the associated maximal expected utilities and solution sets converge to those of  $F_0$ . This is expressed in the following theorem.

Theorem 3. Let  $(F_v)_0^\infty$  be a sequence of distribution functions concentrated on  $R_+^n$  with corresponding solution sets  $(\textcircled{Y}_v)_0^\infty$  and maximal expected utilities  $(M_v)_0^\infty$ . If

$$(12) \quad w\text{-}\lim_{v \rightarrow \infty} F_v = F_0$$

and

$$(13) \quad u(x) \text{ is bounded and increasing,}$$

then

$$(14) \quad \lim_{v \rightarrow \infty} M_v = M_0$$

and

$$(15) \quad \text{every convergent subsequence of } (\underline{y}_v^*)_1^\infty, \underline{y}_v^* \in \textcircled{Y}_v, \\ \text{converges to a point of } \textcircled{Y}_0.$$

Corollary 1. Given  $\varepsilon > 0$  there exists a  $v_0 = v_0(\varepsilon)$  such that all points in

$$\bigcup_{v=v_0}^\infty \textcircled{Y}_v$$

are within a distance  $\varepsilon$  of some point of  $\textcircled{Y}_0$ .

Corollary 2. If  $Y_0$  contains only the point  $\underline{y}_0^*$  then any sequence  $(\underline{y}_v^*)_1^\infty$  is convergent and converges to  $\underline{y}_0^*$ .

4.2 Before turning to the proof of Theorem 3 we first show that the family of functions

$$(16) \quad u(\underline{y} \cdot (\cdot)) : R_+^n \rightarrow R_+^1, \quad \underline{y} \in S_n$$

is equicontinuous ([2] p. 50). (Here  $\underline{y}$  plays the role of a parameter.)

Lemma 1. If  $u(x)$  is increasing and bounded then the family of functions (16) is equicontinuous.

Proof. The conditions of the lemma imply that  $\lim_{x \rightarrow \infty} u(x)$  exists and hence  $u(x)$  is uniformly continuous on  $[0, \infty)$ . Thus given  $\epsilon > 0$  there exists a  $\delta > 0$  such that

$$(17) \quad |x_1 - x_2| < \delta \quad \text{implies} \quad |u(x_1) - u(x_2)| < \epsilon.$$

An application of the Cauchy-Schwarz inequality gives

$$|\underline{y} \cdot \underline{x}_1 - \underline{y} \cdot \underline{x}_2| \leq \sqrt{n} \, Y \, |\underline{x}_1 - \underline{x}_2|$$

where  $|\underline{x}_1 - \underline{x}_2|$  is the usual Euclidean distance between the points  $\underline{x}_1$  and  $\underline{x}_2$  in  $R^n$ . Hence if  $\underline{x}_1$  and  $\underline{x}_2$  satisfy  $|\underline{x}_1 - \underline{x}_2| < \delta / (\sqrt{n} \, Y)$  it follows from (17) that

$$|u(\underline{y} \cdot \underline{x}_1) - u(\underline{y} \cdot \underline{x}_2)| < \epsilon$$

uniformly in  $\underline{y}$ . The family (16) is therefore equicontinuous.

4.3 We now turn to the proof of Theorem 3. We first note that, as  $u(x)$  is increasing and bounded, Theorem 1 implies that the  $(Y_v)_0^\infty$  are non-empty and the  $(M_v)_0^\infty$  are finite.

By Lemma 1 the family (16) is equicontinuous and the boundedness of  $u(x)$  implies

$$\sup_{\underline{y} \in S_n} \left( \sup_{\underline{x} \in R_+^n} u(\underline{y} \cdot \underline{x}) \right) < \infty.$$

It therefore follows ( [2] p. 51) that

$$(18) \quad \lim_{v \rightarrow \infty} \left[ \sup_{\underline{y} \in S_n} \left| \int u(\underline{y} \cdot \underline{x}) F_v(d\underline{x}) - \int u(\underline{y} \cdot \underline{x}) F_0(d\underline{x}) \right| \right] = 0 .$$

We first prove (14). For each  $\underline{y}_0^*$  in  $Y_0$

$$M_v \geq \int u(\underline{y}_0^* \cdot \underline{x}) F_v(d\underline{x})$$

and thus

$$\begin{aligned} \liminf_{v \rightarrow \infty} M_v &\geq \liminf_{v \rightarrow \infty} \int u(\underline{y}_0^* \cdot \underline{x}) F_v(d\underline{x}) \\ &= \int u(\underline{y}_0^* \cdot \underline{x}) F_0(d\underline{x}) \end{aligned}$$

the equality following from (12). We therefore obtain

$$(19) \quad \liminf_{v \rightarrow \infty} M_v \geq M_0 .$$

Suppose now that  $\limsup_{v \rightarrow \infty} M_v > M_0$  . Then there exists a sequence  $(\underline{y}_{v(s)}^*)_{s=1}^\infty$  such that

$$\lim_{s \rightarrow \infty} \int u(\underline{y}_{v(s)}^* \cdot \underline{x}) F_{v(s)}(d\underline{x}) > M_0$$

which by (18) implies

$$\lim_{s \rightarrow \infty} \int u(\underline{y}_{v(s)}^* \cdot \underline{x}) F_0(d\underline{x}) > M_0$$

which is impossible. Hence  $\limsup_{v \rightarrow \infty} M_v \leq M_0$  which together with (19) implies (14).

We now prove (15). Let  $(\underline{y}_{\underline{v}(s)}^*)_{s=1}^{\infty}$  be any convergent sequence with  $\underline{y}_{\underline{v}(s)}^* \in \underline{Y}_{\underline{v}(s)}$ ,  $s = 1, 2, \dots$ . As  $S_n$  is compact there exists a  $\underline{y}_0$  in  $S_n$  such that  $\lim_{s \rightarrow \infty} \underline{y}_{\underline{v}(s)}^* = \underline{y}_0$ .

From (18) we have

$$\lim_{s \rightarrow \infty} \left[ \int u(\underline{y}_{\underline{v}(s)}^* \cdot \underline{x}) F_{\underline{v}(s)}(d\underline{x}) - \int u(\underline{y}_{\underline{v}(s)}^* \cdot \underline{x}) F_0(d\underline{x}) \right] = 0$$

and thus

$$\lim_{s \rightarrow \infty} \int u(\underline{y}_{\underline{v}(s)}^* \cdot \underline{x}) F_0(d\underline{x}) = \lim_{s \rightarrow \infty} M_{\underline{v}(s)} = M_0.$$

As  $u(x)$  is bounded and continuous the Lebesgue dominated convergent theorem implies

$$\lim_{s \rightarrow \infty} \int u(\underline{y}_{\underline{v}(s)}^* \cdot \underline{x}) F_0(d\underline{x}) = \int u(\underline{y}_0 \cdot \underline{x}) F_0(d\underline{x})$$

and hence  $\underline{y}_0 \in \underline{Y}_0$  completing the proof of (15).

4.4 Corollary 2 follows immediately from Corollary 1 which we now prove by indirect means. If the corollary is false the compactness of  $S_n$  implies the existence of a convergent subsequence  $(\underline{y}_{\underline{v}(s)}^*)_{s=1}^{\infty}$ ,  $\underline{y}_{\underline{v}(s)}^* \in \underline{Y}_{\underline{v}(s)}$ , each point of which is at least a distance  $\epsilon$  away from every point of  $\underline{Y}_0$ . This sequence cannot possibly converge to a point in  $\underline{Y}_0$  contradicting (15) and hence Corollary 1 must hold.

4.5 We remark that if  $\underline{Y}_0$  contains more than one point then

the conclusion of Corollary 2 is false.

Example 2. We set  $Y = 1$ ,  $u = 1 - e^{-x}$  and define the random variables  $X_{1n}$  and  $X_{2n}$  by

$$P(X_{1n} = 1) = 1, \quad n = 1, 2, \dots$$

and

$$P(X_{2n} = 1 + (-1)^n) = n^{-1}, \quad P(X_{2n} = 1 + \frac{1}{n}(-1)^n) = 1 - \frac{1}{n}$$

$$n = 1, 2, \dots$$

If  $F_n$  is the joint distribution function of  $X_{1n}$  and  $X_{2n}$  we have

$$w\text{-}\lim_{n \rightarrow \infty} F_n = F_0$$

where  $F_0$  is the distribution function of the unit mass concentrated at the point  $(1,1)$ . The optimal choice  $\underline{y}_n^*$  is given by

$$\underline{y}_{2n}^* = (0,1), \quad \underline{y}_{2n+1}^* = (1,0).$$

$\underline{y}_n^*$  is obviously not convergent.

## 5 Continuity: $u(x)$ unbounded

5.1 If the utility function is unbounded the maximal expected utility is (in general) no longer a continuous function of the joint distribution function of the random returns. Consider the following example.

Example 3. We set  $Y = 1$ ,  $u(x) = \log(1+x)$  and define the random returns  $X_{1n}$  and  $X_{2n}$  by

$$(20) \quad P(X_{1n} = 1) = 1 \quad n = 3, 4, \dots$$

and

$$(21) \quad P(X_{2n} = \frac{1}{2}) = (1 - \frac{1}{\sqrt{\log n}}), \quad P(X_{2n} = n) = \frac{1}{\sqrt{\log n}} \\ n = 3, 4, \dots$$

For  $n \geq 3$  the optimal choice  $y_n^* = (y_n, 1 - y_n)$  is given by

$$(22) \quad y_n = 1 - 2(\frac{1}{\sqrt{\log n}} - \frac{1}{n-1})$$

and the maximal utility  $M_n$  by

$$M_n = (1 - \frac{1}{\sqrt{\log n}}) \log(2 + \frac{1}{n-1} - \frac{1}{\sqrt{\log n}}) + (\log n)^{-1/2} \\ \log(\frac{2(n-1)}{\sqrt{\log n}} - 2) .$$

We therefore have

$$\lim_{n \rightarrow \infty} M_n = \infty .$$

If  $F_n$  is the joint distribution function of  $X_{1n}$  and  $X_{2n}$  it is clear from (20) and (21) that

$$w\text{-}\lim_{n \rightarrow \infty} F_n = F_0$$

where  $F_0$  is the distribution function associated with the unit mass concentrated at the point  $(1, 1/2)$ . We therefore have  $y_0^* = (1, 0)$  and  $M_0 = \log 2$ .



This shows that

$$\lim_{n \rightarrow \infty} M_n \neq M_0 .$$

In spite of this it follows from (22) that

$$\lim_{n \rightarrow \infty} y_n^* = y_0^* .$$

That this is no accident is shown by the theorem given in the next section.

5.2 As the above example shows, it may happen that  $\underline{y}^*$  is a continuous function of  $F$  even when  $M$  is not. We now give a theorem which states, roughly speaking, that this is the normal state of affairs when the utility function  $u(x)$  increases no faster than  $\log x$ .

For reasons of simplicity we impose sufficient conditions on  $u(x)$  and the  $(F_\nu)_0^\infty$  to ensure that the solutions of (1) exist and are unique. We also set  $Y = 1$ .

We give the following theorem.

Theorem 4. Let  $(F_\nu)_0^\infty$  be a sequence of distribution functions concentrated on  $R_+^n$  and with associated solution sets  $(Y_\nu)_0^\infty$ . We suppose

$$(23) \quad w\text{-}\lim_{\nu \rightarrow \infty} F_\nu = F_0 ,$$

$$(24) \quad \text{the random variables } (X_{\nu j})_{j=1}^n \text{ associated with } F_\nu$$

are linearly independent ( $v = 0, 1, 2, \dots$ ),

(25)  $u(x)$  is strictly concave,

(26)  $\max_{1 \leq j \leq n} \left[ \int u(x_j) F_v(dx) \right] < \infty \quad (v = 0, 1, 2, \dots),$

and

(27)  $\sup_{0 \leq x < \infty} |u(\lambda x) - u(x)| < \infty$  for all  $\lambda > 0$ .

Then

(28) each solution set  $\mathcal{Y}_v$  contains exactly one point  $\underline{y}_v^*$  ( $v = 0, 1, 2, \dots$ )

and

(29)  $\lim_{v \rightarrow \infty} \underline{y}_v^* = \underline{y}_0^*$

The proof of the Theorem is somewhat long and we therefore give it in the appendix.

5.3 For utility functions increasing faster than  $\log x$  neither  $\underline{y}^*$  nor  $M$  are in general continuous functions of  $F$ .

Example 4. We take  $Y = 1$ ,  $u(x) = (\log(1+x))^2$  and define  $X_{1n}$  and  $X_{2n}$  by

$$P(X_{1n} = 1) = 1 \quad n = 3, 4, \dots$$

and

$$P(X_{2n} = 0) = 1 - \frac{1}{\log n}, \quad P(X_{2n} = n) = \frac{1}{\log n}, \quad n = 3, 4, \dots$$

The expected utility for the choice  $\underline{y} = (y, 1-y)$  is given by

$$M(y) = \left(1 - \frac{1}{\log n}\right)(\log(1+y))^2 + \frac{1}{\log n}(\log(n+1-y(n-1)))^2.$$

On differentiating with respect to  $y$  we obtain

$$\frac{d}{dy} M(y) \leq 2(\log 2 - \frac{(n-1)\log(n+1)}{n+1(\log n)})$$

which is negative for  $n \geq 6$ .

Thus if  $n \geq 6$   $\underline{y}_n^* = (0, 1)$

and  $M_n = \frac{(\log(n+1))^2}{\log n}.$

If  $F_n$  is the joint distribution function of  $X_{1n}$  and  $X_{2n}$  it is apparent that

$$w\text{-}\lim_{n \rightarrow \infty} F_n = F_0$$

where  $F_0$  is the distribution function associated with the unit mass concentrated at  $(1,0)$ .

Thus  $\underline{y}_0^* = (1, 0)$  and  $M_0 = \log 2$ . It follows that neither of the two equalities

$$\lim_{n \rightarrow \infty} \underline{y}_n^* = \underline{y}_0^*, \quad \lim_{n \rightarrow \infty} M_n = M_0$$

hold.

6      Discussion

Theorem 1 is probably the least interesting from the point of view of economic theory. We confine ourselves to the remark that it is possible to find<sup>a</sup> function  $u(x)$  for which no optimal choice exists i.e. to any given choice there always exists a strictly better one.

Theorem 2 is somewhat more interesting. It implies under fairly weak conditions that the optimal choice is unique for investors who show decreasing risk aversion. We can also deduce the following:

Proposition.      If the random variables  $(X_j)_1^n$  are exchangeable and linearly independent and if  $u(x)$  is strictly concave and satisfies

$$\int u(Yx_1)F(d\underline{x}) < \infty$$

then the optimal choice is given by

$$y_1 = \dots = y_n = \frac{1}{n} Y .$$

If the  $(X_j)_1^n$  have finite variances then this is precisely the choice which minimizes the variance of the return in period 2. The above result is however also valid if the variances do not exist.

The main result of the paper is the continuity (under certain conditions) of both the optimal choice and the expected maximal utility. From the point of view of the investor this

is of course reassuring; if his estimate of the joint distribution function of the random returns is not too far away from the actual distribution, then his optimal choice based on this estimate will be not too far away from the actual optimal choice.

One result which follows from the continuity of portfolio choice is the following. Suppose the random variables  $(X_j)_1^n$  are independent and that  $X_i \sim \Gamma(\mu, \sigma^2)$ ,  $i = 1, \dots, n-1$  and  $X_n \sim \Gamma(\mu-\epsilon, \sigma^2+\delta)$  ( $\sigma, \mu, \epsilon, \delta > 0$ ;  $\Gamma(\mu, \sigma^2)$  = gamma distributed with mean  $\mu$  and variance  $\sigma^2$ ). The asset  $A_n$  is then strictly riskier than any of the assets  $A_1, \dots, A_{n-1}$ . One might therefore expect that this asset would not be considered. However, it follows from the proposition above and the continuity of portfolio choice that as long as  $\epsilon$  and  $\delta$  are small enough some wealth would be invested in  $A_n$ .

It came as something of a surprise to the authors to discover that portfolio choice is "more continuous" than the expected maximal utility (Theorem 4). There seems to be no immediate intuitive reason why this should be so.

Finally we remark that the holding of cash is not excluded. One of the assets  $A_1$  may represent the holding of cash and the corresponding random variable  $X_1$  is then degenerate, taking the value 1 with probability one. The holding of other certain assets may be treated in the same way although in terms of our model, which ignores transaction costs, it only makes sense to include one certain asset, that with the greatest return, the others obviously not entering into the optimal choice.

# Appendix

1 For the proof of Theorem 4 we require the following lemma.

Lemma 2. Let  $(F_v)_0^\infty$  be a sequence of distribution functions concentrated on  $R_+^n$  such that

$$(30) \quad w\text{-}\lim_{v \rightarrow \infty} F_v = F_0$$

and

(31) the random variables  $(X_{0j})_{j=1}^n$  associated with  $F_0$  are linearly independent.

Then there exists an increasing sequence  $(K_s)_1^\infty$  of positive numbers tending to infinity such that

$$(32) \quad \lim_{v \rightarrow \infty} P_{vs} = P_{0s} \quad (s = 1, 2, \dots)$$

and

$$(33) \quad w\text{-}\lim_{v \rightarrow \infty} F_{vs} = F_{0s} \quad (s = 1, 2, \dots)$$

where

$$(34) \quad P_{vs} = \int_{R_+^n(K_s)} F_v(d\underline{x}) \quad (s = 1, 2, \dots, v = 0, 1, \dots)$$

and

$$(35) \quad F_{vs}(x_1, \dots, x_n) = \frac{F_v(\min(x_1, K_s), \dots, \min(x_n, K_s))}{P_{vs}} \\ (s = 1, 2, \dots, v = 0, 1, \dots)$$

Furthermore the sequence  $(K_s)_1^\infty$  may be chosen so that the random variables associated with  $F_{0s}$  are linearly independent,

$s = 1, 2, \dots$

Proof. We denote the boundary of  $R_+^n(K)$  by  $\partial R_+^n(K)$ . It is clear that we can choose an increasing sequence  $(K_s)_1^\infty$  tending to infinity such that

$$\partial R_+^n(K_s) = \bigcup_{j=1}^n \{x : x_j = 0\}$$

has zero  $F_0$ -mass. The assertions (32) and (33) of the lemma then follow from a simply application of Theorem 6.1 on page 40 of [2].

We prove the last part of the lemma by indirect means. If the assertion is false then for all sufficiently large  $s$  there exists a point  $\underline{c}(s)$  satisfying

$$(36) \quad |\underline{c}(s)| = \left( \sum_{j=1}^n c_j(s)^2 \right)^{\frac{1}{2}} = 1$$

such that

$$P \left( \sum_{j=1}^n c_j(s) X_{0j} = 0 \mid \underline{X}_0 \in R^n(K_s) \right) = 1.$$

The set of points  $c$  satisfying (36) is compact and hence there exists a subsequence  $(\underline{c}(s(v)))_{v=1}^\infty$  which converges to a point  $\underline{c}_0$  satisfying (36).

Now

$$\lim_{v \rightarrow \infty} P(\underline{X}_0 \in R^n(K_{s(v)})) = 1$$

and hence

$$\lim_{v \rightarrow \infty} P \left( \sum_{j=1}^n c_j(s(v)) X_{0j} = 0 \right) = 1.$$

The sequence  $\left( \sum_{j=1}^n c_j(s(v)) X_{0j} \right)_{v=1}^\infty$  converges therefore in

probability to zero. This implies there exists a subsequence

$$\left( \sum_{j=1}^n c_j(s(v_i)) X_{Oj} \right)_{i=1}^{\infty}$$

which converges to zero with probability one i.e.

$$\lim_{i \rightarrow \infty} \sum_{j=1}^n c_j(s(v_i)) X_{Oj} = 0 \quad \text{a.s.}$$

This implies

$$\sum_{j=1}^n c_{Oj} X_{Oj} = 0 \quad \text{a.s.}$$

which contradicts (31) and so the last assertion of the lemma must be true.

2 We turn to the proof of the theorem.

We first note that (23) - (26) and Theorem 2 imply (28). It therefore remains to prove (29).

Strictly speaking we must show that  $\lim_{v \rightarrow \infty} \underline{y}_v^*$  exists and is equal to  $\underline{y}_0^*$ . As in Theorem 3 one method would be to show that every convergent subsequence of  $(\underline{y}_v^*)_1^{\infty}$  converges to  $\underline{y}_0^*$ . We shall however assume that  $(\underline{y}_v^*)_1^{\infty}$  itself converges, it being clear that the proof would also hold for any convergent subsequence. In this way we avoid a proliferation of subscripts.

With  $(F_{vs})_{v=0}^{\infty}$ ,  $s = 1, 2, \dots$  as in Lemma 2 we define  $\underline{y}_v^*(s)$  as



a solution (which exists by Theorem 1) of (1) with  $F = F_{vs}$ .

From Theorem 2 and the last part of Lemma 2 it follows that  $\underline{y}_0^*(s)$ ,  $s = 1, 2, \dots$  is unique. Theorem 3 Corollary 2 therefore gives

$$(37) \quad \lim_{v \rightarrow \infty} \underline{y}_v^*(s) = \underline{y}_0^*(s), \quad s = 1, 2, \dots$$

We now prove

$$(38) \quad \lim_{s \rightarrow \infty} \underline{y}_0^*(s) = \underline{y}_0^*.$$

From the definition of  $\underline{y}_0^*(s)$

$$\int u(\underline{y}_0^*(s) \cdot \underline{x}) F_{os}(d\underline{x}) \geq \int u(\underline{y}_0^* \cdot \underline{x}) F_{os}(d\underline{x})$$

and hence

$$\int_{R_+^n(K_s)} u(\underline{y}_0^*(s) \cdot \underline{x}) F_o(d\underline{x}) \geq \int_{R_+^n(K_s)} u(\underline{y}_0^* \cdot \underline{x}) F_o(d\underline{x}).$$

Let  $(\underline{y}_0^*(s(i)))_{i=1}^\infty$  be a convergent subsequence of  $(\underline{y}_0^*(s))_1^\infty$  which converges to a point  $\underline{y}_0$  of  $S_n$ . As

$$0 \leq u(\underline{y} \cdot \underline{x}) \leq \sum_{j=1}^n u(x_j)$$

for all  $\underline{y}$  in  $S_n$  we may, because of (26), apply the Lebesgue theorem to obtain

$$\begin{aligned} \int u(\underline{y}_0^* \cdot \underline{x}) F_o(d\underline{x}) &= \lim_{i \rightarrow \infty} \int_{R_+^n(K_{s(i)})} u(\underline{y}_0^* \cdot \underline{x}) F_o(d\underline{x}) \\ &\leq \lim_{i \rightarrow \infty} \int_{R_+^n(K_{s(i)})} u(\underline{y}_0^*(s(i)) \cdot \underline{x}) F_o(d\underline{x}) \\ &= \int u(\underline{y}_0 \cdot \underline{x}) F_o(d\underline{x}). \end{aligned}$$

Therefore, as  $\underline{y}_0^*$  is unique, we must have  $\underline{y}_0 = \underline{y}_0^*$ . This is true for any convergent subsequence of  $(\underline{y}_0^*(s))$  and hence (38) holds.

From (37) and (38) we have

$$(39) \quad \lim_{s \rightarrow \infty} \left( \lim_{v \rightarrow \infty} \underline{y}_v^*(s) \right) = \underline{y}_0^* .$$

Now as  $\underline{y}_0^* \in S_n$  not all its components are zero. Without loss of generality we may therefore suppose that

$$(40) \quad y_{01} = \dots = y_{0m} = 0, \quad y_{0m+1}, \dots, y_{0n} \geq \eta > 0 .$$

(If no component is zero the proof is even easier). We choose  $\delta > 0$  so small so that  $2n\delta < \eta$  and define

$$(41) \quad \underline{y}(\delta) = (\underbrace{\delta, \dots, \delta}_m, -m\delta, 0, \dots, 0) .$$

We write

$$\underline{y}_0^*(\delta) = \underline{y}_0^* + \underline{y}(\delta)$$

and

$$\underline{y}_v^*(s, \delta) = \underline{y}_v^*(s) + \underline{y}(\delta) .$$

$s, v = 1, 2, \dots$ ).

It is clear from (40) and (41) that  $\underline{y}_0^*(\delta) \in S_n$  and that all the components of  $\underline{y}_0^*(\delta)$  are no smaller than  $\delta$ . From (39) it follows that there exists an  $s_0 = s_0(\delta)$  and a  $v_0 = v_0(s)$  such that for  $v = 0, v \geq v_0(s), s \geq s_0(\delta)$

$$\underline{y}_v^*(s, \delta) \in S_n$$

and

$$y_{vj}^*(s, \delta) \geq \frac{1}{2} \delta > 0, \quad j = 1, \dots, n.$$

For such values of  $v$  and  $s$  we have

$$\begin{aligned} 0 &\leq \int (u(\underline{y}_v^* \cdot \underline{x}) - u(\underline{y}_v^*(s, \delta) \cdot \underline{x})) F_v(d\underline{x}) \\ &= \int_{R_+^n(K_S)} (u(\underline{y}_v^* \cdot \underline{x}) - u(\underline{y}_v^*(s) \cdot \underline{x})) F_v(d\underline{x}) \\ &\quad + \int_{R_+^n(K_S)} (u(\underline{y}_v^*(s) \cdot \underline{x}) - u(\underline{y}_v^*(s, \delta) \cdot \underline{x})) F_v(d\underline{x}) \\ &\quad + \int_{R_+^n - R_+^n(K_S)} (u(\underline{y}_v^* \cdot \underline{x}) - u(\underline{y}_v^*(s, \delta) \cdot \underline{x})) F_v(d\underline{x}). \end{aligned}$$

As  $u(x)$  is increasing

$$\begin{aligned} &u(\underline{y}_v^* \cdot \underline{x}) - u(\underline{y}_v^*(s, \delta) \cdot \underline{x}) \\ &\leq u\left(\sum_{j=1}^n x_j\right) - u\left(\frac{1}{2} \delta \left(\sum_{j=1}^n x_j\right)\right). \end{aligned}$$

We therefore obtain

$$\begin{aligned} 0 &\leq \int_{R_+^n(K_S)} (u(\underline{y}_v^* \cdot \underline{x}) - u(\underline{y}_v^*(s) \cdot \underline{x})) F_v(d\underline{x}) \\ &\quad + \left| \int_{R_+^n(K_S)} (u(\underline{y}_v^*(s) \cdot \underline{x}) - u(\underline{y}_v^*(s, \delta) \cdot \underline{x})) F_v(d\underline{x}) \right| \\ &\quad + \int_{R_+^n - R_+^n(K_S)} \left( u\left(\sum_{j=1}^n x_j\right) - u\left(\frac{1}{2} \delta \left(\sum_{j=1}^n x_j\right)\right) \right) F_v(d\underline{x}). \end{aligned}$$

Now the first term is negative (definition of  $\underline{y}_v^*(s)$ ) whilst the other two are positive. This implies.

$$\begin{aligned} & \left| \int_{R_+^n(K_S)} (u(\underline{y}_v^* \cdot \underline{x}) - u(\underline{y}_v^*(s) \cdot \underline{x})) F_v(d\underline{x}) \right| \\ & \leq \left| \int_{R_+^n(K_S)} (u(\underline{y}_v^*(s) \cdot \underline{x}) - u(\underline{y}_v^*(s, \delta) \cdot \underline{x})) F_v(d\underline{x}) \right| \\ & + \int_{R_+^n - R_+^n(K_S)} (u(\sum_{j=1}^n x_j) - u(\frac{\delta}{2} \cdot (\sum_{j=1}^n x_j))) F_v(d\underline{x}) , \end{aligned}$$

which on using (27) gives

$$\begin{aligned} (42) \quad & \left| \int_{R_+^n(K_S)} (u(\underline{y}_v^* \cdot \underline{x}) - u(\underline{y}_v^*(s) \cdot \underline{x})) F_v(d\underline{x}) \right| \\ & \leq \left| \int_{R_+^n(K_S)} (u(\underline{y}_v^*(s) \cdot \underline{x}) - u(\underline{y}_v^*(s, \delta) \cdot \underline{x})) F_v(d\underline{x}) \right| \\ & + 0 \left( \int_{R_+^n - R_+^n(K_S)} F_v(d\underline{x}) \right) \end{aligned}$$

We remark that the family of functions

$$u(\underline{y} \cdot (\cdot)) : R_+^n(K_S) \rightarrow R_+^1, \quad \underline{y} \in S_n$$

is equicontinuous and uniformly bounded. We therefore obtain (as in Theorem 3)

$$\lim_{v \rightarrow \infty} \left[ \sup_{\underline{y}_1, \underline{y}_2 \in S_n} \left| \int (u(\underline{y}_1 \cdot \underline{x}) - u(\underline{y}_2 \cdot \underline{x})) F_{vS}(\underline{dx}) - \int (u(\underline{y}_1 \cdot \underline{x}) - u(\underline{y}_2 \cdot \underline{x})) F_{oS}(\underline{dx}) \right| \right] = 0$$

which by (32) implies

$$\lim_{v \rightarrow \infty} \left[ \sup_{\underline{y}_1, \underline{y}_2 \in S_n} \left| \int_{R_+^n(K_S)} (u(\underline{y}_1 \cdot \underline{x}) - u(\underline{y}_2 \cdot \underline{x})) F_v(\underline{dx}) - \int_{R_+^n(K_S)} (u(\underline{y}_1 \cdot \underline{x}) - u(\underline{y}_2 \cdot \underline{x})) F_o(\underline{dx}) \right| \right] = 0.$$

Therefore on letting  $v$  tend to infinity in (42) we obtain

(as  $\lim_{v \rightarrow \infty} \underline{y}_v^*(s) = \underline{y}_o^*(s)$ )

$$\begin{aligned} & \left| \int_{R_+^n(K_S)} (u(\lim_{v \rightarrow \infty} \underline{y}_v^*) \cdot \underline{x}) - u(\underline{y}_o^*(s) \cdot \underline{x}) F_o(\underline{dx}) \right| \\ & \leq \left| \int_{R_+^n(K_S)} (u(\underline{y}_o^*(s) \cdot \underline{x}) - u(\underline{y}_o^*(s, \delta) \cdot \underline{x})) F_o(\underline{dx}) \right| \\ & + 0 \left( \int_{R_+^n - R_+^n(K_S)} F_o(\underline{dx}) \right). \end{aligned}$$

If we now let  $s$  tend to infinity we obtain

$$\begin{aligned} & \left| \int (u(\lim_{v \rightarrow \infty} \underline{y}_v^*) \cdot \underline{x}) - u(\underline{y}_o^* \cdot \underline{x}) F_o(\underline{dx}) \right| \\ & \leq \left| \int (u(\underline{y}_o^* \cdot \underline{x}) - u(\underline{y}_o^*(\delta) \cdot \underline{x})) F_o(\underline{dx}) \right| + 0. \end{aligned}$$

The limiting operation is justified by (26). Finally applying the Lebesgue theorem once again and letting  $\delta$  tend to zero we obtain

$$\begin{aligned}
 & \left| \int (u(\lim_{v \rightarrow \infty} \underline{y}_v^*) \cdot \underline{x}) - u(\underline{y}_0^* \cdot \underline{x}) F_0(d\underline{x}) \right| \\
 & \leq \lim_{\delta \rightarrow 0} \left| \int (u(\underline{y}_0^* \cdot \underline{x}) - u(\underline{y}_0^*(\delta) \cdot \underline{x}) F_0(d\underline{x}) \right| \\
 & = 0 .
 \end{aligned}$$

This implies (29) as  $\underline{y}_0^*$  is unique and the proof of the theorem is complete.

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Summary of the paper

*Existence, uniqueness and continuity of  
portfolio choice*

*by*

*Laurie Davies and Gerd Ronning.*

Under fairly weak conditions it is shown that an optimal portfolio choice exists and is unique. It is further shown that this choice is a continuous function of the joint distribution function of the random returns on the assets from which the choice is made.