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Working Paper

Dynamic R&D networks

Working Paper, No. 109

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Suggested Citation: König, Michael D. (2013) : Dynamic R&D networks, Working Paper, No. 109, University of Zurich, Department of Economics, Zurich, https://doi.org/10.5167/uzh-69230

This Version is available at:
https://hdl.handle.net/10419/77510

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Mitglied der Leibniz-Gemeinschaft

University of Zurich

Department of Economics

Working Paper Series

ISSN 1664-7041 (print)
ISSN 1664-705X (online)

Working Paper No. 109

Dynamic R&D Networks

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January 2013

Dynamic R&D Networks ☆

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Abstract

In this paper we analyze R&D collaboration networks in industries where firms are competitors in the product market. Firms' benefits from collaborations arise by sharing knowledge about a cost-reducing technology. By forming collaborations, however, firms also change their own competitive position in the market as well as the overall market structure. We analyze incentives of firms to form R&D collaborations with other firms and the implications of these alliance decisions for the overall network structure. We provide a general characterization of both equilibrium networks and endogenous production choices, and compare it to the efficient network architecture. We also allow for firms to differ in their technological characteristics, investigate how this affects their propensity to collaborate and study the resulting network architecture.

Key words: Bonacich centrality, network formation, R&D networks
JEL: C63, D83, D85, L22

1. Introduction

R&D partnerships have become a widespread phenomenon characterizing technological dynamics, especially in industries [Hagedoorn, 2002] with rapid technological development such as, for instance, the pharmaceutical, chemical and computer industries [see Ahuja, 2000; Powell et al., 2005; Riccaboni and Pammolli, 2002; Roijakkers and Hagedoorn, 2006]. In those industries, firms have become more specialized in specific domains of a technology and they tend to combine their knowledge with that of other firms that are specialized in different domains [Ahuja, 2000; Powell et al., 1996] in order to jointly reduce their production costs.

In this paper we study the incentives of firms to form R&D collaborations with other firms and the implications of these alliance decisions for the overall network

☆ I would like to thank Andrea Montanari, Sanjeev Goyal, Matt Jackson, Ben Golub, Yves Zenou and seminar participants at Stanford University in 2012 for the helpful comments and advice. Financial support from Swiss National Science Foundation through research grant PBEZP1–131169 is gratefully acknowledged.

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structure. Moreover, we investigate the effects of network formation on the level of social welfare generated.

For this purpose, we extend the analysis of R&D intensive industries presented in Goyal and Moraga-Gonzalez [2001],which was restricted to regular graphs and networks comprising of three firms, to take into account general equilibrium structures with an arbitrary number of firms and no ex ante restriction on the collaboration pattern between them. As in Westbrock [2010], we also allow for consumption goods to be imperfect substitutes by adopting the consumer utility maximization approach of Singh and Vives [1984].[1]

Our analysis bears similarities with a number of other recent contributions in as they analyze a similar payoff structure. First, a partial equilibrium analysis has been given in the seminal paper of Ballester et al. [2006]. These authors derive equilibrium outcomes if a condition on the eigenvalue of the underlying network is satisfied, while assuming that the network is exogenously given. Bramoullé et al. [2010] study Nash equilibria for the case of strategic substitutes and an exogenously given network. Differently to both of these contributions, we make the network as well as action choices endogenous.[2] Our approach is a generalization of the endogenous network formation mechanisms proposed in Snijders [2001] and Mele [2010]. Moreover, Cabrales et al. [2010] allow the network to be formed endogenously, but assume that link strengths are proportional to effort levels, while we make the linking decision depending on marginal payoffs. Finally, in the empirical paper by König et al. [2011] a similar market structure is considered, however, the focus lies on developing optimal R&D subsidy strategies and characterizing the firms that are most critical in terms of their contribution to the overall productivity of the economy.

2. The Model

We consider a Cournot oligopoly game in which a set $N = \{1, \ldots, n\}$ of firms is competing in a homogeneous product market.[3] Following Goyal and Moraga-Gonzalez [2001] we assume that firms are not only competitors in the product market, but they can also form pairwise collaborative agreements. These pairwise links involve a commitment to share R&D results and thus lead to lower marginal cost of production of

[1]Note also that differently to Goyal and Joshi [2003], we assume that the cost reduction achieved by an R&D collaboration also depends on the R&D effort of the firms participating in that collaboration.

[2]It is straightforward to see that the results obtained in this paper can be generalized to the payoff structure introduced in Ballester et al. [2006].

[3]Generalizations to Bertrand competition are straight forward [Westbrock, 2010].

the collaborating firms. The amount of this cost reduction depends on the effort the firms invest into R&D. Given the collaboration network G , each firm sets an R&D effort level unilaterally.[4] We assume that firms can only jointly develop a cost reducing technology. Given the effort levels e_i, marginal cost c_i of firm i is given by[5,6]

$$c_i(\mathbf{e}, G) = \bar{c} - \alpha e_i - \beta \sum_{j=1}^{n} a_{ij} e_j, \tag{1}$$

where $a_{ij} = 1$ if firms i and j set up a collaboration (0 otherwise) and $a_{ii} = 0$. The parameter $\alpha \in [0, 1]$ measures the relative cost reduction due to a firms' own R&D effort while the parameter $\beta \in [0, 1]$ measures the relative cost reduction due to the R&D effort of its collaboration partners. In this model, firms are exposed to business stealing effects if their rivals increase their output via cost reducing R&D collaborations. In order to guarantee non-negative marginal costs we assume that $e_i \in [0, \bar{e}]$ and $\bar{c} \geq (n-1)\bar{e}$. This shows that $\bar{c}$ must be of the order of $O(n)$. As in Goyal and Moraga-Gonzalez [2001], throughout the paper we shall assume parameter are constrained such that the second-order conditions hold and equilibria can be characterized in terms of first-order conditions and are interior.

Moreover, we also assume that firms incur a direct cost $\gamma \geq 0$ for their R&D efforts and a fixed cost $\zeta \geq 0$ for each R&D collaboration.[7] The profit of firm i, given the R&D network G and the quantities $\mathbf{q}$ and efforts $\mathbf{e}$, is then given by

$$\pi_i(\mathbf{q}, \mathbf{e}, G) = (p_i - c_i) q_i - \gamma e_i^2 - \zeta d_i.$$

Inserting marginal cost from Equation (1) gives

$$\pi_i(\mathbf{q}, \mathbf{e}, G) = p_i q_i - \bar{c} q_i + \alpha q_i e_i + \beta q_i \sum_{j=1}^{n} a_{ij} e_j - \gamma e_i^2 - \zeta d_i.$$

[4]See also Kamien et al. [1992] for a similar model of competitive RJVs in which firms unilaterally choose their R&D effort levels.

[5]Note that we have neglected spillovers among non-collaborating firms.

[6]This generalizes earlier studies such as the one by D'Aspremont and Jacquemin [1988] where spillovers were assumed to take place between all firms in the industry and no distinction between collaborating and non-collaborating firms was made.

[7]Observe that the direct cost ζ of collaboration is incurred by the firm initiating the collaboration. Therefore, it is the degree d_i of the firm i that appears in its profit function. We assume that R&D collaborations can only be formed if both firms agree to its establishment. One can show that marginal profit of a firm j to which firm i proposes a collaboration is given by $\pi_j(\mathbf{q}, G + ij) - \pi_j(\mathbf{q}, G) = \rho q_i q_j \geq 0$, with a constant $\rho \geq 0$. Since marginal profits are always non-negative, the firm j always accepts the proposed collaboration of i. This is a consequence of the assumption that firms are myopic and will be further discussed in Section 3.2.

The first-order condition with respect to R&D effort e_i is given by

$$\frac{\partial \pi_i(\mathbf{q}, \mathbf{e}, G)}{\partial e_i} = \alpha q_i - 2\gamma e_i = 0. \tag{2}$$

Solving for e_i and taking into account that $e_i \in [0, \bar{e}]$ delivers

$$e_i = \min\{\lambda q_i, \bar{e}\}, \tag{3}$$

where we have denoted by $\lambda = \frac{\alpha}{2\gamma}$. An interior solution hence requires that $q_i \leq \bar{q} \equiv \frac{2\gamma}{\alpha}\bar{e} \leq \frac{2\gamma\bar{c}}{\alpha(n-1)}$ for all $i = 1, \ldots, n$. Since $\bar{c}$ is $O(n)$ we thus require that q_i is $O(1)$. This means that quantities produced do not grow without bound as the number of firms in the industry becomes large. Equation (3) can be viewed as reflecting learning-by-doing effects on R&D efforts. Various empirical studies have found that the R&D effort of a firm is proportional its output or size [Cohen and Klepper, 1996a,b]. We then can write marginal costs from Equation (1) as follows[8]

$$c_i(\mathbf{e}(\mathbf{q}), G) = \bar{c} - \lambda\alpha q_i - \lambda\beta \sum_{j=1}^{n} a_{ij} q_j. \tag{4}$$

Profits can be written as

$$\pi_i(\mathbf{q}, G) = p_i q_i - \bar{c} q_i - \lambda\alpha q_i^2 + \lambda\beta q_i \sum_{j=1}^{n} a_{ij} q_j - \lambda^2 \gamma q_i^2 - \zeta d_i. \tag{5}$$

Next we consider the demand for goods produced by firm i. A representative consumer maximizes [Singh and Vives, 1984]

$$U(I, q_1, \ldots, q_n) = I + a \sum_{i=1}^{n} q_i - \frac{1}{2} \sum_{i=1}^{n} q_i^2 - \frac{b}{2} \sum_{i=1}^{n} \sum_{j \neq i} q_i q_j, \tag{6}$$

with the budget constraint $I + \sum_{i=1}^{n} q_i \leq E$ and endowment E. The parameter a captures the total size of the market, whereas $b \in (0, 1]$, measures the degree of substitutability between products. In particular, $b = 1$ depicts a market of perfect substitutable goods, while $b \to 0$ represents the case of almost independent markets. The constraint is binding and the utility maximization of the representative consumer

[8] We assume that firms always implement the optimal R&D effort level. Since the optimal R&D effort decision only depends on a firm's own output, a firm does not face any uncertainty when implementing this strategy. In Section 3.2 we will, however, introduce noise in the optimal output and collaboration decisions, since these depend on the decisions of all other firms in the industry and their characteristics, which might be harder to observe.

gives the inverse demand function for firm i

$$p_i = a - q_i - b \sum_{j \neq i} q_j. \tag{7}$$

Firms face an inverse linear demand as given in Equation (7). Firm i then sets its quantity q_i in order to maximize its profit π_i given by Equation (5). We also assume that there is a maximum production capacity $\bar{q}$ such that $q_i \leq \bar{q}$ for all $i \in N$. Inserting marginal cost from Equation (4) and inverse demand from Equation (7) we can write firm i's profit as

$$\pi_i(\mathbf{q}, G) = (a - \bar{c})q_i - (1 - \lambda\alpha + \lambda^2\gamma)q_i^2 - bq_i \sum_{j \neq i} q_j + \lambda\beta \sum_{j=1}^{n} a_{ij} q_i q_j - \zeta d_i. \tag{8}$$

We assume that $a > \bar{c}$. Since $\bar{c}$ must be of the order of $O(n)$ this also implies that a is $O(n)$. In the following we will denote by $\eta = (a - \bar{c})/n$ (which is $O(1)$), $\nu = 1 - \lambda\alpha + \lambda^2\gamma$ and $\rho = \lambda\beta$, so that Equation (8) becomes

$$\pi_i(\mathbf{q}, G) = \underbrace{n\eta q_i - \nu q_i^2}_{\text{own concavity}} \quad \underbrace{-bq_i \sum_{j \neq i}^{n} q_j}_{\text{global substitutability}} \quad + \quad \underbrace{\rho q_i \sum_{j=1}^{n} a_{ij} q_j}_{\text{local complementarity}} \quad -\zeta d_i. \tag{9}$$

Firm $i \in N$ sets its quantity q_i and makes profit π_i given by Equation (9). The corresponding first-order conditions are given by

$$\frac{\partial \pi_i(\mathbf{q}, G)}{\partial q_i} = n\eta - 2\nu q_i - b \sum_{j \neq i}^{n} q_j + \rho \sum_{j=1}^{n} a_{ij} q_j = 0. \tag{10}$$

The second-order derivatives for $j \neq i$ are given by

$$\frac{\partial^2 \pi_i(\mathbf{q}, G)}{\partial q_j \partial q_i} = \frac{\partial^2 \pi_i}{\partial q_i \partial q_j} = -b + \rho a_{ij}, \tag{11}$$

which is negative if $b > \rho$, and

$$\frac{\partial^2 \pi_i}{\partial q_i^2} = -2\nu \leq 0, \tag{12}$$

if $\nu \geq 0$. From Equation (10) we can write firm i's best response quantity as

$$q_i = f_i(q_{-i}, G) \equiv \frac{n\eta}{2\nu} - \frac{b}{2\nu} \sum_{j \in N \setminus \{i\}} q_j + \frac{\rho}{2\nu} \sum_{j \in N_i} q_j = \frac{n\eta}{2\nu} - \frac{b}{2\nu} \sum_{j \notin (N_i \cup \{i\})} q_j + \frac{\rho - b}{2\nu} \sum_{j \in N_i} q_j. \tag{13}$$

with the constraint that $0 \leq q_i \leq \bar{q}$. Equation (13) shows that output of i is decreasing in the output of the firms j not connected to i. Moreover, if $\rho < b$, then firm i's output is also decreasing in its neighbors' output. However, if $\rho > b$, i's output is increasing in its neighbors' output. Note that the best response in Equation (13) is similar to the class of local spillover games discussed in Goyal and Joshi [2006].

Let us first consider the case of $\zeta = 0$, such that firms do not incur a fixed cost for an R&D collaboration. Then the best response of firm i can be written as follows

$$q_i = \min\left(\bar{q}, \max\left(0, f_i(q_{-i}, G)\right)\right). \tag{14}$$

Observe that if $f_i(q_{-i}, G) < 0$, then the business stealing effects become so large, that firm i's best response is to leave the market ($q_i = 0$). If $\zeta > 0$ then firm i's best response is

$$q_i = \begin{cases} \min\left(\bar{q}, f_i(q_{-i}, G)\right), & \text{if } \pi_i(f_i(q_{-i}, G), G) > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

The set of networks G for which $q_i > 0$ is starkly reduced if $\zeta > 0$, and if ζ becomes large enough, no firm will operate at positive quantity.

For the remainder of this section we analyze the best response dynamics of quantities in a fixed network G where firms adjust their output levels optimally, given the output levels of all other firms in the industry [Corchon and Mas-Colell, 1996; Weibull, 1997]. We also assume that an interior equilibrium exists. This dynamics is given by

$$\frac{dq_i}{dt} = f_i(q_{-i}, G) - q_i = \frac{n\eta}{2\nu} - \frac{b}{2\nu} \sum_{j \neq i}^{n} q_j + \frac{\rho}{2\nu} \sum_{j=1}^{n} a_{ij} q_j - q_i, \tag{16}$$

with some appropriate initial conditions $\mathbf{q}(0) \geq 0$. The equilibrium quantities for a given network G can be obtained as the fixed points of the best response dynamics. In vector-matrix notation the dynamics can be written as

$$\frac{d\mathbf{q}}{dt} = \frac{n\eta}{2\nu}\mathbf{u} - \frac{1}{2\nu}\left((2\nu - b)\mathbf{I}_n + b\mathbf{u}\mathbf{u}^\top - \rho\mathbf{A}\right)\mathbf{q}. \tag{17}$$

This is an inhomogeneous linear first-order ordinary differential equation with con-

stant coefficients. Let us denote by $\mathbf{U} = \mathbf{u}\mathbf{u}^\top$ and introduce the matrix

$$\mathbf{Q} \equiv \mathbf{I}_n + \frac{b}{2\nu - b}\mathbf{U} - \frac{\rho}{2\nu - b}\mathbf{A}. \tag{18}$$

The solution of Equation (17) is stable if an only if all eigenvalues of $\mathbf{Q}$ have a positive real part. If a stable solution exists and if $\mathbf{Q}$ is invertible, then the steady state $\mathbf{q}^* = \lim_{t\to\infty} \mathbf{q}(t)$ is given by

$$\mathbf{q}^* = \frac{n\eta}{2\nu - b}\mathbf{Q}^{-1}\mathbf{u}, \tag{19}$$

and the solution trajectory is given by

$$\mathbf{q}(t) = \mathbf{q}^* + e^{-\mathbf{Q}t}(\mathbf{q}(0) - \mathbf{q}^*). \tag{20}$$

If $\mathbf{q}(0) = 0$ then we can write

$$\mathbf{q}(t) = \frac{n\eta}{2\nu - b}\left(1 - e^{-\mathbf{Q}t}\right)\mathbf{Q}^{-1}\mathbf{u}. \tag{21}$$

We have that $\mathbf{Q} = \mathbf{I}_n - \frac{\rho}{2\nu-b}\mathbf{A}$, and $\lambda_i(\mathbf{Q}) = 1 - \frac{\rho}{2\nu-b}\lambda_i(\mathbf{A})$. This implies that the stability condition $\lambda_{\min}(\mathbf{Q}) < 0$ is equivalent to $\lambda_{\max}(\mathbf{A}) > \frac{2\nu-b}{\rho}$. The latter condition will be important when analyzing the equilibrium outcomes in the following sections.

3. Stability

In the following we provide an equilibrium analysis of the R&D collaboration game with profits introduced in the previous section. More precisely, in Section 3.1 we analyze equilibrium quantities and payoffs for a given network structure and in Section 3.2 we also allow the network to be endogenously determined by the link incentives of firms.

3.1. Equilibrium Analysis for Exogenous Networks

The profit function introduced in Equation (9) admits a potential game with a corresponding potential function [Monderer and Shapley, 1996]. This is stated in the following proposition.

Proposition 1. *For a given network $G \in \mathcal{G}_n$, the profit function of Equation (9) admits a potential game with potential function $\phi(q, G)\colon \mathbb{R}^n_+ \times \mathcal{G}_n \to \mathbb{R}$ given by*

$$\phi(\mathbf{q}, G) = \sum_{i=1}^{n}(n\eta q_i - \nu q_i^2) - \frac{b}{2}\sum_{i=1}^{n}\sum_{j\neq i} q_i q_j + \frac{\rho}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} q_i q_j. \tag{22}$$

PROOF OF PROPOSITION 1. The potential $\phi(\mathbf{q}, G)$ of Equation (44) has the property that for any $q_i' \neq q_i \in [0, \bar{q}]$ we have that

$$\begin{aligned}\phi(q_i', \mathbf{q}_{-i}, G) - \phi(q_i, \mathbf{q}_{-i}, G) &= n\eta(q_i' - q_i) - \nu(q_i'^2 - q_i^2) - b(q_i' - q_i)\sum_{j \neq i} q_j + \rho(q_i' - q_i)\sum_{j \in N_i} q_j \\ &= \pi_i(q_i', \mathbf{q}_{-i}, G) - \pi_i(q_i, \mathbf{q}_{-i}, G).\end{aligned}$$

□

The existence of a potential function given in Equation (44) allows us to state a condition for all Nash equilibria of our game. For a given network G and no fixed linking costs, $\zeta = 0$, a Nash equilibrium of our game solves the following constrained optimization problem (cf. Bramoullé et al. [2010] and Sandholm [2010, Sec. 3.1.4])

$$\max_{\mathbf{q} \in \mathbb{R}_+^n} \quad \phi(\mathbf{q}, G) \tag{23}$$

$$\text{s.t.} \quad \forall i = 1, \ldots, n$$

$$\frac{\partial \phi}{\partial q_i} = 0 \text{ and } q_i > 0, \text{ or} \tag{24}$$

$$\frac{\partial \phi}{\partial q_i} \leq 0 \text{ and } q_i = 0. \tag{25}$$

We can write the potential function $\phi(\mathbf{q}, G)$ in vector-matrix notation as follows

$$\phi(\mathbf{q}, G) = n\eta \mathbf{u}^\top \mathbf{q} - \frac{1}{2}\mathbf{q}^\top \underbrace{\left((2\nu - b)\mathbf{I}_n + b\mathbf{U} - \rho\mathbf{A}\right)}_{=(2\nu - b)\mathbf{Q}} \mathbf{q}. \tag{26}$$

The Hessian of the potential is given by $\Delta\phi(\mathbf{q}, G) = \left(\frac{\partial^2 \phi(\mathbf{q}, G)}{\partial q_i \partial q_j}\right)_{i,j \in N} = -(2\nu - b)\mathbf{Q}$. If $2\nu > b$ and the matrix $\mathbf{Q}$ is positive definite then $\Delta\phi(\mathbf{q}, G) < 0$ is negative definite.[9] The matrix $\mathbf{Q}$ is positive definite if and only if the matrix $\mathbf{B} = \mathbf{I}_n - \frac{\rho}{2\nu - b}\mathbf{A}$ is positive definite. If $\mathbf{B}$ is positive definite then its inverse $\mathbf{B}^{-1}$ exists and is positive definite. $\mathbf{B}^{-1}$ exists if and only if the following eigenvalue condition is satisfied

$$\frac{\rho}{2\nu - b} < \frac{1}{\lambda_{\max}(\mathbf{A})}, \tag{27}$$

[9] The $n \times n$ matrix $\mathbf{Q}$ is positive definite if and only if for all $q \in \mathbb{R}_+^n$ we have that $\mathbf{q}^\top Q \mathbf{q} > 0$. If $\mathbf{Q}$ is positive definite, then all its eigenvalues are positive.

where $\lambda_{\max}(\mathbf{A})$ is the largest (real) eigenvalue of the (real and symmetric) adjacency matrix $\mathbf{A}$. If the inequality in (27) is satisfied, then the maximization of $\phi(q, G)$ as stated in (25), for a given G, is a linear-quadratic programming problem [Boyd and Vandenberghe, 2004; Lee et al., 2005], where $\phi(\mathbf{q}, G)$ is a concave function of $\mathbf{q}$, and this optimization problem has a unique solution.

In the following we assume that the inequality in (27) is satisfied. Then we can obtain firms' equilibrium quantities and profits as follows:

Proposition 2. *Denote by* $\varphi = \frac{\rho}{2\nu - b} = \frac{\alpha\beta}{2(2-b)\gamma - \alpha^2}$ *and consider a network* $G \in \mathcal{G}_n$ *satisfying* $\varphi < 1/\lambda_{\max}(\mathbf{A})$.

(i) If $\zeta = 0$, *equilibrium output and profit are given by*

$$q_i = \frac{n\eta}{2\nu + b(\|\mathbf{b}(G, \varphi)\| - 1)} b_i(G, \varphi) \tag{28}$$

and

$$\pi_i = \frac{\nu n^2 \eta^2}{(2\nu + b(\|\mathbf{b}(G, \varphi) - 1\|))^2} b_i^2(G, \varphi), \tag{29}$$

for all $i = 1, \ldots, n$.

(ii) If $\zeta > 0$ *and* $\pi_i > \zeta d_i$ *for all* $i = 1, \ldots, n$ *in Equation (29) then equilibrium quantities are given by Equation (28) and equilibrium profits are given by Equation (29) less the cost of collaboration* ζd_i.

PROOF OF PROPOSITION 2. Equation (56) can be written as

$$q_i - \frac{\rho}{2\nu - b} \sum_{j=1}^{n} a_{ij} q_j = \frac{n\eta}{2\nu - b} - \frac{b}{2\nu - b} \|q\| \tag{30}$$

where $\|q\| = \mathbf{u}^\top \mathbf{q}$. Let us denote by $\varphi = \frac{\rho}{2\nu - b}$, $A = \frac{n\eta}{2\nu - b}$ and $B = \frac{b}{2\nu - b}$. Then, in vector-matrix notation, the above equation can then be written as

$$(\mathbf{I}_n - \varphi \mathbf{A})\, \mathbf{q} = (A - B\|q\|)\mathbf{u}. \tag{31}$$

If $\varphi < \frac{1}{\lambda_{\max}}(\mathbf{A})$ then the matrix $\mathbf{I}_n - \varphi\mathbf{A}$ is invertible, and we obtain

$$\mathbf{q} = (A - B\|q\|)\, (\mathbf{I}_n - \varphi \mathbf{A})^{-1}\, \mathbf{u}. \tag{32}$$

Noting that

$$(\mathbf{I}_n - \varphi \mathbf{A})^{-1}\, \mathbf{u} = \mathbf{b}(G, \varphi), \tag{33}$$

where $\mathbf{b}(G, \varphi)$ is the vector of Bonacich centralities with parameter φ [Bonacich, 1987], we obtain

$$\mathbf{q} = (A - B\|\mathbf{q}\|)\mathbf{b}(G, \varphi). \tag{34}$$

With

$$\|\mathbf{q}\| = (A - B\|\mathbf{q}\|)\|b(G,\varphi)\| \tag{35}$$

we obtain

$$\|\mathbf{q}\| = \frac{A\|b(G,\varphi)\|}{1 + B\|b(G,\varphi)\|)} \tag{36}$$

and it follows that

$$\mathbf{q} = \frac{A}{1 + B\|\mathbf{b}(G,\varphi)\|)}\mathbf{b}(G,\varphi) = \frac{n\eta}{2\nu + b(\|\mathbf{b}(G,\varphi)\| - 1)}\mathbf{b}(G,\varphi). \tag{37}$$

Next, we compute equilibrium profits. Let us denote by $C = \frac{n\eta}{2\nu + b(\|\mathbf{b}(G,\varphi)\| - 1)}$, so that $q_i = Ab_i(G,\varphi)$. Profit of firm i from Equation (9) can then be written as

$$\begin{aligned}\pi_i = n\eta C b_i(G,\varphi) - (\nu - b)C^2 b_i(G,\varphi)^2 - bC^2\|\mathbf{b}(G,\varphi)\|b_i(G,\varphi) \\ + \rho C^2 b_i(G,\varphi)\sum_{j=1}^{n} a_{ij}b_j(G,\varphi) - \zeta d_i.\end{aligned} \tag{38}$$

Using the fact that

$$b_i(G,\varphi) = 1 + \frac{\rho}{2\nu - b}\sum_{j=1}^{n} a_{ij}b_j(G,\varphi) \tag{39}$$

we obtain

$$\begin{aligned}\pi_i &= n\eta C b_i(G,\varphi) - (\nu - b)C^2 b_i(G,\varphi)^2 - bC^2\|\mathbf{b}(G,\varphi)\|b_i(G,\varphi) \\ &\quad + \rho C^2(2\nu - b)b_i(G,\varphi)(b_i(G,\varphi) - 1) - \zeta d_i. \\ &= \nu C^2 b_i(G,\varphi)^2 - \zeta d_i.\end{aligned} \tag{40}$$

which gives

$$\pi_i = \frac{n^2\eta^2\nu}{(2\nu + b(\|\mathbf{b}(G,\varphi)\| - 1))^2}b_i^2(G,\varphi) - \zeta d_i. \tag{41}$$

□

Observe that, in the limit $\varphi \uparrow \lambda_{\max}^{-1}$, the normalized Bonacich centrality converges to the eigenvector centrality $\mathbf{v}$, where $\mathbf{A}\mathbf{v} = \lambda_{\max}\mathbf{v}$. This implies that

$$\lim_{\varphi\uparrow\lambda_{\max}^{-1}} q_i = \frac{n\eta}{b}v_i \tag{42}$$

and

$$\lim_{\varphi\uparrow\lambda_{\max}^{-1}} \pi_i = \frac{n\eta^2\nu}{b^2}v_i^2 - \zeta d_i, \tag{43}$$

for all $i = 1, \ldots, n$.

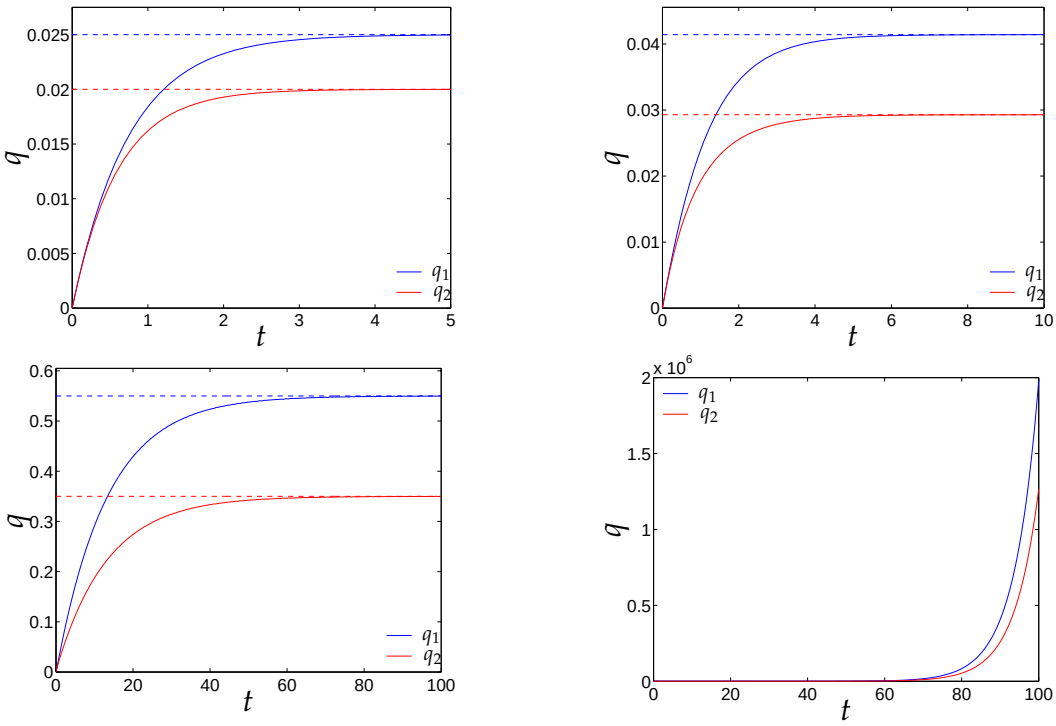

Figure 1: A star network with $n = 3$, $\lambda_{\max}(A) = \sqrt{2} = 1.41$ and no linking costs, i.e. $\zeta = 0$. The top left panel shows the evolution $q(t)$ for $\varphi = 1/3 < \lambda_{\max}(A)^{-1} = 0.70$, the top right panel for $\varphi = \lambda_{\max}(A)^{-1} = 0.70$, the bottom left panel for $\varphi = 4/3 > \lambda_{\max}(A)^{-1} = 0.70$, and the bottom right panel for $\varphi = 3/2 > \lambda_{\max}(A)^{-1} = 0.70$. The dashed lines indicate the solutions from Equation (28).

Note that if the eigenvalue condition (27) is not satisfied, then corner solutions must be considered.[10]

In Figure 1 we give an example of the evolution of $\mathbf{q}(t)$ from Equation (17) for the star network $K_{1,n-1}$ with $n = 3$. The stationary state $\mathbf{q}^*$ for values of $\varphi \leq \lambda_{\max}(\mathbf{A})^{-1}$ is correctly described by Equation (28). Interestingly, this solution is also correct for values of $\varphi > \lambda_{\max}(\mathbf{A})^{-1}$ (see bottom left panel in Figure 1), unless the equilibrium quantities from Equation (28) explode (see bottom right panel in Figure 1), and $q_i(t)$ grows to its capacity constraint $\bar{q}$.

3.2. Equilibrium Analysis with Endogenous Networks

Similar to the analysis in the previous section, we can provide a potential function that not only accounts for quantity adjustments but also for the linking strategies.

Proposition 3. *Assume that both, quantities and links can be changed according to a myopic profit maximizing rationale of firms. Then the profit function of Equation (9) admits a potential*

[10]See Bramoullé et al. [2010] for the case of strategic substitutes and Cabrales et al. [2010] for the case of strategic complements and linking strengths proportional to socialization effort.

game with potential function $\Phi\colon \mathbb{R}^n_+ \times \mathcal{G}_n \to \mathbb{R}$ *given by*

$$\Phi(\mathbf{q}, G) = \sum_{i=1}^{n}(n\eta q_i - \nu q_i^2) - \frac{b}{2}\sum_{i=1}^{n}\sum_{j\neq i} q_i q_j + \frac{\rho}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} q_i q_j - \zeta m = \phi(\mathbf{q}, G) - \zeta m. \quad (44)$$

where m *is the number of links in* G.

PROOF OF PROPOSITION 3. The potential $\Phi(\mathbf{q}, G)$ has the property that

$$\Phi(\mathbf{q}, G + ij) - \Phi(\mathbf{q}, G) = \rho q_i q_j - \zeta = \pi_i(\mathbf{q}, G + ij) - \pi_i(\mathbf{q}, G). \quad (45)$$

From the properties of $\pi_i(\mathbf{q}, G)$ it also follows that $\Phi(q_i', \mathbf{q}_{-i}, G) - \Phi(q_i, \mathbf{q}_{-i}, G) = \phi(q_i', \mathbf{q}_{-i}, G) - \phi(q_i, \mathbf{q}_{-i}, G) = \pi_i(q_i', \mathbf{q}_{-i}, G) - \pi_i(q_i, \mathbf{q}_{-i}, G)$. □

In this section we allow the network to be formed endogenously, based on the profit maximizing decisions of firms with whom to collaborate, and share knowledge about a cost reducing technology. The precise definition of the dynamics of quantity adjustment and network evolution is given in the following:

Definition 1. *The evolution of the industry is characterized by a sequence of states* $(\boldsymbol{\omega}_t)_{t\in\mathbb{R}_+}$, $\omega_t \in \Omega$, *where each state* $\boldsymbol{\omega}_t = (\mathbf{q}, G_t)$ *consists of a vector of firms' quantities* $\mathbf{q}_t \in [0, \bar{q}]^n$ *and a network of collaborations* $G_t \in \mathcal{G}_n$. *In a short time interval* $[t, t + \Delta t)$, $t \in \mathbb{R}_+$, *one of the following events happens:*

Action adjustment *At rate* νn *a firm* $i \in N$ *is selected at random and given a revision opportunity of its current output level* q_i. *When firm* i *receives such a revision opportunity, it draws a quantity* $q_i' \in [0, \bar{q}]$ *uniformly at random (with probability* $1/\bar{q}$*) and evaluates its marginal profits from changing its quantity to* q_i'. *The computation of marginal profits is perturbed by an additive i.i.d. shock* ε_{it}, *so that the probability that we observe a switch from output level* q_i *to* q_i' *is given by*

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (q_i', \mathbf{q}_{-it}, G_t) | \boldsymbol{\omega}_t = (q_i, \mathbf{q}_{-it}, G_t)\right) &= \frac{\nu}{\bar{q}}\mathbb{P}\left(\pi_i(q_i', \mathbf{q}_{-i}, G_t) - \pi_i(q_i, \mathbf{q}_{-i}, G_t) + \varepsilon_{it} > 0\right)\Delta t \\ &= \frac{\nu}{\bar{q}}\mathbb{P}\left(\Phi(q_i', \mathbf{q}_{-i}, G_t) - \Phi(q_i, \mathbf{q}_{-i}, G_t) + \varepsilon_{it} > 0\right)\Delta t.\end{aligned}$$

where ϑ *is a scale parameter measuring the extent of noise relative to profit maximization, and we have used the fact that* $\pi_i(q_i', \mathbf{q}_{-i}, G_t) - \pi_i(q_i, \mathbf{q}_{-i}, G_t) = \Phi(q_i', \mathbf{q}_{-i}, G_t) - \Phi(q_i, \mathbf{q}_{-i}, G_t)$.

Link formation *With rate* $\lambda > 0$ *a pair of firms* ij *which is not already connected receives an opportunity to form a link. The formation of a link depends on the marginal profit the firms receive from the link plus an additive pairwise i.i.d. error term* $\varepsilon_{ij,t}$. *The probability that link* ij *is created is then given by*

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (\mathbf{q}_t, G_t + ij) | \boldsymbol{\omega}_{t-1} = (\mathbf{q}, G_t)\right) &= \lambda\mathbb{P}\left(\{\pi_i(\mathbf{q}_t, G_t + ij) - \pi_i(\mathbf{q}_t, G_t) + \varepsilon_{ij,t} > 0\}\right. \\ &\quad \left.\cap\{\pi_j(\mathbf{q}_t, G_t + ij) - \pi_j(\mathbf{q}_t, G_t) + \varepsilon_{ij,t} > 0\}\right)\Delta t \\ &= \lambda\mathbb{P}\left(\Phi(\mathbf{q}_t, G_t + ij) - \Phi(\mathbf{q}_t, G_t) + \varepsilon_{ij,t} > 0\right)\Delta t,\end{aligned}$$

where we have used the fact that $\pi_i(\mathbf{q}_t, G_t + ij) - \pi_i(\mathbf{q}_t, G_t) = \pi_j(\mathbf{q}_t, G_t + ij) - \pi_j(\mathbf{q}_t, G_t) = \Phi(\mathbf{q}_t, G_t + ij) - \Phi(\mathbf{q}_t, G_t)$.

Link removal *With rate* $\xi > 0$ *a pair of connected firms* ij *receives an opportunity to terminate their connection. The link is removed if at least one firm find this profitable. The marginal profits from removing the link* ij *are perturbed by an additive pairwise i.i.d. error term* $\varepsilon_{ij,t}$. *The probability that link* ij *is removed is then given by*

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (\mathbf{q}_t, G_t - ij) | \boldsymbol{\omega}_t = (\mathbf{q}, G_t)\right) &= \xi \mathbb{P}\left(\{\pi_i(\mathbf{q}_t, G_t - ij) - \pi_i(\mathbf{q}_t, G_t) + \varepsilon_{ij,t} > 0\}\right. \\ &\quad \left.\cup \{\pi_j(\mathbf{q}_t, G_t - ij) - \pi_j(\mathbf{q}_t, G_t) + \varepsilon_{ij,t} > 0\}\right) \Delta t \\ &= \xi \mathbb{P}\left(\Phi(\mathbf{q}_t, G_t - ij) - \Phi(\mathbf{q}_t, G_t) + \varepsilon_{ij,t} >\right) \Delta t,\end{aligned}$$

where we have used the fact that $\pi_i(\mathbf{q}_t, G_t - ij) - \pi_i(\mathbf{q}_t, G_t) = \pi_j(\mathbf{q}_t, G_t - ij) - \pi_j(\mathbf{q}_t, G_t) = \Phi(\mathbf{q}_t, G_t - ij) - \Phi(\mathbf{q}_t, G_t)$.

In the following we make a specific assumption on the distribution of the random shocks. In particular, we assume that these shocks are independent and identically exponentially distributed with parameter $\vartheta \geq 0$. We then can write[11]

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (q_i', \mathbf{q}_{-it}, G_t) | \boldsymbol{\omega}_t = (q_i, \mathbf{q}_{-it}, G_t)\right) &= \mathbb{P}\left(-\varepsilon_{it} < \Phi(q_i', \mathbf{q}_{-it}, G_t) - \Phi(q_i, \mathbf{q}_{-i}, G_t)\right) \\ &= \frac{\nu}{\bar{q}} \frac{e^{\vartheta \Phi(q_i', \mathbf{q}_{-i}, G_t)}}{e^{\vartheta \Phi(q_i', \mathbf{q}_{-it}, G_t)} + e^{\vartheta \Phi(q_i, \mathbf{q}_{-it}, G_t)}} \Delta t, \quad (46)\end{aligned}$$

and similarly we obtain for the creation of the link ij

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (\mathbf{q}_t, G_t + ij) | \boldsymbol{\omega}_t = (\mathbf{q}_t, G_t)\right) &= \mathbb{P}\left(-\varepsilon_{ij,t} < \Phi(\mathbf{q}_t, G_t + ij) - \Phi(\mathbf{q}_t, G_t)\right) \\ &= \lambda \frac{e^{\vartheta \Phi(\mathbf{q}_t, G_t + ij)}}{e^{\vartheta \Phi(\mathbf{q}_t, G_t + ij)} + e^{\vartheta \Phi(\mathbf{q}_t, G_t)}} \Delta t, \quad (47)\end{aligned}$$

and the removal of the link ij

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (\mathbf{q}_t, G_t - ij) | \boldsymbol{\omega}_t = (\mathbf{q}_t, G_t)\right) &= \mathbb{P}\left(-\varepsilon_{ij,t} < \Phi(\mathbf{q}_t, G_t - ij) - \Phi(\mathbf{q}_t, G_t)\right) \\ &= \xi \frac{e^{\vartheta \Phi(\mathbf{q}_t, G_t - ij)}}{e^{\vartheta \Phi(\mathbf{q}_t, G_t - ij)} + e^{\vartheta \Phi(\mathbf{q}_t, G_t)}} \Delta t. \quad (48)\end{aligned}$$

Let $\mathcal{F}$ denote the smallest σ-algebra generated by $\boldsymbol{\varpi}(\boldsymbol{\omega}_t : t \in \mathbb{R}_+)$. The filtration is the non-decreasing family of sub-σ-fields $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ on the measure space $(\Omega, \mathcal{F})$, with the property that $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_t \subseteq \cdots \subseteq \mathcal{F}$. The probability space is

[11] Let ν be i.i. logistically distributed with mean 0 and scale parameter ϑ, i.e. $F_\nu(x) = \frac{e^{\vartheta x}}{1+e^{\vartheta x}}$. Consider the random variable $\varepsilon = g(\nu) = -\nu$. Since g is monotonic decreasing, and ν is a continuous random variable, the distribution of ε is given by $F_\varepsilon(y) = 1 - F_\nu(g^{-1}(y)) = \frac{e^{\vartheta y}}{1+e^{\vartheta y}}$.

given by the triple $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}\colon \mathcal{F} \to [0,1]$ is the probability measure satisfying $\int_\Omega \mathbb{P}(\boldsymbol{\omega}) d\mu(\boldsymbol{\omega}) = 1$. The sequence of states $(\boldsymbol{\omega}_t)_{t\in\mathbb{R}_+}$, $\boldsymbol{\omega}_t \in \Omega$ induces an irreducible and aperiodic (i.e. ergodic) Markov chain. The one step transition probability $P : \Omega^2 \to [0,1]$ from a state $\omega \in \Omega$ to a state $\omega' \in \Omega$ is given by

$$\begin{aligned}\mathbb{P}(\boldsymbol{\omega}_{t+\Delta t} = \boldsymbol{\omega}' | \mathcal{F}_t = \sigma(\boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_t = \boldsymbol{\omega})) &= \mathbb{P}(\boldsymbol{\omega}_{t+\Delta t} = \boldsymbol{\omega}' | \boldsymbol{\omega}_t = \boldsymbol{\omega}) \\ &= p(\boldsymbol{\omega}' | \boldsymbol{\omega}) \Delta t,\end{aligned}$$

where $p(\boldsymbol{\omega}'|\boldsymbol{\omega})$ is the transition rate from state $\boldsymbol{\omega}$ to state $\boldsymbol{\omega}'$. Observe that any function $f : \Omega \to \mathbb{R}$ of the state variables $\omega \in \Omega$ is a Carathéodory function since $f(\mathbf{q}, \cdot)$ is continuous for each $\mathbf{q} \in [0, \bar{q}]^n$ and $f(\cdot, G)$ is $(\mathcal{G}_n, \mathcal{B}_G)$ measurable [Aliprantis and Border, 2006].

In vector-matrix notation we can write $\Phi(\mathbf{q}, G) = \phi(\mathbf{q}, G) - \frac{\zeta}{2}\mathbf{u}^\top \mathbf{A}\mathbf{u}$. With the potential function $\Phi(\mathbf{q}, G)$ we then can state the following proposition.

Proposition 4. *The dynamic process $(\boldsymbol{\omega}_t)_{t\in\mathbb{R}_+}$ induces an irreducible and aperiodic Markov chain with a unique stationary distribution $\mu^\vartheta\colon [0,\bar{q}]^n \times \mathcal{G}_n \to [0,1]$ such that $\lim_{t\to\infty} \mathbb{P}(\boldsymbol{\omega}_t = (\mathbf{q}, G) | \boldsymbol{\omega}_0 = (\mathbf{q}_0, G_0)) = \mu^\vartheta(\mathbf{q}, G)$. The (Carathéodory) probability measure μ^ϑ is given by*

$$\mu^\vartheta(\mathbf{q}, G) = \frac{e^{\vartheta\Phi(\mathbf{q},G) - m\ln\left(\frac{\xi}{\lambda}\right)}}{\sum_{G'\in\mathcal{G}_n} \int_{[0,\bar{q}]^n} e^{\vartheta\Phi(\mathbf{q}',G') - m'\ln\left(\frac{\xi}{\lambda}\right)} d\mathbf{q}'}. \tag{49}$$

In the limit of vanishing noise $\vartheta \to \infty$, the (stochastically stable) states in the support of μ^ϑ are given by [Kandori et al., 1993]

$$\lim_{\vartheta\to\infty} \mu^\vartheta(\mathbf{q}, G) \begin{cases} > 0, & \text{if } \Phi(\mathbf{q}, G) \geq \Phi(\mathbf{q}', G'), \ \ \forall \mathbf{q}' \in [0,\bar{q}]^n, \ \ G' \in \mathcal{G}_n, \\ = 0, & \text{otherwise.} \end{cases} \tag{50}$$

Proof of Proposition 4. In the following we show that the stationary distribution $\mu^\vartheta(\boldsymbol{\omega})$ satisfies the detailed balance condition

$$\mu^\vartheta(\boldsymbol{\omega}) p(\boldsymbol{\omega}'|\boldsymbol{\omega}) = \mu^\vartheta(\boldsymbol{\omega}') p(\boldsymbol{\omega}|\boldsymbol{\omega}') \tag{51}$$

where $p(\boldsymbol{\omega}'|\boldsymbol{\omega})$ denotes the transition rate of the Markov chain from state $\boldsymbol{\omega}$ to $\boldsymbol{\omega}'$. Observe that the detailed balance condition is trivially satisfied if $\boldsymbol{\omega}'$ and $\boldsymbol{\omega}$ differ in more than one link or more than one quantity level. Hence, we consider only the case of link creation $G' = G + ij$ (and removal $G' = G - ij$) or and adjustment in quantity $q_i' \neq q_i$ for some $i \in N$. For the case of link creation with a transition from $\boldsymbol{\omega} = (\mathbf{q}, G)$

to $\boldsymbol{\omega}' = (\mathbf{q}, G + ij)$ we can write the detailed balance condition as follows

$$e^{\vartheta\Phi(\mathbf{q},G)-m\ln\left(\frac{\xi}{\lambda}\right)}\frac{e^{\vartheta\Phi(\mathbf{q},G+ij)}}{e^{\vartheta\Phi(\mathbf{q},G+ij)}+e^{\vartheta\Phi(\mathbf{q},G)}}\lambda = e^{\vartheta\Phi(\mathbf{q},G+ij)-(m+1)\ln\left(\frac{\xi}{\lambda}\right)}\frac{e^{\vartheta\Phi(\mathbf{q},G)}}{e^{\vartheta\Phi(\mathbf{q},G)}+e^{\vartheta\Phi(\mathbf{q},G+ij)}}\xi.$$

This equality is trivially satisfied. A similar argument holds for the removal of a link with a transition from $\boldsymbol{\omega} = (\mathbf{q}, G)$ to $\boldsymbol{\omega} = (\mathbf{q}, G - ij)$ where the detailed balance condition reads

$$e^{\vartheta\Phi(\mathbf{q},G)-m\ln\left(\frac{\xi}{\lambda}\right)}\frac{e^{\vartheta\Phi(\mathbf{q},G-ij)}}{e^{\vartheta\Phi(\mathbf{q},G-ij)}+e^{\vartheta\Phi(\mathbf{q},G)}}\xi = e^{\vartheta\Phi(\mathbf{q},G+ij)-(m-1)\ln\left(\frac{\xi}{\lambda}\right)}\frac{e^{\vartheta\Phi(\mathbf{q},G)}}{e^{\vartheta\Phi(\mathbf{q},G)}+e^{\vartheta\Phi(\mathbf{q},G-ij)}}\lambda.$$

For a change in the output level with a transition from $\boldsymbol{\omega} = (q_i, \mathbf{q}_{-i}, G)$ to $\boldsymbol{\omega}' = (q_i', \mathbf{q}_{-i}, G)$ we get for the detailed balance condition

$$\begin{aligned} e^{\vartheta\Phi(q_i,\mathbf{q}_{-i},G)-m\ln\left(\frac{\xi}{\lambda}\right)}&\frac{e^{\vartheta\Phi(q_i',\mathbf{q}_{-i},G)}}{e^{\vartheta\Phi(q_i',\mathbf{q}_{-i},G)}+e^{\vartheta\Phi(q_i,\mathbf{q}_{-i},G)}}\frac{\nu}{\bar{q}} \\ &= e^{\vartheta\Phi(q_i',\mathbf{q}_{-i},G)-(m+1)\ln\left(\frac{\xi}{\lambda}\right)}\frac{e^{\vartheta\Phi(q_i,\mathbf{q}_{-i},G)}}{e^{\vartheta\Phi(q_i,\mathbf{q}_{-i},G)}+e^{\vartheta\Phi(q_i',\mathbf{q}_{-i},G)}}\frac{\nu}{\bar{q}}, \end{aligned}$$

Hence, the probability measure $\mu^{\vartheta}(\boldsymbol{\omega})$ satisfies a detailed balance condition and therefore is the stationary distribution of the Markov chain with transition rates $p(\boldsymbol{\omega}'|\boldsymbol{\omega})$. □

Note that we could also allow quantity adjustments of Definition 1 to follow a noisy directional learning process as in Anderson et al. [1998, 2002, 2004].[12] Quantity adjustments then follow a logit dynamics with continuous strategies such that

$$\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (q_i', \mathbf{q}_{-it}, G_t)|\boldsymbol{\omega}_t = (q_i, \mathbf{q}_{-it}, G_t)\right) = \nu\frac{e^{\vartheta\pi_i(q_i',\mathbf{q}_{-it},G_t)}}{\int_{[0,\bar{q}]}e^{\vartheta\pi_i(q,\mathbf{q}_{-it},G_t)}dq}\Delta t.$$

However, this alternative definition would give rise to the same stationary distribution μ^{ϑ} as in Proposition 4.

In the following we will set $\lambda = \xi$. The stationary distribution $\mu^{\vartheta}(\mathbf{q}, G)$ can be further analyzed by computing the partition function [13]

$$\mathscr{Z}_{\vartheta} = \sum_{G\in\mathcal{G}_n}\int_{[0,\bar{q}]^n} e^{\vartheta\Phi(\mathbf{q},G)}d\mathbf{q}, \tag{52}$$

[12] Mattsson and Weibull [2002] provide a motivation from boundedly rational choices with implementation costs.

[13] See Park and Newman [2004] for an excellent discussion in the context of exponential random graphs.

so that we can write $\mu^{\vartheta}(\mathbf{q}, G) = e^{\vartheta\Phi(\mathbf{q},G)} / \mathscr{Z}_{\vartheta}$. Observe that the potential can be written as

$$\Phi(\mathbf{q}, G) = \underbrace{\sum_{i=1}^{n} \left(a - \bar{c} - \nu q_i - \frac{b}{2} \sum_{j \neq i} q_j \right) q_i}_{\psi(\mathbf{q})} + \sum_{i=1}^{n} \sum_{j=i+1}^{n} a_{ij} \underbrace{(\rho q_i q_j - \zeta)}_{\sigma_{ij}}.$$

We then have that

$$e^{\vartheta\Phi(\mathbf{q},G)} = e^{\vartheta\psi(\mathbf{q})} e^{\vartheta \sum_{i<j}^{n} a_{ij}\sigma_{ij}}.$$

Observe that only the second factor in the above expression is network dependent. We then can use the fact that

$$\sum_{G \in \mathcal{G}_n} e^{\vartheta \sum_{i<j}^{n} a_{ij}\sigma_{ij}} = \prod_{i<j} \left(1 + e^{\vartheta\sigma_{ij}}\right)$$

to obtain

$$\begin{aligned} \sum_{G \in \mathcal{G}_n} e^{\vartheta\Phi(\mathbf{q},G)} &= e^{\vartheta\psi(\mathbf{q})} \prod_{i<j} \left(1 + e^{\vartheta\sigma_{ij}}\right) \\ &= \prod_{i=1}^{n} e^{\vartheta\left(a - \bar{c} - \nu q_i - \frac{b}{2}\sum_{j \neq i} q_j\right) q_i} \prod_{i<j} \left(1 + e^{\vartheta(\rho q_i q_j - \zeta)}\right). \end{aligned}$$

We can use this expression to compute the marginal distribution

$$\mu^{\vartheta}(\mathbf{q}) = \frac{1}{\mathscr{Z}_{\vartheta}} \sum_{G \in \mathcal{G}_n} e^{\vartheta\Phi(\mathbf{q},G)} = \frac{1}{\mathscr{Z}_n^{\vartheta}} \prod_{i=1}^{n} e^{\vartheta\left(a - \bar{c} - \nu q_i - \frac{b}{2}\sum_{j \neq i} q_j\right) q_i} \prod_{i<j} \left(1 + e^{\vartheta(\rho q_i q_j - \zeta)}\right).$$

Moreover, it allows us to compute the probability of observing a network G given a specified output distribution $\mathbf{q}$.[14]

Proposition 5. *The probability of observing a network $G \in \mathcal{G}_n$, given an output distribution $\mathbf{q} \in [0, \bar{q}]^n$ is determined by conditional distribution*

$$\mu^{\vartheta}(G|\mathbf{q}) = \prod_{i<j} \frac{e^{\vartheta a_{ij}(\rho q_i q_j - \zeta)}}{1 + e^{\vartheta(\rho q_i q_j - \zeta)}}, \tag{53}$$

which is equivalent to the probability of observing an inhomogeneous random graph with link

[14]For a discussion of inhomogeneous random graphs see Bollobás et al. [2001]; Van Der Hofstad [2009] and the "hidden variables" model studied in Boguñá and Pastor-Satorras [2003]. Observe that the complementary problem of determining the distribution of random variables that depend only on their neighbors in a given network G is associated with a *Markov random field*, whose distribution is a *Gibbs measure* (by the Hammersley-Clifford theorem), and can be decomposed into a sum over all cliques in G (Besag [1974] and Kolaczyk [2009, Chap. 8] as well as Rue and Held [2005]).

probability

$$p_{ij} = \frac{e^{\vartheta(\rho q_i q_j - \zeta)}}{1 + e^{\vartheta(\rho q_i q_j - \zeta)}}. \tag{54}$$

PROOF OF PROPOSITION 5. The conditional distribution is given by

$$\begin{aligned}
\mu^{\vartheta}(G|\mathbf{q}) = \frac{\mu^{\vartheta}(\mathbf{q}, G)}{\mu^{\vartheta}(\mathbf{q})} &= \frac{e^{\vartheta\Phi(\mathbf{q},G)}}{\prod_{i=1}^{n} e^{\vartheta\left(a-\bar{c}-\nu q_i - \frac{b}{2}\sum_{j\neq i} q_j\right)q_i} \prod_{i<j}\left(1 + e^{\vartheta(\rho q_i q_j - \zeta)}\right)} \\
&= \frac{e^{\vartheta \sum_{i<j}^{n} a_{ij}(\rho q_i q_j - \zeta)}}{\prod_{i<j}\left(1 + e^{\vartheta(\rho q_i q_j - \zeta)}\right)} \\
&= \prod_{i<j} \frac{e^{\vartheta a_{ij}(\rho q_i q_j - \zeta)}}{1 + e^{\vartheta(\rho q_i q_j - \zeta)}} \\
&= \prod_{i<j} \left(\frac{e^{\vartheta(\rho q_i q_j - \zeta)}}{1 + e^{\vartheta(\rho q_i q_j - \zeta)}}\right)^{a_{ij}} \left(1 - \frac{e^{\vartheta(\rho q_i q_j - \zeta)}}{1 + e^{\vartheta(\rho q_i q_j - \zeta)}}\right)^{1-a_{ij}} \\
&= \prod_{i<j} p_{ij}^{a_{ij}} \left(1 - p_{ij}\right)^{1-a_{ij}}.
\end{aligned}$$

□

Proposition 5 has a number of important implications. Numerous empirical studies have shown that the distribution of output levels among firms tends to follow a power-law distribution (Zipf's law) [Axtell, 2001; Gabaix, 1999; Growiec et al., 2008; Stanley et al., 1996]. If the output levels q_i and q_j in the link probability of Equation (54) are distributed according to a power-law, then we obtain the so called *fitness model* analyzed in Boguñá and Pastor-Satorras [2003]; Caldarelli et al. [2002]. This models also refer to random threshold graphs [Diaconis et al., 2008; Ide and Konno, 2007; Ide et al., 2010]. It has been shown that this model produces a number of interesting characteristics, such as a power-law degree distribution. This distribution has been documented in various empirical studies of R&D networks [Gay and Dousset, 2005; Powell et al., 2005].

A special case is one in which all firms produce at the same fixed output level $q_i = q_0 \leq \bar{q}$ for all $i = 1, \ldots, n$ and there are no substitutability effects, $b = 0$.

Proposition 6. *Assume that all output levels are fixed and identically given by $q_i = q_0$ with $q_0 \in [0, \bar{q}]$ for all $i = 1, \ldots, n$ and there are no substitutability effects, $b = 0$. Then the stochastically stable network is given by the complete graph K_n if $\rho q_0^2 > \zeta$ and it is given by the empty graph if $\rho q_0^2 < \zeta$.*

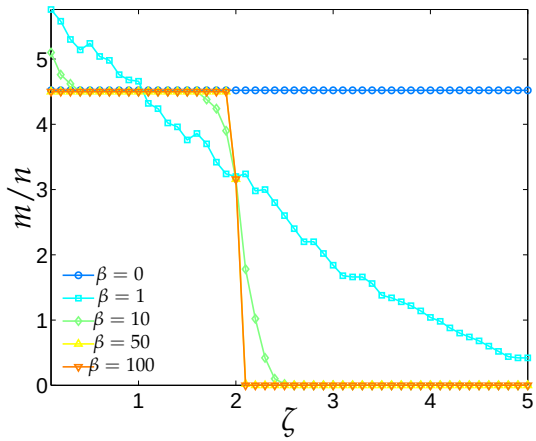

Figure 2: The degree $\bar{d}^+ = \bar{d}^- = m/n$ as a function of the linking cost ζ for a fixed, homogeneous output distribution $q_i = 1$ for all $i = 1, \ldots, n$ with $b = 0$, $\rho = 2$ and $n = 10$. The critical linking cost is $\zeta^* = \rho q_0^2 = 2$.

PROOF OF PROPOSITION 6. When $b = 0$ then we can write the potential function as

$$\Phi(\mathbf{q}, G) = \sum_{i=1}^{n} (n\eta q_i - \nu q_i^2) + \frac{\rho}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} q_i q_j - \zeta m,$$

where we have used the fact that the number of links in G can be written as $m = \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} d_i^- = \frac{1}{2} \sum_{i=1}^{n} d_i = \frac{1}{2} \mathbf{u}^\top \mathbf{A} \mathbf{u}$. For $\mathbf{q} = q_0 \mathbf{u}$ we can write this as

$$\Phi(\mathbf{q}, G) = n^2 \eta q_0 - \nu n q_0^2 + \frac{1}{2} (\rho q_0^2 - \zeta) \mathbf{u}^\top \mathbf{A} \mathbf{u}.$$

From this expression we see that $\Phi(\mathbf{q}, G)$ is maximized for $G = K_n$ if $\rho q_0^2 > \zeta$ and $G = \bar{K}_n$ if $\rho q_0^2 < \zeta$. □

The phase transition from the empty to the complete graph that occurs at $\zeta^* = \rho q_0^2$ is shown in Figure 2 for different values of ζ and β for $n = 10$ nodes.

Proposition 7. *Assume that the output levels are fixed and given by $q_i \in [0, \bar{q}]$ for all $i = 1, \ldots, n$ and there are no substitutability effects, $b = 0$. Then the stochastically stable network is given by a nested split graph with adjacency matrix $\mathbf{A}$ whose $1 \leq i, j \leq n$ elements are given by*

$$a_{ij} = \begin{cases} 1, & \text{if } \rho q_i q_j > \zeta, \\ 0, & \text{if } \rho q_i q_j < \zeta. \end{cases}$$

PROOF OF PROPOSITION 7. When $b = 0$ and the output levels are fixed and given by $q_i \in [0, \bar{q}]$ for all $i = 1, \ldots, n$ then we can write the potential as

$$\Phi(\mathbf{q}, G) = \sum_{i=1}^{n} (n\eta q_i - \nu q_i^2) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\rho q_i q_j - \zeta).$$

The second term in the above expression for $\Phi(\mathbf{q}, G)$ is a sum over positive terms with $a_{ij} = 1$ if $\rho q_i q_j > \zeta$ and negative otherwise. Hence, $\Phi(\mathbf{q}, G)$ is maximized if $a_{ij} = 1$ for all $\rho q_i q_j > \zeta$ and $a_{ij} = 0$ for all $\rho q_i q_j < \zeta$. □

The marginal distribution for $G \in \mathcal{G}_n$ can be obtained from

$$\mu^{\vartheta}(G) = \frac{1}{\mathscr{Z}_{\vartheta}} \int_{[0,\bar{q}]^n} e^{\vartheta \Phi(G,\mathbf{q})} d\mathbf{q}^n.$$

Using a Laplace expansion [Wong, 2001] around the equilibrium values $\mathbf{q}^*$ of Equation (28) we can write

$$\mu^{\vartheta}(G) = \frac{1}{\mathscr{Z}_{\vartheta}} \int_{[0,\bar{q}]^n} e^{\vartheta \Phi(G,\mathbf{q})} \approx \frac{1}{\mathscr{Z}_{\vartheta}} \left(\frac{\vartheta}{2\pi}\right)^{\frac{n}{2}} \left| \left(\frac{\partial^2 \Phi(G,\mathbf{q})}{\partial q_i \partial q_j} \right)_{\mathbf{q}=\mathbf{q}^*} \right|^{-\frac{1}{2}} e^{\vartheta(\Phi(G,\mathbf{q}^*))},$$

and the conditional distribution is given by

$$\mu^{\vartheta}(\mathbf{q}|G) = \frac{\mu^{\vartheta}(G,\mathbf{q})}{\mu^{\vartheta}(G)} \approx \left(\frac{\vartheta}{2\pi}\right)^{\frac{n}{2}} \left| \left(\frac{\partial^2 \Phi(G,\mathbf{q})}{\partial q_i \partial q_j} \right)_{\mathbf{q}=\mathbf{q}^*} \right|^{-\frac{1}{2}} e^{\vartheta(\Phi(G,\mathbf{q}) - \Phi(G,\mathbf{q}^*))}.$$

We next analyze the partition sum $\mathscr{Z}_{\vartheta}$ in more detail. Note that

$$\mathscr{Z}_{\vartheta} = \int_{[0,\bar{q}]^n} \prod_{i=1}^{n} e^{\vartheta\left(a - \bar{c} - \nu q_i - \frac{b}{2}\sum_{j \neq i} q_j\right) q_i} \prod_{i<j} \left(1 + e^{\vartheta(\rho q_i q_j - \zeta)}\right) d\mathbf{q}.$$

Next, we introduce the Hamiltonian defined by

$$\mathscr{H}(\mathbf{q}) \equiv \sum_{i=1}^{n} \left(n\eta q_i - \nu q_i^2 + \sum_{j>i} \left(\frac{1}{\vartheta} \ln\left(1 + e^{\vartheta(\rho q_i q_j - \zeta)}\right) - b q_i q_j, \right) \right)$$

so that $\sum_{G \in \mathcal{G}_n} e^{\Phi(\mathbf{q},G)} = e^{\mathscr{H}(\mathbf{q})}$. Then we can write the partition function as

$$\mathscr{Z}_{\vartheta} = \int_{[0,\bar{q}]^n} e^{\vartheta \mathscr{H}(\mathbf{q})} d\mathbf{q}.$$

In the following we make a Laplace approximation of the partition function as follows [Wong, 2001]

$$\mathscr{Z}_{\vartheta} \approx \left(\frac{2\pi}{\vartheta}\right)^{\frac{n}{2}} \left| \left(\frac{\partial^2 \mathscr{H}}{\partial q_i \partial q_j} \right)_{q_i = q^*} \right|^{-\frac{1}{2}} e^{\mathscr{H}(\mathbf{q}^*)}, \tag{55}$$

where $\mathbf{q}^* = \text{argmax}_{\mathbf{q}\in[0,\bar{q}]^n} \mathscr{H}(\mathbf{q})$, and the Hessian $\frac{\partial^2 \mathscr{H}}{\partial q_i \partial q_j}$ for $1 \leq i, j \leq n$. We have that

$$\frac{\partial \mathscr{H}(\mathbf{q})}{\partial q_i} = n\eta - 2\nu q_i + \sum_{j\neq i} \left(\frac{\rho}{2} \left(1 + \tanh\left(\frac{\vartheta}{2} \left(\rho q_i q_j - \zeta \right) \right) \right) - b \right) q_j.$$

For large n, the first order conditions imply that

$$\eta = \frac{1}{n} \sum_{j\neq i} \left(b - \frac{\rho}{2} \left(1 + \tanh\left(\frac{\vartheta}{2} \left(\rho q_i q_j - \zeta \right) \right) \right) \right) q_j.$$

Assuming symmetry $q_i = q$ for all $i = 1, \ldots, n$ we get

$$bq - \eta = \frac{\rho}{2} \left(1 + \tanh\left(\frac{\vartheta}{2} \left(\rho q^2 - \zeta \right) \right) \right) q. \tag{56}$$

In the limit of $\vartheta \to \infty$ we obtain from the FOC of Equation (56) that

$$bq - \eta = \begin{cases} \rho q, & \text{if } \zeta < \rho q^2, \\ 0, & \text{if } \rho q^2 < \zeta. \end{cases}$$

This shows that the right hand side of Equation (56) has a point of discontinuity at $\sqrt{\frac{\zeta}{\rho}}$ (cf. Figure 3). It then follows that, in the limit of $\vartheta \to \infty$ (the stochastically stable equilibrium), we have

$$q = \begin{cases} \frac{\eta}{b-\rho}, & \text{if } \zeta < \frac{\rho\eta^2}{b^2}, \\ \left\{ \frac{\eta}{b-\rho}, \frac{\eta}{\rho} \right\}, & \text{if } \frac{\rho\eta^2}{b^2} < \zeta < \frac{\rho\eta^2}{(b-\rho)^2}, \\ \frac{\eta}{\rho}, & \text{if } \frac{\rho\eta^2}{(b-\rho)^2} < \zeta, \end{cases}$$

which is increasing in ρ and η, and decreasing in ζ and b (cf. Figures 3 and 4).

We further have that

$$\frac{\partial^2 \mathscr{H}}{\partial q_i \partial q_j} = \begin{cases} -2\nu + \frac{\vartheta\rho^2}{4} \sum_{j\neq i} \left(1 - \tanh\left(\frac{\vartheta}{2} \left(\rho q_i q_j - \zeta \right) \right)^2 \right) q_j^2, & \text{if } i = j, \\ \frac{\rho}{2} \left(1 + \tanh\left(\frac{\vartheta}{2} \left(\rho q_i q_j - \zeta \right) \right) \right) \left(1 + \vartheta q_i q_j \frac{\rho}{2} \tanh\left(\frac{\vartheta}{2} \left(\rho q_i q_j - \zeta \right) \right) \right) - b, & \text{if } i \neq j. \end{cases}$$

In the symmetric equilibrium $q_i = q$ for all $i = 1, \ldots, n$ this is

$$\frac{\partial^2 \mathscr{H}}{\partial q_i \partial q_j} = \begin{cases} \vartheta(n-1)(bq - \eta)(\eta - (b-\rho)q), & \text{if } i = j, \\ \vartheta(bq - \eta)(\eta - (b-\rho)q) - \frac{\eta}{q}, & \text{if } i \neq j. \end{cases}$$

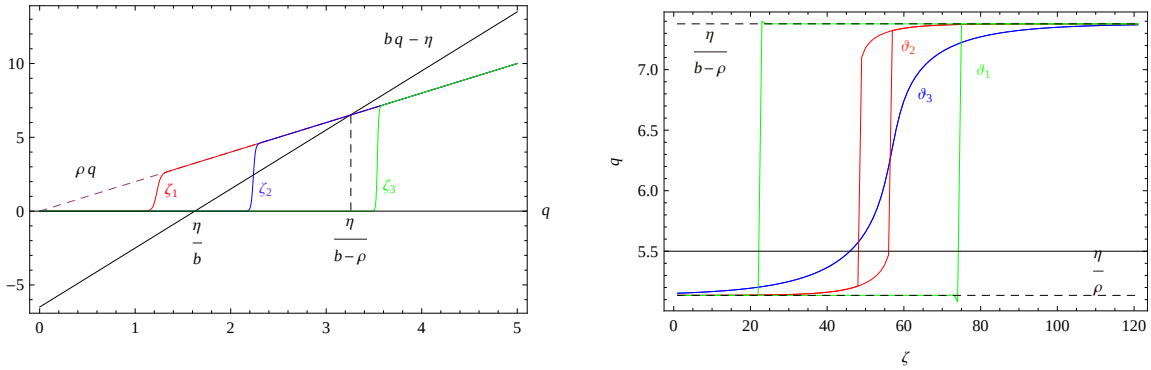

Figure 3: (Left panel) The right hand side of Equation (56) for different values of $\zeta_1 = 25$, $\zeta_2 = 10$, $\zeta_3 = 3$ and $b = 4$, $\rho = 2$, $\eta = 6.5$ and $\vartheta = 10$. (Right panel) The values of q solving Equation (56) for different values of ζ with $b = 1.48$, $\rho = 0.45$ and $\vartheta_1 = 49.5$, $\vartheta_2 = 0.495$, $\vartheta_3 = 0.2475$.

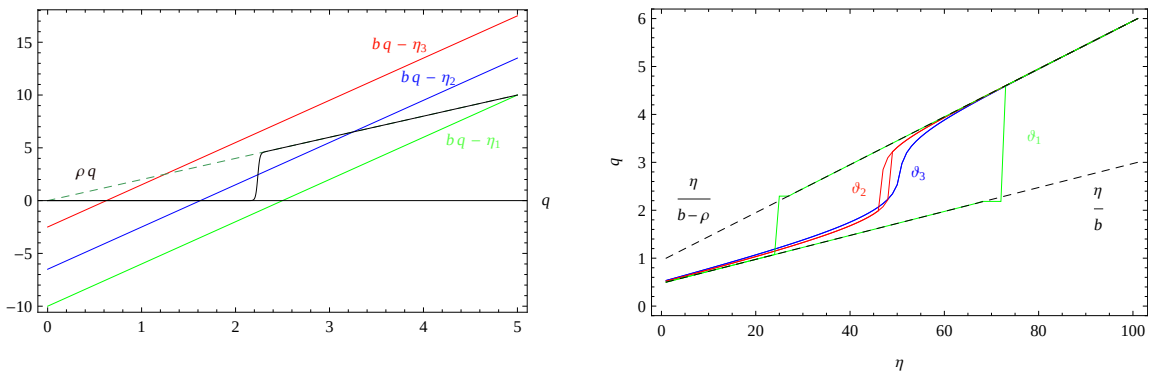

Figure 4: (Left panel) The right hand side of Equation (56) for different values of $\eta_1 = 2.5$, $\eta_2 = 6.5$, $\eta_3 = 10$ and $b = 4$, $\rho = 2$, $\zeta = 10$ and $\vartheta = 10$. (Right panel) The values of q solving Equation (56) for different values of η with $b = 4$, $\rho = 2$ and $\vartheta_1 = 10$, $\vartheta_2 = 0.26$, $\vartheta_3 = 0.2$.

Using the fact that

$$\begin{vmatrix} a & b & b & \dots \\ b & a & b & \dots \\ b & b & a & \\ \vdots & \vdots & & \ddots \end{vmatrix} = (a-b)^{n-1}(a+(n-1)b),$$

which is a special case of a circulant matrix and the determinant follows from the general formula, we obtain

$$\begin{aligned} \left| \left(\frac{\partial^2 \mathscr{H}}{\partial q_i \partial q_j} \right)_{q_i = q} \right| &= \frac{1}{q^n} \left(\vartheta(n-2)(bq-\eta)(\eta-(b-\rho)q)q+\eta \right)^{n-1} \\ &\quad \times \left(2\vartheta(n-1)(bq-\eta)(\eta-(b-\rho)q)q-(n-1)\eta \right). \end{aligned}$$

In the symmetric case $q_i = q$ for all $i = 1, \dots, n$ the Laplace approximation of Equation (55) can be written as

$$\mathscr{Z}_\vartheta \approx \left(\frac{2\pi}{\vartheta} \right)^{\frac{n}{2}} \left| \left(\frac{\partial^2 \mathscr{H}}{\partial q_i \partial q_j} \right)_{q_i = q^*} \right|^{-\frac{1}{2}} e^{\vartheta n \mathscr{H}(q^*)},$$

where

$$\begin{aligned} \mathscr{H}(q) &\equiv n\eta q - \nu q^2 + (n-1)\left(\frac{1}{\vartheta} \ln\left(1 + e^{\vartheta(\rho q^2 - \zeta)} \right) - bq^2 \right) \\ &\approx n^2 \left(\eta q + \frac{1}{\vartheta} \ln\left(1 + e^{\vartheta(\rho q^2 - \zeta)} \right) - bq^2 \right), \end{aligned}$$

for large n. We also introduce the free energy $\mathscr{F}_\vartheta \equiv -\ln \mathscr{Z}_\vartheta$, which allows us to write the expected number of links as follows

$$\begin{aligned} \mathbb{E}_\vartheta(m) &= \sum_{G \in \mathcal{G}_n} \int_{[0,\bar{q}]^n} m\mu^\vartheta(\mathbf{q}, G) d\mathbf{q} = \frac{1}{\mathscr{Z}_\vartheta} \sum_{G \in \mathcal{G}_n} \int_{[0,\bar{q}]^n} \underbrace{m e^{\vartheta \Phi(\mathbf{q}, G)}}_{-\frac{1}{\vartheta}\frac{\partial}{\partial \zeta} e^{\vartheta \Phi(\mathbf{q}, G)}} d\mathbf{q} \\ &= -\frac{1}{\vartheta} \frac{1}{\mathscr{Z}_\vartheta} \frac{\partial \mathscr{Z}_\vartheta}{\partial \zeta} = \frac{1}{\vartheta} \frac{\partial \mathscr{F}_\vartheta}{\partial \zeta}. \end{aligned}$$

From the Laplace approximation of the partition function we find that

$$\frac{\partial \mathscr{F}_\vartheta}{\partial \zeta} \approx -\vartheta \frac{\partial \mathscr{H}(\mathbf{q})}{\partial \zeta} + \frac{1}{2} \mathrm{tr} \left(\left(\frac{\partial^2 \mathscr{H}}{\partial q_i \partial q_j} \right)^{-1} \frac{\partial}{\partial \zeta} \left(\frac{\partial^2 \mathscr{H}}{\partial q_i \partial q_j} \right) \right).$$

We further have that

$$\frac{\partial \mathscr{H}}{\partial \zeta} = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j>i}^{n}\left(1+\tanh\left(\frac{\vartheta}{2}\left(\rho q_i q_j - \zeta\right)\right)\right),$$

and in the symmetric equilibrium this is

$$\left(\frac{\partial \mathscr{H}}{\partial \zeta}\right)_{q_i=q} = -\frac{n(n-1)}{2}\left(1+\tanh\left(\frac{\vartheta}{2}\left(\rho q^2 - \zeta\right)\right)\right).$$

Moreover we find that

$$\frac{\partial^2 \mathscr{H}}{\partial q_i^2} = \frac{\vartheta^2\rho^2}{4}\sum_{j\neq i}^{n}\tanh\left(\frac{\vartheta}{2}\left(\rho q_i q_j - \zeta\right)\right)\left(1-\tanh\left(\frac{\vartheta}{2}\left(\rho q_i q_j - \zeta\right)\right)^2\right)q_j^2,$$

and

$$\frac{\partial^2 \mathscr{H}}{\partial q_i \partial q_j} = \frac{\vartheta\rho}{4}\left(1-\tanh\left(\frac{\vartheta}{2}\left(\rho q_i q_j - \zeta\right)\right)^2\right)\left(\vartheta\rho q_i q_j \tanh\left(\frac{\vartheta}{2}\left(\rho q_i q_j - \zeta\right)\right)-1\right).$$

Assuming symmetry this is

$$\left(\frac{\partial^2 \mathscr{H}}{\partial q_i^2}\right)_{q_i=q} = \frac{\vartheta^2(n-1)}{\rho q}\left(2(bq-\eta)-\rho q\right)(bq-\eta)(\eta-(b-\rho)q),$$

and

$$\left(\frac{\partial^2 \mathscr{H}}{\partial q_i \partial q_j}\right)_{q_i=q} = \frac{\vartheta}{\rho q^2}(bq-\eta)(\eta-(b-\rho)q)\left(2\vartheta q(bq-\eta)-\vartheta\rho q^2-1\right).$$

After some simplifications we then can write the expected number of links as follows

$$\mathbb{E}_\vartheta(m) \approx n(n-1)\frac{bq-\eta}{\rho q} + \frac{n}{2}\vartheta\frac{(2b-\rho)q-2\eta}{\rho q} = \frac{n(n-1)}{2}\left(1+\tanh\left(\frac{\vartheta}{2}\left(\rho q^2-\zeta\right)\right)\right)+O(n),$$

where q derives from Equation (56). Hence, the expected number of links is increasing in ρ, q and η, and decreasing in ζ and b (by reducing the equilibrium quantity q). Note that the above expression becomes exact as n becomes large.

With the partition function $\mathscr{Z}_\theta$ in Equation (55) we are able to compute the marginal

$$\mu^\vartheta(\mathbf{q}) = \sum_{G\in\mathcal{G}_n}\mu^\vartheta(G,\mathbf{q}) = \frac{1}{\mathscr{Z}_\theta}\prod_{i=1}^{n}e^{\vartheta\left(a-\bar{c}-\nu q_i-\frac{b}{2}\sum_{j\neq i}q_j\right)q_i}\prod_{i<j}\left(1+e^{\vartheta(\rho q_i q_j-\zeta)}\right),$$

and the joint distribution can then be written as

$$\mu^{\vartheta}(G, \mathbf{q}) = \mu^{\vartheta}(G|\mathbf{q})\mu^{\vartheta}(\mathbf{q}),$$

where the condition distribution $\mu^{\vartheta}(G|\mathbf{q})$ is given in Equation (53).

3.3. Endogenous Networks with Heterogeneous Spillovers among Firms

In this section we allow for unobserved heterogeneity among firms in terms of their technological abilities. Let θ_i denote the technology level of firm i. We assume that technology spillovers between firms are imperfect, and depend on their relative technological difference. We introduce a function $f : [0,1] \times [0,1] \to [0,1]$ capturing the potential technology transfer between any pairs of firms. Let θ_i and θ_j be the technology levels of firms i and j. A simple choice for the function f would then be

$$f(\theta_i, \theta_j) = |\theta_i - \theta_j|.$$

Other functional forms have been suggested in the literature [see e.g. Baum et al., 2009],[15] such as

$$f(\theta_i, \theta_j) = a_1|\theta_i - \theta_j| - a_2|\theta_i - \theta_j|^2,$$

with $a_1, a_2 \geq 0$. Let us denote by $f_{ij} \equiv f(\theta_i, \theta_j)$. Given the spillover function f, the marginal cost of production of a firm i becomes

$$c_i = \bar{c} - \alpha e_i - \beta \sum_{j=1}^{n} a_{ij} f_{ij} e_j$$

and profits of firm i are given by

$$\pi_i = (a - \bar{c})q_i - q_i^2 - bq_i \sum_{j \neq i} q_j + \alpha q_i e_i + \beta q_i \sum_{j=1}^{n} a_{ij} f_{ij} e_j - \gamma e_i^2 - \zeta d_i.$$

The FOC with respect to effort e_i is given by

$$\frac{\partial \pi_i}{\partial e_i} = \alpha q_i - 2\gamma e_i = 0,$$

from which it follows that

$$e_i = \frac{\alpha}{2\gamma} q_i = \lambda q_i.$$

[15] See also Cowan and Jonard [2008, 2009]; Jonard et al. [2009]; Müller et al. [2009].

Inserting into profits yields

$$\pi_i = (a - \bar{c})q_i - (1 - \lambda\alpha + \lambda^2\gamma)q_i^2 - bq_i \sum_{j \neq i} q_j + \lambda\beta q_i \sum_{j=1}^{n} a_{ij} f_{ij} q_j - \zeta d_i$$
$$= (a - \bar{c})q_i - \nu q_i^2 - bq_i \sum_{j \neq i} q_j + \rho q_i \sum_{j=1}^{n} a_{ij} f_{ij} q_j - \zeta d_i.$$

We can then obtain a potential function given by

$$\Phi(\mathbf{q}, G) = \sum_{i=1}^{n}((a - \bar{c})q_i - \nu q_i^2) - \frac{b}{2}\sum_{i=1}^{n} q_i \sum_{j \neq i} q_j + \sum_{i=1}^{n} q_i \sum_{j=1}^{n} a_{ij} f_{ij} q_j - \zeta m.$$

The stationary distribution is given by

$$\mu^{\vartheta}(\mathbf{q}, G) = \frac{e^{\vartheta\Phi(\mathbf{q},G)}}{\sum_{H \in \mathcal{G}_n} \int_{[0,\bar{q}]^n} e^{\vartheta\Phi(\mathbf{s},H)} d\mathbf{s}}.$$

The probability of observing a network $G \in \mathcal{G}_n$, given an output distribution $\mathbf{q} \in [0, \bar{q}]^n$ is determined by conditional distribution

$$\mu^{\vartheta}(G|\mathbf{q}) = \prod_{i<j} \frac{e^{\vartheta a_{ij}(\rho f_{ij} q_i q_j - \zeta)}}{1 + e^{\vartheta(\rho f_{ij} q_i q_j - \zeta)}}, \tag{57}$$

which is equivalent to the probability of observing an inhomogeneous random graph with link probability

$$p_{ij} = \frac{e^{\vartheta(\rho f_{ij} q_i q_j - \zeta)}}{1 + e^{\vartheta(\rho f_{ij} q_i q_j - \zeta)}}. \tag{58}$$

Note that an inhomogeneous random graph with a link probability similar to the one in Equation (58) has been analyzed in Boguna et al. [2004]. The authors show that if the technology levels are drawn from a multivariate uniform distribution a number of network characteristics can be computed which closely reproduce the empirically observed patterns of R&D networks.

3.4. Endogenous Networks with Heterogeneous Marginal Costs

We consider ex ante heterogeneity among firms in the variable cost $\bar{c}_i \geq 0$ [see also Banerjee and Duflo, 2005], expressing their different technological and organizational

capabilities.[16] The marginal cost of production of firm $i \in N$ is then given by

$$c_i(\mathbf{e}, G) = \bar{c}_i - \alpha e_i - \beta \sum_{j=1}^{n} a_{ij} e_j, \tag{59}$$

where $e_i \in [0, \bar{e}_i]$ and $\alpha, \beta \in [0, 1]$. Requiring that $c_i \geq 0$ we must have that $\bar{c}_i \geq \sum_{j=1}^{n} e_j = n\bar{e}$ for all $i = 1, \ldots, n$. Hence, $\bar{c}_i$ is $O(n)$. Similarly, as in the previous sections, the first-order conditions for efforts imply that $e_i = \max\{\frac{\alpha}{2\gamma} q_i, \bar{e}_i\}$. The non-negativity of marginal cost in the case of an interior equilibrium then requires that $q_i \leq \frac{2\gamma}{\alpha} \bar{e}_i$ for all $i = 1, \ldots, n$. We further assume that $a > \max_{1 \leq i \leq n}\{\bar{c}_i\}$. We denote by $\lambda = \frac{\alpha}{2\gamma}$. Profits of firm i from Equation (8) then become

$$\pi_i = \underbrace{(a - \bar{c}_i)}_{n\eta_i} q_i - b \sum_{j \neq i}^{n} q_i q_j + \alpha q_i e_i + \beta \sum_{j=1}^{n} a_{ij} q_i e_j - \gamma e_i^2 - \zeta d_i. \tag{60}$$

Using the fact that in the interior equilibrium $e_i = \lambda q_i$, we can write firm i's profit as

$$\pi_i = n\eta_i q_i - \nu q_i^2 - b q_i \sum_{j \neq i} q_j + \rho q_i \sum_{j=1}^{n} a_{ij} q_j - \zeta d_i.$$

We first compute the equilibrium quantities for a given network G. The FOC can be written as

$$\frac{\partial \pi_i}{\partial q_i} = n\eta_i - 2\nu q_i - b \sum_{j \neq i} q_j + \rho \sum_{j=1}^{n} a_{ij} q_j = n\eta_i - (2\nu - b) q_i - b\|\mathbf{q}\| + \rho \sum_{j=1}^{n} a_{ij} q_j = 0.$$

Let $\mathbf{J} \equiv (\eta_1, \ldots, \eta_n)^\top$. Then in vector-matrix notation this is

$$n\mathbf{J} = ((2\nu - b)\mathbf{I}_n - \rho \mathbf{A})\mathbf{q} + b\mathbf{u}\|q\|.$$

Denoting by $\bar{} \equiv \frac{n}{2\nu - b}\mathbf{J}$, $\phi \equiv \frac{\rho}{2\nu - b}$ and $\kappa \equiv \frac{b}{2\nu - b}$ this is

$$\mathbf{q} = (\mathbf{I} - \phi \mathbf{A})^{-1}(\bar{} - \kappa\|\mathbf{q}\|)\mathbf{u}.$$

Following Calvó-Armengol et al. [2009] we define the $\boldsymbol{\mu}$-weighted Bonacich centrality

[16] Blundell et al. [1995] argued that because the main source of unobserved heterogeneity in models of innovation lies in the different knowledge stocks with which firms enter a sample, a variable that approximates the build-up of firm knowledge at the time of entering the sample is a particularly good control for unobserved heterogeneity.

as

$$\mathbf{b}\text{-}(G,\phi) = (\mathbf{I}_n - \phi\mathbf{A})^{-1}\mathbf{-} = \sum_{k=0}^{\infty} \phi^k \mathbf{A}^k\mathbf{-}, \tag{61}$$

where suitable conditions have to be imposed on the vector $\boldsymbol{\mu}$, the parameter ϕ and the eigenvalue $\lambda_{\mathrm{PF}}(G)$. The Bonacich centrality is then simply given by $\mathbf{b}(G,\phi) = \mathbf{b}_{\mathbf{u}}(G,\phi)$. Then we can write

$$\mathbf{q} = \mathbf{b}\text{-}(G,\phi) - \kappa\|\mathbf{q}\|\mathbf{b}_{\mathbf{u}}(G,\phi).$$

Multiplying from the left with $\mathbf{u}^\top$ gives

$$\mathbf{u}^\top\mathbf{q} = \|\mathbf{q}\| = \|\mathbf{b}\text{-}(G,\phi)\| - \nu\|\mathbf{q}\|\|\mathbf{b}_{\mathbf{u}}(G,\phi)\|.$$

from which we get

$$\|\mathbf{q}\| = \frac{\|\mathbf{b}\text{-}(G,\phi)\|}{1+\kappa\|\mathbf{b}_{\mathbf{u}}(G,\phi)\|}.$$

It follows that equilibrium quantity can be written as

$$\mathbf{q} = \mathbf{b}\text{-}(G,\phi) - \frac{\kappa\|\mathbf{b}\text{-}(G,\phi)\|}{1+\kappa\|\mathbf{b}_{\mathbf{u}}(G,\phi)\|}\mathbf{b}_{\mathbf{u}}(G,\phi).$$

Note that the Hamiltonian in the partition function $\mathscr{Z}_\vartheta = \int_{[0,\bar{q}_1]}\int_{[0,\bar{q}_2]}\cdots\int_{[0,\bar{q}_n]} e^{\vartheta\mathscr{H}(\mathbf{q})}d\mathbf{q}$ in the case of heterogeneous marginal costs is given by

$$\mathscr{H}(\mathbf{q}) = \sum_{i=1}^{n}\left(n\eta_i q_i - \nu q_i^2 + \sum_{j>i}\left(\frac{1}{\vartheta}\ln\left(1+e^{\vartheta(\rho q_i q_j - \zeta)}\right) - b q_i q_j\right)\right).$$

When $\vartheta \to \infty$ we can write

$$\lim_{\vartheta\to\infty}\mathscr{H}(\mathbf{q}) = \sum_{i=1}^{n}\left(n\eta_i q_i - \nu q_i^2 + \sum_{j>i}(\rho q_i q_j - \zeta)\,\mathbb{1}_{\{\rho q_i q_j > \zeta\}} - b q_i q_j\right).$$

From the maximization of this expression we find that if the capacity constraints $\bar{q}_i$ are binding, then the stochastically stable state will be a threshold graph (nested split graph) in which a link ij is present if and only if $\bar{q}_{(i)}\bar{q}_{(j)} > \frac{\zeta}{\rho}$ and quantities are given by the ordered vector $(\bar{q}_{(1)}, \bar{q}_{(2)}, \ldots, \bar{q}_{(k)}, 0, \ldots, 0)$ where $k = \max\{1 \le j \le n : \bar{q}_{(1)}\bar{q}_{(j)} > \frac{\zeta}{\rho}\}$. In the case of finite ϑ we obtain a generalized threshold graph as they have been studied in Boguñá and Pastor-Satorras [2003]; Diaconis et al. [2008]; Ide et al. [2010, 2009]; Söderberg [2002].

4. Efficiency

For a given network G, social welfare $W(G)$ is given by the sum of consumer surplus and firms' profits. When firms compete in a homogeneous product oligopoly then social welfare is given by [Goyal and Moraga-Gonzalez, 2001][17]

$$\begin{aligned} W(\mathbf{q}, G) &= \frac{1}{2}\left(\sum_{i=1}^{n} q_i\right)^2 + \sum_{i=1}^{n} \pi_i(\mathbf{q}, G) \\ &= \frac{1}{2}\left(\sum_{i=1}^{n} q_i\right)^2 + \sum_{i=1}^{n}\left(\eta q_i - \nu q_i^2 - b\sum_{j\neq i} q_i q_j + \rho q_i \sum_{j=1}^{n} a_{ij} q_i q_j\right) - 2\zeta m. \end{aligned} \tag{62}$$

Note that welfare $W(\mathbf{q}, G)$ is related to the potential $\Phi(\mathbf{q}, G)$ as follows

$$W(\mathbf{q}, G) = \frac{1}{2}\left(\sum_{i=1}^{n} q_i\right)^2 - \sum_{i=1}^{n} q_i(\eta - \nu q_i) + 2\Phi(\mathbf{q}, G). \tag{63}$$

Hence, the states maximizing the potential $\Phi(\mathbf{q}, G)$ are not necessarily identical to the ones maximizing welfare $W(\mathbf{q}, G)$. However, the only network dependent part in $W(\mathbf{q}, G)$ is the potential function $\Phi(\mathbf{q}, G)$. For a given vector of outputs $\mathbf{q}$ the network that maximizes the potential is the threshold graph G where each link $ij \in G$ if and only if $\rho q_i q_j > \zeta$. Hence, we can write welfare reduced to this class of networks as follows

$$W(\mathbf{q}) = \eta \sum_{i=1}^{n} q_i + \frac{1-2b}{2}\left(\sum_{i=1}^{n} q_i\right)^2 - (\nu - b)\sum_{i=1}^{n} q_i^2 + \sum_{i=1}^{n}\sum_{j\neq i}(\rho q_i q_j - \zeta)\mathbb{1}_{\{\rho q_i q_j > \zeta\}}.$$

From the form of $W(\mathbf{q})$ we see that we can distinguish two types of firms: those that are connected, and those that are not. Let N_1 denote the set of the first, and N_2 the set of the latter. The FOC for $i \in N_1$ is given by

$$\frac{\partial W(\mathbf{q})}{\partial q_i} = \eta + (1-2b)(n_1 q_1 + n_2 q_2) - 2(\nu - b)q_1 + 2\rho(n_1 - 1)q_1 = 0,$$

[17] In the empirical paper by König et al. [2011] this welfare analysis is extended to account for R&D subsidies. Moreover, the authors characterize the firms that are most critical in terms of their contribution to the aggregate productivity of the economy.

network	q_1	q_2
empty graph	$q_1 = q_2 = 0$	$q_1 = q_2 = 0$
dominant group	$q_1 > 0$	$q_2 = 0$
dominant group	$q_1 > 0$	$q_2 > 0$
complete graph	$q_1 = q_2 > 0$	$q_1 = q_2 > 0$

Table 1: Summary of efficient networks and quantities.

while the FOC for $i \in N_2$ is

$$\frac{\partial W(\mathbf{q})}{\partial q_i} = \eta + (1-2b)(n_1 q_1 + n_2 q_2) - 2(\nu - b) q_2 = 0.$$

The FOCs for all firms in the same set are identical, so their quantities must be identical too. We denote by q_1 the optimal quantity level of the firms in N_1 and by q_2 the optimal quantity level of the firms in N_2. Moreover, let $n_1 = |N_1|$ and $n_2 = |N_2| = n - n_1$. For $0 \leq q_1, q_2 \leq \bar{q}$ we then we have that

$$\begin{aligned} q_1(n_1, n_2) &= \frac{\eta(b-\nu)}{(b-\nu)(2(\nu-b) + n(2b-1)) + (n_1 - 1)(2(\nu - b) + (2b-1)n_2)\rho}, \\ q_2(n_1, n_2) &= \frac{\eta(b - \nu + (n_1 - 1)\rho)}{(b-\nu)(2(\nu-b) + n(2b-1)) + (n_1 - 1)(2(\nu - b) + (2b-1)n_2)\rho}, \end{aligned} \tag{64}$$

and welfare can be written as as a function of $0 \leq n_1 \leq n$ (since $n_2 = n - n_1$) as follows

$$\begin{aligned} W(n_1, n_2(n_1)) = \eta(n_1 q_1 + n_2 q_2) + \frac{1-2b}{2}(n_1 q_1 + n_2 q_2)^2 \\ - (\nu - b)(n_1 q_1^2 + n_2 q_2^2) + n_1(n_1 - 1)(\rho q_1^2 - \zeta). \end{aligned} \tag{65}$$

The above discussion can be summarized in the following proposition (see also Table 1).

Proposition 8. *The efficient network G^* maximizing welfare is either (i) the empty network, (ii) the complete network, or (iii) has the dominant group architecture. The optimal quantities $\mathbf{q}^* = (q_1, \ldots, q_2, \ldots)$ are given by Equation (64) subject to $0 \leq q_1, q_2 \leq \bar{q}$, where the optimal size n_1 of the dominant group maximizes Equation (65) and $n_2 = n - n_1$.*

In the following we discuss two special cases. First, for $q_2 = 0$ we find that

$$W(q_1) = \eta n_1 q_1 + \frac{1-2b}{2} n_1^2 q_1^2 - (\nu - b) n_1^2 q_1^2 + n_1(n_1 - 1)(\rho q_1^2 - \zeta),$$

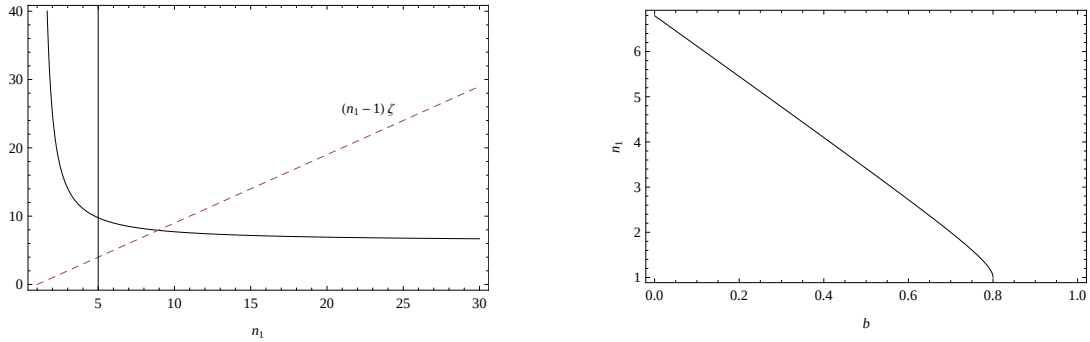

Figure 5: (Left panel) The optimal size n_1 solving Equation (66) when $q_2 = 0$ with $\rho = 0.3$ and $\nu = 0.8$. (Right panel) The optimal size n_1 for $\zeta = 0$ from Equation (67) with $\rho = 0.4$, $\nu = 0.5$ and $\eta = \sqrt{10}n$.

and from the FOC $\frac{\partial W(q_1)}{\partial n_1} = 0$ we obtain

$$q_1 = \frac{\eta}{(2(\nu-\rho)-1)n_1 + 2\rho}.$$

Inserting into welfare gives

$$W(n_1) = \frac{n_1}{2}\left(\frac{\eta^2}{2\rho + (2(\nu-\rho)-1)n_1} - 2(n_1-1)\zeta\right).$$

We take n_1 as a continuous variable, so that the FOC of $W(n_1)$ with respect to n_1 leads us to the condition

$$\frac{\eta^2\rho}{(2\rho + (2(\nu-\rho)-1)n_1)^2} = (n_1-1)\zeta. \tag{66}$$

The optimal size n_1 solving this equality is illustrated in Figure 5. Next, for $\zeta = 0$ we obtain

$$W(n_1) = \frac{\eta^2(((b-\nu)n + (n-n_1)(n_1-1)\rho}{2(b-\nu)(2b(n-1)-n+2\nu) + 2(n_1-1)(2b(n-n_1-1)-n-n_1+2\nu)\rho}.$$

Note that in this case $q_2 = 0$. From the FOC $\frac{\partial W(q_1)}{\partial n_1} = 0$ we get

$$n_1 = \frac{1}{\rho}\left(\rho + \nu - b + \sqrt{(\nu-b)(\nu-b+\rho)}\right). \tag{67}$$

The optimal size n_1 for $\zeta = 0$ is illustrated in Figure 5.

5. Future Work

Two important avenues are left for future work. First, it would be interesting to study entry and exit dynamics in the current framework. It has been argued that entry and exit play an important role in shaping the distribution of firm sizes [Acemoglu and Cao, 2010; Luttmer, 2007]. A promising approach seems to be a union of the model proposed in this paper and the one in Garlaschelli et al. [2007]. Second, it would be interesting to analyze the dynamics of technological change and convergence and their relation with firm and network dynamics in the current model. Such an extension could shed light on the coevolution of R&D networks and the knowledge portfolios of firms König and Montanari [2011]. Finally, an empirical application of the model to real-world R&D networks could help to shed light on the often significant differences between sectors and, in particular, why the biotech sector has witnessed a steady increase in the number of collaborations while other sectors have experienced a less sustained development.

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