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Working Paper

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Working Paper, No. 16

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Suggested Citation: Ewerhart, Christian (2013) : Cournot games with biconcave demand, Working Paper, No. 16, University of Zurich, Department of Economics, Zurich, <http://dx.doi.org/10.5167/uzh-51520>

This Version is available at:

<http://hdl.handle.net/10419/77479>

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University of
Zurich^{UZH}

University of Zurich
Department of Economics

Working Paper Series

ISSN 1664-7041 (print)
ISSN 1664-705X (online)

Working Paper No. 16

Cournot Games with Biconcave Demand

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Revised version, April 2013

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This version: April 18, 2013, 7:47

*) For useful comments and discussions, I would like to thank Rabah Amir, Luis Corchon, Paul Klemperer, Dan Kovenock, Koji Okuguchi, Burkhard Schipper, Armin Schmutzler, and Pierre von Mouche. Valuable advice has been received from an advisory editor and three anonymous referees. Earlier versions of this paper have been presented at St. Gallen, at the EEA meetings 2010 in Glasgow, at the Econometric Society World Congress 2010 in Shanghai, and at the Oligo2013 Workshop in Budapest. For excellent research assistance, I am grateful to Carlo Possenti and Roman Smirnov.

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Cournot Games with Biconcave Demand

Abstract. Biconcavity is a simple condition on inverse demand that corresponds to the ordinary concept of concavity after simultaneous parameterized transformations of price and quantity. The notion is employed here in the framework of the homogeneous-good Cournot model with potentially heterogeneous firms. The analysis leads to unified conditions, respectively, for the existence of a pure-strategy equilibrium via nonincreasing best-response selections, for existence via quasiconcavity, and for uniqueness of the equilibrium. The usefulness of the generalizations is illustrated in cases where inverse demand is either “nearly linear” or isoelastic. It is also shown that commonly made assumptions regarding large outputs are often redundant.

Keywords. Cournot games; existence and uniqueness of a pure-strategy Nash equilibrium; generalized concavity; supermodularity.

JEL-Codes. C72 - Noncooperative games; L13 - Oligopoly and other imperfect markets; C62 - Existence and stability conditions of equilibrium.

1. PRELIMINARIES

1.1. Introduction

This paper employs expanded notions of concavity to review the main conditions for existence and uniqueness of a pure-strategy Nash equilibrium in Cournot's (1838) homogeneous-good oligopoly with potentially heterogeneous firms.¹ Central to the approach is a family of monotone transformations given by $\varphi_\alpha(x) = x^\alpha/\alpha$ if $\alpha \neq 0$ and by $\varphi_\alpha(x) = \ln(x)$ if $\alpha = 0$. An inverse demand function $P = P(Q)$ is then called (α, β) -biconcave if P becomes concave (in the interval where inverse demand is positive) after transforming the price scale by φ_α and, simultaneously, the quantity scale by φ_β , where $\alpha, \beta \in \mathbb{R}$.²

Many of the concavity assumptions used in the literature can be expressed conveniently in terms of biconcavity. See Figure 1 for an overview. Concavity of inverse demand, as assumed by Szidarovszky and Yakowitz (1977, 1982), corresponds to $(1, 1)$ -biconcavity. Selten (1970) and Murphy et al. (1982), respectively, impose concavity conditions on industry revenues that correspond to strict and non-strict variants of $(1, -1)$ -biconcavity. Novshek's (1985) marginal revenue condition corresponds to $(1, 0)$ -biconcavity. Amir's (1996) log-concavity of inverse demand corresponds to $(0, 1)$ -biconcavity. Finally, Deneckere and Kovenock (1999) use a condition that corresponds to a strict variant of $1/P$ being convex, i.e., to $(-1, 1)$ -biconcavity.³ Thus, the notion of biconcavity provides a simple framework for organizing the main conditions in the literature.⁴

¹For the symmetric case, see McManus (1962, 1964), Roberts and Sonnenschein (1976), and Amir and Lambson (2000). Kukushkin (1993) allows for heterogeneous capacity constraints in an otherwise symmetric setting.

²Thus, P is (α, β) -biconcave if $\varphi_\alpha \circ P \circ \varphi_\beta^{-1}$ is concave, where φ_β^{-1} is the inverse of φ_β .

³Still another condition requires P to be log-concave in $\ln(Q)$, which corresponds to $(0, 0)$ -biconcavity. Until very recently, however, that condition has been more familiar in other contexts, such as imperfect competition (Caplin and Nalebuff, 1991a) or monopoly pricing (van den Berg, 2007).

⁴Given this perspective, it is natural to seek unified conditions. However, having a well-rounded

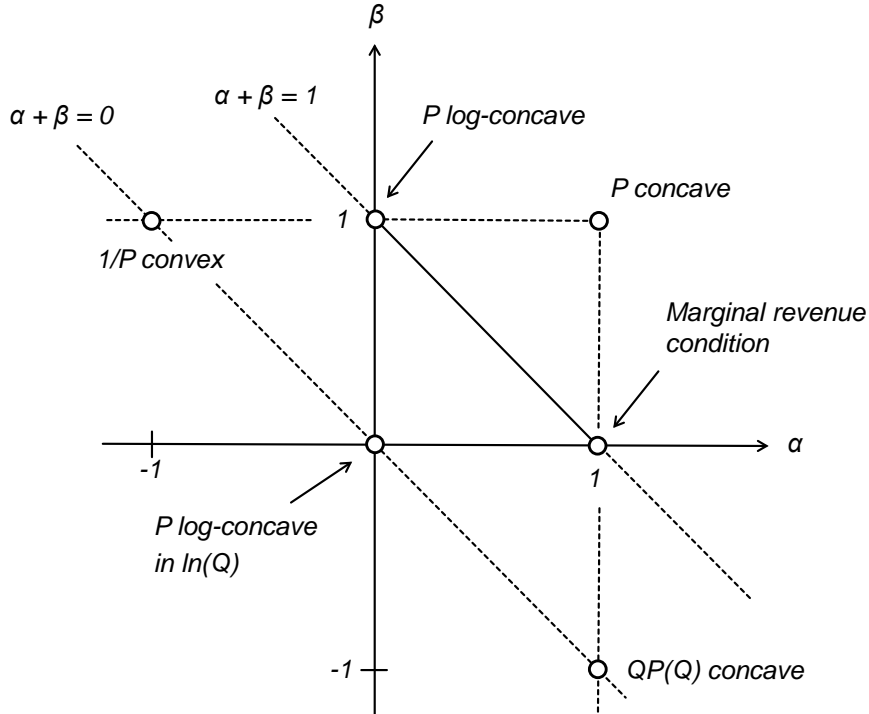


Figure 1: Special cases of biconcavity

The analysis reviews conditions in three areas. A first topic is equilibrium existence via nonincreasing best-response selections. In this case, the literature offers two main conditions, due to Novshek (1985) and Amir (1996), respectively. A partial consolidation of these conditions may be achieved by considering cross-partials of monotone transformations of the profit function.⁵ The present paper considers instead monotone transformations of the revenue function. This has some advantages. Specifically, cross-partial conditions may be replaced by simpler biconcavity conditions, and costs may be general (i.e., nondecreasing and lower semi-continuous) rather than linear. Moreover, exploiting the intuitive interpretation of biconcavity,

theory is desirable also because the Cournot model features prominently as an example for some broader classes of games, such as games with strategic complementarities (Milgrom and Roberts, 1990; Vives, 1990), surplus sharing games (Watts, 1996), and aggregative games with strategic substitutes (Corchón, 1994; Dubey et al., 2006; Jensen, 2006, 2010).

⁵For a discussion of the scope and limitations of this approach, see Amir (2005).

assumptions for large outputs will be shown to be redundant.

The second topic of the paper is equilibrium existence via quasiconcavity or even concavity of the profit functions, in the tradition of Friedman (1971) and Okuguchi (1976). While this approach does impose restrictions on costs, it generates alternative conditions for existence in the case where monotone best-response selections cannot be ascertained. Again, a number of results in the literature can be subsumed. Moreover, straightforward variants of the conditions considered, with weak inequalities replaced by strict ones, turn out to be very useful both for the discussion of specific examples and for the later analysis of uniqueness.

The third and final topic is, consequently, the uniqueness of the pure-strategy equilibrium, both in games admitting nonincreasing best-response selections and in games with quasiconcave profit functions. Intuitively, the assumption of biconcavity is employed here to ensure the famous “necessary and sufficient” conditions that result from the index approach to uniqueness (Kolstad and Mathiesen, 1987). For convenience, however, the formal analysis will be based upon Selten’s (1970) “backward mapping” approach and its subsequent developments by Szidarovszky and Yakowitz (1977) and Gaudet and Salant (1991).⁶ Maybe interestingly, the analysis identifies a single additional condition,

$$(\alpha + \beta)P' - C_i'' < 0 \quad (i = 1, \dots, N), \tag{1}$$

that implies the uniqueness of the pure-strategy equilibrium in the smooth model with or without capacity constraints (here C_i denotes, of course, firm i ’s cost function). In fact, as will become clear, condition (1) and its weak-form variant jointly consolidate an entire “catalog” of uniqueness conditions that have been proposed over the past decades.

⁶Useful descriptions can be found in Friedman (1982) and in Bischi et al. (2010).

Quite obviously, the present analysis draws heavily upon a strand of literature that has emphasized the role of expanded notions of concavity for economic theory in general, and for the analysis of imperfect competition in particular. Most notably, Caplin and Nalebuff (1991a) defined ρ -concavity via parameterized transformations of the quantity variable, and thereby introduced the notion of generalized concavity (together with the Prékopa-Borell theorem) to the economics literature.⁷ More closely related to the present analysis is the paper by Anderson and Renault (2003), who apply generalized concavity to derive efficiency and surplus bounds in the Cournot framework. Further applications include, for example, price discrimination (Cowan, 2007, 2012; Aguirre et al., 2010), auctions (Mares and Swinkels, 2011a, 2011b), hazard-rate conditions (Ewerhart, 2012), public goods (Myatt and Wallace, 2009), taxation (Moyes, 2003), and density estimation (Koenker and Mizera, 2010). In contrast to all those contributions, however, the present analysis employs *simultaneous parameterized* transformations of price and quantity. To the best of my knowledge, this particular concept has not been applied before.

The rest of the paper is structured as follows. The following two subsections introduce the notion of biconcavity and the set-up. Section 2 derives conditions for existence via nonincreasing best-response selections. Conditions for quasiconcave payoffs are stated in Section 3. Section 4 deals with uniqueness. Section 5 concludes. All proofs can be found in an Appendix.

1.2. *Biconcavity*

This subsection introduces the notion of biconcavity more formally, and derives some of its elementary properties.⁸

⁷See also Caplin and Nalebuff (1991b).

⁸The definition with simultaneous parameterized transformations captures an extension briefly mentioned in Avriel (1972). Subsequent studies, such as Ben-Tal (1977) and Moyes (2003), allow for more general transformations (e.g., continuous and increasing), but do not deal explicitly with parameterized transformations.

Consider the parameterized family of transformations $\{\varphi_\alpha\}_{\alpha \in \mathbb{R}}$ defined in the Introduction. Given arbitrary parameters $\alpha, \beta \in \mathbb{R}$, an (inverse demand) function $P = P(Q) \geq 0$, possibly unbounded at $Q = 0$, will be called (α, β) -biconcave [(α, β) -biconvex] if the domain $I_P = \{Q > 0 : P(Q) > 0\}$ is an interval and $\varphi_\alpha(P(Q))$ is a concave [convex] function of $\varphi_\beta(Q)$ over the domain where $Q \in I_P$. Clearly, the condition on I_P holds trivially when P is nonincreasing, which will be assumed essentially everywhere in the paper.

The following useful result extends a well-known ranking property of ρ -concavity (cf. Caplin and Nalebuff, 1991a) to the case of simultaneous parameterized transformations.

Theorem 1.1. *Let $\alpha', \beta' \in \mathbb{R}$ with $\alpha' \leq \alpha$ and $\beta' \leq \beta$. If P is nonincreasing and (α, β) -biconcave, then P is also (α', β') -biconcave.*

Thus, returning to the graphical representation of biconcavity (see Figure 1), the condition of (α, β) -biconcavity implies also all biconcavity conditions corresponding to points southwest of (α, β) . For example, $(1, 1)$ -biconcavity is more stringent than $(1, 0)$ -biconcavity, which in turn is more stringent than $(0, 0)$ -biconcavity. The property captured by Theorem 1.1 is clearly intuitive because a lower value of either α or β makes it easier for the transformed function to be concave. It is essential here, however, that P is nonincreasing. Without this assumption, the ranking result regarding β would not hold in general.⁹

The following immediate property of biconcavity translates conditions on direct demand $D = D(p) \geq 0$, possibly unbounded at $p = 0$, into conditions on inverse demand (and vice versa), in the spirit of Deneckere and Kovenock (1999).

Lemma 1.2. *Let $P = P(Q)$ and $D = D(p)$ be continuous and nonincreasing,*

⁹Indeed, if inverse demand were to be upward-sloping, e.g., due to general equilibrium effects, then applying a concave transformation to the quantity scale would make the transformed function more convex rather than more concave.

with $D(P(Q)) = Q$ over I_P . Then P is (α, β) -biconcave if and only if D is (β, α) -biconcave.

For example, $(0, 1)$ -biconcavity of direct demand corresponds to $(1, 0)$ -biconcavity of inverse demand, etc.

Finally, it is often convenient to work with the following second-order characterization of biconcavity.

Lemma 1.3. *Assume that P is nonincreasing, and twice differentiable on I_P . Then P is (α, β) -biconcave if and only if $\Delta_{\alpha, \beta}^P \leq 0$ holds on I_P , where*

$$\Delta_{\alpha, \beta}^P(Q) = (\alpha - 1)QP'(Q)^2 + QP(Q)P''(Q) + (1 - \beta)P(Q)P'(Q). \quad (2)$$

Intuitively, the criterion captured by Lemma 1.3 puts a bound on a weighted sum of the elasticity, $e_P = -QP'/P$, and the curvature, $e_{P'} = -QP''/P'$, of inverse demand. Indeed, if $P' < 0$, then condition (2) is easily seen to be equivalent to the inequality $(\alpha - 1)e_P + e_{P'} \leq 1 - \beta$.

The lemma above is straightforward to apply. E.g., linear inverse demand, $P(Q) = \max\{1 - Q; 0\}$, is (α, β) -biconcave if and only if $\alpha \leq 1$ and $\beta \leq 1$. For another example, isoelastic inverse demand, defined through $P(Q) = Q^{-\eta}$ for $\eta > 0$, is (α, β) -biconcave if and only if $\alpha\eta + \beta \leq 0$. Further examples will be given in Section 2.

1.3. Set-up

The following set-up will be used throughout the paper. There is an industry composed of $N \geq 2$ firms. Each firm $i = 1, \dots, N$ produces a quantity $q_i \in T_i$ of the homogeneous good, where $T_i \subseteq \mathbb{R}_+$ denotes the set of output levels that are technologically feasible for firm i . Aggregate output $Q = \sum_{i=1}^N q_i$ determines *inverse demand* $P(Q) \geq 0$.¹⁰ Firm i 's profit is $\Pi_i(q_i, Q_{-i}) = R(q_i, Q_{-i}) - C_i(q_i)$, where

¹⁰In all what follows, P may be infinite at $Q = 0$ provided that $\lim_{Q \rightarrow 0} QP(Q) = 0$.

$Q_{-i} = \sum_{j \neq i} q_j$ is the joint output of firm i 's competitors, $R = R(q_i, Q_{-i}) \equiv q_i P(q_i + Q_{-i})$ is the revenue function, and $C_i = C_i(q_i)$ is firm i 's cost function. Firm i 's *best-response correspondence* \hat{r}_i is given by

$$\hat{r}_i(Q_{-i}) = \{q_i \in T_i : \Pi_i(q_i, Q_{-i}) \geq \Pi_i(\tilde{q}_i, Q_{-i}) \text{ for all } \tilde{q}_i \in T_i\}, \quad (3)$$

where $Q_{-i} \geq 0$. Should $\hat{r}_i(Q_{-i})$ be a singleton for a range of $Q_{-i} \geq 0$, then the *best-response function* that maps Q_{-i} to the unique element of $\hat{r}_i(Q_{-i})$ will be denoted by $r_i = r_i(Q_{-i})$. A *pure-strategy Nash equilibrium* is a vector $(q_1, \dots, q_N) \in T_1 \times \dots \times T_N$ such that $q_i \in \hat{r}_i(Q_{-i})$ for $i = 1, \dots, N$.

2. EXISTENCE VIA NONINCREASING BEST-RESPONSE SELECTIONS

2.1. Existence theorem

This section deals with the issue of existence in the most general case, i.e., when firms are not necessarily symmetric and profit functions are not necessarily quasiconcave. Novshek (1985) observed for this case that, if marginal revenues are nonincreasing in rivals' aggregate output, then a firm's best-response correspondence satisfies a downward monotonicity property that can be exploited to prove existence.¹¹ Following this route, the first existence result of the present paper employs conditions that ensure that a firm's smallest best response is well-defined and nonincreasing in rivals' aggregate output. The monotonicity property is established here using the ordinal variant of supermodularity (Milgrom and Shannon, 1994). More specifically, the proof of the theorem below extends Amir's (1996) argument for log-concave inverse demand functions by showing that an entire family of biconcavity conditions implies the crucial dual single-crossing condition for general cost specifications.

¹¹See also Novshek (1984) and Bamon and Frayssé (1985). The original argument has been developed into a convenient fixed point theorem by Kukushkin (1994). In fact, this is the result that will be used in the proof. Existence theorems for more general classes of aggregative games can be found, in particular, in Dubey et al. (2006) and Jensen (2006, 2010).

The following theorem is the first main existence result of the present paper.

Theorem 2.1. *Assume that P is continuous, nonincreasing, non-constant, and $(\alpha, 1 - \alpha)$ -biconcave for some $\alpha \in [0, 1]$. Assume also that T_i is nonempty and closed, and that C_i is lower semi-continuous and nondecreasing, for $i = 1, \dots, N$. Then, a pure-strategy Nash equilibrium exists.*

2.2. Discussion

Theorem 2.1 embeds the two main approaches to existence that require neither symmetry nor quasiconcave payoffs. Indeed, the second-order characterization of $(\alpha, 1 - \alpha)$ -biconcavity reduces to Novshek's (1985) marginal revenue condition

$$P'(Q) + QP''(Q) \leq 0 \quad (Q \in I_P) \quad (4)$$

at $\alpha = 1$, and to Amir's (1996) log-concavity assumption

$$P(Q)P''(Q) - P'(Q)^2 \leq 0 \quad (Q \in I_P) \quad (5)$$

at $\alpha = 0$. In Figure 1, the family of biconcavity conditions employed in Theorem 2.1 corresponds to a straight line connecting the points $(1, 0)$ and $(0, 1)$. Thus, as discussed in the Introduction, the theorem above may be seen as convexifying the two main approaches to existence for general cost specifications.¹²

The additional generality achieved by Theorem 2.1 might even be of some applied value, as the following example with “nearly linear” demand suggests.

Example 2.2. Consider $P(Q) = \max\{(1 - Q^\delta)^{1/\gamma}; 0\}$, where $\gamma \approx 1$ and $\delta \approx 1$. To use the second-order characterization of biconcavity, one calculates

$$\Delta_{\alpha, \beta}^P(Q) = \frac{\delta}{\gamma^2} Q^{\delta-1} (1 - Q^\delta)^{\frac{2-2\gamma}{\gamma}} \{(\alpha - \gamma)\delta Q^\delta + (\beta - \delta)\gamma(1 - Q^\delta)\}. \quad (6)$$

¹²Obviously, Theorem 2.1 also accounts for the fact that convexity of choice sets is not essential for equilibrium existence via monotone best-response selections (cf. Dubey et al., 2006).

Since the expression in the curly brackets is linear in Q^δ , it suffices to check the sign of $\Delta_{\alpha,\beta}^P(Q)$ for $Q \rightarrow 0$ and for $Q \rightarrow 1$. It follows that P is (α, β) -biconcave if and only if $\alpha \leq \gamma$ and $\beta \leq \delta$.

The point of this example is that if γ and δ are marginally smaller than unity, then inverse demand becomes practically indistinguishable from the linear specification, yet neither (4) nor (5) holds. In contrast, all biconcavity conditions corresponding to values of α with $1 - \delta \leq \alpha \leq \gamma$ are satisfied.

2.3. Large outputs

Theorem 2.1 does away with the commonly made assumption that output levels above some threshold are suboptimal. To see why this is possible, consider an inverse demand function P that is (α, β) -biconcave for some $\alpha, \beta \in \mathbb{R}$. If P is nonincreasing and non-constant, the same is true for the transformed function, so the graph with transformed scales has a negative slope somewhere. Provided $\alpha > 0$ and $\beta \geq 0$, concavity implies that the market price reaches zero at some finite $Q_0 > 0$. Hence, a firm has never a strict incentive to operate at an output level of Q_0 or higher. It is shown in the Appendix that a similar bound exists when $\alpha = 0$ and $\beta > 0$. Thus, the game is effectively compact provided $\alpha \geq 0$ and $\beta \geq 0$ with $\alpha + \beta > 0$, which strictly includes the cases considered in Theorem 2.1. In particular, assumptions for large outputs made in Novshek (1985) and Amir (1996) are seen to be redundant.¹³

2.4. Other values of α and β

It is immediate from Figure 1 that Theorem 2.1 applies more generally provided P is (α, β) -biconcave, where $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta \geq 1$. However, if any of these three constraints is marginally relaxed, keeping the respective other two, then best-response correspondences need not allow a nonincreasing selection, and an equilibrium

¹³Biconcavity has also implications for small output levels. Given the same restrictions on the values of α and β as before, the biconcavity assumption implies $\lim_{Q \rightarrow 0} QP(Q) = 0$.

may fail to exist. The following example establishes this fact for the case in which the constraint $\alpha + \beta \geq 1$ is relaxed.

Example 2.3. Consider $P(Q) = \max\{(1 - Q^\delta)^{1/\gamma}; 0\}$ for $\gamma > 0$, $\delta > 0$, and $\gamma + \delta < 1$. Let costs be zero. Then, because P is (γ, δ) -biconcave and $P' < 0$, profits are strongly pseudoconcave in the range where $q_i + Q_{-i} \in (0, 1)$; see Section 3. The monopoly output is $Q^M \equiv r_i(0) = (\frac{\gamma}{\gamma+\delta})^{1/\delta}$. Implicit differentiation of the first-order condition at $Q_{-i} = 0$ shows that $r'_i(0) = \frac{1-\gamma-\delta}{\gamma+\delta} > 0$, so that r_i is indeed locally upward-sloping. Moreover, Q^M is a “potentially optimal output” in the sense of Novshek (1985, Theorem 4), whereas the marginal revenue condition fails at Q^M , i.e., $P'(Q^M) + Q^M P''(Q^M) > 0$. Therefore, an equilibrium may fail to exist for general (i.e., nondecreasing, lower semi-continuous) cost specifications.

Similar examples of non-existence may be constructed if one of the other two constraints is relaxed.¹⁴ Thus, for general cost specifications, the parameter restrictions in Theorem 2.1 are indeed just as tight as possible.

3. EXISTENCE VIA QUASICONCAVE PROFITS

3.1. Another existence theorem

This section considers environments in which a firm’s profit function is quasiconcave in own output. The property is of interest, in particular, because it ensures the existence of a pure-strategy equilibrium when profit functions are continuous and effective choice sets are non-empty compact intervals. Since quasiconcavity is, however, neither necessary nor sufficient for the availability of a nonincreasing best-response

¹⁴Here is a brief outline of these examples. When the constraint $\alpha \geq 0$ is relaxed, consider $P(Q) = (1 + Q^\delta)^{1/\gamma}$ with $\gamma < 0$, $|\gamma|$ small, $\delta \geq 1$, and zero costs. Then, $r'_i(Q_{-i}) > 0$ for large Q_{-i} . Similarly, when the constraint $\beta \geq 0$ is relaxed, consider $P(Q) = \max\{(Q^\delta - 1)^{1/\gamma}; 0\}$ with $\gamma \geq 1$, $\delta < 0$, $|\delta|$ small, and constant marginal costs $c_i \geq 0$. Then, $r'_i(0) > 0$ when c_i is large. Note also that, as a consequence of Theorem 1.1, there are no other cases to be considered.

selection, the analysis leads to conditions for existence that differ from (but overlap with) the conditions considered in the previous section.

For convenience, the subsequent discussion will focus on the smooth case, as captured by the following assumption.

Assumption 3.1. *P is continuous and nonincreasing on \mathbb{R}_+ , as well as twice continuously differentiable on I_P ; for any $i = 1, \dots, N$, either $T_i = \mathbb{R}_+$ or $T_i = [0, k_i]$ with $0 \leq k_i < \infty$, and C_i is nondecreasing and twice continuously differentiable over T_i .¹⁵*

The next assumption captures the effective compactness of the Cournot game.

Assumption 3.2. *There is some finite $\bar{Q} > 0$ such that for any $i = 1, \dots, N$, any $q_i \in T_i$, $q_i > \bar{Q}$, and any $Q_{-i} \geq 0$, there is some $\tilde{q}_i \leq \bar{Q}$ such that $\Pi_i(\tilde{q}_i, Q_{-i}) \geq \Pi_i(q_i, Q_{-i})$.*

Of course, this assumption is required only if at least one firm has unbounded capacity and inverse demand is everywhere positive. Even then, as explained in Section 2, the assumption will often be redundant.

The following result provides biconcavity conditions sufficient for a firm's profit function to be quasiconcave in own output. Thereby, a second main existence result is obtained.¹⁶

Theorem 3.3. *Impose Assumptions 3.1 and 3.2. Let $\alpha \leq 1$, $\beta \leq 1$ such that (i) $\Delta_{\alpha, \beta}^P \leq 0$, and (ii) $(\alpha + \beta)P' - C_i'' \leq 0$ for any $i = 1, \dots, N$. Then, a pure-strategy Nash equilibrium exists.*

This theorem subsumes a variety of existence conditions. Indeed, taking Assumptions 3.1 and 3.2 for granted, any of the conditions sufficient for quasiconcavity listed in

¹⁵In particular, at $q_i = 0$, the first two directional derivatives of C_i exist and are finite, and similarly at $q_i = k_i$ if T_i is bounded. As before, P may be unbounded at $Q = 0$ provided that $\lim_{Q \rightarrow 0} QP(Q) = 0$.

¹⁶Here and in the sequel, obvious constraints on Q , q_i , and Q_{-i} will be omitted. E.g., the use of the derivative of P is meant to indicate a restriction to $Q \in I_P$, etc.

Table I is covered by Theorem 3.3.¹⁷

3.2. Strong pseudoconcavity

By strengthening the assumptions of Theorem 3.3 somewhat, one may ensure that profit functions are strictly quasiconcave or even strongly pseudoconcave in the relevant domain.¹⁸

Theorem 3.4. *Impose the conditions of the previous theorem and assume, in addition, that either condition (i) holds strictly with $\alpha + \beta < 2$, or that condition (ii) holds strictly. Then, for any $Q_{-i} \geq 0$ with $P(Q_{-i}) > 0$, the function $\Pi_i(\cdot, Q_{-i})$ is strictly quasiconcave over the interval where $P(q_i + Q_{-i}) > 0$, and even strongly pseudoconcave over the interval where $q_i + Q_{-i} \in I_P$.*

It follows that, under the assumptions of the theorem, $r_i(Q_{-i})$ is well-defined whenever $P(Q_{-i}) > 0$. Moreover, the first-order condition holding with equality at some $q_i \in T_i$ with $q_i + Q_{-i} \in I_P$ is sufficient for a unique global maximum at q_i , where the second-order condition at q_i is then satisfied with strict inequality.

4. UNIQUENESS

4.1. Conditions for uniqueness

This section derives biconcavity conditions sufficient for the existence of a unique pure-strategy Nash equilibrium. The assumptions of smoothness and effective compactness from the previous section will be kept. Note, however, that smoothness is

¹⁷As a technical innovation, the proof of Theorem 3.3 uses $\partial \Pi_i / \partial q_i > 0 \implies \partial^2 \Pi_i / \partial q_i^2 \leq 0$ as a condition sufficient for quasiconcavity over an open interval. While intuitive, this condition does not appear to be widely known, so that a self-contained proof will be given in the Appendix. To be sure, we also remind the reader that $\partial \Pi_i / \partial q_i = 0 \implies \partial^2 \Pi_i / \partial q_i^2 \leq 0$ does not guarantee quasiconcavity.

¹⁸A twice continuously differentiable function $f = f(x)$ is strongly pseudoconcave over an open interval X if and only if $f'(x) = 0$ implies $f''(x) < 0$. When X has a non-empty boundary, then strong pseudoconcavity requires in addition that, if the (right or left) directional derivative is zero at a boundary point, then f decreases quadratically in a neighborhood in the direction of the derivative. See Diewert et al. (1981) for further details.

no longer assumed for convenience only.¹⁹ The following additional assumption will be imposed.

Assumption 4.1. *For any $(q_1, \dots, q_N) \in T_1 \times \dots \times T_N$ with $P(Q) = 0$, there is some $i = 1, \dots, N$ such that $C_i(q_i) > C_i(0)$.*

The sole purpose of this assumption is it to exclude the possibility of pathological equilibria in which the market price is zero, yet any individual firm is unable to generate a positive price by reducing its output. The following result is the main uniqueness theorem of the present paper.

Theorem 4.2. *Impose Assumptions 3.1, 3.2, and 4.1. Assume that P is (α, β) -biconcave with $0 \leq \alpha \leq 1$ and $\alpha + \beta \leq 1$. Assume also that*

$$(\alpha + \beta)P' - C_i'' < 0. \quad (i = 1, \dots, N) \quad (7)$$

Then, there is precisely one pure-strategy Nash equilibrium. Moreover, condition (7) may be replaced by a weak inequality (simultaneously for all $i = 1, \dots, N$) provided that $\Delta_{\alpha, \beta}^P < 0$ and $\alpha + \beta < 1$.

It is important to acknowledge that, under the conditions of the theorem, necessarily $P' - C_i'' < 0$.²⁰ In particular, this excludes trivial multiplicity with inactive firms, as described by Amir (2002).

4.2. Discussion

The theorem above offers a unifying perspective on numerous sufficient criteria for uniqueness in the literature. Table II provides an overview. Indeed, taking again

¹⁹Differentiability of inverse demand is needed, in fact, to avoid multiple equilibria. See Szidarovszky and Yakowitz (1982), amongst others.

²⁰For example, when condition (7) holds weakly only, then $\Delta_{\alpha, \beta}^P < 0$ implies $P' < 0$ over I_P , so that $P' - C_i'' < 0$ follows from $(\alpha + \beta)P' - C_i'' \leq 0$ and $\alpha + \beta < 1$. A similar argument applies when condition (7) holds strictly.

Assumptions 3.1 and 3.2 for granted, one may check by careful inspection that any of the listed conditions is essentially covered by the theorem.²¹

Theorem 4.2 also adds some flexibility to existing conditions, as the following example illustrates.

Example 4.3. Consider isoelastic inverse demand $P(Q) = Q^{-\eta}$, with $0 < \eta < 1$, and assume finite capacities $k_i > 0$, for $i = 1, \dots, N$. Note that the condition for small output levels holds, i.e., $\lim_{Q \rightarrow 0} QP(Q) = 0$. Given that P is (α, β) -biconcave if and only if $\alpha\eta + \beta \leq 0$, the tightest condition available from Theorem 4.2 is $(1 - \eta)P' - C_i'' < 0$. Thus, cost functions may be strictly concave, whereas applicable conditions from Table II would all require convex costs.²²

5. CONCLUSION

This paper has used expanded notions of concavity to review conditions for existence and uniqueness of a pure-strategy Nash equilibrium in the homogeneous-good Cournot model. While a number of potentially useful generalizations and simplifications have been obtained, the most immediate benefit of the approach is probably its unifying character. In particular, conditions on inverse and direct demand have been

²¹The conditions of Theorem 4.2 may differ, however, in the formalization of strictness. I.e., $P' < 0$ replaces the assumption that P is strictly decreasing, $C_i'' > 0$ replaces the assumption that C_i is strictly convex, etc. There are also three conditions that require elaboration. To start with, Szidarovszky and Okuguchi (1997) interpret the standard lottery contest as a Cournot game with unit-elastic inverse demand. Even though the condition for small outputs is not met, uniqueness follows from the proof of Theorem 4.2 for $(\alpha, \beta) = (0, 0)$, because $Q = 0$ is not an equilibrium. Grandmont's (1993) assumptions imply (α, β) -biconcavity with α and β small in absolute terms, but possibly negative. That result is also covered because positive marginal costs relax the biconcavity assumption needed in the proof of Theorem 4.2. Finally, Jensen's (2006) analysis of Cournot games with strict strategic substitutes roughly corresponds to the case discussed in Section 2 of the present paper. If $q_i P'' + C_i'' > 0$ is assumed in addition to $\Delta_{\alpha, 1-\alpha}^P \leq 0$, where $\alpha \in [0, 1]$, then profits are strongly pseudoconcave in the relevant domain. Adding the second-order condition yields $P' - C_i'' < 0$ along the best response, which suffices to complete the proof of Theorem 4.2 in this case.

²²This type of example might prove useful in applications of quantity competition in which the assumption of strategic substitutes would be too restrictive, as in Bulow et al. (1985), while increasing returns to scale cannot be ruled out a priori.

integrated in a natural way, which addresses a concluding request in Deneckere and Kovenock (1999). Further consolidation is desirable, of course. For example, the theorem of Nishimura and Friedman (1981) has not been reviewed here. McLennan et al. (2011) manage to subsume that result and Novshek's (1985) existence theorem in the duopoly case, yet the general relationship still seems to be unexplored. Moreover, as the discussion in Section 4 has shown, there is a lack of conditions (on the primitives of the model) that imply uniqueness even if profit functions are not quasiconcave. Last but not least, further applications of biconcavity appear desirable, both within the framework of the Cournot model and beyond.

APPENDIX: PROOFS

Proof of Theorem 1.1. For $x, \hat{x} > 0$, $\lambda \in [0, 1]$, and $\rho \in \mathbb{R}$, write $M_\rho(x, \hat{x}, \lambda) = \varphi_\rho^{-1}((1 - \lambda)\varphi_\rho(x) + \lambda\varphi_\rho(\hat{x}))$, where φ_ρ^{-1} is the inverse of φ_ρ . Then, by definition, P is (α, β) -biconcave if and only if $M_\alpha(P(Q), P(\hat{Q}), \lambda) \leq P(M_\beta(Q, \hat{Q}, \lambda))$ for all $Q, \hat{Q} \in I_P$ and all $\lambda \in [0, 1]$. By Hölder's inequality (cf. Hardy et al., 1934), the generalized mean $M_\rho(x, \hat{x}, \lambda)$ is nondecreasing in ρ . Thus, the condition of (α, β) -biconcavity becomes more stringent as α increases, and if P is nonincreasing, also as β increases. \square

Proof of Lemma 1.2. If P is (α, β) -biconcave, then the function that maps $\varphi_\beta(Q)$ to $\varphi_\alpha(P(Q))$ is concave (in the interval where $Q \in I_P$). Since P is strictly declining on I_P , also the function that maps $\varphi_\alpha(P(Q))$ to $\varphi_\beta(Q)$ is concave. Substituting $P(Q)$ by p , and Q by $D(p)$, shows that D is (β, α) -biconcave. The converse is similar. \square

Proof of Lemma 1.3. The function that maps $\varphi_\beta(Q)$ to $\varphi_\alpha(P(Q))$ is concave over the interval where $Q \in I_P$ if and only if

$$\frac{d\varphi_\alpha(P(Q))}{d\varphi_\beta(Q)} = \frac{\varphi'_\alpha(P(Q))P'(Q)}{\varphi'_\beta(Q)} \quad (8)$$

is nonincreasing over I_P . Differentiating (8) with respect to Q leads to (2). \square

Proof of Theorem 2.1. By Lemma A.1 below, w.l.o.g., $T_i \subseteq [0, \bar{Q}]$ for $i = 1, \dots, N$, where $\bar{Q} > 0$ is finite. Since $\Pi_i(\cdot, Q_{-i})$ is u.s.c., the minimum best response, $\min \hat{r}_i$, is well-defined. Take $\hat{Q}_{-i} > Q_{-i}$, and suppose $\hat{q}_i \equiv \min \hat{r}_i(\hat{Q}_{-i}) > \min \hat{r}_i(Q_{-i}) \equiv q_i$. Since $q_i \in \hat{r}_i(Q_{-i})$, it follows that $\Pi_i(q_i, Q_{-i}) \geq \Pi_i(\hat{q}_i, Q_{-i})$. Moreover, $P(\hat{q}_i + \hat{Q}_{-i}) > 0$ because $\hat{q}_i > 0$ is a minimum best response. Thus, by Lemma A.2, $\Pi_i(q_i, \hat{Q}_{-i}) \geq \Pi_i(\hat{q}_i, \hat{Q}_{-i})$, contradicting $q_i < \hat{q}_i$. Thus, $\min \hat{r}_i$ is nonincreasing. But \hat{r}_i is u.h.c. because $\Pi_i(q_i, Q_{-i})$ is both u.s.c. in q_i for any Q_{-i} , and continuous in Q_{-i} for any $q_i \in T_i$. Existence follows now from Kukushkin (1994). \square

The lemma below is used to verify the effective compactness of the Cournot game.

Lemma A.1. Assume that P is nonincreasing, non-constant, and (α, β) -biconcave for $\alpha \geq 0$, $\beta \geq 0$ such that $\alpha + \beta > 0$. Then there is a finite $\bar{Q} > 0$ such that $R(\cdot, Q_{-i})$ is nonincreasing in the interval $[\bar{Q}; \infty)$ for any $Q_{-i} \geq 0$.

Proof. The case $\alpha > 0$ has been dealt with in Section 2. Consider now $\alpha = 0$. One may clearly assume w.l.o.g. that $P > 0$. Then, for almost any $Q \geq 0$,

$$\frac{\partial \ln R(q_i, Q_{-i})}{\partial \varphi_\beta(q_i)} = \frac{\partial \ln P(Q)}{\partial \varphi_\beta(Q)} \frac{\partial \varphi_\beta(Q)}{\partial \varphi_\beta(q_i)} + \frac{\partial \ln q_i}{\partial \varphi_\beta(q_i)}. \quad (9)$$

Since P is nonincreasing and non-constant, $\frac{\partial \ln P(Q)}{\partial \varphi_\beta(Q)}|_{Q=Q^\#} \equiv s < 0$ for some $Q^\# \geq 0$. But P is $(0, \beta)$ -biconcave, hence $\frac{\partial \ln P(Q)}{\partial \varphi_\beta(Q)} \leq s$ for almost any $Q \geq Q^\#$. Note also that $\frac{\partial \varphi_\beta(Q)}{\partial \varphi_\beta(q_i)} = (1 + \frac{Q_{-i}}{q_i})^\beta \geq 1$, and that $\frac{\partial \ln q_i}{\partial \varphi_\beta(q_i)} = q_i^{-\beta} < |s|$ for all sufficiently large q_i . Thus, (9) is negative for almost any sufficiently large q_i , regardless of Q_{-i} . \square

The following lemma establishes the dual single-crossing property of Cournot profits.

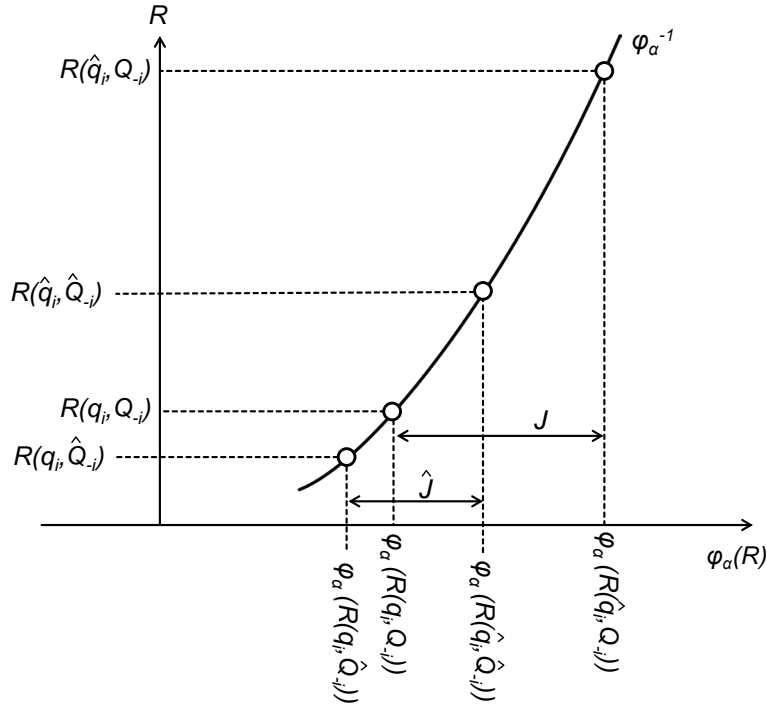


Figure 2: Extending Amir's (1996) key argument.

Lemma A.2. *Let P be nonincreasing and $(\alpha, 1 - \alpha)$ -biconcave for some $\alpha \in [0, 1]$. Assume also that C_i is nondecreasing. Then, for any $\widehat{q}_i > q_i$ and $\widehat{Q}_{-i} > Q_{-i}$ such that $P(\widehat{q}_i + \widehat{Q}_{-i}) > 0$ and $\Pi_i(q_i, Q_{-i}) \geq \Pi_i(\widehat{q}_i, Q_{-i})$, it follows that $\Pi_i(q_i, \widehat{Q}_{-i}) \geq \Pi_i(\widehat{q}_i, \widehat{Q}_{-i})$.*

Proof. Suppose $\Pi_i(q_i, \widehat{Q}_{-i}) < \Pi_i(\widehat{q}_i, \widehat{Q}_{-i})$. Then, $R(q_i, \widehat{Q}_{-i}) < R(\widehat{q}_i, \widehat{Q}_{-i})$, and the interval $\widehat{J} = [\varphi_\alpha(R(q_i, \widehat{Q}_{-i})), \varphi_\alpha(R(\widehat{q}_i, \widehat{Q}_{-i}))]$ is non-degenerate. By Lemma A.3 below, $J = [\varphi_\alpha(R(q_i, Q_{-i})), \varphi_\alpha(R(\widehat{q}_i, Q_{-i}))]$ is at least as wide as \widehat{J} . Moreover, the left endpoint of J weakly exceeds the left endpoint of \widehat{J} , as in Figure 2. Applying the convex inverse φ_α^{-1} to J and \widehat{J} yields $R(\widehat{q}_i, Q_{-i}) - R(q_i, Q_{-i}) \geq R(\widehat{q}_i, \widehat{Q}_{-i}) - R(q_i, \widehat{Q}_{-i})$. Hence, $\Pi_i(\widehat{q}_i, Q_{-i}) - \Pi_i(q_i, Q_{-i}) \geq \Pi_i(\widehat{q}_i, \widehat{Q}_{-i}) - \Pi_i(q_i, \widehat{Q}_{-i}) > 0$, a contradiction. \square

The next lemma extends an argument in Novshek (1985) and enters the proof above.

Lemma A.3. *Let P be nonincreasing and $(\alpha, 1 - \alpha)$ -biconcave for some $\alpha \geq 0$. Then for any $\widehat{q}_i > q_i$ and $\widehat{Q}_{-i} > Q_{-i}$ such that $P(\widehat{q}_i + \widehat{Q}_{-i}) > 0$,*

$$\varphi_\alpha(R(\widehat{q}_i, Q_{-i})) - \varphi_\alpha(R(q_i, Q_{-i})) \geq \varphi_\alpha(R(\widehat{q}_i, \widehat{Q}_{-i})) - \varphi_\alpha(R(q_i, \widehat{Q}_{-i})). \quad (10)$$

Proof. By Lemma A.4 below, $P(q_i + \widetilde{Q}_{-i})$ is $(\alpha, 1 - \alpha)$ -biconcave in q_i , for any $\widetilde{Q}_{-i} \in [Q_{-i}, \widehat{Q}_{-i}]$. Therefore, for almost any $\widetilde{Q}_{-i} \in [Q_{-i}, \widehat{Q}_{-i}]$, the inequality

$$\frac{\partial \varphi_\alpha(P(q_i + \widetilde{Q}_{-i}))}{\partial \varphi_{1-\alpha}(q_i)} \geq \frac{\partial \varphi_\alpha(P(\widehat{q}_i + \widetilde{Q}_{-i}))}{\partial \varphi_{1-\alpha}(\widehat{q}_i)} \quad (11)$$

is well-defined and holds. Using (8) and the functional form of φ_α ,

$$\frac{\partial \varphi_\alpha(P(q_i + \widetilde{Q}_{-i}))}{\partial \varphi_{1-\alpha}(q_i)} = \frac{\varphi'_\alpha(P(q_i + \widetilde{Q}_{-i}))P'(q_i + \widetilde{Q}_{-i})}{\varphi'_{1-\alpha}(q_i)} = \frac{\partial \varphi_\alpha(R(q_i, \widetilde{Q}_{-i}))}{\partial \widetilde{Q}_{-i}}. \quad (12)$$

Integrating over the interval $[Q_{-i}, \widehat{Q}_{-i}]$ yields

$$\varphi_\alpha(R(q_i, \widehat{Q}_{-i})) - \varphi_\alpha(R(q_i, Q_{-i})) = \int_{Q_{-i}}^{\widehat{Q}_{-i}} \frac{\partial \varphi_\alpha(R(q_i, \widetilde{Q}_{-i}))}{\partial \widetilde{Q}_{-i}} d\widetilde{Q}_{-i}. \quad (13)$$

Since (12) and (13) hold likewise with q_i replaced by \widehat{q}_i , inequality (10) follows. \square

The next lemma, used in the proof above, generalizes a result in Murphy et al. (1982).

Lemma A.4. *Assume that P is $(\alpha, 1 - \alpha)$ -biconcave and nonincreasing, for $\alpha \geq 0$. Then $P(q_i + Q_{-i})$ is $(\alpha, 1 - \alpha)$ -biconcave in q_i , for any $Q_{-i} \geq 0$.*

Proof. Suppose $\varphi_\alpha(P(Q))$ is concave and nonincreasing in $\varphi_{1-\alpha}(Q)$ over the domain where $Q \in I_P$. Using Lemma 1.3, $\varphi_{1-\alpha}(Q) \equiv \varphi_{1-\alpha}(q_i + Q_{-i})$ is easily seen to be convex in $\varphi_{1-\alpha}(q_i)$ if $Q_{-i} \geq 0$. Hence, $\varphi_\alpha(P(q_i + Q_{-i}))$ is concave in $\varphi_{1-\alpha}(q_i)$ over the domain where $Q \in I_P$. \square

Proof of Theorem 3.3. To apply Lemma A.5 below, suppose $\partial\Pi_i(q_i, Q_{-i})/\partial q_i > 0$, where $Q \in I_P$. Then, inequality (14) holds. Using $\beta \leq 1$ and $q_i/Q \leq 1$ yields $q_i P''(Q) + P'(Q)(2 - \alpha - \beta) \leq 0$. Adding $P'(Q)(\alpha + \beta) - C_i''(q_i) \leq 0$, one obtains $\partial^2\Pi_i(q_i, Q_{-i})/\partial q_i^2 \leq 0$. Thus, $\Pi_i(\cdot, Q_{-i})$ is quasiconcave over the domain where $Q \in I_P$. Since $C_i' \geq 0$, and by continuity, $\Pi_i(\cdot, Q_{-i})$ is quasiconcave over the whole of T_i . Existence now follows from Assumptions 3.1 and 3.2 (cf. Friedman, 1971). \square

The proof of the following lemma is adapted from Diewert et al. (1981).

Lemma A.5. *Assume that $f = f(x)$ is twice continuously differentiable on an open interval $X \subseteq \mathbb{R}$. Then f is quasiconcave over X if $f'(x) > 0$ implies $f''(x) \leq 0$.*

Proof. Suppose f is not quasiconcave. Then, there are $x_1 < x_* < x_2$ such that $f(x_*) < \min\{f(x_1), f(x_2)\}$. Take some $\tilde{x}_1 \in (x_1, x_*)$ with $f'(\tilde{x}_1) < 0$, and some $\tilde{x}_2 \in (x_*, x_2)$ with $f'(\tilde{x}_2) > 0$. Denote by x_0 the largest element in the interval $(\tilde{x}_1, \tilde{x}_2)$ such that $f'(x_0) = 0$. By Taylor's theorem, there is some $x^* \in (x_0, \tilde{x}_2)$ with $f(\tilde{x}_2) = f(x_0) + f'(x_0)(\tilde{x}_2 - x_0) + (1/2)f''(x^*)(\tilde{x}_2 - x_0)^2$. Using $f'(x_0) = 0$ and $f(\tilde{x}_2) > f(x_0)$ shows $f''(x^*) > 0$. Yet $x_0 < x^* < \tilde{x}_2$ implies $f'(x^*) > 0$. \square

The following lemma is needed for Theorems 3.3, 3.4, and 4.2.

Lemma A.6. *Let $Q \in I_P$, and assume that $\Delta_{\alpha,\beta}^P(Q) \leq 0$, where $\alpha \leq 1$ and $\beta \in \mathbb{R}$.*

Then, $q_i P'(Q) + P(Q) \geq 0$ implies

$$q_i P''(Q) + P'(Q) \leq \left(\alpha - \frac{q_i}{Q}(1 - \beta)\right) P'(Q). \quad (14)$$

Proof. To obtain (14), one multiplies $q_i P'(Q) + P(Q) \geq 0$ through with $(1 - \alpha)P'(Q) \leq 0$, and subsequently adds $(q_i/Q)\Delta_{\alpha,\beta}^P(Q) \leq 0$. \square

Proof of Theorem 3.4. Assume first $\Delta_{\alpha,\beta}^P < 0$ with $\alpha + \beta < 2$. Let $q_i \in T_i$ such that $Q \in I_P$, and suppose $\partial \Pi_i(q_i, Q_{-i})/\partial q_i = 0$. Then, for $q_i > 0$, the proof of Theorem 3.3 shows that $\partial^2 \Pi_i(q_i, Q_{-i})/\partial q_i^2 < 0$. For $q_i = 0$, the second-order condition is $2P'(Q_{-i}) - C_i''(0) < 0$, which follows from $(\alpha + \beta)P' - C_i'' \leq 0$ and $\alpha + \beta < 2$ because $\Delta_{\alpha,\beta}^P < 0$ implies $P' < 0$ over I_P . Thus, $\Pi_i(\cdot, Q_{-i})$ is strong pseudoconcave over the range where $Q \in I_P$, and by continuity, strictly quasiconcave over the range where $P(Q) > 0$. The case where $(\alpha + \beta)P' - C_i'' < 0$ is analogous. \square

Proof of Theorem 4.2. Existence follows from Theorem 3.3. As for uniqueness, note first that $P(Q) > 0$ in any equilibrium, by Assumption 4.1. Assume next that $q_i > 0$ for some firm i in an equilibrium (q_1, \dots, q_N) . Then, by strict quasiconcavity, $\Pi_i(q_i, 0) \geq \Pi_i(q_i, Q_{-i}) > \Pi_i(0, Q_{-i}) = \Pi_i(0, 0)$, so that $Q = 0$ is not a second equilibrium. Consider, finally, $\chi(Q) = \sum_{i=1}^N \chi_i(Q)$, where $\chi_i(Q)$ is defined in Lemma A.7 below. Since $\chi(Q) = Q$ holds in any equilibrium, it suffices to show (Hagood and Thomson, 2006) that the right-derivative of χ , denoted by $D^+\chi$, satisfies $D^+\chi < 1$. Write $B(Q) = \{i : D^+\chi_i(Q) > 0\}$. Then,

$$D^+\chi(Q) = \sum_{i=1}^N D^+\chi_i(Q) \leq \sum_{i \in B(Q)} D^+\chi_i(Q) = \sum_{i \in B(Q)} \frac{q_i P''(Q) + P'(Q)}{C_i''(q_i) - P'(Q)}. \quad (15)$$

Note that $q_i P'(Q) + P(Q) \geq 0$ for any $i \in B(Q)$. Indeed, if $q_i P'(Q) + P(Q) < 0$, then $\chi_i(Q) = 0$ also if Q is marginally raised, which would contradict $i \in B(Q)$. Hence,

by Lemma A.6,

$$D^+\chi(Q) \leq \sum_{i \in B(Q)} \frac{(\alpha - \frac{q_i}{Q}(1 - \beta))P'(Q)}{C_i''(q_i) - P'(Q)} \leq \sum_{i \in \tilde{B}(Q)} \frac{(\alpha - \frac{q_i}{Q}(1 - \beta))P'(Q)}{C_i''(q_i) - P'(Q)}, \quad (16)$$

where $\tilde{B}(Q) = \{i \in B(Q) : \alpha - \frac{q_i}{Q}(1 - \beta) < 0\}$. If now either $P'(Q) = 0$ or $\tilde{B}(Q) = \emptyset$, then (16) implies $D^+\chi(Q) \leq 0$. Otherwise, i.e., if $P'(Q) < 0$ and $|\tilde{B}(Q)| \geq 1$, then necessarily $\alpha + \beta < 1$, and

$$D^+\chi(Q) < \sum_{i \in \tilde{B}(Q)} \frac{-\alpha + \frac{q_i}{Q}(1 - \beta)}{1 - \alpha - \beta} \leq \frac{1 - \alpha|\tilde{B}(Q)| - \beta}{1 - \alpha - \beta} \leq 1. \quad (17)$$

Thus, $D^+\chi < 1$ in any case, and there is precisely one equilibrium. \square

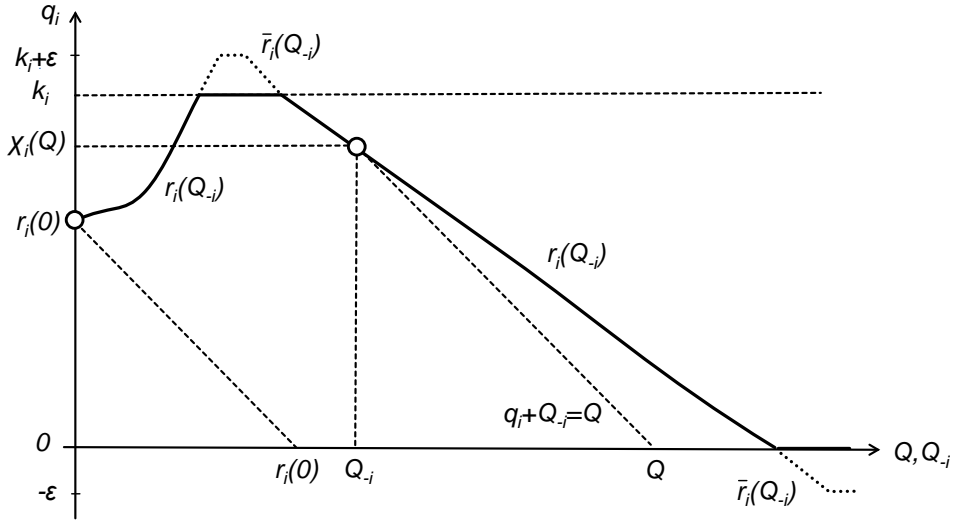


Figure 3: The slope of the best-response function strictly exceeds -1 .

The following lemma is needed for the argument above. See also Figure 3.

Lemma A.7. *Impose the assumptions of Theorem 4.2. Then, for any $Q \in I_P$, the equation $q_i = r_i(Q - q_i)$ has a unique solution $q_i \equiv \chi_i(Q) \in [0; Q]$ if $Q \geq r_i(0)$, and no solution if $Q < r_i(0)$. Moreover,*

$$D^+\chi_i(Q) = \frac{q_i P''(Q) + P'(Q)}{C_i''(q_i) - P'(Q)} \mathcal{I}_{M_i}(Q), \quad (18)$$

where \mathcal{I}_{M_i} is the indicator function of a measurable set $M_i \subseteq I_P$.

Proof. Let $\bar{\Pi}_i(q_i, Q_{-i}) = q_i P(q_i + Q_{-i}) - \Gamma_i(q_i)$, where Γ_i is twice continuously differentiable over \mathbb{R} , and $\Gamma_i(q_i) = C_i(q_i)$ over T_i . By Theorem 3.4, for any $Q_{-i} \in I_P$, the function $\bar{\Pi}_i(\cdot, Q_{-i})$ is strongly pseudoconcave over the subinterval of T_i where $P(q_i + Q_{-i}) > 0$. Hence, for any $Q_{-i}^0 \in I_P$, there is some $\varepsilon > 0$, and a neighborhood \mathcal{U} of Q_{-i}^0 such that $\bar{\Pi}_i(\cdot, Q_{-i})$ is strongly pseudoconcave over the corresponding subinterval of $T_i^\varepsilon = [-\varepsilon, \infty)$ if $k_i = \infty$, and of $T_i^\varepsilon = [-\varepsilon, k_i + \varepsilon]$ if $k_i < \infty$, for any $Q_{-i} \in \mathcal{U}$. By making $\varepsilon > 0$ sufficiently small, $\bar{r}_i(Q_{-i}) = \arg \max_{q_i \in T_i^\varepsilon} \bar{\Pi}_i(q_i, Q_{-i})$ is well-defined on any given compact subset of I_P . Since, locally, either $r_i(Q_{-i}) = \max\{0; \bar{r}_i(Q_{-i})\}$ or $r_i(Q_{-i}) = \min\{\bar{r}_i(Q_{-i}); k_i\}$,

$$D^+ r_i(Q_{-i}) = -\frac{P'(Q) + q_i P''(Q)}{2P'(Q) + q_i P''(Q) - C_i''(q_i)} \mathcal{I}_{M_i^0}(Q_{-i}) \quad (19)$$

for some measurable set $M_i^0 \subseteq I_P$; see Dem'yanov and Malozemov (1971). Now $P' - C_i'' < 0$ implies $D^+ r_i > -1$. Thus, $\psi_i(Q_{-i}) \equiv Q_{-i} + r_i(Q_{-i})$ is continuous and strictly increasing, with $\psi_i(0) = r_i(0)$ and $\psi_i(Q) \geq Q$, proving the first assertion. As $D^+ \psi_i = 1 + D^+ r_i > 0$, the directional inverse function theorem (Pang et al., 2003) implies $D^+(\psi_i^{-1}) = 1/(1 + D^+ r_i)$. Hence, $D^+ \chi_i(Q) = D^+ r_i(Q)/(1 + D^+ r_i(Q))$, and (18) holds with $M_i = \psi_i(M_i^0)$. \square

REFERENCES

- Aguirre, I., Cowan, S., Vickers, J., 2010, Monopoly price discrimination and demand curvature, *American Economic Review* 100, 1601-1615.
- Amir, R., 1996, Cournot oligopoly and the theory of supermodular games, *Games and Economic Behavior* 15, 132-148.
- Amir, R., 2002, Market structure, scale economies and industry performance, CORE working paper.
- Amir, R., 2005, Ordinal versus cardinal complementarity: The case of Cournot oligopoly, *Games and Economic Behavior* 53, 1-14.
- Amir, R., Lambson, V., 2000, On the effects of entry in Cournot games, *Review of Economic Studies* 67, 235-254.
- Anderson, S., Renault, R., 2003, Efficiency and surplus bounds in Cournot competition, *Journal of Economic Theory* 113, 253-264.
- Avriel, M., 1972, r -convex functions, *Mathematical Programming* 2, 309-323.
- Bamon, R., Frayssé, J., 1985, Existence of Cournot equilibrium in large markets, *Econometrica* 53, 587-597.
- Ben-Tal, A., 1977, On generalized means and generalized convex functions, *Journal of Optimization Theory and Applications* 21, 1-13.
- Bischi, G., Chiarella, C., Kopel, M., Szidarovszky, F., 2010, *Nonlinear Oligopolies: Stability and Bifurcations*, Springer, Berlin.
- Bulow, J., Geanakoplos, J., Klemperer, P., 1985, Holding idle capacity to deter entry, *Economic Journal* 95, 178-182.
- Caplin, B., Nalebuff, C., 1991a, Aggregation and imperfect competition: On the existence of equilibrium, *Econometrica* 59, 25-59.

- Caplin, B., Nalebuff, C., 1991b, Aggregation and social choice: A mean voter theorem, *Econometrica* 59, 1-23.
- Corchón, L., 1994, Comparative statics for aggregative games: The strong concavity case, *Mathematical Social Sciences* 28, 151-165.
- Cournot, A., 1838, *Recherches sur les Principes Mathématiques de la Théorie des Richesses*. English edition (translated by N. Bacon), *Researches into the Mathematical Principles of the Theory of Wealth*, New York: Macmillan, 1897.
- Cowan, S., 2007, The welfare effects of third-degree price discrimination with nonlinear demand functions, *Rand Journal of Economics* 38, 419-428.
- Cowan, S., 2012, Third-degree price discrimination and consumer surplus, *Journal of Industrial Economics* 60, 333-345.
- Dem'yanov, V., Malozemov, V., 1971, On the theory of non-linear minimax problems, *Russian Mathematical Surveys* 26, 57-115.
- Deneckere, R., Kovenock, D., 1999, Direct demand-based Cournot existence and uniqueness conditions, unpublished manuscript.
- Diewert, W., Avriel, M., Zang, I., 1981, Nine kinds of quasiconcavity and concavity, *Journal of Economic Theory* 25, 397-420.
- Dubey, P., Haimanko, O., Zapechelnyuk, B., 2006, Strategic complements and substitutes, and potential games, *Games and Economic Behavior* 54, 77-94.
- Ewerhart, C., 2012, Regular type distributions in mechanism design and ρ -concavity, *Economic Theory*, doi 10.1007/s00199-012-0705-3.
- Friedman, J., 1971, A non-cooperative equilibrium for supergames, *Review of Economic Studies* 38, 1-12.
- Friedman, J., 1982, "Oligopoly Theory." In Arrow K., Intrilligator M. (Eds.), *Handbook of Mathematical Economics*, Vol. 2.

- Gaudet, G., Salant, S., 1991, Uniqueness of Cournot equilibrium: New results from old methods, *Review of Economic Studies* 58, 399-404.
- Grandmont, J.-M., 1993, Behavioural heterogeneity and Cournot oligopoly, *Ricerche Economiche* 47, 167-187.
- Hagood, J., Thomson, B., 2006, Recovering a function from its Dini derivative, *American Mathematical Monthly* 113, 34-46.
- Hardy, G., Littlewood, J., Polya, G., 1934, *Inequalities*, Cambridge University Press.
- Jensen, M., 2006, Existence, comparative statics, and stability in games with strategic substitutes, manuscript, University of Birmingham.
- Jensen, M., 2010, Aggregative games and best-reply potentials, *Economic Theory* 43, 45-66.
- Koenker, R., Mizera, I., 2010, Quasi-concave density estimation, *Annals of Statistics* 38, 2998-3027.
- Kolstad, C., Mathiesen, L., 1987, Necessary and sufficient conditions for uniqueness in a Cournot equilibrium, *Review of Economic Studies* 54, 681-690.
- Kukushkin, N., 1993, Cournot equilibrium with 'almost' identical convex costs, mimeo, Russian Academy of Sciences.
- Kukushkin, N., 1994, A fixed-point theorem for decreasing mappings, *Economics Letters* 46, 23-26.
- Mares, V., Swinkels, J., 2011a, Near-optimality of second price mechanisms in a class of asymmetric auctions, *Games and Economic Behavior* 72, 218-241.
- Mares, V., Swinkels, J., 2011b, On the analysis of asymmetric first price auctions, working paper, Northwestern University.
- McManus, M., 1962, Numbers and size in Cournot oligopoly, *Yorkshire Bulletin of Social and Economic Research* 14, 14-22.

McManus, M., 1964, Equilibrium, numbers and size in Cournot oligopoly, *Yorkshire Bulletin of Social and Economic Research* 16, 68-75.

McLennan, A., Monteiro, P., Tourky, R., 2011, Games with discontinuous payoffs: A strengthening of Reny's existence theorem, *Econometrica* 79, 1643-1664.

Milgrom, P., Roberts, J., 1990, Rationalizability, learning, and equilibrium in games with strategic complementarities, *Econometrica* 58, 1255-1277.

Milgrom, P., Shannon, D., 1994, Monotone comparative statics, *Econometrica* 62, 157-180.

Moyes, P., 2003, Redistributive effects of minimal equal sacrifice taxation, *Journal of Economic Theory* 108, 111-140.

Murphy, F., Sherali, H., Soyster, B., 1982, A mathematical programming approach for determining oligopolistic market equilibrium, *Mathematical Programming* 24, 92-106.

Myatt, D., Wallace, C., 2009, Evolution, teamwork and collective action: Production targets in the private provision of public goods, *Economic Journal* 119, 61-90.

Nishimura, K., Friedman, J., 1981, Existence of Nash equilibrium in n person games without quasiconcavity, *International Economic Review* 22, 637-648.

Novshek, W., 1984, Finding all n -firm Cournot equilibria, *International Economic Review* 25, 61-70.

Novshek, W., 1985, On the existence of Cournot equilibrium, *Review of Economic Studies* 52, 85-98.

Okuguchi, K., 1976, *Expectations and Stability in Oligopoly Models*, Berlin, Springer.

Pang, J.-S., Sun, D., Sun, J., 2003, Semismooth homeomorphisms and strong stability of semidefinite and Lorentz complementarity problems, *Mathematics of Operations Research* 28, 39-63.

- Roberts, J., Sonnenschein, H., 1976, On the existence of Cournot equilibrium without concave profit functions, *Journal of Economic Theory* 13, 112-117.
- Selten, R., 1970, *Preispolitik der Mehrproduktenunternehmung in der statischen Theorie*, Springer, Berlin.
- Szidarovszky, F., Okuguchi, Y., 1997, On the existence and uniqueness of pure Nash equilibrium in rent-seeking games, *Games and Economic Behavior* 18, 135-140.
- Szidarovszky, F., Yakowitz, S., 1977, A new proof of the existence and uniqueness of the Cournot equilibrium, *International Economic Review* 18, 787-789.
- Szidarovszky, F., Yakowitz, S., 1982, Contributions to Cournot oligopoly theory, *Journal of Economic Theory* 28, 51-70.
- Van Long, N., Soubeyran, A., 2000, Existence and uniqueness of Cournot equilibrium: a contraction mapping approach, *Economics Letters* 67, 345-348.
- Vives, X., 1990, Nash equilibrium with strategic complementarities, *Journal of Mathematical Economics* 19, 305-321.
- Vives, X., 1999, *Oligopoly Pricing*, MIT Press, Cambridge.
- van den Berg, G., 2007, On the uniqueness of optimal prices set by monopolistic sellers, *Journal of Econometrics* 141, 482-491.
- von Mouche, P., Quartieri, F., 2012, On the uniqueness of Cournot equilibrium in case of concave integrated price flexibility, *Journal of Global Optimization*, forthcoming.
- Watts, A., 1996, On the uniqueness of equilibrium in Cournot oligopoly and other games, *Games and Economic Behavior* 13, 269-285.

Table I: Conditions sufficient for firm i 's profits to be quasiconcave in own output[†]

Reference	Conditions	Implication [‡]	(α, β)
Sel70, Ch. 9	P twice continuously differentiable and positive, with $P' < 0$ and $QP''(Q) + 2P'(Q) < 0$; $T_i = [0, k_i]$ for $k_i \geq 0$; costs C_i twice continuously differentiable with $C_i'' \geq 0$	$\partial^2 \Pi_i(q_i, Q_{-i}) / \partial q_i^2 < 0$	$(1, -1)$
MSS82, Lemma 1	P nonincreasing or convex [strictly decreasing or strictly convex]; $QP(Q)$ concave	$R(\cdot, Q_{-i})$ concave [strictly concave if $Q_{-i} > 0$]	$(1, -1)$
DK99, proof of Theorem 1	$P : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ continuous; there is a finite $Q_0 > 0$ such that $P(Q) = 0$ for all $Q \geq Q_0$; P is C^2 on $[0, Q_0)$ with $P'(Q) < 0$; $2P'(Q) + QP''(Q) < 0$ for all $Q \in (0, Q_0)$; costs $C_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are C^2 , strictly increasing, and convex	$\Pi_i(\cdot, Q_{-i})$ strictly concave on $[0, Q_0]$	$(1, -1)$
Am96, criterion used in Ex. 3.3	$P''P - 2P'^2 < 0$, with linear costs	$\Pi_i(\cdot, Q_{-i})$ quasiconcave	$(-1, 1)$
DK99, proof of Theorem 2	$D : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ continuous; there is a finite $p_0 > 0$ such that $D(p) = 0$ for all $p \geq p_0$; further, D is C^2 on $(0, p_0]$ with $D'(p) < 0$; finally, $2D'(p) + pD''(p) < 0$ for all $p \in (0, p_0)$; costs $C_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are C^2 , strictly increasing, and convex	$p \cdot [D(p) - Q_{-i}] - C_i(D(p) - Q_{-i})$ strictly concave in p over the relevant interval (see the original paper)	$(-1, 1)$
vMQ12, Theorem 4	P positive and differentiable; $L_P(Q) \equiv Q \ln P(Q) - \int_0^Q \ln P(\xi) d\xi$ concave; costs C_i are continuous, nondecreasing, and convex	$\Pi_i(\cdot, Q_{-i})$ quasiconcave	$(0, 0)$
Viv99, Section 4.2	P is C^2 with $P' < 0$ and $P' + q_i P'' \leq 0$ when $P > 0$; costs C_i are C^2 with $C_i''(q_i) > P'(Q)$	$\Pi_i(\cdot, Q_{-i})$ strictly concave over the interval in which $P > 0$	$(1, 0)$
Viv99, Section 4.2	P is C^2 with $P' < 0$ when $P > 0$; P is log-concave; costs C_i are C^2 with $C_i''(q_i) > P'(Q)$	$\Pi_i(\cdot, Q_{-i})$ strictly quasiconcave over the interval in which $P > 0$	$(0, 1)$

[†] The notation and terminology of the present paper is used throughout.

[‡] The implication is meant to hold for any $Q_{-i} \geq 0$.

Table II: Conditions sufficient for the uniqueness of a pure-strategy equilibrium[†]

Reference	Conditions [‡]	# equ.	(α, β)
SY77, Theorem 1	There is a finite $Q_0 > 0$ such that $P(Q) = 0$ for all $Q \geq Q_0$; P twice differentiable on $[0, Q_0]$ with $P'(Q) < 0$ and $P''(Q) \leq 0$; all C_i twice differentiable with $C'_i > 0$ and $C''_i \geq 0$	= 1	$(1, 1)$ [§]
SY82, Theorem 2	P differentiable, nonincreasing, and concave; $T_i = [0, k_i]$ for $k_i > 0$ finite; all C_i continuous, nondecreasing, and convex; either $P' < 0$, or all C_i strictly convex	= 1	$(1, 1)$ [§]
Sel70, Ch. 9	P twice continuously differentiable and positive, with $P' < 0$ and $QP''(Q) + 2P'(Q) < 0$; $T_i = [0, k_i]$ for $k_i \geq 0$; all C_i are twice continuously differentiable with $C''_i \geq 0$, $C'_i(0) = 0$, and $C'_i(k_i) > P(k_i)$	= 1	$(1, -1)$
MSS82, Lemma 5	P continuously differentiable and strictly decreasing; $QP(Q)$ concave for $Q \geq 0$; all C_i continuously differentiable, nondecreasing, and convex; either all C_i are strictly convex, or $QP(Q)$ is strictly concave, or P is concave [‡]	≤ 1	$(1, -1)$
Wat96, Cor. 1	$QP(Q)$ strictly concave; all C_i convex; $QP(Q) - C_i(Q)$ decreasing for Q large enough	= 1	$(1, -1)$
DK99, Theorem 1	$P : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ continuous; there is a finite $Q_0 > 0$ such that $P(Q) = 0$ for all $Q \geq Q_0$; further, P is C^2 on $[0, Q_0]$ with $P'(Q) < 0$; finally, $2P'(Q) + QP''(Q) < 0$ for all $Q \in (0, Q_0)$; all $C_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are C^2 , strictly increasing, and convex	= 1	$(1, -1)$
SO97, Theorem 1	$P(Q) = 1/Q$ for $Q > 0$, and $\Pi_i(0, 0) = 0$; all C_i twice differentiable, $C_i(0) = 0$, $C'_i > 0$, $C''_i > 0$	= 1	See text
vMQ12, Theorem 3	Either $T_i = \mathbb{R}$ or $T_i = [0, k_i]$ for $k_i > 0$ finite; P positive, nonincreasing, and differentiable on $T \setminus \{0\}$, where $T \equiv \sum_{i=1}^N T_i$ is the Minkowski sum; $\tilde{L}_P(Q) \equiv Q \ln P(Q) - \int_{\xi_0}^Q \ln P(\xi) d\xi$ concave, where $\xi_0 \in T \setminus \{0\}$; all C_i nondecreasing and convex; either $\tilde{L}_P(Q)$ is strictly concave, or all C_i are strictly convex	≤ 1	$(0, 0)$
Gra93, Proposition	Individual demand functions with sufficient heterogeneity of characteristics (see the original paper); cost functions linear and sufficiently increasing	= 1	See text
GS91, remark following Assumption 5	There is a finite $Q_0 > 0$ such that $P(Q) > 0$ for $Q \in [0, Q_0]$ and $P(Q) = 0$ for $Q \in [Q_0, \infty)$; inverse demand P is C^2 with $P'(Q) \leq 0$ on $[0, Q_0]$; all C_i are C^2 , and $C'_i(q_i) > 0$ for any $q_i > 0$; for all $Q \in [0, Q_0]$, there is some $\sigma_i(Q) < 0$ such that $P'(Q) - C''_i(q_i) \leq \sigma_i(Q) < 0$ for every $q_i \geq 0$; finally, $q_i P''(Q) + P'(Q) \leq 0$ for all $Q \in [0, Q_0]$	= 1	$(1, 0)$
Viv99, Section 4.2	P is C^2 with $P' < 0$ (in the interval for which $P > 0$); $P' + q_i P'' \leq 0$; all C_i are C^2 with $C''_i(q_i) > P'(Q)$	= 1	$(1, 0)$
VLS00	There is some $Q_0 > 0$ such that $P(Q) > 0$ for $Q \in [0, Q_0]$, and $P(Q) = 0$ for $Q \geq Q_0$; P is C^2 with $P'(Q) < 0$ for $Q \in [0, Q_0]$; all C_i are C^2 , with $C'_i(q_i) > 0$ for all $q_i \in (0, Q_0]$; $C_i(0) = 0$, $C'_i(0) = 0$, and $C''_i(q_i) > 0$ for all $q_i \in (0, Q_0]$; $q_i P''(Q) + P'(Q) < 0$ for all $Q \in [0, Q_0]$ and $q_i \in (0, Q]$; there is some τ such that $-P'(Q) > \tau > 0$ for all $Q \in [0, Q_0]$; there is some $b > 0$ such that $C''_i(q_i) < b$ for all $q_i \in (0, Q_0]$	= 1	$(1, 0)$
Am96, Th. 2.3	$N = 2$; inverse demand P is strictly decreasing and log-concave; all C_i left-continuous, strictly increasing, and convex; there exists $\bar{Q} > 0$ such that $QP(Q) - C_i(Q) < 0$ for all $Q > \bar{Q}$	= 1	$(0, 1)$
DK99, Theorem 7	$D : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ continuous; there is a $p_0 > 0$ such that $D(p) = 0$ for all $p \geq p_0$; further, D is C^2 on $(0, p_0]$ with $D'(p) < 0$; finally, $D'(p) + pD''(p) < 0$ for all $p \in (0, p_0]$; all C_i are C^2 , strictly increasing, and convex	= 1	$(0, 1)$
Viv99, Section 4.2	P is C^2 with $P' < 0$ (in the interval for which $P > 0$); P is log-concave; all C_i are C^2 with $C''_i(q_i) > P'(Q)$	= 1	$(0, 1)$
Jen06, Example 9	Smooth Cournot model with strict strategic substitutes; $T_i = [0, k_i]$ for $k_i > 0$ finite; $q_i P''(Q) + C''_i(q_i) > 0$	= 1	See text

[†] The notation and terminology of the present paper is used throughout.

[‡] Unless indicated otherwise, inverse demand and cost functions are defined on \mathbb{R}_+ . Assumptions on choice sets T_i and costs C_i are generally imposed on all firms $i = 1, \dots, N$.

[§] To replicate the result using Theorem 4.2, choose $(\alpha, \beta) = (1, 0)$, and exploit Theorem 1.1.

[‡] Murphy et al. (1982) write “convex.”