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An Algorithm for Constructing Hadamard Matrices

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Abstract

We define a new class of Hadamard matrices and present some examples obtained by an algorithm using a computer program.

Introduction

A Hadamard matrix \mathbf{H}_n is an orthogonal $(n \times n)$ -matrix in which all elements are either $+1$ or -1 . If $n > 2$ then $n \equiv 0 \pmod{4}$ is a necessary condition that a \mathbf{H}_n exists. In this paper we define a class of Hadamard matrices and give a list of examples obtained by a computer program.

Hadamard matrices play an important in various field of statistics and mathematics. E.g. in statistics: incomplete balanced block designs, orthogonal arrays, factorial designs, Taguchi methods and Hotellings weighing problem; e.g. in mathematics: geometry, combinatorics, approximation theory and coding theory. There exists a wide-spread literature on Hadamard matrices, especially on the existence problem. A survey may be found in various text books of combinatorial theory.

1 Hadamard Schemes

To $m \in \mathbb{N}$ (\mathbb{N} includes 0) we write

$$\begin{aligned}\mathbb{N}_m &= \{0, 1, \dots, m-1, m\}, \\ \mathbb{Z}_m &= \{-m, -(m-1), \dots, -1, 0, 1, \dots, m-1, m\}.\end{aligned}$$

To $u \in \mathbb{Z}$ let $\rho(u)$ denote the additive residual class of u modulo \mathbb{Z}_m . (So, to $u \in \mathbb{Z}$ there exists exactly one $z \in \mathbb{Z}$ with $\rho(u) = u + z(2m+1) \in \mathbb{Z}_m$.) Let

$$M := \{1, i, -1, -i\}.$$

Here i is the imaginary unit. M is a multiplicative cyclic group of order 4 generated by i . For the elements of M we write '+' instead of 1, '-' instead of -1 and 'j' instead of $-i$. We use the following notation: Small boldface letters denote vectors, vectors in $M^{\mathbb{Z}_m}$ carry an asterisk; matrices are denoted by boldface capital letters.

If $\hat{\mathbf{A}}$ is an $(n \times m)$ -matrix whose elements $A_{i,j}$ are $(p \times q)$ -matrices, then we call $\hat{\mathbf{A}}$ a *block-matrix*. Block matrices are marked by carrying a hat accent. k -fold iterated block matrices are also marked with a hat accent, (but with one hat only). If $\hat{\mathbf{A}}$ is an (iterated) block-matrix, then the matrix obtained from $\hat{\mathbf{A}}$ by stripping off all the block boundaries is called the *deblocked* matrix of $\hat{\mathbf{A}}$. The deblocked matrix of $\hat{\mathbf{A}}$ is denoted by deleting the hat accent. E.g. if $\hat{\mathbf{A}} = [A_{i,j}]_{i=1\dots n, j=1\dots m}$ is an $(n \times m)$ -block-matrix whose elements $\mathbf{A}_{i,j} = [a_{i,j;u,v}]_{u=1,\dots,p, v=1,\dots,q}$, $i = 1, \dots, n, j = 1, \dots, m$ are $(p \times q)$ -matrices, then

$$\mathbf{A} = [a_{(i-1)p+u, (j-1)q+v}]_{\substack{i=1,\dots,n, u=1,\dots,p, \\ j=1,\dots,m, v=1,\dots,q}}$$

is the deblocked matrix of $\hat{\mathbf{A}}$.

If $z = z_1 + iz_2 \in \mathbb{C}$ is a complex number then $\bar{z} = z_1 - iz_2$ is its conjugate.

The mapping $\sigma : M \times M \longrightarrow \{-1, 0, 1\}$ is defined by

$$\sigma(u, v) := \begin{cases} u \cdot v & \text{if } u = v, \\ -u \cdot v & \text{if } u = -v, \\ 0 & \text{if } u \neq \pm v. \end{cases} \quad (1)$$

To $k \in \mathbb{Z}$, $\mathbf{a}^*, \mathbf{c}^* \in M^{\mathbb{Z}_m}$ we define

$$\chi_k(\mathbf{a}^*, \mathbf{c}^*) := \sum_{u \in \mathbb{Z}_m} \sigma(a_u, c_{\rho(u+k)}). \quad (2)$$

Then

$$\chi_0(\mathbf{a}^*, \mathbf{a}^*) = \sum_{u \in \mathbb{N}_m} \sigma(a_u, a_u) = 2m + 1, \quad (3)$$

$$\chi_k(\mathbf{a}^*, \mathbf{c}^*) = \chi_{\rho(k)}(\mathbf{a}^*, \mathbf{c}^*), \quad (4)$$

$$\chi_k(\mathbf{a}^*, \mathbf{c}^*) = \chi_{-k}(\mathbf{c}^*, \mathbf{a}^*). \quad (5)$$

So $\chi_k(\mathbf{a}^*, \mathbf{c}^*)$, $k \in \mathbb{Z}$ is completely determined by the indices $k = 1, \dots, m$. To $\mathbf{a}^*, \mathbf{c}^*$ we define the χ -vector of $\mathbf{a}^*, \mathbf{c}^*$:

$$\chi(\mathbf{a}^*, \mathbf{c}^*) := [\chi_k(\mathbf{a}^*, \mathbf{c}^*)]_{k \in \mathbb{N}_m}.$$

Definition 1.

a) $\mathbf{a}^* \in M^{\mathbb{Z}_m}$ is *symmetric*, if $a_u = a_{-u}$ for every $u \in \mathbb{N}_m$.

b) If $\mathbf{a} \in M^{\mathbb{N}_m}$ then

$$\epsilon(\mathbf{a}) := [a_m, a_{m-1}, \dots, a_{-1}, a_0, a_1, \dots, a_{m-1}, a_m]$$

is the *symmetric extension* of \mathbf{a} .

c) If $\mathbf{a}, \mathbf{b} \in M^{\mathbb{N}_m}$ then

$$\chi(\mathbf{a}, \mathbf{b}) := \chi(\epsilon(\mathbf{a}), \epsilon(\mathbf{b})). \quad \blacksquare$$

Definition 2. Suppose $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in M^{\mathbb{N}_m}$.

a) $[\mathbf{a}, \mathbf{b}]$ is a *Hadamard pair* (of size m), if

$$\chi(\mathbf{a}, \mathbf{a}) + \chi(\mathbf{b}, \mathbf{b}) = [1, 0, \dots, 0]. \quad (6)$$

b) Suppose that $[\mathbf{a}, \mathbf{b}], [\mathbf{c}, \mathbf{d}]$ are Hadamard pairs.

$$\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix} \quad (7)$$

is a *Hadamard scheme (of size m)*, if

$$\chi(\mathbf{a}, \mathbf{c}) + \chi(\mathbf{b}, \mathbf{d}) = \mathbf{0}. \quad \blacksquare$$

$\mathbf{HP}(m)$ symbolizes a Hadamard pair, $\mathbf{HS}(m)$ a Hadamard scheme (of size m) and $\mathbf{HM}(n)$ an $(n \times n)$ -Hadamard matrix.

Definition 3. If $k \in \mathbb{Z}, \mathbf{x}^*, \mathbf{y}^* \in M^{\mathbb{Z}_m}, d \in M$ then

$$I(k, \mathbf{x}^*, \mathbf{y}^*; d) := \{u \in \mathbb{Z}_m : \sigma(x_u, y_{\rho(u+k)}) = d\} .$$

If $k, \mathbf{x}^*, \mathbf{y}^*$ are fixed then $I(d) := I(k, \mathbf{x}^*, \mathbf{y}^*; d)$.

To $\mathbf{x}, \mathbf{y} \in M^{\mathbb{N}_m}, d \in M$ define $I(k, \mathbf{x}, \mathbf{y}; d) := I(k, \epsilon(\mathbf{x}), \epsilon(\mathbf{y}); d)$. ■

So for every fixed $k, \mathbf{x}^*, \mathbf{y}^*$

$$\mathbb{Z}_m = I(+) + I(i) + I(-) + I(j) \quad (8)$$

is a partition of \mathbb{Z}_m .

Lemma 1. Using the notation of definition 2 we obtain

a) $[\mathbf{a}, \mathbf{b}]$ is a $\mathbf{HP}(m)$ if and only if

$$\begin{aligned} \#I(k, \mathbf{a}, \mathbf{a}; +) + \#I(k, \mathbf{b}, \mathbf{b}; +) &= \#I(k, \mathbf{a}, \mathbf{a}; -) + \#I(k, \mathbf{b}, \mathbf{b}; -), \\ \#I(k, \mathbf{a}, \mathbf{a}; i) + \#I(k, \mathbf{b}, \mathbf{b}; i) &= \#I(k, \mathbf{a}, \mathbf{a}; j) + \#I(k, \mathbf{b}, \mathbf{b}; j) \end{aligned} \quad (9)$$

for $k = 1, \dots, m$.

b) $\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$ is a $\mathbf{HS}(m)$ if and only if $[\mathbf{a}, \mathbf{b}]$ and $[\mathbf{c}, \mathbf{d}]$ are $\mathbf{HP}(m)$'s and

$$\begin{aligned} \#I(k, \mathbf{a}, \mathbf{c}; +) + \#I(k, \mathbf{b}, \mathbf{d}; +) &= \#I(k, \mathbf{a}, \mathbf{c}; -) + \#I(k, \mathbf{b}, \mathbf{d}; -), \\ \#I(k, \mathbf{a}, \mathbf{c}; i) + \#I(k, \mathbf{b}, \mathbf{d}; i) &= \#I(k, \mathbf{a}, \mathbf{c}; j) + \#I(k, \mathbf{b}, \mathbf{d}; j). \end{aligned} \quad (10)$$

for $k = 0, 1, \dots, m$.

Proof.

$$\chi_k(\mathbf{a}, \mathbf{a}) + \chi_k(\mathbf{b}, \mathbf{b}) = 0$$

\Leftrightarrow

$$\sum_{u \in \mathbb{Z}_m} \sigma(a_u, a_{\rho(u+k)}) + \sigma(b_u, b_{\rho(u+k)}) = 0$$

\Leftrightarrow

$$\sum_{d \in M} \left(\sum_{u \in I(k, \mathbf{a}, \mathbf{a}; d)} \sigma(a_u, a_{\rho(u+k)}) + \sum_{u \in I(k, \mathbf{b}, \mathbf{b}; d)} \sigma(b_u, b_{\rho(u+k)}) \right) = 0$$

\Leftrightarrow

$$\begin{aligned} & (+) \cdot \#I(k, \mathbf{a}, \mathbf{a}; +) + (-) \cdot \#I(k, \mathbf{a}, \mathbf{a}; -) \\ & + (i) \cdot \#I(k, \mathbf{a}, \mathbf{a}; i) + (j) \cdot \#I(k, \mathbf{a}, \mathbf{a}; j) \\ & + (+) \cdot \#I(k, \mathbf{b}, \mathbf{b}; +) + (-) \cdot \#I(k, \mathbf{b}, \mathbf{b}; -) \\ & + (i) \cdot \#I(k, \mathbf{b}, \mathbf{b}; i) + (j) \cdot \#I(k, \mathbf{b}, \mathbf{b}; j) = 0 \end{aligned}$$

\Leftrightarrow

$$\begin{aligned} & (+) \cdot \#I(k, \mathbf{a}, \mathbf{a}; +) + (-) \cdot \#I(k, \mathbf{a}, \mathbf{a}; -) \\ & + (+) \cdot \#I(k, \mathbf{b}, \mathbf{b}; +) + (-) \cdot \#I(k, \mathbf{b}, \mathbf{b}; -) = 0 \end{aligned}$$

and

$$\begin{aligned} & + (i) \cdot \#I(k, \mathbf{a}, \mathbf{a}; i) + (j) \cdot \#I(k, \mathbf{a}, \mathbf{a}; j) \\ & + (i) \cdot \#I(k, \mathbf{b}, \mathbf{b}; i) + (j) \cdot \#I(k, \mathbf{b}, \mathbf{b}; j) = 0. \end{aligned}$$

b) Part b follows by a completely analogous calculation. ■

The following lemma gives a representation of $\chi(\mathbf{a})$, using indices in \mathbb{N}_m only.

Lemma 2. Suppose $\mathbf{a} \in M^{\mathbb{N}_m}$ and $k \in 1, \dots, m$. Then

$$\chi_k(\mathbf{a}) = 1 + 2 \cdot \sum_{u=1}^{\lceil \frac{k+1}{2} \rceil - 1} \sigma(a_{-(u - \lceil \frac{k+1}{2} \rceil)}, a_{u + \lceil \frac{k}{2} \rceil})$$

$$\begin{aligned}
& + 2 \cdot \sum_{u=\lceil \frac{k+1}{2} \rceil}^{m-\lfloor \frac{k}{2} \rfloor} \sigma(a_{u-\lceil \frac{k+1}{2} \rceil}, a_{u+\lfloor \frac{k}{2} \rfloor}) \\
& + 2 \cdot \sum_{u=m-\lfloor \frac{k}{2} \rfloor+1}^m \sigma(a_{u-\lceil \frac{k+1}{2} \rceil}, a_{-(u+\lfloor \frac{k}{2} \rfloor)+(2m+1)}).
\end{aligned}$$

Proof. The mapping $k \rightarrow \rho(2k)$ is bijective on \mathbb{Z}_m . We denote its inverse mapping by $\cdot//2$, i.e. $k \rightarrow k//2$ for every $k \in \mathbb{Z}_m$. Then

$$k//2 = \begin{cases} k/2 & : k \in \mathbb{N}_m, \text{ } k \text{ even,} \\ (k-1)/2 - m & : k \in \mathbb{N}_m, \text{ } k \text{ odd.} \end{cases} \quad (11)$$

It follows

$$\begin{aligned}
\chi_k(\mathbf{a}) & = \chi_k(\epsilon(\mathbf{a})) \\
& = \sum_{u=-m}^m \sigma(a_u, a_{\rho(u+k)}) \\
& = \sum_{u=-m}^m \sigma(a_u, a_{\rho(u+k//2+k/2)}) \\
& = \sum_{u=-m}^m \sigma(a_{\rho(u-k//2)}, a_{\rho(u+k//2)}) \\
& = \sigma(a_{-k//2}, a_{k//2}) + 2 \cdot \sum_{u=1}^m \sigma(a_{\rho(u-|k//2|)}, a_{\rho(u+|k//2|)}) \\
& = 1 + 2 \cdot \sum_{u=1}^m \sigma(a_{|\rho(u-|k//2|)|}, a_{|\rho(u+|k//2|)|}). \quad (12)
\end{aligned}$$

If k is even, then $\rho(k//2) > 0$, if k odd, then $\rho(k//2) < 0$. Using this and (11) we determine $|\rho(u - |k//2|)|$ and $|\rho(u + |k//2|)|$.

$$|\rho(u - |k//2|)| = \begin{cases} -(u - k/2) & k \text{ even, } u \leq k/2 - 1, \\ u - k/2 & k \text{ even, } u \geq k/2, \\ -(u - (m - (k-1)/2)) & k \text{ odd, } u \leq m - (k-1)/2, \\ u - (m - (k-1)/2) & k \text{ odd, } u \geq m - (k+1)/2; \end{cases}$$

$$|\rho(u + |k//2|)| = \begin{cases} u + k/2 & k \text{ even}, u \leq m - k/2, \\ -(u + k/2) + (2m + 1) & k \text{ even}, u \geq m - k/2 + 1, \\ u + m - (k - 1)/2 & k \text{ odd}, u \leq (k - 1)/2, \\ -u + m + (k + 1)/2 & k \text{ odd}, u \geq (k + 1)/2. \end{cases}$$

Inserting these values (7) and using Gauss-brackets leads to the equation of the lemma. \blacksquare

To $\mathbf{a} \in M^{\mathbb{N}_m}$, $k \in \{1, \dots, m\}$ define

$$\psi_k(\mathbf{a}) := \frac{1}{2}(\chi_k(a) - 1). \quad (13)$$

and $\psi(\mathbf{a}) := [\psi_k(\mathbf{a})]_{k=1, \dots, m}$. Then we obtain the following version of Lemma 1:

Lemma 3. Suppose $\mathbf{a} \in M^{\mathbb{N}_m}$ and $k \in 1, \dots, m$. Then

$$\psi_k(\mathbf{a}) = \sum_{\substack{u+v=k \\ 1 \leq u < v}} \sigma(a_u, a_v) + \sum_{u=0}^{m-k} \sigma(a_u, a_{u+k}) + \sum_{\substack{u+v=2m+1-k \\ u < v \leq m}} \sigma(a_u, a_v). \quad \blacksquare$$

From (6) follows

Lemma 4. $[\mathbf{a}, \mathbf{b}]$ is a **HP**(m) if and only if

$$\psi(\mathbf{a}) + \psi(\mathbf{b}) := -\mathbf{1} \quad (14)$$

(where $-\mathbf{1} = -[1, \dots, 1]$). \blacksquare

To obtain a Hadamard scheme one needs two Hadamard pairs $[\mathbf{a}, \mathbf{b}]$, $[\mathbf{c}, \mathbf{d}]$ satisfying $\chi(\mathbf{a}, \mathbf{c}) + \chi(\mathbf{b}, \mathbf{d}) = \mathbf{0}$. The following lemma is a crucial support for the construction of Hadamard schemes, for it delivers to a given Hadamard pair a second one so that they together form a Hadamard scheme.

Lemma 5. If $[\mathbf{a}, \mathbf{b}]$ is a **HP**(m) then

$$\begin{bmatrix} \mathbf{a}, & \mathbf{b} \\ \bar{\mathbf{b}}, & -\bar{\mathbf{a}} \end{bmatrix}$$

is a **HS**(m).

Proof. a) We show first, that $[\bar{\mathbf{b}}, -\bar{\mathbf{a}}]$ is a Hadamard pair:

$$\begin{aligned}
& \chi_k(\bar{\mathbf{b}}, \bar{\mathbf{b}}) + \chi_k((-\bar{\mathbf{a}}), (-\bar{\mathbf{a}})) \\
&= \sum_{u \in \mathbb{Z}_m} (\sigma(\bar{b}_u, \bar{b}_{\rho(u+k)}) + \sigma(-\bar{a}_u, -\bar{a}_{\rho(u+k)})) \\
&= \overline{\sum_{u \in \mathbb{Z}_m} (\sigma(b_u, b_{\rho(u+k)} + a_u, a_{\rho(u+k)})} \\
&= \overline{\chi_k(\sigma(\mathbf{b}, \mathbf{b})) + \chi_k(\sigma(\mathbf{a}, \mathbf{a}))} \\
&= \bar{0} = 0.
\end{aligned}$$

b) Now we verify the condition b of definition 3.

$$\begin{aligned}
& \chi_k(\mathbf{a}, \bar{\mathbf{b}}) + \chi_k(\mathbf{b}, -\bar{\mathbf{a}}) \\
&= \sum_{u \in \mathbb{Z}_m} (\sigma(a_u, \bar{b}_{\rho(u+k)}) + \sigma(b_u, -\bar{a}_{\rho(u+k)})) \\
&= \sum_{u \in \mathbb{Z}_m} (\sigma(a_u, b_{\rho(u+k)}) - \sigma(b_u, a_{\rho(u+k)})) \\
&= \sum_{u \in \mathbb{Z}_m} \sigma(a_u, b_{\rho(u+k)}) - \sum_{u \in \mathbb{Z}_m} \sigma(b_{\rho(-u+k)}, a_{-u}) \\
&= \sum_{u \in \mathbb{Z}_m} \sigma(a_u, b_{\rho(u+k)}) - \sum_{v \in \mathbb{Z}_m} \sigma(b_{\rho(v+k)}, a_v) \\
&= 0. \quad \blacksquare
\end{aligned}$$

To obtain a **HS**(m) it is therefore sufficient to construct a **HP**(m).

It is sometimes desirable to confine the investigation of **HP**(m) to special normalized cases. To define them requires the following preparations: On M introduce the linear order relation $+ > i > - > j$. Then $>$ introduces canonically the lexicographical order relation $>$ on $M^{\mathbb{N}_m}$: $\mathbf{a} = [a_0, \dots, a_m] > \mathbf{b} = [b_0, \dots, b_m]$ if there exists $r \in \mathbb{N}_m$ with $a_r > b_r$ and $a_s = b_s$, $s = 0, \dots, r-1$.

$x \in \mathbb{N}_m$ is *i-leading*, if x has the following property: If $D := \{u \in \mathbb{N}_m : x_u \in \{i, j\}\} \neq \emptyset$, and \min is the now well defined minimal index with

$x_{min} \in \{i, j\}$, then $x_{min} = i$.

Definition 4. $[\mathbf{a}, \mathbf{b}]$, $\mathbf{a}, \mathbf{b} \in M^{\mathbb{Z}_m}$ is a *normalized pair* if:

- a) $\mathbf{a} \geq \mathbf{b}$,
- b) $a_0 = b_0 = +$,
- c) \mathbf{a} and \mathbf{b} are i -leading. ■

The following lemma is trivial:

Lemma 6. To every pair $[\tilde{\mathbf{a}}, \tilde{\mathbf{b}}]$, $\tilde{\mathbf{a}} \leq \tilde{\mathbf{b}} \in \mathbb{Z}_m$ there exists a normalized pair $[\mathbf{a}, \mathbf{b}]$, $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_m$ with $\chi(\mathbf{a}, \mathbf{a}) = \chi(\tilde{\mathbf{a}}, \tilde{\mathbf{a}})$. ■

2 The Construction of Hadamard Matrices

Define

$$\mathbf{H}_+ = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{H}_i = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{H}_- = -\mathbf{H}_+, \quad \mathbf{H}_j = -\mathbf{H}_i.$$

The matrices \mathbf{H}_u , $u \in M$ are symmetric $\mathbf{HM}(2)$. To

$$\mathbf{K} = [k_{p,q}]_{\substack{p=0,\dots,a-1 \\ q=0,\dots,b-1}}, \quad k_{p,q} \in M$$

define

$$\hat{\mathbf{H}}_{\mathbf{K}} := [\mathbf{H}_{k_{p,q}}]_{\substack{p=0,\dots,a-1 \\ q=0,\dots,b-1}}.$$

So the blocked matrix $\hat{\mathbf{H}}_{\mathbf{K}}$ is obtained by replacing each element $k_{p,q} \in M$ of \mathbf{K} by the corresponding (2×2) - matrix $\mathbf{H}_{k_{p,q}}$.

Definition 5. If $\mathbf{a}^* \in M^{\mathbb{Z}_m}$, then the $((2m+1) \times (2m+1))$ -matrix

$$\mathbf{Z}(\mathbf{a}^*) := [a_{\rho(j-i)}]_{i,j \in M^{\mathbb{Z}_m}}$$

is the *circulant matrix* of \mathbf{a}^* . If $\mathbf{a} \in M^{\mathbb{N}_m}$ then $\mathbf{Z}(\mathbf{a}) := \mathbf{Z}(\epsilon(\mathbf{a}))$. ■

Lemma 7. Suppose $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in M^{\mathbb{N}^m}$. If

$$\begin{bmatrix} \mathbf{a}, & \mathbf{b} \\ \mathbf{c}, & \mathbf{d} \end{bmatrix}$$

is a $\mathbf{HS}(m)$, then define the two-fold blocked (2×2) -matrix

$$\hat{\mathbf{H}} = \begin{bmatrix} \hat{\mathbf{H}}_{\mathbf{Z}(\mathbf{a})} & \hat{\mathbf{H}}_{\mathbf{Z}(\mathbf{b})} \\ \hat{\mathbf{H}}_{\mathbf{Z}(\mathbf{c})} & \hat{\mathbf{H}}_{\mathbf{Z}(\mathbf{d})} \end{bmatrix}.$$

Then the $(4(2m+1) \times 4(2m+1))$ deblocked matrix \mathbf{H} of $\hat{\mathbf{H}}$ is a $\mathbf{HM}(8m+4)$.

Proof. In a matrix $\mathbf{R} = [r_{i,j}]_{i=0,\dots,s-1, j=0,\dots,t-1}$ the submatrix consisting of the rows i_1, \dots, i_w , $0 \leq i_1 < \dots < i_w < s$ is denoted by $\mathbf{R}(i_1, \dots, i_w)$. If $0 \leq g \leq 4m$ is even, then for $\mathbf{x} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ we obtain

$$\hat{\mathbf{H}}_{\mathbf{Z}(\mathbf{x})}(g, g+1) = [\mathbf{H}_{x_{\rho(-m-g)}}, \dots, \mathbf{H}_{x_{\rho(0-g)}}, \dots, \mathbf{H}_{x_{\rho(m-g)}}].$$

To $0 \leq p < 8m+4$ define $g(p) = 2 \cdot \lfloor p/2 \rfloor$. Then $\mathbf{H}(p)$ is a row of the deblocked $(2 \times (8m+4))$ -submatrix $\mathbf{H}(g(p), g(p)+1)$ of

$$\begin{aligned} \hat{\mathbf{H}}(g(p), g(p)+1) = & \\ & [\mathbf{H}_{x_{\rho(-m-g(p))}}, \dots, \mathbf{H}_{x_{\rho(0-g(p))}}, \dots, \mathbf{H}_{x_{\rho(m-g(p))}}, \\ & \mathbf{H}_{y_{\rho(-m-g(p))}}, \dots, \mathbf{H}_{y_{\rho(0-g(p))}}, \dots, \mathbf{H}_{y_{\rho(m-g(p))}}] \end{aligned}$$

where $[\mathbf{x}, \mathbf{y}] = [\mathbf{a}, \mathbf{b}]$, if $0 \leq p < 4m+2$ and $[\mathbf{x}, \mathbf{y}] = [\mathbf{c}, \mathbf{d}]$ if $4m+2 \leq p < 8m+4$.

Now suppose $0 \leq p < q < 8m+4$. We show that the rows number p and q of \mathbf{H} are orthogonal, thereby proving the theorem. If $g(p) = g(q)$, then p is even, $q = p+1$ and so $\mathbf{H}(g(p), g(p)+1)$ consists of these two rows. From

$$\mathbf{H}(g(p), g(p)+1) \cdot \mathbf{H}(g(p), g(p)+1)' = (8m+4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we trivially obtain their orthogonality.

Now if $g(p) \neq g(q)$, then the two $(2 \times (4m+2))$ -submatrices $\mathbf{H}(g(p), g(p)+1)$ and $\mathbf{H}(g(q), g(q)+1)$ of \mathbf{H} have no common row. Define

$$\mathbf{A}(p, q) = \mathbf{H}(g(p), g(p)+1) \cdot \mathbf{H}(g(q), g(q)+1)'$$

We show

$$\mathbf{A}(p, q) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

thereby proving the lemma.

Define

$$[\mathbf{w}, \mathbf{x}] = \begin{cases} [\mathbf{a}, \mathbf{b}] & : 0 \leq p < 4m+2, \\ [\mathbf{c}, \mathbf{d}] & : 4m+2 \leq p < 8m+4, \end{cases}$$

and likewise

$$[\mathbf{y}, \mathbf{z}] = \begin{cases} [\mathbf{a}, \mathbf{b}] & : 0 \leq q < 4m+2, \\ [\mathbf{c}, \mathbf{d}] & : 4m+2 \leq q < 8m+4. \end{cases}$$

We use the partition (8) to obtain canonically a partition of $\mathbf{A}(p, q)$ as follows:

$$\begin{aligned} \mathbf{A}(p, q) &= \sum_{d \in M} \sum_{v \in I(k, \mathbf{w}, \mathbf{x}; d)} \mathbf{H}_{\mathbf{w}_v} \times \mathbf{H}'_{\mathbf{x}_{\rho(v+k)}} \\ &+ \sum_{d \in M} \sum_{v \in I(k, \mathbf{y}, \mathbf{z}; d)} \mathbf{H}_{\mathbf{y}_v} \times \mathbf{H}'_{\mathbf{z}_{\rho(v+k)}} \\ &= \sum_{v \in I(k, \mathbf{w}, \mathbf{x}; +)} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \sum_{v \in I(k, \mathbf{y}, \mathbf{z}; +)} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ &+ \sum_{v \in I(k, \mathbf{w}, \mathbf{x}; -)} \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} + \sum_{v \in I(k, \mathbf{y}, \mathbf{z}; -)} \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \\ &+ \sum_{v \in I(k, \mathbf{w}, \mathbf{x}; i)} \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} + \sum_{v \in I(k, \mathbf{y}, \mathbf{z}; i)} \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
& + \sum_{v \in I(k, \mathbf{w}, \mathbf{x}; j)} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} + \sum_{v \in I(k, \mathbf{y}, \mathbf{z}; j)} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \\
& = (\#I(k, \mathbf{w}, \mathbf{x}; +) + \#I(k, \mathbf{y}, \mathbf{z}; +)) \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\
& + (\#I(k, \mathbf{w}, \mathbf{x}; -) + \#I(k, \mathbf{y}, \mathbf{z}; -)) \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \\
& + (\#I(k, \mathbf{w}, \mathbf{x}; i) + \#I(k, \mathbf{y}, \mathbf{z}; i)) \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \\
& + (\#I(k, \mathbf{w}, \mathbf{x}; j) + \#I(k, \mathbf{y}, \mathbf{z}; j)) \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \\
& = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

where zeros in the last equation follow from lemma 1. ■

3 Examples

3.1 An Algorithmic Construction of Hadamard Matrices

If a $\mathbf{HP}(m) = [\mathbf{a}, \mathbf{b}]$ exists, a $\mathbf{HM}(8m + 4)$ can be obtained by the following steps:

Step 1. Build the $\mathbf{HS}(m)$

$$\mathbf{HS}(m) = \begin{bmatrix} \mathbf{a}, & \mathbf{b} \\ \overline{\mathbf{b}}, & -\overline{\mathbf{a}} \end{bmatrix}$$

Step 2. Build the cyclic extension

$$\mathbf{HS}^*(m) = \begin{bmatrix} \epsilon(\mathbf{a}), & \epsilon(\mathbf{b}) \\ \epsilon(\overline{\mathbf{b}}), & \epsilon(-\overline{\mathbf{a}}) \end{bmatrix}$$

Step 3. Build the cyclic $((2m + 1) \times (2m + 1))$ -matrices $\mathbf{Z}(\mathbf{a}), \mathbf{Z}(\mathbf{b}), \mathbf{Z}(\overline{\mathbf{b}}), \mathbf{Z}(-\overline{\mathbf{a}})$.

Step 4. Using the four cyclic $(2m + 1) \times (2m + 1)$ -block matrices $\hat{\mathbf{H}}_{\mathbf{Z}(\mathbf{a})}, \hat{\mathbf{H}}_{\mathbf{Z}(\mathbf{b})}, \hat{\mathbf{H}}_{\mathbf{Z}(\bar{\mathbf{b}})}, \hat{\mathbf{H}}_{\mathbf{Z}(-\bar{\mathbf{a}})}$, having as elements (blocks) (2×2) -matrices \mathbf{H}_u , $u \in M$ build the twofold blocked matrix

$$\hat{\mathbf{H}} = \begin{bmatrix} \hat{\mathbf{H}}_{\mathbf{Z}(\mathbf{a})} & \hat{\mathbf{H}}_{\mathbf{Z}(\mathbf{b})} \\ \hat{\mathbf{H}}_{\mathbf{Z}(\bar{\mathbf{b}})} & \hat{\mathbf{H}}_{\mathbf{Z}(-\bar{\mathbf{a}})} \end{bmatrix}$$

Step 5. Build the $((8m + 4) \times (8m + 4))$ -matrix deblocked $\{+1, -1\}$ matrix \mathbf{H} of $\hat{\mathbf{H}}$. Then \mathbf{H} is a $\mathbf{HM}(8m + 4)$. ■

3.2 Two Examples

We apply these steps to the sizes $m = 0$ and $m = 1$.

$$m = 0.$$

The only normalized pair is $[[+], [+]]$, obtained from $\mathbf{a} = [a_0] = [+]$ and $\mathbf{b} = [b_0] = [+]$ is a $\mathbf{HP}(0)$.

Step 1. Obviously,

$$\begin{bmatrix} + & + \\ + & - \end{bmatrix},$$

is a $\mathbf{HS}(0)$, (which after deblocking accidentally is itself a $\mathbf{HM}(2)$.)

Step 2. Here $\epsilon(\mathbf{a}) = \mathbf{a}, \epsilon(\mathbf{b}) = \mathbf{b}$. So $\mathbf{HS}^*(0) = \mathbf{HS}(0)$.

Step 3. $\mathbf{Z}([\mathbf{a}]) = \mathbf{Z}([\mathbf{b}]) = \mathbf{Z}([\bar{\mathbf{b}}]) = \mathbf{Z}(+) = [+], \mathbf{Z}([-\bar{\mathbf{a}}]) = \mathbf{Z}([-]) = [-]$

Step 4.

$$\hat{\mathbf{H}} = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \end{bmatrix}$$

Step 5. Writing "+" instead of 1 and "-" instead of -1 we obtain the matrix

$$\mathbf{H} = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

This is a $\mathbf{HM}(8 \cdot 0 + 4)$. ■

$$m = 1.$$

Define $\mathbf{a} = [+ , -]$, $\mathbf{b} = [+ , i]$. Then according to section 2 $[+ , -]$, $[+ , i]$ is a normalized $\mathbf{HS}(1)$.

Following again steps 1 to 6 gives us a $\mathbf{HM}(12)$.

(Indeed, from the symmetric extensions $\epsilon(\mathbf{a}) = [- , + , -]$, $\epsilon(\mathbf{b}) = [i , + , i]$ of \mathbf{a} and \mathbf{b} we obtain $\chi_1(\mathbf{a}) = \chi_1(\epsilon(\mathbf{a})) = (-1) + 1 + (-1) = -1$, $\chi_1(\mathbf{b}) = \chi_1(\epsilon(\mathbf{b})) = 0 + 1 + 0 = 1$. This implies $\psi_1(a) + \psi_1(b) = \frac{1}{2}((\chi_k(a) - 1) + (\chi_k(b) - 1)) = -1$. So, as it be checked also easily directly $[[1, -], [1, i]]$ is a $\mathbf{HP}(1)$.)

Step 1. The correponding $\mathbf{HS}(1)$ is

$$\mathbf{HS}(1) = \left[\begin{array}{c} \left[\begin{array}{cc} + & - \\ + & j \end{array} \right] \left[\begin{array}{cc} + & i \\ - & + \end{array} \right] \end{array} \right].$$

Step 2.

$$\mathbf{HS}^*(1) = \left[\begin{array}{c} \left[\begin{array}{cc} - & + \\ j & + \end{array} \right] \left[\begin{array}{cc} i & + \\ + & - \end{array} \right] \end{array} \right].$$

Step 3.

$$\mathbf{Z}(\mathbf{a}) = \begin{bmatrix} - & + & - \\ - & - & + \\ + & - & - \end{bmatrix}, \quad \mathbf{Z}(\mathbf{b}) = \begin{bmatrix} i & + & i \\ i & i & + \\ + & i & i \end{bmatrix},$$

$$\mathbf{Z}(\bar{\mathbf{b}}) = \begin{bmatrix} j & + & j \\ j & j & + \\ + & j & j \end{bmatrix}, \quad \mathbf{Z}(-\bar{\mathbf{a}}) = \begin{bmatrix} + & - & + \\ + & + & - \\ - & + & + \end{bmatrix}.$$

Step 4.

$$\mathbf{H}_K = \left[\begin{array}{c} \left[\begin{array}{c} \left[\begin{array}{cc} - & - \\ - & + \\ - & - \\ - & + \\ + & + \\ + & - \\ + & - \\ + & - \\ + & - \\ + & + \\ + & - \end{array} \right] \\ \left[\begin{array}{cc} + & + \\ + & - \\ - & - \\ - & + \\ - & - \\ - & + \\ + & + \\ + & - \\ - & - \\ + & - \\ - & - \end{array} \right] \\ \left[\begin{array}{cc} - & - \\ - & + \\ + & + \\ + & - \\ - & - \\ - & + \\ + & + \\ + & - \\ + & - \\ - & - \\ - & - \end{array} \right] \\ \left[\begin{array}{cc} - & + \\ + & + \\ - & + \\ + & + \\ + & - \\ + & + \\ + & - \\ + & - \\ - & - \\ - & + \end{array} \right] \\ \left[\begin{array}{cc} + & + \\ + & - \\ - & + \\ + & + \\ - & + \\ - & - \\ - & + \\ + & + \\ + & - \\ + & + \\ + & - \end{array} \right] \\ \left[\begin{array}{cc} - & + \\ + & + \\ - & + \\ + & + \\ - & + \\ - & - \\ - & + \\ + & + \\ + & - \\ + & + \\ + & - \end{array} \right] \\ \left[\begin{array}{cc} - & + \\ + & + \\ - & + \\ + & + \\ - & + \\ - & - \\ - & + \\ + & + \\ + & - \\ + & + \\ + & - \end{array} \right] \end{array} \right]$$

Step 5.

$$\mathbf{H}_K = \begin{bmatrix} - & - & + & + & - & - & - & + & + & + & - & + \\ - & + & + & - & - & + & + & + & + & - & + & + \\ - & - & - & - & + & + & - & + & - & + & + & + \\ - & + & - & + & + & - & + & + & + & + & + & - \\ + & + & - & - & - & - & + & + & - & + & - & + \\ + & - & - & + & - & + & + & - & + & + & + & + \\ + & - & + & + & + & - & + & + & - & - & + & + \\ - & - & + & - & - & - & + & - & - & + & + & - \\ + & - & + & - & + & + & + & + & + & + & - & - \\ - & - & - & - & + & - & + & - & + & - & - & + \\ + & + & + & - & + & - & - & - & + & + & + & + \\ + & - & - & - & - & - & - & + & + & - & + & - \end{bmatrix}$$

3.3 Tables of all normalized $\mathbf{HP}(m)$ of sizes m for $0 \leq m \leq 7$

In the following tables on the left entry the Hadamard-pairs are listed. The right entry contains the two corresponding Ψ -vectors. Every $\mathbf{HP}(m)$ generates a $\mathbf{HM}(8m + 4)$.

m=0

[]	[]
[]	[]

m=1

[i]	[0]
[-]	[-1]

m=2

[i j]	[-1 -1]
[- -]	[0 0]
[- i]	[-1 0]
[i -]	[0 -1]

m=3

[i j j]	[0 0 -2]
[+ - +]	[-1 -1 1]
[i i j]	[0 -2 0]
[- + +]	[-1 1 -1]
[- i j]	[-2 -1 0]
[i - -]	[1 0 -1]
[+ i j]	[0 -1 0]
[i + -]	[-1 0 -1]
[i - j]	[0 -2 -1]
[- i -]	[-1 1 0]
[i + j]	[0 0 -1]
[- i +]	[-1 -1 0]
[i j i]	[-2 0 0]
[+ + -]	[1 -1 -1]
[- - i]	[0 -1 1]
[i j -]	[-1 0 -2]
[+ - i]	[0 -1 -1]
[i j +]	[-1 0 0]

$$\mathbf{m} = 4$$

$\begin{bmatrix} i & j & j & j \\ - & + & + & - \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & -1 & -1 \\ -2 & -2 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} - & i & j & j \\ i & + & + & - \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} - & i & j & j \\ i & - & + & + \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} i & i & i & j \\ + & - & + & - \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & -1 & 1 \\ -2 & 0 & 0 & -2 \end{bmatrix}$
$\begin{bmatrix} i & - & i & j \\ + & i & + & - \end{bmatrix}$	$\begin{bmatrix} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
$\begin{bmatrix} i & - & i & j \\ - & i & + & + \end{bmatrix}$	$\begin{bmatrix} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
$\begin{bmatrix} - & - & i & j \\ i & j & - & - \end{bmatrix}$	$\begin{bmatrix} -1 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 \end{bmatrix}$
$\begin{bmatrix} - & i & - & j \\ i & - & j & - \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & -2 & 1 \\ 0 & -1 & 1 & -2 \end{bmatrix}$
$\begin{bmatrix} i & - & - & j \\ - & i & j & - \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & -2 & 0 \\ -2 & 0 & 1 & -1 \end{bmatrix}$
$\begin{bmatrix} i & j & + & j \\ - & - & i & + \end{bmatrix}$	$\begin{bmatrix} -1 & 1 & 0 & -1 \\ 0 & -2 & -1 & 0 \end{bmatrix}$
$\begin{bmatrix} i & i & + & j \\ - & + & i & - \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & 0 & -1 \\ -2 & 0 & -1 & 0 \end{bmatrix}$
$\begin{bmatrix} - & i & + & j \\ i & + & i & - \end{bmatrix}$	$\begin{bmatrix} -1 & -2 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$
$\begin{bmatrix} i & j & i & i \\ - & - & + & + \end{bmatrix}$	$\begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & -2 & 0 & -2 \end{bmatrix}$
$\begin{bmatrix} + & - & i & i \\ i & j & + & - \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -1 & 0 \\ -2 & -1 & 0 & -1 \end{bmatrix}$
$\begin{bmatrix} i & j & + & i \\ + & - & i & - \end{bmatrix}$	$\begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & -2 \end{bmatrix}$
$\begin{bmatrix} + & - & + & i \\ i & j & j & - \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \end{bmatrix}$
$\begin{bmatrix} - & + & + & i \\ i & j & j & - \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \end{bmatrix}$

$m=5$

$\begin{bmatrix} + & - & i & j & j \\ i & i & - & + & - \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & -2 & -1 & 0 \\ -1 & 0 & 1 & 0 & -1 \end{bmatrix}$
$\begin{bmatrix} - & i & - & j & j \\ i & + & j & - & + \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & -2 & 0 & -1 \\ -1 & -2 & 1 & -1 & 0 \end{bmatrix}$
$\begin{bmatrix} - & + & i & i & j \\ i & i & + & - & - \end{bmatrix}$	$\begin{bmatrix} -2 & -1 & -2 & 1 & 0 \\ 1 & 0 & 1 & -2 & -1 \end{bmatrix}$
$\begin{bmatrix} + & - & + & i & j \\ i & j & j & - & - \end{bmatrix}$	$\begin{bmatrix} -2 & -1 & 0 & 1 & -1 \\ 1 & 0 & -1 & -2 & 0 \end{bmatrix}$
$\begin{bmatrix} i & j & - & + & j \\ - & - & i & i & + \end{bmatrix}$	$\begin{bmatrix} -2 & 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & -1 & 0 \end{bmatrix}$
$\begin{bmatrix} - & i & j & - & i \\ i & + & + & j & - \end{bmatrix}$	$\begin{bmatrix} -2 & -1 & 1 & 0 & 0 \\ 1 & 0 & -2 & -1 & -1 \end{bmatrix}$
$\begin{bmatrix} + & - & i & - & i \\ i & j & + & i & - \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & -1 & -1 & 0 \\ -1 & -2 & 0 & 0 & -1 \end{bmatrix}$
$\begin{bmatrix} - & + & i & - & i \\ i & i & - & j & + \end{bmatrix}$	$\begin{bmatrix} -2 & 1 & 1 & -1 & 0 \\ 1 & -2 & -2 & 0 & -1 \end{bmatrix}$
$\begin{bmatrix} i & - & - & + & i \\ + & i & j & i & - \end{bmatrix}$	$\begin{bmatrix} 0 & -2 & -1 & 1 & 1 \\ -1 & 1 & 0 & -2 & -2 \end{bmatrix}$

m=6

$\begin{bmatrix} + & - & i & j & j & j \\ i & j & - & - & + & - \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -1 \\ -2 & -1 & 0 & -1 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} - & i & + & i & j & j \\ i & - & i & + & - & - \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & -1 & -3 & -1 & 0 \\ 0 & -1 & 0 & 2 & 0 & -1 \end{bmatrix}$
$\begin{bmatrix} i & - & + & i & j & j \\ - & i & i & + & - & - \end{bmatrix}$	$\begin{bmatrix} -1 & -1 & 1 & -2 & -1 & -2 \\ 0 & 0 & -2 & 1 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} i & - & - & j & i & j \\ + & i & i & - & + & - \end{bmatrix}$	$\begin{bmatrix} -1 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}$
$\begin{bmatrix} i & + & i & - & i & j \\ - & i & + & j & - & - \end{bmatrix}$	$\begin{bmatrix} -1 & 1 & -1 & 0 & 0 & -1 \\ 0 & -2 & 0 & -1 & -1 & 0 \end{bmatrix}$
$\begin{bmatrix} - & i & i & j & - & j \\ i & - & - & + & i & + \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & 0 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} i & i & - & i & - & j \\ - & + & i & - & j & + \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ -2 & -2 & -1 & 0 & 0 & -1 \end{bmatrix}$
$\begin{bmatrix} - & + & + & i & - & j \\ i & i & j & + & j & + \end{bmatrix}$	$\begin{bmatrix} -1 & -2 & -2 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 & -1 \end{bmatrix}$
$\begin{bmatrix} i & - & - & + & - & j \\ - & i & i & i & j & + \end{bmatrix}$	$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{bmatrix}$
$\begin{bmatrix} i & + & - & - & + & j \\ - & i & j & j & j & - \end{bmatrix}$	$\begin{bmatrix} -1 & -1 & 0 & -2 & -2 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 \end{bmatrix}$
$\begin{bmatrix} + & + & i & - & j & i \\ i & j & + & j & + & - \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 1 & 0 & -2 & -1 \\ -2 & 1 & -2 & -1 & 1 & 0 \end{bmatrix}$
$\begin{bmatrix} i & j & - & + & j & i \\ + & - & i & i & - & - \end{bmatrix}$	$\begin{bmatrix} -3 & -1 & -1 & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$
$\begin{bmatrix} - & i & j & j & - & i \\ i & + & + & - & i & - \end{bmatrix}$	$\begin{bmatrix} -1 & -2 & -2 & 1 & -1 & 1 \\ 0 & 1 & 1 & -2 & 0 & -2 \end{bmatrix}$
$\begin{bmatrix} + & - & - & i & - & i \\ i & j & i & + & j & - \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 0 \\ -2 & -1 & -1 & 1 & -2 & -1 \end{bmatrix}$
$\begin{bmatrix} - & + & + & i & - & i \\ i & j & j & - & i & - \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 & -1 \end{bmatrix}$
$\begin{bmatrix} - & - & - & i & + & i \\ i & j & i & - & j & + \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -2 \\ -2 & -1 & -1 & -1 & -2 & 1 \end{bmatrix}$
$\begin{bmatrix} + & - & - & i & + & i \\ i & j & i & - & j & - \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & -2 & 0 & 1 & 0 \\ -2 & 1 & 1 & -1 & -2 & -1 \end{bmatrix}$

m=7 (continued)

$\begin{bmatrix} i & + & j & + & + & - & j \\ - & i & - & j & j & i & + \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & -1 & 1 & 0 & -1 \\ -1 & -1 & -2 & 0 & -2 & -1 & 0 \end{bmatrix}$
$\begin{bmatrix} i & i & - & + & + & - & j \\ - & + & i & i & i & j & + \end{bmatrix}$	$\begin{bmatrix} 0 & -2 & 1 & 0 & -1 & -1 & -3 \\ -1 & 1 & -2 & -1 & 0 & 0 & 2 \end{bmatrix}$
$\begin{bmatrix} i & j & i & i & i & + & j \\ + & - & + & + & - & i & - \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -3 & 0 & 0 & 2 & 0 \\ -1 & -1 & 2 & -1 & -1 & -3 & -1 \end{bmatrix}$
$\begin{bmatrix} i & j & i & i & - & + & j \\ + & - & + & + & i & i & - \end{bmatrix}$	$\begin{bmatrix} -2 & 0 & -1 & -2 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$
$\begin{bmatrix} - & i & - & - & + & + & j \\ i & + & j & j & i & j & + \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & -1 & 0 & -1 & -3 & -1 \\ -1 & 0 & 0 & -1 & 0 & 2 & 0 \end{bmatrix}$
$\begin{bmatrix} - & i & - & - & + & + & j \\ i & + & i & j & j & i & + \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & -1 & 0 & -1 & -3 & -1 \\ -1 & 0 & 0 & -1 & 0 & 2 & 0 \end{bmatrix}$
$\begin{bmatrix} i & j & j & j & i & j & i \\ + & - & - & + & - & - & - \end{bmatrix}$	$\begin{bmatrix} -2 & 0 & -2 & -2 & -2 & 2 & 0 \\ 1 & -1 & 1 & 1 & 1 & -3 & -1 \end{bmatrix}$
$\begin{bmatrix} - & + & - & i & + & j & i \\ i & i & i & - & j & + & + \end{bmatrix}$	$\begin{bmatrix} -4 & -1 & 0 & 1 & -1 & -1 & 0 \\ 3 & 0 & -1 & -2 & 0 & 0 & -1 \end{bmatrix}$
$\begin{bmatrix} i & + & - & - & + & j & i \\ - & i & i & i & j & + & + \end{bmatrix}$	$\begin{bmatrix} -2 & -2 & 0 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 0 & 0 & 0 & -2 \end{bmatrix}$
$\begin{bmatrix} i & j & i & j & j & i & i \\ + & - & - & + & - & - & - \end{bmatrix}$	$\begin{bmatrix} -2 & 0 & -2 & -2 & -2 & 2 & 0 \\ 1 & -1 & 1 & 1 & 1 & -3 & -1 \end{bmatrix}$
$\begin{bmatrix} + & - & + & i & j & i & i \\ i & i & j & + & + & - & - \end{bmatrix}$	$\begin{bmatrix} -2 & 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & -2 & -1 & -2 & -1 & 0 & -1 \end{bmatrix}$
$\begin{bmatrix} i & - & + & + & j & i & i \\ + & i & j & i & + & - & - \end{bmatrix}$	$\begin{bmatrix} 0 & -2 & 0 & -1 & 0 & -1 & 3 \\ -1 & 1 & -1 & 0 & -1 & 0 & -4 \end{bmatrix}$
$\begin{bmatrix} + & - & i & j & - & i & i \\ i & j & - & + & j & - & - \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 & -1 \end{bmatrix}$
$\begin{bmatrix} + & i & j & j & i & - & i \\ i & + & - & + & + & i & - \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 1 & -2 & -2 & -2 & 0 \\ -1 & 0 & -2 & 1 & 1 & 1 & -1 \end{bmatrix}$
$\begin{bmatrix} i & + & - & j & + & - & i \\ + & i & j & + & i & i & - \end{bmatrix}$	$\begin{bmatrix} -2 & 0 & -1 & -3 & -1 & 1 & 0 \\ 1 & -1 & 0 & 2 & 0 & -2 & -1 \end{bmatrix}$
$\begin{bmatrix} + & + & i & - & + & - & i \\ i & j & - & i & i & j & + \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 & -1 & 1 & -2 & -1 \\ -1 & -2 & -2 & 0 & -2 & 1 & 0 \end{bmatrix}$
$\begin{bmatrix} i & - & j & + & + & - & i \\ + & i & + & i & j & i & - \end{bmatrix}$	$\begin{bmatrix} 0 & -4 & -1 & -1 & -1 & 0 & 1 \\ -1 & 3 & 0 & 0 & 0 & -1 & -2 \end{bmatrix}$
$\begin{bmatrix} - & + & i & i & j & + & i \\ i & i & - & - & + & j & + \end{bmatrix}$	$\begin{bmatrix} -2 & -1 & -1 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & -1 & -1 & -1 \end{bmatrix}$
$\begin{bmatrix} i & + & i & - & j & + & i \\ - & i & - & i & + & j & + \end{bmatrix}$	$\begin{bmatrix} 0 & -2 & -1 & 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & -2 & -1 & -1 & -2 \end{bmatrix}$

3.4 One normalized HP(8)

m=8

[- - + - i - j j]	[-1 0 1 -2 0 0 1 0]
[i j i i + j - -]	[0 -1 -2 1 -1 -1 -2 -1]
[i i - j - j i j]	[-1 1 -1 -1 -1 -2 -2 1]
[- - i - i + + -]	[0 -2 0 0 0 1 1 -2]

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