Dittmann, Ingolf

Working Paper

Error correction models for fractionally cointegrated time series

Technical Report, SFB 475: Komplexitätsreduktion in Multivariaten Datenstrukturen, Universität Dortmund, No. 2000,02

Provided in Cooperation with:
Collaborative Research Center 'Reduction of Complexity in Multivariate Data Structures' (SFB 475), University of Dortmund

Suggested Citation: Dittmann, Ingolf (2000) : Error correction models for fractionally cointegrated time series, Technical Report, SFB 475: Komplexitätsreduktion in Multivariaten Datenstrukturen, Universität Dortmund, No. 2000,02

This Version is available at:
http://hdl.handle.net/10419/77144

Terms of use:
Documents in EconStor may be saved and copied for your personal and scholarly purposes.
You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.
If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.
Error Correction Models for
Fractionally Cointegrated Time Series

Ingolf Dittmann

December 1999

Abstract:
This note provides a proof of Granger's (1986) error correction model for fractionally cointegrated variables and points out a necessary assumption that has not been noted before. Moreover, a simpler, alternative error correction model is proposed which can be employed to estimate fractionally cointegrated systems in three steps.

JEL Classification Code: C32

Keywords: error correction model; fractional cointegration; Granger Representation Theorem

# I like to thank Christian Kleiber and Walter Krämer for helpful discussions and comments. Financial support by DAAD (Gemeinsames Hochschulsonderprogramm III von Bund und Ländern) is gratefully acknowledged.

§ University of California at San Diego, Department of Economics, 9500 Gilman Drive, La Jolla, CA 92093-0508. Tel.: (858) 534-5476, fax: (858) 534-7040, e-mail: idittman@weber.ucsd.edu
1. Introduction

Fractional cointegration has become an important and relevant topic in time series econometrics in recent years (see, among others, Cheung and Lai, 1993, Baillie and Bollerslev, 1994, Booth and Tse, 1995, and Dittmann, 1998, 1999). A vector $x_t$ of I(1) time series is called fractionally cointegrated if there is a linear combination $a'x_t$ that is a long-memory process or, more formally, I(d) with $0 < d < 1$. In a side remark, Granger (1986) provided an error correction model for this case, which has been widely cited in the fractional cointegration literature. Interestingly enough, this error correction model has never been explicitly proven in the econometrics literature.

This note provides a proof for Granger's error correction model and points out a necessary assumption that seems to have been neglected in the literature so far, namely that all cointegrating relationships share the same long-memory parameter $d$. A detailed interpretation of Granger's error correction model concludes that this model is far too complicated to be useful in practice. Therefore, a simpler error correction model is proposed which can be employed to estimate fractionally cointegrated systems in three steps.

2. Fractional Cointegration and Cointegration Rank

Definition (fractional cointegration): Let $x_t$ be an n-dimensional vector of I(1) processes. $x_t$ is called fractionally cointegrated if there is an $a \in \mathbb{R}^n$, $a \neq 0$, such that $a'x_t \sim I(d)$ with $0 < d < 1$. We call $d$ the equilibrium long-memory parameter and write $x_t \sim FCI(d)$.

Compared to classical cointegration, where $d = 0$, defining the cointegration rank is more difficult for fractionally cointegrated systems, because different cointegrating relationships need not have the same long-memory parameter.

Definition (cointegration rank): Let $x_t$ be an n-dimensional vector of I(1) time series. If there are exactly $r$ linearly independent cointegrating vectors $a_1, a_2, ..., a_r \in \mathbb{R}^n$ with $A = (a_1, a_2, ..., a_r)$ and $A'x_t \sim I(d)$, $d \in (0, 1)$, $r$ is called the cointegration rank with respect to the equilibrium long-memory parameter $d$. We write $x_t \sim FCI(d, r)$. Moreover, if there is a
fixed \( \tilde{d} \in (0, 1) \) so that any linear combination of \( x_i \) is either \( \text{I}(\tilde{d}) \) or \( \text{I}(1) \), \( x_i \) is called *purely fractionally cointegrated with cointegration rank* \( r \).

Note that \( x_i \sim \text{FCI}(d, r) \) implies that \( r \leq (n – 1) \). However, if \( x_i \sim \text{FCI}(d_1, r_1) \) and \( x_i \sim \text{FCI}(d_2, r_2) \), \( r_1 + r_2 \) can be larger than \( (n – 1) \). Hence, the number of independent cointegrating relationships can be larger than \( (n – 1) \). Consider for instance \( x_{1t} \sim \text{I}(1), x_{2t} = x_{1t} + y_t, x_{3t} = x_{2t} + z_t \) with \( y_t \sim \text{I}(d_1), z_t \sim \text{I}(d_2) \) and \( d_1 > d_2 \). Then there are three cointegrating relationships: \((-1, 1, 0) x_t \sim \text{I}(d_1), (-1, 0, 1) x_t \sim \text{I}(d_1) \) and \((0, -1, 1) x_t \sim \text{I}(d_2) \).

### 3. Granger’s Error Correction Model

We consider the multivariate Wold representation of \( \Delta x_i \):

\[
(1 – B) x_t = \delta + C(B) \varepsilon_t
\]

with the (possibly infinite order) moving average polynomial \( C(B) = I_n + C_1 B + C_2 B^2 + C_3 B^3 + ... \) and a zero-mean white-noise vector process \( \varepsilon_t \) (see, e.g., Brockwell and Davis, 1991, p. 187).

**Proposition 1 (Granger Representation Theorem):** If \( x_i \) is purely FCI(d, r) with full-rank \( n \times r \) cointegrating matrix \( A \), then \( A'\delta = 0 \) and \( A'C(1) = 0 \). Furthermore, there exist linear filters \( A(B) \) and \( d(B) \) and an \( n \times r \) matrix \( \Gamma \) of rank \( r \) such that

\[
A(B) (1 – B) x_t = -\left[1 – (1 – B)^{1 – d}\right] (1 – B)^d \Gamma z_t + d(B) \varepsilon_t
\]

with \( z_t = A'x_t \) and \( A(0) = I_n \), where \( d(B) \) is a scalar filter.

**Proof:** As \( z_t = A'x_t \) is \( \text{I}(d) \), \( A'\delta = 0 \) and \( A'\Delta x_t = A'C(1) = 0 \). The sum of the Wold coefficients of a long-memory process with negative long-memory parameter is zero, so that \( A'C(1) = 0 \).

Note that \( C(B) \) can be rewritten as

\[
C(B) = C(1) + (1 – B)^{1 – d} C'(B).
\]

\( C'(B) \) does not contain any fractional differencing filters, so that the corresponding matrix polynomial \( C'(z) \) with complex argument \( z \) is well-defined on the unit-disk with \( C'(1) \neq 0 \).
Using Engle and Granger's (1987) Lemma, which provides the determinant and adjoint of a singular matrix polynomial, we obtain:

\[ \det(C(B)) = (1 - B)^{r(1-d)} d(B) I_n, \]  

and

\[ \text{Adj}(C(B)) = (1 - B)^{(r-1)(1-d)} A^*(B). \]

Multiplying (1) by (5) gives:

\[ (1 - B)^{1+(r-1)(1-d)} A^*(B) x_i = (1 - B)^{r(1-d)} d(B) \varepsilon_i + (1 - B)^{(r-1)(1-d)} A^*(B) \delta \]

\[ \Leftrightarrow \]

\[ (1 - B)^d A^*(B) x_i = d(B) \varepsilon_i + (1 - B)^{d-1} A^*(B) \delta \]

I first show that A^*(1) = \Gamma A'. Then (1 - B)^{d-1} A^*(B) \delta = 0, as A^*(B) \delta = A^*(1) \delta = 0.

Multiplication of (1) by A^*(B) leads to

\[ (1 - B) A^*(B) x_i = A^*(B) C(B) \varepsilon_i + A^*(B) \delta \]

Substituting with (7) gives

\[ (1 - B)^{1-d} [d(B) \varepsilon_i + (1 - B)^{d-1} A^*(B) \delta] = A^*(B) C(B) \varepsilon_i + A^*(B) \delta \]

\[ \Leftrightarrow \]

\[ (1 - B)^{1-d} d(B) \varepsilon_i = A^*(B) C(B) \varepsilon_i . \]

With B = 1 we obtain A^*(1)C(1) = 0. Hence, A^*(1) is a linear combination of the cointegrating vectors A': A^*(1) = \Gamma A'.

Consequently, (7) reduces to

\[ (1 - B)^d A^*(B) x_i = d(B) \varepsilon_i . \]

Factorization of A^*(B) = A^*(1) + (1 - B)^{1-d} A^**(B) gives (together with A^*(1) = \Gamma A'):

\[ (1 - B)^d [(1 - B)^{1-d} (A^*(1) + A^**(B)) + [1 - (1 - B)^{1-d}] A^*(1)] x_i = d(B) \varepsilon_i \]

\[ \Leftrightarrow \]

\[ [A^*(1) + A^**(B)] (1 - B) x_i = - [1 - (1 - B)^{1-d}] (1 - B)^d \Gamma A^* x_i + d(B) \varepsilon_i \]

Choosing A(B) := A^*(1) + A^**(B), we obtain (due to the definition of A^*(B)) A(0) = A^*(1) + A^**(0) = A^*(0). It remains to show that A^*(0) = I_n. With B = 0 in (10) we obtain A^*(0) C(0) = d(0). As C(0) = I_n and d(0) is the determinant of C(0), we get A^*(0) = I_n.

Note that Proposition 1 only holds if x_i is purely fractionally cointegrated. If x_i is multiply fractionally cointegrated, i.e., x_i \sim FCI(d, r) and x_i \sim FCI(d_2, r_2) with d_2 \neq d, the error correction model (2) cannot be a representation of the system x_i. The reason is that (2) takes into account only the information contained in the r cointegrating relationships of order I(d), whereas the information contained in all the other cointegrating relationships is
neglected. Formally, the above proof does not work any more, because $C'(B)$ in (3) is no longer well-defined on the unit disk with $C'(1) \neq 0$, so that the Lemma of Engle and Granger (1987) cannot be applied. In order to see this, let $A$ and $\tilde{A}$ be the two cointegrating matrices with $A'x_t \sim I(d)$ and $\tilde{A}'x_t \sim I(d_j)$ and consider

$$(1 - B)^{d_2} \tilde{A}'x_t = (1 - B)^{d_2 - 1} \tilde{A}' \Delta x_t = (1 - B)^{d_2 - 1} \tilde{A}' C(B) \epsilon_t = (1 - B)^{d_2 - d} \tilde{A}' C'(B) \epsilon_t$$

(14)

As $(1 - B)^{d_2} \tilde{A}'x_t \sim I(0)$, $C'(B)$ contains the factor $(1 - B)^{d - d_2}$. If $d_2 < d$, $C'(1) = 0$ and, if $d_2 > d$, $C'(1)$ does not exist. Note that multiple fractional cointegration can only occur if the dimension $n$ of the vector $x_t$ is larger than two.

Equation (2) is a special case of the formula given in Granger (1986). Granger (1986) considers the more general case that $x_t$ is not $I(1)$, but $I(D)$ with some $D > d$. The present paper uses $D = 1$, because this is the special case commonly associated with fractional cointegration. Nevertheless, the proof given above also works for the more general case. The error correction model (2) then becomes

$$A(B)(1 - B)^D x_t = -[1 - (1 - B)^{D - d}](1 - B)^d \Gamma z_{t-1} + d(B) \epsilon_t$$

(2')

Note that (2) becomes the "traditional" error correction model (see, e.g., Engle and Granger, 1987) for $d = 0$, i.e., in the case of classical cointegration. In this case, only yesterday's equilibrium error $z_{t-1}$ enters the model (2), indicating that today's changes $\Delta x_t$ are influenced by yesterday's equilibrium error $z_{t-1}$ but not by older equilibrium errors. If $d = 0$ and $r = 1$, (2) can be used for estimation of $A(B)$ and $d(B)$ after $\Gamma$ has been estimated by an ordinary least squares regression.

If $d > 0$, (2) becomes more complicated. Let us first rewrite the filter in front of the equilibrium error $\Gamma z_t$ by expanding the fractional differencing filter (see Granger and Joyeux, 1980, or Hosking, 1981):

$$[1 - (1 - B)^{1-d}](1 - B)^d = (1 - B)^d - 1 + B = (1 - d)B + \sum_{j=2}^{\infty} B^j$$

(15)

with $d_1 = -d$ and $d_j = j - 1 - d \frac{d_j - 1}{j} d_{j-1}$. Note that all $d_j$ with $j \geq 1$ are negative with $|d_j|$ decreasing in $j$ and $\sum_{j=2}^{\infty} |d_j| = 1 - d$. 


As all the terms in (15) contain the factor B, only past equilibrium errors $z_{t-1}, z_{t-2}, z_{t-3}$\ldots appear in the error correction model (2). Hence, today's change $\Delta x_t$ can be explained by past changes $\Delta x_{t-1}, \Delta x_{t-2}, \ldots$, past equilibrium errors $z_{t-1}, z_{t-2}, \ldots$ and a stationary error $d(B)\varepsilon_t$. With increasing equilibrium long-memory parameter $d$, the influence of yesterday's equilibrium error $z_{t-1}$ decreases. If $d$ converges to 1, the error correction term vanishes (as (15) converges to zero), so that there is no error correction in the limit.

If $d \in (0, 1)$, the error correction model (2) contains infinitely many past values not only of the equilibrium error $z_t$, but also of the changes $\Delta x_t$ and of the errors $\varepsilon_t$, because, in general, $A(B)$ and $d(B)$ also contain fractional differencing filters. Hence, (2) is unsuitable for estimation.

4. An Alternative Error Correction Model

If $x_t$ is purely FCI($d, r$), the cointegrating matrix $A$ can be partitioned as $A = (A_1, A_2)$ with regular $(r, r)$-matrix $A_1$ and $(r, n-r)$-matrix $A_2$. Moreover, we can normalize $A_1 = I_r$ without loss of generality. Thus, with $z_t = A'x_t \sim I(d)$ and correspondingly partitioned $x_t$, we obtain

$$x_{1t} = -A_2x_{2t} + z_t. \quad (16)$$

This is the Phillips's (1991) triangular representation of the system. It shows that the system contains only $n-r$ random walks $x_{2t}$.

**Proposition 2:** If $x_t$ is purely FCI($d, r$) with cointegrating matrix $A = (I_r \mid A_2)'$, then there exists an $n$-dimensional nondeterministic $I(0)$-process $u_t$ such that

$$\Delta x_t = \begin{pmatrix} I_r & -A_2 \\ 0 & I_{n-r} \end{pmatrix} u_t - \begin{pmatrix} I_r \\ 0 \end{pmatrix} \left[ (1 - (1-B)^{1-d}) (1 - B)^d \right] A' x_t. \quad (17)$$

□
Proof: By taking differences in (16), we obtain
\[
\Delta x_t = \begin{pmatrix} \Delta x_{1t} \\ \Delta x_{2t} \end{pmatrix} = \begin{pmatrix} -A_2 \Delta x_{2t} + (1 - B) z_t \\ \Delta x_{2t} \end{pmatrix}
\] 
(18)

Let \( u_{1t} = (1 - B)^d z_t \) and \( u_{2t} = \Delta x_{2t} \). Then
\[
\Delta x_t = \begin{pmatrix} I_r - A_2 \\ 0 \\ I_{n-r} \end{pmatrix} \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} - \begin{pmatrix} I_r \\ 0 \end{pmatrix} u_{1t} + \begin{pmatrix} I_r \\ 0 \end{pmatrix} (1 - B) z_t
\] 
(19)

and (17) follows immediately.

As in Proposition 1, the assumption that \( x_t \) is purely fractionally cointegrated is necessary. If \( x_t \) is multiply fractionally cointegrated as described above, \( u_t \) in (17) is not I(0) but instead some elements of \( u_t \) are antipersistent (i.e., I(d') with d' < 0).

In contrast to Granger's error correction model, (17) can be used for estimation of the system in three steps if \( r = 1 \): In the first step, \( A_2 \) is estimated by an ordinary least squares regression of \( x_{1t} \) on \( x_{2t} \). Cheung and Lai (1993) show that this estimator is consistent if \( n = 2 \) and \( r = 1 \). In the second step, the long-memory parameter \( d \) of the cointegrating equilibrium \( A'x_t \) is estimated, employing a periodogram regression (cf. Geweke and Porter-Hudak, 1983). In the third step, \( \hat{u}_t \) is calculated from equation (17) and the estimates of the first two stages and modeled as an ARMA-process.
References


