Grammig, Joachim; Hujer, Reinhard; Kokot, Stefan

Working Paper

Tackling boundary effects in nonparametric estimation of intra-day liquidity measures

Working Paper Series: Finance & Accounting, No. 77

Provided in Cooperation with:
Faculty of Economics and Business Administration, Goethe University Frankfurt

Suggested Citation: Grammig, Joachim; Hujer, Reinhard; Kokot, Stefan (2001) : Tackling boundary effects in nonparametric estimation of intra-day liquidity measures, Working Paper Series: Finance & Accounting, No. 77, Johann Wolfgang Goethe-Universität Frankfurt am Main, Fachbereich Wirtschaftswissenschaften, Frankfurt a. M., https://nbn-resolving.de/urn:nbn:de:hebis:30-18531

This Version is available at:
https://hdl.handle.net/10419/76896

JOHANN WOLFGANG GOETHE-UNIVERSITÄT
FRANKFURT AM MAIN

FACHBEREICH WIRTSCHAFTSWISSENSCHAFTEN

Joachim Grammig, Reinhard Hujer and Stefan Kokot

Tackling Boundary Effects in Nonparametric Estimation of Intra-Day Liquidity Measures

July 2001

WORKING PAPER

Tackling Boundary Effects in Nonparametric Estimation of Intra-Day Liquidity Measures[1]

Joachim Grammig[2,3] Reinhard Hujer[3] and Stefan Kokot[3]

Abstract

We investigate methods to estimate intra-day liquidity measures which take into account boundary bias problems affecting the open and closing trading period. In a simulation study we demonstrate the severity of boundary effects when using standard kernel approaches and find that local linear as well as variable kernel estimators offer a much improved performance. In an empirical application using financial transactions data we find a striking asymmetry between the open and close of the New York stock exchange trading process that standard kernel smoothers fail to detect.

Keywords: Liquidity, nonparametric estimation, boundary effects, financial transactions data, local linear estimation, variable kernel methods.

JEL classification: C14, C41, D4.

[1]The authors thank the associate editor and referees for comments that greatly improved the paper. Financial support from the Landeszentralbank Hessen is gratefully acknowledged. We retain scientific responsibility for all remaining errors.

[2]Center for Operations Research and Econometrics, 34 Voie du Roman Pays, B-1348 Louvain-la-Neuve.

[3]Faculty of Economics and Business Administration, Johann Wolfgang Goethe-University, Mertonstr. 17, D-60054 Frankfurt.

Contents

1. Introduction

The notion of liquidity is a crucial one in economics and finance. While there is probably little doubt about the importance of this concept for any theory of asset allocation, at the same time there is no operational, generally accepted definition that indicates how to *measure* the degree of an asset's liquidity. As Grossman and Miller (1988) put it: "*The T-bond Futures pit at the Chicago Board of Trade is surely more liquid than the local market for residential housing. But how much more*?"

Broadly speaking, liquidity can be considered as an asset's ability to be traded in large *quantities* at reasonable *prices* given the demand and supply conditions at the time of the trade. While this definition is certainly intuitively appealing, it does not provide us with an unambiguous measure of liquidity that could be used e.g. for comparison between assets or markets. Yet it emphasizes that volume, price and time to order execution are the principal factors that contribute to the liquidity of an asset. Commonly used measures of liquidity such as the bid-ask spread, the liquidity ratio or the variance ratio fail to account for these three factors appropriately and none of these liquidity indicators explicitly incorporates the time factor.[1] This drawback was addressed by Gouriéroux, Jasiak, and Le Fol (1999), henceforth referred to as GJL, who introduced a new class of intra-day market liquidity measures. By distinguishing different trading events their methods account for all three liquidity components. The measures take the trade arrival process as a starting point, but are easily modified to take account of other characteristics associated with each trade event. So called *weighted durations* focus on the time until a prespecified volume or value has been traded, or the time until a given price change has occurred. GJL show how their liquidity measures can be used for a variety of comparative financial analyses, such as assessing investment strategies based on liquidity orderings between assets or parallel markets.

The GJL approach relies on nonparametric estimation techniques. More precisely, a Gaussian kernel function is used to estimate conditional intensity, density and survivor functions. In this paper, we argue that GJL's use of standard kernel methods implies one important drawback, in that it leads to severely biased estimates of liquidity measures near the open and the close of the trading day. Adapting alternative estimators for density functions with bounded support recently proposed by Lejeune and Sarda (1992), Jones

[1] For a general critique of these liquidity measures see Grossman and Miller (1988) and Schwartz (1992).

(1993), Jones and Foster (1996) and Chen (1999a, 1999b) we propose effective methods to estimate intra day liquidity measures that avoid this drawback.

In the empirical section we illustrate the practical relevance of our alternative approaches. We focus on estimating intra day liquidity measures of selected stocks traded at the New York stock exchange (NYSE). It is a well known stylized fact that trading activity at the NYSE evolves during the regular trading day (i.e. when the initial batch auction phase is neglected) according to a U-shaped pattern, i.e. it shows peaks at the opening and the close. When the auction period is included, the trading process features an asymmetry in the sense, that trading intensity is increasing and takes on the U-shape after completion of the batch auction. While the estimators proposed in this paper are able to detect this idiosyncratic pattern, GJL's standard methods fail to do so.

This paper is organized as follows. In Section 2 we summarize the main statistical properties of GJL's trading activity model and review nonparametric estimation of the implied liquidity measures. Section 3 addresses the inherent boundary bias and discusses methods to alleviate this effect. In section 4 we present simulation results that investigate the severity of boundary effects when estimating liquidity measures and compare the performance of bias reduction methods. The empirical application is presented in section 5. Section 6 concludes the paper.

2. Estimation of intra-daily market activity measures

2.1. A statistical model of intra-day market activity

Following GJL, we assume that trade arrivals evolve randomly in time according to a time dependent Poisson process with intensity function $\lambda(t)$ that depends on the clock time of day. The counting process $N(t)$ gives the number of trades observed from the beginning of the trading day until t. Trades occur according to the following probability law:

$$P[N(t+dt) - N(t) = 0] = 1 - \lambda(t) \cdot dt + o(dt), \tag{1}$$

$$P[N(t+dt) - N(t) = 1] = \lambda(t) \cdot dt + o(dt), \tag{2}$$

which implies

$$P[N(t+dt) - N(t) > 1] = o(dt). \tag{3}$$

Thus, the probability of observing a trade in the interval $(t, t+dt)$ depends on the length of the interval dt and on the intensity function $\lambda(t)$, which gives the instantaneous rate of trading per unit of time.

Based on the theory of self-exciting point processes that dates back to Cox and Lewis (1966), Engle and Russell (1998) define an intensity rate conditional on the complete history of the trading process since the beginning of the day at t^{min},

$$\lambda(t \mid \mathcal{F}(t)) = \lim_{dt \to 0} \frac{P[N(t+dt) - N(t) > 0 \mid \mathcal{F}(t)]}{dt}, \tag{4}$$

where $\mathcal{F}(t) = \{N(t), t_1, \ldots, t_{N(t)}\}$. GJL restrict the conditioning information set so that it contains only the time of day, $\mathcal{F}(t) = t$, which yields[2]

$$\lambda(t) = \lim_{dt \to 0} \frac{P[N(t+dt) - N(t) > 0 \mid t]}{dt}. \tag{5}$$

The intensity function of the pure trading process is an important liquidity measure for an investor whose only concern is the ability to trade a unit share quickly. Since the intensity rate $\lambda(t)$ is proportional to the probability of trading in the next instant of time, conditional on the time of day, an asset a can be considered as being more liquid at time t than asset b if and only if the following relation holds

$$\begin{array}{rrcl} & P_a[N(t+dt) - N(t) = 1] & > & P_b[N(t+dt) - N(t) = 1] \\ \Leftrightarrow & \lambda_a(t) & > & \lambda_b(t). \end{array} \tag{6}$$

Liquidity orderings between one asset at different times or different markets may also be conducted using estimates of liquidity measures that are based on the traded volume and value, as we will discuss in Section 2.3..

2.2. Nonparametric estimation of the trading intensity

Given availability of a financial transactions data set consisting of trading times and associated trade characteristics for a number of M trading days, a simple estimate of the intensity function can be obtained by

$$\hat{\lambda}(i) = \frac{1}{M} \cdot \frac{\sum_{m=1}^{M} n_{i,m}}{\Delta t}, \tag{7}$$

[2]This rules out any dependency of the intensity rate on quantities other than the time of day. Engle and Russell (1998) and Hamilton and Jorda (1999) have introduced parametric models that allow more general intensity rate dynamics.

where $n_{i,m}$ counts the number of trades in interval i on day m, and Δt is the predetermined interval length (Cox and Lewis, 1966). This estimator has a number of drawbacks. First, it uses sample information inefficiently, since only the counts in the fixed interval are taken into account. Second, the fact that (7) is a step-function is inconsistent with the continuity of the trading process. Third, the estimator is sensitive toward local variations caused by random noise contained in the data. However, in the context of this paper it offers the advantage of being robust against boundary effects. Hence, it provides a crude, yet unbiased estimate.

A smooth nonparametric estimator of the intensity function (5) has been proposed by Ramlau-Hansen (1983), extending an earlier contribution by Watson and Leadbetter (1964).[3] The basic idea is to conduct a kernel smoothing of the observed occurrence rates $\frac{1}{Y(t_n)}$ where $Y(t_n)$ is the number of statistical units that are exposed to the risk of going through a transition at the time t_n,

$$\hat{\lambda}(t) = \frac{1}{h_I} \cdot \sum_{n=1}^{N} K_I \left(\frac{t - t_n}{h_I} \right) \cdot \frac{1}{Y(t_n)}, \tag{8}$$

h_I is a bandwidth parameter and $K_I(\cdot)$ an appropriate kernel function. $t_1 < t_2 < \ldots < t_N$ denotes the sequence of jump times of the underlying count process. In the context of this paper this refers to the time of day at which an asset is being traded. Since we restrict our attention to a single asset we always have $Y(t_n) = 1$. The estimator $\hat{\lambda}(t)$ is a weighted average of the observed increments of the count process over the range $[t - h_I, t + h_I]$, i.e. only observations in this interval contribute to the sum. This holds true for all symmetric kernels except the Gaussian, which is usually not bounded, but gives negligible weights to observations that are more than $4h_I$ away from t. Gouriéroux, Jasiak, and Le Fol (1999) propose to treat each trading day in the sample as a distinct, independent realization of the counting process and thus to estimate the intensity rate as an average of the Ramlau-Hansen estimators for separate days. Denote the n-th observed trading time on day m as $t_n(m)$. For a total of M trading days the estimator is given by

$$\hat{\lambda}(t) = \frac{1}{M} \cdot \sum_{m=1}^{M} \left[\frac{1}{h_I} \cdot \sum_{n=1}^{N_m} K_I \left(\frac{t - t_n(m)}{h_I} \right) \right]. \tag{9}$$

[3] See also Andersen, Borgan, Gill, and Keiding (1992), ch. IV.2.1.

2.3. Estimation of liquidity measures based on weighted durations

An extension of the liquidity measure introduced above that takes into account trading volume involves construction of *weighted durations*. A volume duration $x(t,v)$ is defined as the time required to trade a prespecified volume v, given the current time of day t. Formally,

$$x(t,v) = \inf\{x : V(t+x) \geq V(t) + v\}. \tag{10}$$

$V(t) = \sum_{n=1}^{N(t)} v_n$ denotes the trading volume cumulated over a total of $N(t)$ transactions and v_n is the volume of the n-th transaction.[4] So called *trade durations* are obtained by setting v equal to one in equation (10). Volume durations have the natural appeal to account for the quantity dimension of liquidity.

Hence, besides the trade intensity, two additional measures were proposed by Gouriéroux, Jasiak, and Le Fol (1999) in order to characterize intra daily liquidity. The first is the *conditional survivor function* $S(x|t)$, which gives the probability of waiting at least x seconds for the arrival of the next relevant event (e.g. a cumulative traded volume of v shares), given that the last observed trade has occurred at time t. A kernel based estimator is given by

$$\hat{S}(x|t) = \frac{1}{M} \cdot \sum_{m=1}^{M} \frac{\sum_{n=1}^{N_m} I\left(x\left(t_{n-1}(m), v\right) > x\right) \cdot K_I\left(\frac{t_{n-1}(m)-t}{h_I}\right)}{\sum_{n=1}^{N_m} K_I\left(\frac{t_{n-1}(m)-t}{h_I}\right)}. \tag{11}$$

The corresponding *conditional density* $f(x|t)$ can be estimated by

$$\hat{f}(x|t) = \frac{1}{M} \cdot \sum_{m=1}^{M} \frac{1}{h_D} \cdot \frac{\sum_{n=1}^{N_m} K_D\left(\frac{x(t_{n-1}(m),v)-x}{h_D}\right) \cdot K_I\left(\frac{t_{n-1}(m)-t}{h_I}\right)}{\sum_{n=1}^{N_m} K_I\left(\frac{t_{n-1}(m)-t}{h_I}\right)}, \tag{12}$$

where $I(\cdot)$ denotes the indicator function, K_D is the kernel function that is used to smooth the duration time series, K_I the kernel function used to compute the intensity estimates and h_D and h_I denote the corresponding bandwidth parameters.

[4] It is also possible to consider durations weighted with respect to other observed characteristics of a trade event, such as price, value or the prevailing bid-ask spread.

3. Addressing the boundary bias problem

3.1. Boundary effects in liquidity measure estimation

The *boundary bias* arises as a consequence of using a fixed bandwidth h together with a symmetric kernel function $K(\cdot)$ to estimate densities and related quantities, such as the intensity function, with compact support. For example, the widely used Epanechnikov kernel is defined as

$$K_E(u) = 0.75 \cdot \left(1 - u^2\right) \cdot I\left(|u| \leq 1\right). \tag{13}$$

This kernel assigns positive weights to any value of $u \equiv \left(\frac{t - t_n(m)}{h_I}\right)$ that is smaller than one in absolute value, while all $|u| > 1$ receive a weight of zero. Since the trading times always lie in a closed interval $[t^{min}, t^{max}]$, the Epanechnikov kernel implicitly gives weight to values of u *outside* the admissible range in a neighborhood of the bounds. Because there are no observations outside the range, observations in the neighborhood of the bounds will receive weights that are too small. Hence, the amount of smoothing will be systematically higher than during the rest of the day. As a consequence, estimates of the intensity function based on equation (9) will be biased towards zero in any of the boundary regions $[t^{min}, t^{min} + h_I]$ and $[t^{max} - h_I, t^{max}]$. As shown in Marron and Ruppert (1994), the expected value of a standard kernel density estimate at the lower boundary of support $\hat{f}\left(t^{min}\right)$ is approximately equal to $\frac{1}{2} \cdot f\left(t^{min}\right)$ with a similar bias at the upper boundary.

To illustrate the significance of the boundary bias problem we replicate the simulation design used by GJL and generate a sequence of 600 equally spaced trading times.[5] The simulation mimics a 6.5 hour trading day which starts at 9:30 a.m. and ends at 4:00 p.m. Based on the simulated data, we estimate the conditional intensity function according to equation (9). Following GJL we use a Gaussian kernel

$$K_G(u) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-0.5u^2\right), \tag{14}$$

and set the bandwidth parameter equal to 22 minutes. Additionally, we employ the Epanechnikov and two other standard kernel functions, namely the uniform $K_U(u)$ and the quartic $K_Q(u)$,

$$K_U(u) = 0.5 \cdot I\left(|u| \leq 1\right), \tag{15}$$

$$K_Q(u) = \frac{15}{16}\left(1 - u^2\right)^2 \cdot I\left(|u| \leq 1\right). \tag{16}$$

[5] In Section 4 we conduct a simulation study to investigate boundary effects on estimates of liquidity measures in greater detail.

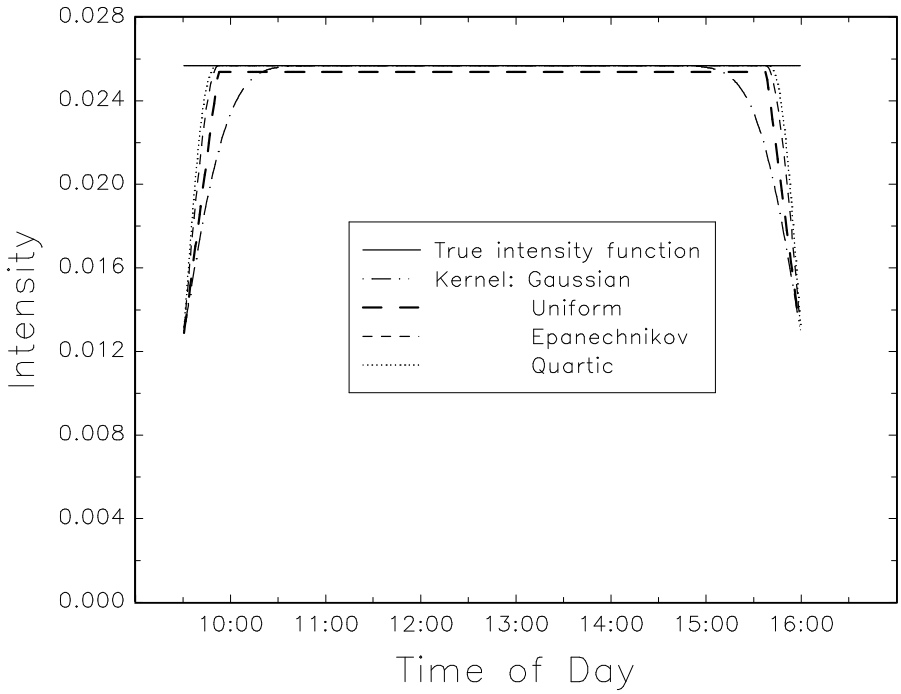

FIGURE 1: Estimated intensity rates for 600 equally spaced trading times.

The true intensity function for the simulated data is a straight line with a value of 0.0257, which is equal to the inverse of the mean duration of 38.94 seconds. As figure 1 shows, all kernel functions do a good job during the central part of the trading day, regardless of the choice of kernel. However, a strong downward bias is clearly visible near the open and close: The trading intensity is severely underestimated. The Gaussian is clearly the kernel that is most affected by the boundary bias. This is an expected result, since the Gaussian is the only kernel with unbounded domain.[6]

Since the conditional survivor function (11) and density function (12) depend on the trading intensity it is likely that the use of standard kernel methods yields estimates that are also affected by boundary bias. An additional source

[6] We have also employed triangular, triweight and cosinus kernels, but the results were not qualitatively different. Also, varying the bandwidth does not improve the results. For all kernels, the region affected by the bias grows with increasing h. All variations of the bandwidth confirmed Marron and Ruppert's result in the sense that the intensity estimate at the upper and lower boundary was approximately equal to one half of the true value.

of boundary effects can be expected for the conditional density (12) because the support of the duration variable defined in (10) is bounded by zero. The use of a standard kernel for $K_D(\cdot)$ is therefore also inappropriate.

3.2. Alternative approaches to boundary bias correction

Previous research on estimation techniques for densities with bounded support includes the *transformation technique* introduced by Copas and Fryer (1980), the *reflection technique* as proposed by Schuster (1985) and Silverman (1986). Alternative approaches to the boundary problem in density estimation as well as in the context of regression analysis have been proposed by Rice (1984), Gasser, Müller, and Mammitzsch (1985), Müller (1991), Hall and Wehrly (1991), Marron and Ruppert (1994), Cowling and Hall (1996) and Cheng, Fan, and Marron (1997) among others [see also Silverman (1986), pp. 29-33 and Jones (1993)].

The probably most obvious remedy for the boundary bias is to transform the data according to $y = h(x)$, where the function $h(.)$ is such that the support of the transformed data y is given by the whole real line. One proceeds to estimate the density of the transformed data using standard kernel estimators, and then re-transforms the density estimates back, employing the well known relation

$$\hat{f}_T(x) = \left| \frac{dh(x)}{dx} \right| \hat{f}(y).$$

This *transformation technique* has been used by Copas and Fryer (1980) in the context of density estimation for nonnegative random variables. They proposed to use the logarithmic transformation, yielding the estimator $\hat{f}_T(x) = \frac{1}{x}\hat{f}(\log x)$. If the support of the density is a finite interval $[a, b]$ Silverman (1986) proposes to use the transformation

$$y = H^{-1}\left(\frac{x-a}{b-a}\right),$$

where H^{-1} is the inverse function of any strictly increasing cumulative distribution defined on the real line.

In general, using this technique leads to an estimator of the density, that will be less smooth near the boundary than in the interior of the support. In the example of nonnegative random variables this effect is caused by the presence of the multiplier $\frac{1}{x}$ that goes to infinity as x approaches zero, resulting in extremely unstable behaviour of the estimate near zero. In cases, where

the density is restricted to a finite interval, this unstable behaviour may well carry over to a substantial proportion of the support, depending on the chosen value of the bandwidth parameter.

The *reflection technique* introduced in Schuster (1985) consists of augmenting the sample by adding *reflections* of the observed data points around the boundaries of the support. In the case of a random variable distributed on the interval $[a, b]$ this yields a data set of size $3n$ given by

$$\{x_1 - a, x_2 - a, \ldots, x_n - a, x_1, x_2, \ldots, x_n, x_1 + b, x_2 + b, \ldots, x_n + b\},$$

where n is the original sample size. An estimate for the density is then obtained by first computing a kernel estimate $\hat{f}$ from the augmented data set of size $3n$, and then taking

$$\hat{f}_R(x) = \begin{cases} 3\hat{f}(x) & \text{if} \quad x \in [a, b] \\ 0 & \text{if} \quad x \notin [a, b] \end{cases}, \tag{17}$$

as the estimate of the density based on the original data. If felt necessary, the technique may be further refined, by augmenting the original data with k additional reflections $\{x_j - ka, x_j - (k-1)a, \ldots, x_j + (k-1)b, x_j + kb\}$, $j = 1, \ldots, n$, thus yielding a sample of size $(2k+1)n$.

The reflection method has the drawback, that the resulting estimator will always have zero derivative at the boundaries, provided that a symmetric and differentiable kernel function has been used for implementation. A related technique, called *negative reflection* by Silverman (1986) consists of giving reflected points weight equal to -1 instead of the usual $\frac{1}{h}$ in the calculation of the estimate, thus forcing the density itself, rather than its derivative, to be equal to zero at the boundary points. This estimator has the additional drawback, that it will not integrate to 1. For these reasons, both estimators do not seem to qualify as serious competitors for the beta kernel estimator, which may reproduce arbitrary features of the density near the boundaries and yields a density estimate that will always integrate to unity. Improved variants of the reflection method are discussed in Hall and Wehrly (1991) and in Cowling and Hall (1996).

Marron and Ruppert (1994) proposed a *three-step estimation technique*, that constitutes a combination of transformation and reflection techniques. In the first step, a transformation $y = h(x)$ is applied to the sample. The transformation $h(.)$ is selected from a parametric family, so that the density of the transformed data y has first derivative of (approximately) zero at the boundaries of its support. In the second stage a standard kernel estimator is

applied to the reflected data set. The last step involves retransformation of the estimates of $\hat{f}(y)$ according to the formula introduced earlier.

The method involves two estimation problems. First an appropriate transformation function $h(.)$ must be found, so that the first derivative condition is met (otherwise the reflection technique yields biased estimates). Estimation of the density of y is then performed conditional on the first step estimate. In general, an optimal transformation function is given by the cumulative density function of the original data $F(x)$. In this case, they show, that the asymptotic bias of $\hat{f}(x)$ will be of the same order in the boundary region and in the interior of its support, thus effectively removing any boundary effects from the resulting estimates.

Since the quantity $F(x)$ is not known in practice, Marron and Ruppert (1994) develop four different algorithms for the estimation of the transformation function from a predetermined family of functions.[7] Optimal choice of the algorithm depends on some features of the true density, e.g. whether it has poles at the boundaries or multiple modes in the interior. Although by using their estimators $\hat{h}(.)$ one can achieve the same effect as by using the optimal transformation $F(x)$, the dependence of the proposed algorithms on the shape of the true density as well as the complexity of implementation make this approach unfavorable in practice.

3.3. Local linear estimators for liquidity measures

The *local linear estimator* is an advanced method to avoid boundary effects, that has been developed in Lejeune and Sarda (1992) and Jones (1993) for densities with nonnegative support. Local linear density estimation involves repeated evaluation of the function $a_s(t,h)$, which is defined as

$$\tilde{a}_s(t,h) = \int_{-1}^{\min(\omega,1)} u^s K(u)\, du,$$

[7] On the topic of transformation function choice see also Wand, Marron, and Ruppert (1991).

where $\omega = \frac{t}{h}$ and $K(\cdot)$ is any standard kernel function with support in $[-1,1]$ when the support of t is nonnegative (Jones (1993)) and

$$\breve{a}_s(t,h) = \begin{cases} \int_{-1}^{\min(\omega,1)} u^s K(u)\, du & \text{if} \quad t \in [0, 1-h] \\ \int_{\max\left(\omega-\frac{1}{h},-1\right)}^{1} u^s K(u)\, du & \text{if} \quad t \in (1-h, 1] \end{cases},$$

when t has support in $[0,1]$, see Chen (1999a).[8] A local linear version of the intensity estimator is given by

$$\hat{\lambda}_L(t) = \frac{1}{M}\sum_{m=1}^{M}\left[\frac{1}{h_I}\sum_{n=1}^{N_m} K_L\left(t, h_I, \frac{t - t_n(m)}{h_I}\right)\right], \tag{18}$$

with boundary adapted kernel function $K_L(t,h,y)$ equal to

$$K_L(t,h,y) = \frac{a_2(t,h) - a_1(t,h)\cdot y}{a_0(t,h)\cdot a_2(t,h) - a_1^2(t,h)} \cdot K(y), \tag{19}$$

In the context of our paper the domain of the trading times is given by the interval $[t^{min}, t^{max}]$. We can account for this by replacing $t_n(m)$ on the right hand side of (18) by standardized trading times $z_n(m) \equiv \frac{t_n(m) - t^{min}}{\Delta}$ and t by $z \equiv \frac{t - t^{min}}{\Delta}$, where $\Delta \equiv t^{max} - t^{min}$. This leads to the following estimator

$$\hat{\lambda}_L(t) = \frac{1}{M}\sum_{m=1}^{M}\left[\frac{1}{h}\sum_{n=1}^{N_m} K_L\left(z, h_z, \frac{z - z_n(m)}{h_z}\right)\right], \tag{20}$$

where $h_z \equiv \frac{h_I}{\Delta}$.

A drawback of the estimators (18) and (20) is that they may produce negative intensity estimates. This is caused by the appearance of the two differences in (19). In order to circumvent this problem, Jones and Foster (1996) proposed a straightforward modification that ensures nonnegativity in density estimation. Their method is easily transferred to the context of intensity estimation. A *nonnegative local linear estimator* of the intensity is obtained by applying the following transformation to $\hat{\lambda}_L(t)$,

$$\hat{\lambda}_{NL}(t) = \frac{\hat{\lambda}(t)}{a_0(t, h_I)} \cdot \exp\left[\frac{\hat{\lambda}_L(t)\cdot a_0(t,h)}{\hat{\lambda}(t)} - 1\right], \tag{21}$$

[8] Note that the expressions for the limits of the integral appearing in the definition of $\breve{a}_s(t,h)$ given in Chen (1999a) are erroneous.

where $\hat{\lambda}$ is the standard kernel estimator of equation(9).

Local linear estimators of the conditional survivor function and the conditional density function are obtained by replacing $K_I\left(\frac{t_{n-1}(m)-t}{h_I}\right)$ in equations (11) and (12) by $K_L\left(z, h_z, \frac{z-z_n(m)}{h_z}\right)$, and $K_D\left(\frac{x(t_{n-1}(m),v)-x}{h_D}\right)$ by $K_L\left(x, h_D, \frac{x(t_{n-1}(m),v)-x}{h_D}\right)$. Nonnegative versions can be obtained by straightforward modifications of equation (21).

Nonnegative local linear estimators of density functions and related quantities have the virtue to eliminate boundary effects in the sense that they lead to estimates with asymptotic bias of the same order in the boundary region as in the interior of the support. The estimators proposed above are easy to implement provided that a kernel function is employed for which analytical expressions of its integral are available so that the function $a_s(t,h)$ can easily be evaluated. We follow Jones and Foster (1996) and Chen (1999a) and employ the quartic kernel of equation (16).

3.4. Variable kernel estimators for liquidity measures

Alternative estimators for liquidity measures designed to reduce boundary effects can be derived from *variable kernel* methods. The basic idea is the use of a kernel function with varying shape that, unlike standard fixed kernels, naturally accommodates to the support of the data. Chen (1999a) proposed to employ the beta p.d.f. as a suitable kernel function for densities with bounded support and Chen (1999b) recommended the gamma p.d.f. for densities with nonnegative support. In the following we will adapt both approaches for the estimation of intraday liquidity measures.

Using Chen's beta type-I kernel yields the following estimator for the conditional intensity function:

$$\hat{\lambda}_{B_I}(t) = \frac{1}{M} \cdot \sum_{m=1}^{M} \left[\frac{1}{N_m \cdot \Delta} \cdot \sum_{n=1}^{N_m} B_{\frac{z}{h_z}+1, \frac{1-z}{h_z}+1}\left(z_n(m)\right) \right], \tag{22}$$

where $B_{p,q}(z)$ denotes the p.d.f. of a standard beta random variable $z \in (0,1)$ with parameters p and q

$$B_{p,q}(z) = \frac{\Gamma(p+q)}{\Gamma(p) \cdot \Gamma(q)} \cdot z^{p-1} \cdot (1-z)^{q-1}. \tag{23}$$

$\Gamma(\cdot)$ denotes the Gamma function. Standardized trading times $z_n(m)$ and the bandwidth parameter h_z are defined as in the previous section. Chen

(1999a) also proposes a second version of the beta kernel density estimator (referred to as beta type II) that is designed to reduce bias in finite samples. Applied to intensity estimation, we get

$$\hat{\lambda}_{B_{II}}(t) = \frac{1}{M} \cdot \sum_{m=1}^{M} \left[\frac{1}{N_m \cdot \Delta} \cdot \sum_{n=1}^{N_m} B^*_{z,h_z}\left(z_n(m)\right) \right], \tag{24}$$

where $B^*_{z,h_z}(.)$ is a modified beta kernel defined as

$$B^*_{z,h_z}(z) = \begin{cases} B_{\rho_B(z,h_z),\frac{1-z}{h_z}} & \text{if} \quad z_n(m) \in [0, 2h_z) \\ B_{\frac{z}{h_z},\frac{1-z}{h_z}} & \text{if} \quad z_n(m) \in [2h_z, 1-2h_z] \\ B_{\frac{z}{h_z},\rho_B(1-z,h_z)} & \text{if} \quad z_n(m) \in (1-2h_z, 1] \end{cases} \tag{25}$$

and

$$\rho_B(z, h_z) = 2.5 + 2h_z^2 - \sqrt{2.25 + 4h_z^4 + 6h_z^2 - z^2 - \frac{z}{h_z}}. \tag{26}$$

Beta kernels can be used for estimation of the conditional survivor function by replacing $K_I(\cdot)$ in equation (11) by $B_{\frac{z}{h_z}+1,\frac{1-z}{h_z}+1}(\cdot)$ or $B^*_{z,h_z}(\cdot)$, respectively.

In order to obtain estimates of the conditional density function $f(x|t)$ we propose to combine beta and gamma kernels. The gamma kernel is an adequate choice for duration data which are nonnegative by definition, see equation (10). In the context of conventional density estimation Chen (1999b) has proposed two versions of the gamma kernel. The first version is given by

$$K_{G_I}\left(x, h, x(t_{n-1}(m), v)\right) = G_{\frac{x}{h}+1,h}\left(x(t_{n-1}(m), v)\right), \tag{27}$$

where $G_{p,q}(x)$ denotes the gamma p.d.f.

$$G_{p,q}(x) = \frac{x^{p-1} \cdot \exp\left(-\frac{x}{q}\right)}{q^p \cdot \Gamma(p)}. \tag{28}$$

An improved performance in terms of the asymptotic mean squared error (AMISE) can be achieved by using Chen's type II gamma kernel, given by

$$K_{G_{II}}\left(x, h, x(t_{n-1}(m), v)\right) = G_{\rho_G(x,h),h}\left(x(t_{n-1}(m), v)\right), \tag{29}$$

where the function $\rho_G(x, h)$ is given by

$$\rho_G(x, h) = \begin{cases} \frac{x}{h} & \text{if} \quad x \geq 2h \\ \left(\frac{x}{2h}\right)^2 + 1 & \text{if} \quad x \in [0, 2h) \end{cases}. \tag{30}$$

A variable kernel estimator of the conditional density function is obtained by replacing $K_D(\cdot)$ in equation (12) by either (27) or (29), and substituting a beta kernel for $K_I(\cdot)$, as described above.

Variable kernel methods offer a number of advantages. The resulting estimates are always nonnegative[9] and the kernels match the support of the data by construction. The effective sample size used in the estimation is equal to the total sample, which leads to a lower finite sample variance. Because of the flexible shape of the kernels, the amount of smoothing is automatically altered as the trading times approach one of the boundary regions without explicitly altering the value of the bandwidth parameter. Asymptotic reasoning and Monte Carlo evidence reveal that for both beta and gamma kernel the type II versions are preferable since they offer smaller bias and variance (Chen (1999b), Chen (1999a) and Bouezmarni and Rolin (2001)).

3.5. Bandwidth choice

When selecting the value of the bandwidth parameter in the estimators discussed in the previous section two aspects have to be considered. First, as shown in Chen (1999b) and Chen (1999a), the optimal value (in terms of AMISE) of the bandwidth parameter for variable kernel estimators is of order $O\left(N^{-\frac{2}{5}}\right)$ instead of $O\left(N^{-\frac{1}{5}}\right)$ for standard kernels, where N is the sample size. Second, since we want to compare the performance of GJL's estimators, which are based on the Gaussian kernel, with the quartic kernel based local linear estimators we have to ensure bandwidth comparability.

With regard to bandwidth choice we use modified versions of Silverman's rule of thumb. This familiar plug-in selector is derived assuming that the true density function belongs to the family of normal distributions and the kernel used for estimation is the Gaussian. The optimal bandwidth h_{opt} is selected by

$$h_{opt} = 0.9 \cdot \hat{A} \cdot N^{-\frac{1}{5}}, \tag{31}$$

where $\hat{A}$ is the minimum of the sample estimates of the standard deviation and the interquartile range divided by 1.34 (Silverman (1986)). To account for the different orders of h_{opt} for fixed and variable kernels we adapt this bandwidth selector accordingly. The bandwidth choices for the Gaussian

[9] This is an advantage over the class of boundary kernel functions considered by Müller (1991).

kernel are [10]

$$h_I = 0.9 \cdot \hat{\sigma}_t \cdot N^{-\frac{1}{5}}, \tag{32}$$

$$h_D = 0.9 \cdot \hat{\sigma}_x \cdot N^{-\frac{1}{5}}, \tag{33}$$

where $\hat{\sigma}_t$ denotes the sample standard deviation of the trading times $t_n(m)$ and $\hat{\sigma}_x$ the corresponding estimate for durations $x(t, v)$. We follow the approach in Chen (1999a,b) and use the same bandwidth selector for variable kernels that we used for standard kernels, except for the replacement of the exponent of the sample size N from $-\frac{1}{5}$ to $-\frac{2}{5}$.

In order to choose the appropriate bandwidth for local linear estimators, we adopt the concept of *canonical kernels*, developed by Marron and Nolan (1989). This allows to transform a bandwidth value h_i derived for a specific kernel $K_i(.)$ into a value h_j that will produce an equivalent estimate using kernel $K_j(.)$ instead.[11] The approach involves translation of the bandwidth h_i according to

$$h_j = \frac{\delta_j}{\delta_i} \cdot h_i, \tag{34}$$

where δ_j and δ_i are kernel specific constants which are given in Marron and Nolan (1989) for commonly used kernel functions. We therefore use relation (34) to translate the optimal Gaussian bandwidths (33) and (33) into values that are appropriate for the quartic kernel. This comes down to multiply Gaussian bandwidths with the constant $\frac{\delta_{Quartic}}{\delta_{Gaussian}} = 2.6226$. The following simulation study will show that these adaptations of Silverman's rule for local linear and variable kernel estimation yield quite satisfactory results.

4. Simulation study

In order to compare the performance of the estimators discussed in the previous sections, we simulate a time-dependent Poisson process that mimics the typical U-shaped intraday activity pattern documented in recent studies (e.g. Engle and Russell (1998), Engle (2000)).[12] We assume that the conditional

[10] We will use only the standard deviation instead of $\hat{A}$ in equation (31), since it turns out to be the smaller quantity in all of our applications.

[11] Equivalence means that both kernel estimates will have the same AMISE.

[12] U-shaped patterns have also been found for volume, volatility and the bid-ask spread in numerous studies based on NYSE transaction data. For a summary of these findings see Goodhart and O'Hara (1997).

intensity $\lambda(t)$ is given by the harmonic oscillation

$$\lambda(t) = a + b \cdot \cos\left(\frac{2\pi\,(t - t^{min})}{c}\right), \tag{35}$$

where $a > b$ is a shift parameter that ensures nonnegativity, b is the amplitude, c is the period of the cycle, and t^{min} is the time (measured in seconds after midnight) at which trading begins. Setting $a = 0.035$, $b = 0.025$, $c = 23400$, and $t^{min} = 34200$, we obtain a U-shaped time of day pattern with a period that is chosen to equal the length $t^{max} - t^{min} = 23400$ (= 6.5 hours) of the NYSE trading day. Local maxima of $\lambda(t)$ are located at the beginning and the end of the trading day ($\lambda(t) = 0.06$) and the minimum is achieved at the middle ($\lambda(t) = 0.01$). The intensity pattern is depicted in Figure 2.

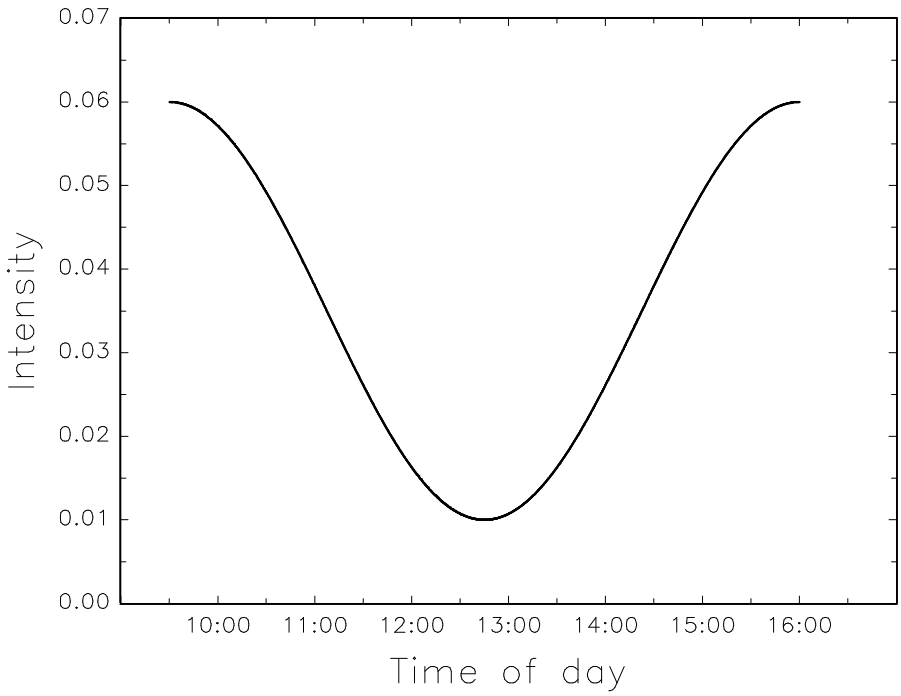

FIGURE 2: True time of day intensity function.

A random sample of trading times is constructed in the following manner: Starting at $t = t^{min}$ we determine the value of the assumed time of day function $\lambda(t^{min})$. We then draw a random number x_1 from an exponential distribution with p.d.f. given by $\lambda(t^{min}) \cdot \exp(-\lambda(t^{min}) \cdot x)$ and obtain $t_1 =$

$t^{min} + x_1$ as the time of the first trade. In order to determine the time of the next trade, another exponential random number x_2, with distribution parameter now set to $\lambda(t_1)$, is drawn. For simplicity, we assume that the trading volume is equal to one unit for each trade. This procedure is repeated until we reach the end of the first trading day. To generate a sample of M independent trading days, the algorithm is then started again at time $t = t^{min}$ for each of the remaining $M-1$ days. This simulation setup has the advantage that the true conditional density and survivor function of the durations x are readily given by $f(x|t) = \lambda(t) \cdot \exp(-\lambda(t) \cdot x)$ and $S(x|t) = \exp(-\lambda(t) \cdot x)$, when the prespecified trading volume v in equation (10) is equal to one.[13] We generated a sample consisting of $M = 50$ trading days. Figure 3 depicts the resulting sample of $N = 40800$ trade durations, sorted by time of day. Note that, as expected, long durations tend to be clustered around 12:45 a.m. and short durations are concentrated at the open and close.

We have estimated transaction intensities based on the simulated data using GJL's Gaussian as well as the nonnegative local linear (NL) and the beta type II estimators. Bandwidths were chosen as outlined in section 3.5.. The resulting estimates are depicted in Figure 4. While all estimators do a good job during the central part of the trading day, the poor performance of the Gaussian estimator within the half hour intervals after the open and before the close is striking. The intensity estimates at the boundaries are approximately half of the true value, which is consistent with the theoretical result of Marron and Ruppert (1994).[14] On the other hand, both the NL and the beta kernel estimators are quite successful in recovering the true time of day activity pattern without any visible boundary effect. Compared to the beta kernel, the NL estimator offers a slightly smaller variance at the bounds.

We obtain a similar result for the estimates of the conditional density function for trade durations. We compute GJL's Gaussian, the NL and the variable kernel estimator, which combines beta type II and gamma type II kernels. Figure 5 depicts the corresponding conditional density estimates at four arbitrarily chosen time points during the trading day. Bandwidth selection has been carried out as discussed in Section 3.5..

Irrespective of the time of day at which the density estimate is evaluated GJL's Gaussian estimator is strongly biased downwards for small durations,

[13] In the general case, where the volume of each trade is a random variable and the prespecified trading volume v is greater than one unit, $f(x|t)$ and $S(x|t)$ would also depend on the distribution of the volume.

[14] Varying the bandwidth and using any other standard kernel function did not remove the boundary effects.

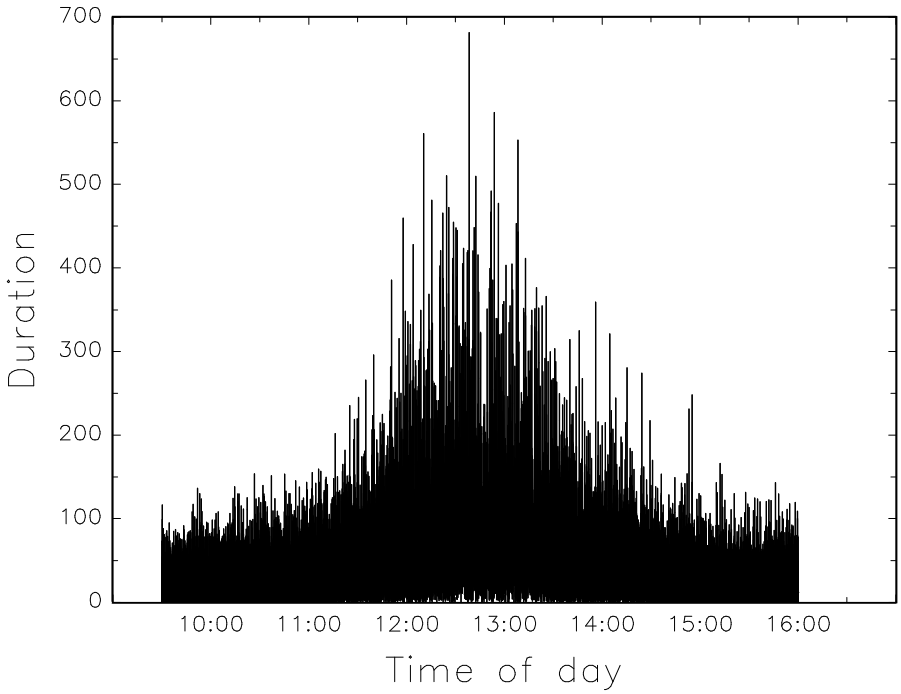

FIGURE 3: Simulated trade durations by time of day.

while the NL and the variable kernel estimators produce conditional densities which are extremely close to the true (exponential) distribution. As an additional illustration, Figure 6 depicts the joint density function of trading times and trade durations and the corresponding Gaussian, NL and variable kernel estimates. It is clearly visible that the boundary effect of the Gaussian estimate is present throughout the trading day. The NL estimates are slightly less noisy than the variable kernel estimates, an effect that becomes more pronounced as we move closer towards the middle of the trading day.

Figure 7 shows that the performance of the three estimators of the conditional survivor function is comparable and quite satisfactory. The small differences between the Gaussian estimate on one side, and the variable kernel and LN on the other, lead us to the conclusion that the boundary effects visible in the conditional density function estimate are mainly caused by the inappropriate use of the Gaussian kernel for the nonnegative duration data.

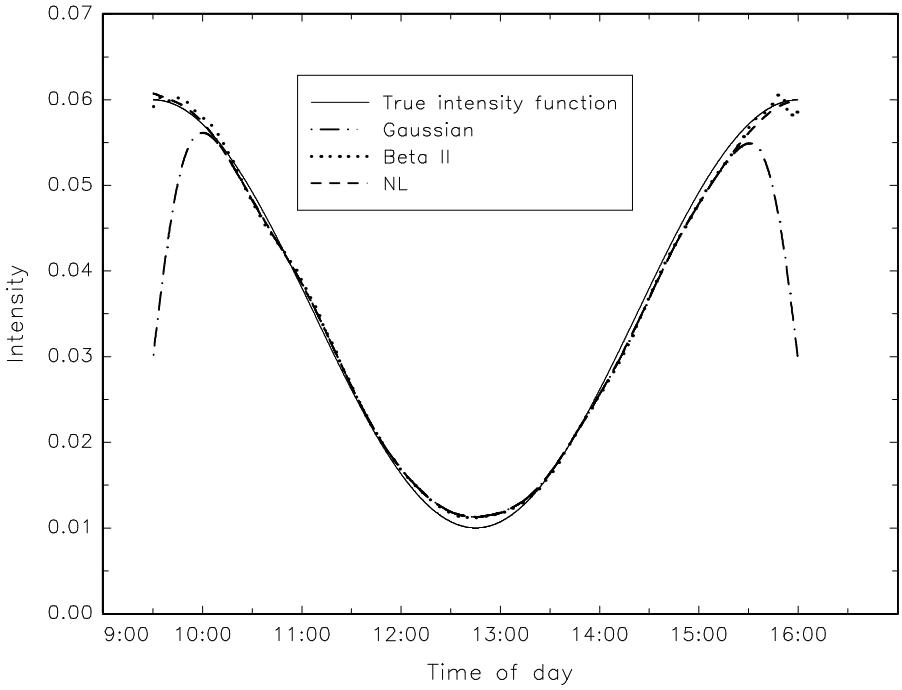

FIGURE 4: Estimates of the trade intensity function for simulated transaction times.

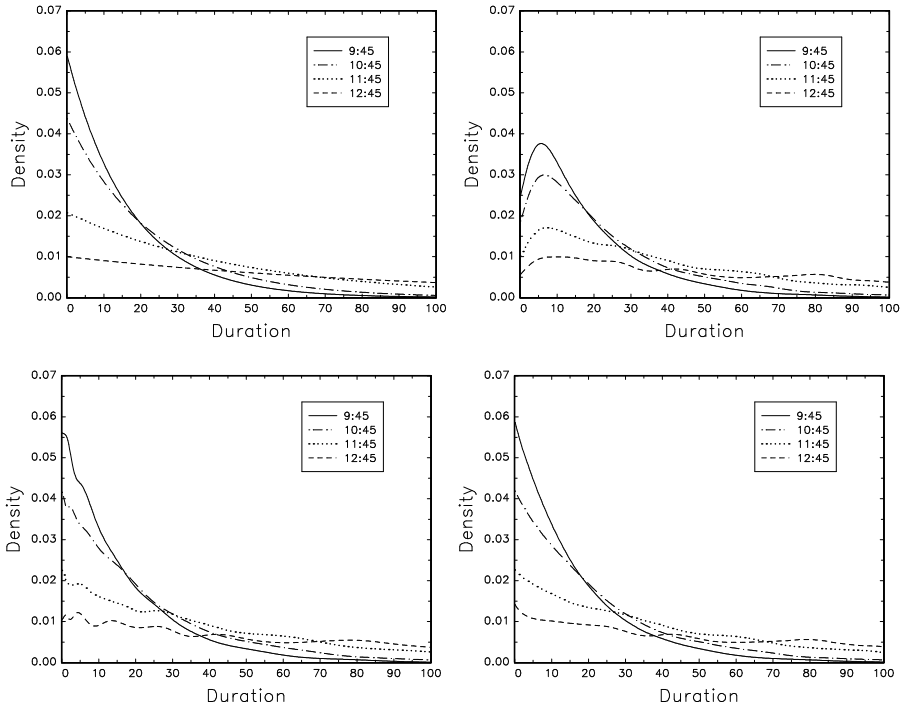

FIGURE 5: Conditional density estimates for simulated trade durations (volume weighted durations with $v = 1$). Top left: True conditional density, top right: Gaussian estimate, bottom left: variable kernel estimate, bottom right: nonnegative local linear estimate.

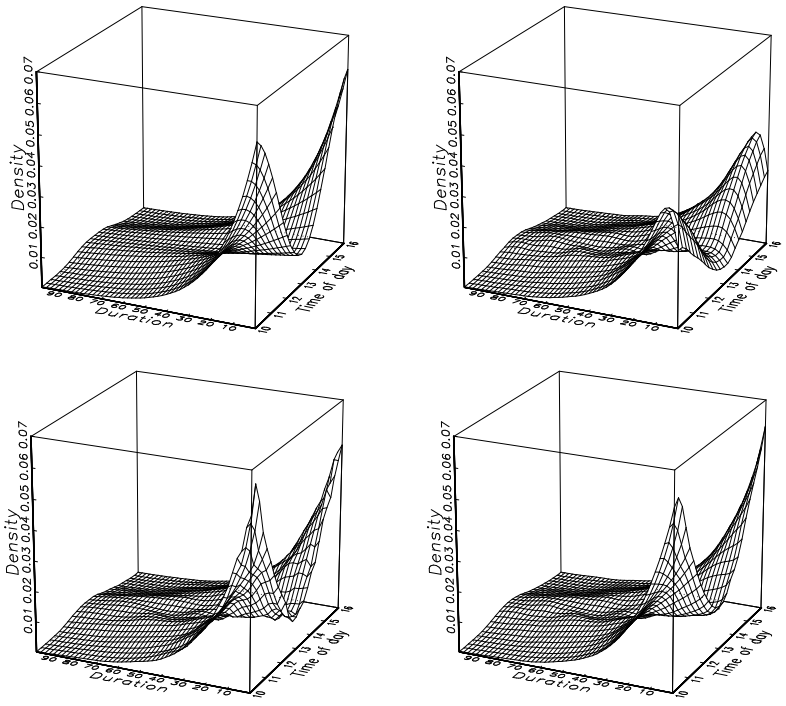

FIGURE 6: Joint density function estimates for simulated trade durations (volume weighted durations with $v = 1$). Top left: True density, top right: Gaussian estimate, bottom left: variable kernel estimate, bottom right: non-negative local linear estimate.

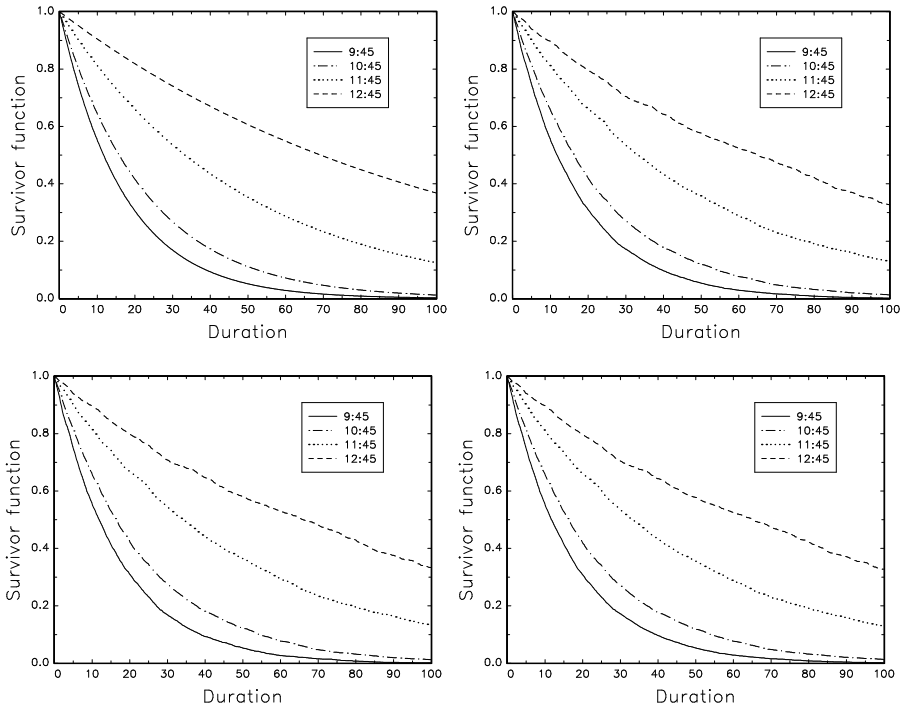

FIGURE 7: Conditional survivor function estimates for simulated trade durations (volume weighted durations with $v = 1$). Top left: True conditional survivor function, top right: Gaussian estimate, bottom left: variable kernel estimate, bottom right: nonnegative local linear estimate.

5. Empirical results

We use transactions data for a selection of four NYSE traded stocks, Disney, IBM, Exxon and Boeing, from the Trades And Quotes (TAQ) data set.[15] The TAQ data set contains information about the timing of the trades, transaction prices and volumes as well as every revision of best bid and ask prices and corresponding volumes. We extract all trades during the regular trading hours (9:30 a.m. until 4:00 p.m.) between 6/3/96 and 12/31/96, excluding observations on the 7/5/96 and the 11/29/96 because of the afternoon closure of the NYSE on these two days. Follwing Engle and Russell (1998), consecutive trades that were recorded with trade durations equal to zero were aggregated. Sample descriptive statistics are reported in Table I.

We decided to include the batch auction period which takes place during the first minutes of each trading day in the data sets used for estimation. During the opening auction, the designated market maker sets a price in order to maximize the transaction volume. After the price is fixed and orders are executed, transactions are recorded. Once this is done, continuous trading begins. Because of a delay of the open that may occur from time to time, the first recorded transaction duration can be relatively long. Hence, we expect an initially low intensity at the open that increases quickly before the U-shape pattern is assumed.

We first present estimates of transaction intensities for our sample of four stocks using GJL's Gaussian, the nonnegative local linear and the beta type II estimator. The bandwidths were chosen as outlined in Section 3.5. and reported in Table I. The resulting estimates for the IBM stock, including, as a reference, the simple intensity estimator of equation (7), are depicted in Figure 8. For the latter, we used a fixed interval length of five minutes. Estimates for the three other stocks are collected in Figure 9. We find that all estimates are generally compatible with the idiosyncratic feature of the NYSE open. On the other hand, while simple, NL and variable kernel estimators predict a sharp *increase* of the trading intensity towards the close, the Gaussian estimator predicts a sharp *decrease* of the trading intensity during the half hour before the end of the trading day. This is clearly a result of the boundary effect.

Taking into account the quantity aspect of liquidity, we finally estimate conditional densities of volume weighted IBM durations, where the cumulated

[15] This selection of stocks was also used in the papers by Bauwens et al. (2000) and Grammig and Maurer (2000).

TABLE I: Sample descriptives and bandwidths

	Boeing	Disney	Exxon	IBM
Trade events	56717	74972	63178	133505
Trading times				
Mean	45311.07	45458.14	45455.30	45512.73
Standard error	7279.43	7279.52	7303.59	7502.78
Skewness	0.13	0.11	0.11	0.09
Excess kurtosis	-1.36	-1.36	-1.38	-1.43
Minimum	34229.00	34224.00	34228.00	34219.00
1st Quartile	38559.00	38756.00	38654.00	38375.00
Median	44656.00	44820.50	44821.00	44993.00
3rd Quartile	52077.00	52274.00	52355.50	52735.75
Maximum	57599.00	57599.00	57599.00	57599.00
Trade durations				
Mean	57.36	43.38	51.38	24.42
Standard error	75.15	55.96	66.49	35.63
Skewness	3.18	3.18	3.33	4.73
Excess kurtosis	15.64	16.53	20.54	40.53
Minimum	1.00	1.00	1.00	1.00
1st Quartile	11.00	8.00	10.00	6.00
Median	31.00	24.00	28.00	12.00
3rd Quartile	72.00	56.00	65.00	29.00
Maximum	1021.00	929.00	1597.00	967.00
Traded volume				
Mean	2.38	1.98	2.58	2.56
Standard error	6.46	4.82	5.80	5.69
Skewness	24.02	10.12	16.98	29.79
Excess kurtosis	1413.21	187.10	694.00	2723.12
Minimum	0.10	0.10	0.10	0.10
1st Quartile	0.20	0.20	0.30	0.50
Median	0.90	0.50	1.00	1.00
3rd Quartile	2.00	2.00	3.00	2.60
Maximum	569.90	209.70	400.00	711.80
Bandwidths for intensity estimation				
Variable kernel	82.20	67.40	67.40	60.15
Gaussian kernel	733.83	694.01	720.55	637.33
Local linear	1924.56	1820.12	1889.73	1671.47

Trading times are measured in seconds since midnight, *trade durations* in seconds and *traded volume* in 1000 shares.

trading volume v in equation (10) is set to 30000 (being roughly equal to the average IBM trading volume in 15 minutes) and 5000 shares for comparison. Applying the bandwidth selection rules of Section 3.5. to the volume weighted durations using $v = 5000$ ($v = 30000$) yields a bandwidth for the Gaussian estimate equal to 11.42 (32.01), for the local linear estimate equal

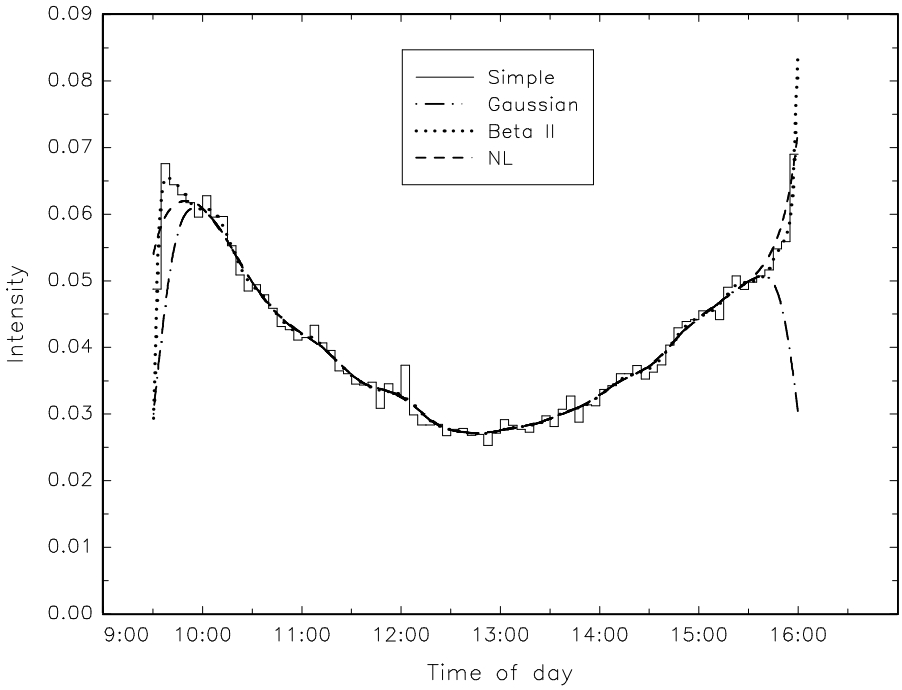

FIGURE 8: IBM intensity estimates.

to 29.95 (83.95), and for the gamma kernel equal to 1.09 (3.06). See Table 1 for the bandwidth w.r.t the trading times. In order to avoid effects of right censoring (i.e. weighted durations that are not completed until the end of the trading day), we restrict our attention to trading times until 3.30 p.m. Figure 10 shows that for $v = 30000$ the variable kernel, nonnegative local linear and the standard Gaussian estimates do not differ to a great extend. The three conditional density estimates are clearly hump shaped. The similarities can be explained by the fact that weighted durations setting $v = 30000$ contain only a small number of durations close to zero. Hence, the boundary effects affecting the Gaussian estimate are reduced. Differences between the three estimates are more pronounced when setting $v = 5000$. In this case a larger number of small durations (closer to the lower bound) are found in the data.

Whilst all estimators produce humped shaped densities, the variable kernel density estimate at small durations is the steepest. The hump of the density close to origin is least pronounced in case of the LN estimate. Similarly, as can be seen in figures 8 and 9, the variable kernel intensity estimator

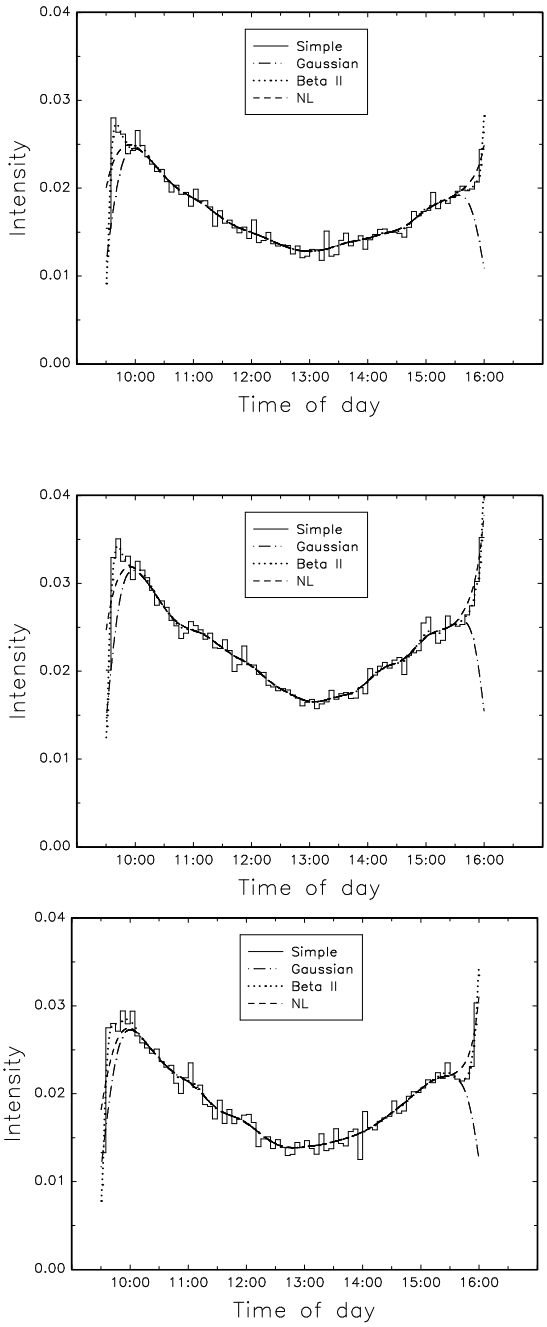

FIGURE 9: Intensity estimates. Top to bottom: Boeing, Disney, Exxon.

seems to capture the highly nonlinear curvature of the intensity function during the first five minutes of the trading day better than the local linear

estimator. This can be seen by comparing the estimates with the crude, yet robust simple estimator. These results let us conclude that the variable kernel estimate is preferable in case of highly nonlinear shapes of intensity and density functions near the boundary.

$v = 30000$ shares. $v = 5000$ shares.

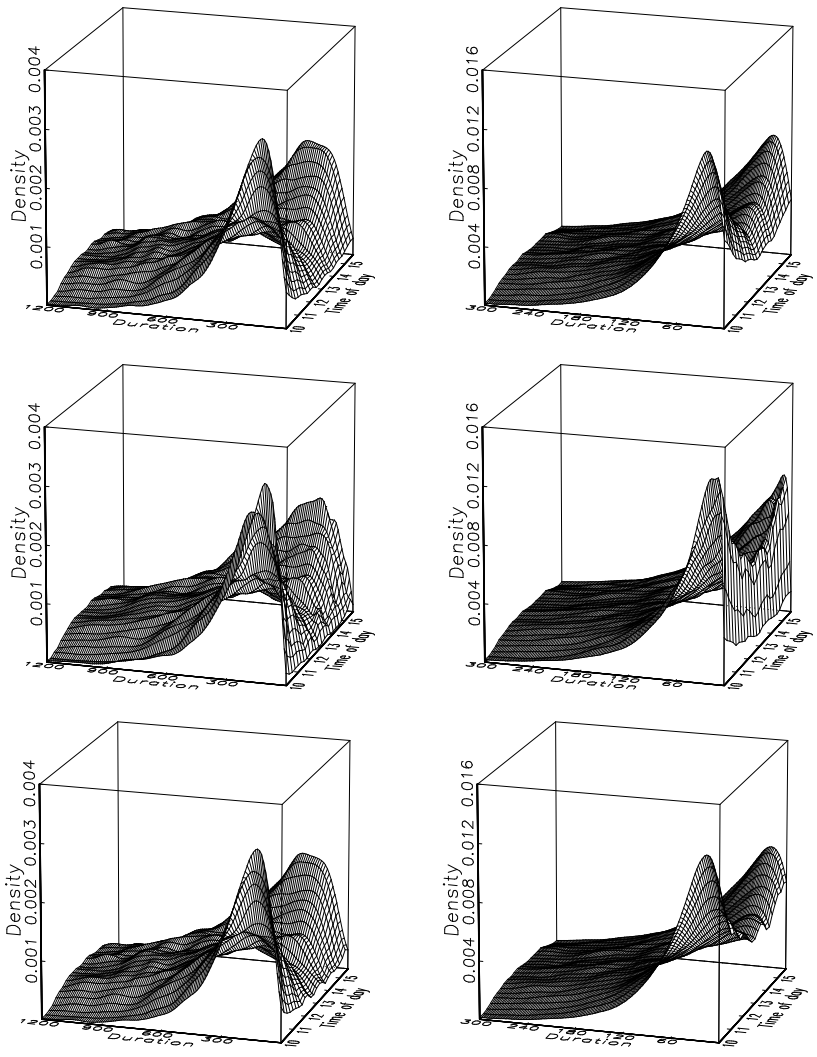

FIGURE 10: Joint density function estimates for volume weighted IBM durations. First row: Gaussian estimate, second row: variable kernel estimate, third row: nonnegative local linear estimate.

6. Conclusions

A meaningful measure of liquidity provides an important input for an investor's asset allocation and trading strategies. Yet there is no operational, generally accepted definition that indicates how to measure the degree of an asset's liquidity. Recognizing that liquidity can be interpreted as the ability to trade an asset quickly in large quantities without causing large price jumps, Gouriéroux, Jasiak, and Le Fol (1999) have introduced a new class of intra-day market activity measures that are designed to account for these three components of liquidity simultaneously.

In this paper we have argued that the standard nonparametric estimators that have been employed in GJL's seminal paper suffer from a serious deficiency. We have shown in a simulation study and using NYSE trade data that these standard methods produce estimates of the intra-day liquidity measures that are severely affected by a boundary bias. We have proposed straightforward adaptions of variable kernel and local linear estimation methods that alleviate this problem.

In an empirical application we have found that the boundary bias works like a disguise, hiding intriguing features of the trading process near the opening and close: First, the non monotonic shape of the intensity function at the opening is not an artefact caused by the boundary bias, but an economically plausible effect attributable to the opening auction. Second, the apparent decrease of trading activity before the close suggested by standard kernel estimators is solely due to the boundary bias: The alternative estimators discussed in this paper clearly indicate an unambiguous increase of trading activity near the close. This asymmetry of the NYSE trading process between the open and the close is not detectable by standard methods.

REFERENCES

Andersen, P. K., O. Borgan, R. D. Gill, and N. Keiding (1992): *Statistical Models Based on Counting Processes*. Springer, New York.

Bauwens, L., P. Giot, J. Grammig, and D. Veredas (2000): "A comparison of financial duration models via density forecasts," CORE Discussion Paper 0060, Université Catholique de Louvain.

Bouezmarni, T., and J. Rolin (2001): "Consistency of beta kernel density function estimator," Discussion Paper 0108, Institut de Statistique, Université Catholique de Louvain.

Chen, S. X. (1999a): "Beta kernel estimators for density functions," *Computational Statistics & Data Analysis*, 31(2), 131–145.

Chen, S. X. (1999b): "Probability density function estimation using gamma kernels," Discussion Paper, School of Statistical Science, La Trobe University.

Cheng, M. Y., J. Fan, and J. S. Marron (1997): "On automatic boundary corrections," *Annals of statistics*, 25(4), 1691–1708.

Copas, J., and M. Fryer (1980): "Density estimation and suicide risks in psychiatric treatment," *Journal of the Royal Statistical Society, Series A*, 143(2), 167–176.

Cowling, A., and P. Hall (1996): "On pseudodata methods for removing boundary effects in kernel density estimation," *Journal of the Royal Statistical Society, Series B*, 58(3), 551–563.

Cox, D., and P. A. W. Lewis (1966): *The Statistical Analysis of Series of Events*. Chapman and Hall, London.

Engle, R. F. (2000): "The econometrics of ultra high frequency data," *Econometrica*, 68(1), 1–22.

Engle, R. F., and J. R. Russell (1998): "Autoregressive conditional duration: A new model for irregulary spaced transaction data," *Econometrica*, 66(5), 1127–1162.

Gasser, T., H.-G. Müller, and V. Mammitzsch (1985): "Kernels for nonparametric curve estimation," *Journal of the Royal Statistical Society, Series B*, 47(2), 238–252.

Goodhart, C. A. E., and M. O'Hara (1997): "High frequency data in financial markets: Issues and applications," *Journal of Empirical Finance*, 4(2-3), 73–114.

Gouriéroux, C., J. Jasiak, and G. Le Fol (1999): "Intra-day market activity," *The Journal of Financial Markets*, 2(2), 193–226.

Grammig, J., and K.-O. Maurer (2000): "Non-monotonic hazard functions and the autoregressive conditional duration model," *The Econometrics Journal*, 3, 16–38.

Grossman, S. J., and M. H. Miller (1988): "Liquidity and Market Structure," *The Journal of Finance*, 43(3), 617–633.

Hall, P., and T. Wehrly (1991): "A geometrical method for removing edge effects from kernel-type nonparametric regression estimators," *Journal of the American Statistical Association*, 86(415), 665–672.

Hamilton, J. D., and O. Jorda (1999): "A model for the federal funds rate target," University of California at San Diego and University of Califonia at Davis.

Jones, M. (1993): "Simple boundary corrections for kernel density estimation," *Statistics and Computing*, 3, 135–146.

Jones, M. C., and P. J. Foster (1996): "A simple nonegative boundary correction method for kernel density estimation," *Statistica Sinica*, 6, 1005–1013.

Lejeune, M., and P. Sarda (1992): "Smooth estimation of distribution and density functions," *Computational Statistics & Data Analysis*, 6, 457–471.

Marron, J., and D. Nolan (1989): "Canonical kernels for density estimation," *Statistics & Probability Letters*, 7, 195–199.

Marron, J.-S., and D. Ruppert (1994): "Transformation to reduce boundary bias in kernel estimation," *Journal of the Royal Statistical Society, Series B*, 56, 653–671.

Müller, H.-G. (1991): "Smooth optimum kernel estimators near endpoints," *Biometrika*, 78, 521–530.

Ramlau-Hansen, H. (1983): "Smoothing Counting Process Intensities by Means of Kernel Functions," *The Annals of Statistics*, 11(2), 453–466.

Rice, J. (1984): "Boundary modifications for kernel regression," *Communications in Statistics, Theory & Methods*, 13, 893–900.

Schuster, E. (1985): "Incorporating support constrains into nonparametric estimators of densities," *Communications in Statistics, Theory & Methods*, 14, 1123–1136.

Schwartz, R. A. (1992): "Market Liquidity," in *The New Palgrave Dictionary of Money and Finance*, ed. by P. Newman, M. Milgate, and J. Eatwell, pp. 666–667. Macmillan, London.

Silverman, B. W. (1986): *Density Estimation for Statistics and Data Analysis*. Chapman and Hall, London.

Wand, M., J. Marron, and D. Ruppert (1991): "Transformations in density estimation (with discussion)," *Journal of the American Statistical Association*, 86(414), 343–361.

Watson, G. S., and M. R. Leadbetter (1964): "Hazard Analysis I," *Biometrika*, 51(1-2), 175–184.