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On the Frequency of Interior Cournot-Nash Equilibria in a Public Good Economy

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ON THE FREQUENCY OF INTERIOR COURNOT-NASH EQUILIBRIA IN A PUBLIC GOOD ECONOMY

Abstract

In a public good economy the distribution of initial income is an important determinant of how many individuals contribute to the public good. For the case when all individuals have identical preferences in this paper a simple formula is derived that describes the proportion of all income distributions for which an interior Cournot-Nash equilibrium will result in which every agent makes a strictly positive contribution to the public good. This formula is then applied to a standard Cobb-Douglas utility function showing that the likelihood of interior Cournot-Nash equilibria falls dramatically when the number of individuals is increased. The implications this result might have for the significance of Shibata-Warr neutrality are finally discussed.

Keywords: private provision of public goods, Cournot-Nash equilibria, Shibata-Warr neutrality.

JEL Code: D3, H4.
1 Introduction

The theory of private provision of a public good typically focuses on interior Cournot-Nash equilibria in which all agents make a strictly positive contribution to the public good. It is, however, well known that such interior solutions are not the only possible outcome of the Cournot-Nash provision game. Rather, corner solutions may occur in which some agents are complete free riders who do not contribute at all to the public good. Whether an interior or corner solution will prevail as a Cournot-Nash equilibrium is largely determined by the distribution of initial income, the preferences of the agents, as well as by the size of the economy. So it has been shown that corner solutions are more likely when incomes or preferences are heterogeneous (see Bergstrom et al., 1986) and the number of agents is high (see Andreoni, 1988). In this paper we will analyse how these determinants interact. In order to simplify the exposition and to emphasise the relationship with the famous Shibata-Warr neutrality (see Shibata, 1971, Warr, 1983, and Cornes and Sandler, 1996) we will concentrate the analysis on economies in which individuals have identical preferences. We first derive a simple general formula that indicates the proportion of all possible income distributions for which an interior Cournot-Nash equilibrium will result. We then apply this formula to generate numerical simulations when the preferences are Cobb-Douglas. It turns out that the likelihood of interior Cournot-Nash equilibria falls dramatically as the number of individuals is increased. When e.g. the private and the public good have equal weight in the utility function interior Cournot-Nash equilibria will almost certainly not arise even if there are only five or six individuals. We conclude by discussing the adverse consequences that this observation might have for Shibata-Warr neutrality.

2 The Range of Income Distributions Leading to Interior Cournot-Nash equilibrium: General Analysis

Let \( u(x_i,G) \) be the common utility function where \( x_i \) is private consumption of individual \( i \) and \( G \) the provision level of a pure public good. Utility is strictly quasi-concave, twice continuously differentiable and strictly monotone increasing in both variables. Additionally we assume that both goods are strictly normal. All individuals have the same linear technology for producing the public good. Without loss of generality the marginal rate of transformation
between the private and the public good then can be normalised to one. We want to consider economies of different population sizes, denoted by \( n \). The average (per person) private good endowment \( \bar{y} \), however, is always the same.

Let \( e(G) \) be the income expansion path which connects all points in the \( x_i - G \)-space in which the marginal rate of substitution between the public and the private good is equal to one. It follows from strict normality that \( e(G) \) is well defined and strictly increasing in \( G \), and that \( e(0) = 0 \) holds. The inverse function of \( e(G) \) then exists and is denoted by \( G(x_i) \).

In order to characterise Cournot-Nash equilibria in an easy way, we will use functions \( r(G, y_i) \) which, for any \( (G, y_i) \) describes how much an individual with endowment \( y_i \) would voluntarily contribute to the public good in a Cournot-Nash equilibrium in which aggregate public good supply were \( G \). Formally, these "replacement functions" are given by

\[
(1) \quad r(G, y_i) = \max(y_i - e(G), 0).
\]

By strict normality such a function is strictly monotone decreasing if \( G < G(y_i) \) and truncated at \( G = G(y_i) \). Note that in the case \( G < G(y_i) \) an individual with income \( y_i \) can only be in a Cournot-Nash equilibrium when her position is on the income expansion path and she makes a strictly positive contribution to the public good. In the case \( G \geq G(y_i) \), however, she has no incentive to make a positive contribution to the public good by her own when the other individuals provide \( G \). Many other fruitful applications of the replacement function can be found in Cornes and Hartley (2003).

For any distribution \( (y_1, \ldots, y_n) \) of aggregate initial income among the \( n \) agents the public good level \( G^N_n \) in the Cournot-Nash equilibrium then is given by the consistency requirement

\[
(2) \quad G^N_n = \sum_{i=1}^{n} r(G^N_n, y_i).
\]
By $\hat{G}_n$ we now denote public good supply which, in an economy consisting of $n$ agents, is obtained when income is equally distributed such that every individual has the same income $\bar{y}$ and makes the same contribution to the public good. Then (2) becomes

\[(3) \quad \hat{G}_n = nr(\hat{G}_n, \bar{y}).\]

As $\hat{G}_n > 0$ this Cournot-Nash equilibrium obviously is an interior one where each individual makes a strictly positive contribution to the public good. From Shibata-Warr neutrality it is well known that $\hat{G}_n$ also describes public good supply in any interior Cournot-Nash equilibrium, independent of the distribution of aggregate initial income leading to an interior solution. Then $\hat{x}_n := e(\hat{G}_n)$ gives each individual’s private consumption not only in the symmetric case but also in any interior Cournot-Nash equilibrium. Now it is possible to describe the set of all income distribution leading to an interior Cournot-Nash equilibrium.

**Proposition 1**: Given a certain distribution $(y_1, \ldots, y_n)$ of aggregate income $n\bar{y}$ the corresponding Cournot-Nash equilibrium is interior if and only if $y_i > \bar{y}_n := \hat{x}_n$ for every individual $i = 1, \ldots, n$.

**Proof**: Individual $i$ with income $y_i$ makes a strictly positive contribution to the public good if and only if $r(\hat{G}_n, y_i) > 0$ which (by (1)) is equivalent to $y_i > e(\hat{G}_n) = \hat{x}_n$. QED.

At first sight the condition on income distributions for having interior Cournot-Nash equilibria provided by Proposition 1 might seem rather innocuous. Nevertheless it implies that the range of income distributions leading to interior Cournot-Nash equilibria shrinks rapidly as $n$ increases. This will turn out to be a consequence of the following result:

**Proposition 2**: In a public goods economy with population size $n$ the proportion of all income distributions leading to an interior Cournot-Nash equilibrium is
Proof: In the $R^n$-space all feasible income distributions are described by the $(n-1)$-dimensional simplex $P_n(0)$ the vertices of which are given by the $n$ vectors $(\overline{y},0,...,0), (0,\overline{y},0,...,0), ...$ and $(0,...,0,\overline{y})$. The set of all income distributions $(y_1,...,y_n)$ for which $y_i > \overline{y}_n$ holds for every $i=1,...,n$ is the interior of the $(n-1)$-dimensional sub-simplex $P_n(\overline{y}_n)$ which has the $n$ vertices $(\overline{y}_n,\overline{y}_n,...,\overline{y}_n), (\overline{y}_n,\overline{y}_n,\overline{y}_n,...,\overline{y}_n), ...$ and $(\overline{y}_n,\overline{y}_n,...,\overline{y}_n,\overline{z}_n)$ where $\overline{z}_n := n\overline{y}-(n-1)\overline{y}_n$. The sub-simplex $P_n(\overline{y})$ is obtained from $P_n(0)$ by a linear contraction with the point $(\overline{y},...,\overline{y})$ as the centroid and $1-\frac{\overline{y}_n}{\overline{y}}$ as the contraction factor. As it follows from $\tilde{g}_n = \frac{\tilde{G}_n}{n} = \overline{y} - \overline{y}_n$ that $\frac{\tilde{g}_n}{\overline{y}} = 1-\frac{\overline{y}_n}{\overline{y}}$ the volume of $P_n(\overline{y}_n)$ in the $(n-1)$-dimensional space then is $\left(\frac{\tilde{g}_n}{\overline{y}}\right)^{n-1}$ times the volume of $P_n(0)$ which gives (4). QED.

In Figure 1 the idea lying behind the proof of Proposition 2 is illustrated for $n=3$, where the height of the triangle $ABC$ is $\overline{y}$. 
In Figure 1 every point in the equilateral triangle $ABC$ (which has been projected from the three-dimensional into the two-dimensional space) represents a certain income distribution where the income levels of the three agents are given by the vertical distances to the sides of the triangle. The equilateral sub-triangle $DEF$ then represents the set of income distributions which according to Proposition 1 imply an interior Cournot-Nash equilibrium. The normalised side length of $DEF$ is $\frac{\tilde{g}_1}{\bar{y}} = 1 - \frac{\bar{y}_3}{\bar{y}}$ times the side length of the original triangle such that 

\[ \left( \frac{\tilde{g}_1}{\bar{y}} \right)^2 \] 

is the ratio of the areas of both triangles.

The basic formula described by Proposition 1 now makes it possible to describe how $p_n$ evolves when $n$ is increased.

**Proposition 3**: The proportion $p_n$ of income distributions leading to an interior Cournot-Nash equilibrium converges to zero when population size goes to infinity.

**Proof**: By (3) and (4)

\[ (5) \quad \tilde{g}_n = r(n\tilde{g}_n, \bar{y}) \]

holds for any $n$. As the replacement function is strictly decreasing when $G < G(\bar{y})$ and $\tilde{G}_n < G(\bar{y})$ for any $n$ it follows from (5) that $\tilde{g}_n$ is decreasing in $n$. Then it is implied by (4) that $p_n$ is bounded from above by $p_{2^{-1}}^n = \left( \frac{\tilde{g}_2}{\bar{y}} \right)^{n-1}$ which converges to zero as $p_2 < 1$.

QED.

When preferences for the public good as expressed by the utility function $u$ are strong, then $\tilde{g}_n$ will be large. Then, by (4), the share parameter $p_n$ will be relatively high for every $n$, too, but $p_n$ will nevertheless converge to zero when $n$ grows. Conversely, when preferences for the public good are weak the share parameter will be low, too, and the proportion of income distributions will fall as $n$ rises.
3 A Numerical Simulation for the Cobb-Douglas Case

We now illustrate these results using a specific numerical example. Let individual preferences be described by the general Cobb-Douglas utility function \( u(x,G) = x^\rho G \) where \( \rho > 0 \). The income expansion path \( e(G) \) then is given by \( e(G) = pG \). Then, by calculating the replacement function in this case and then applying condition (3)

\[
\tilde{G}_n = \frac{n\bar{y}}{1+n\rho}
\]

and

\[
\tilde{g}_n = \frac{\bar{y}}{1+n\rho}
\]

is obtained for every \( n \geq 2 \). Then it immediately follows from formula (4) of Proposition 2 that

\[
p_n(\rho) = \left( \frac{1}{1+n\rho} \right)^{n-1}
\]

holds for any \( n \) and every average endowment level \( \bar{y} \). A calculation of \( p_n(\rho) \) for certain preference parameters \( \rho \) and different numbers of individuals \( n \) gives the following table:

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>9.1740^{-1}</td>
<td>5.8810^{-1}</td>
<td>3.6440^{-1}</td>
<td>1.9740^{-1}</td>
<td>9.3240^{-2}</td>
<td>4.1740^{-2}</td>
<td>1.6310^{-2}</td>
<td>5.8810^{-3}</td>
<td>1.9540^{-3}</td>
</tr>
<tr>
<td>1</td>
<td>3.3340^{-1}</td>
<td>6.2540^{-1}</td>
<td>8.0040^{-3}</td>
<td>7.7140^{-4}</td>
<td>5.9440^{-5}</td>
<td>3.8240^{-6}</td>
<td>2.0940^{-7}</td>
<td>1.0040^{-8}</td>
<td>4.2440^{-10}</td>
</tr>
<tr>
<td>2</td>
<td>2.0040^{-1}</td>
<td>2.0440^{-2}</td>
<td>1.3740^{-3}</td>
<td>6.8340^{-5}</td>
<td>2.6940^{-6}</td>
<td>8.7740^{-8}</td>
<td>2.440^{-9}</td>
<td>5.8940^{-11}</td>
<td>1.2640^{-12}</td>
</tr>
</tbody>
</table>
As \( \rho < 1 \) is not plausible you would need - figuratively speaking - an "electron microscope" to detect interior Cournot-Nash equilibria even if the economy is not very large.

How small the domain of income distribution leading to interior Cournot-Nash equilibria really is also becomes obvious when we compare the proportion \( p_n \) with the proportion of income distributions that imply standalone Cournot-Nash equilibria in which only a single individual makes a strictly positive contribution to the public good. Here, we restrict our considerations to the case \( \rho = 1 \), and we assume that individual 1 is to be the sole contributor.

If \( y_i > \frac{2}{3}n\bar{y} \) holds for a certain income distribution \((y_1, \ldots, y_n)\) then it is implied that only individual 1 will make a strictly positive contribution to the public good. Therefore the \((n-1)\)-dimensional simplex with the vertices \((n\bar{y},0,\ldots,0)\), \(\left(\frac{2}{3}n\bar{y},\frac{1}{3}n\bar{y},0,\ldots,0\right)\), \(\left(\frac{2}{3}n\bar{y},0,\frac{1}{3}n\bar{y},0,\ldots,0\right)\), \ldots and \(\left(\frac{2}{3}n\bar{y},0,\ldots,0,\frac{1}{3}n\bar{y}\right)\) describes a subset of income distributions for which the Cournot-Nash equilibrium has individual 1 as only contributor. This simplex is obtained from the original simplex \(P_n(0)\) by a contraction for which \((n\bar{y},0,\ldots,0)\) is the centroid and \(\frac{1}{3}\) is the contraction factor. In Figure 2 this is described for the case \(n = 3\) where for all income distributions lying in \(GHC\) individual 1 is the sole contributor.

**Figure 2**
Thus, the proportion $m_n^1$ of income distributions leading to a standalone Cournot-Nash equilibrium of individual 1 is bounded from above by $\left( \frac{1}{3} \right)^{n-1}$, which implies for $\rho = 1$

\begin{align*}
\frac{m_n^1}{p_n} & \leq \left( \frac{n+1}{3} \right)^{n-1} \\
(9)
\end{align*}

which is evaluated for $n = 2, \ldots, 10$ in the following Table 2.

**Table 2**

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_n^1/p_n$</td>
<td>1.00</td>
<td>1.77</td>
<td>4.63</td>
<td>16.00</td>
<td>69.16</td>
<td>359.59</td>
<td>2187</td>
<td>15242</td>
<td>119796</td>
</tr>
</tbody>
</table>

This result has the following disturbing consequence. When there are random choices of an income distribution from the set of all income distributions the chance that we will get an income distribution leading to a Cournot-Nash equilibrium with individual 1 as sole contributor is more than 100,000 as large as the chance for obtaining an income distribution for which the Cournot-Nash equilibrium is an interior one when population size is only $n = 10$.

Let $m_n$ denote the proportion of income distributions for which *some* individual is the only contributor in an $n$ person economy. Then obviously, $m_n = nm_n^1$ and, e.g., in the case $n = 10$ we obtain $m_{10}/p_{10} > 10^6$. This means that in an economy consisting of only 10 individuals with a relatively high preference for the public good, standalone equilibria will occur, by a random choice of the income distribution, more than 1 million times as often than an interior Cournot-Nash equilibrium.
4 Conclusion

Shibata-Warr neutrality is one of the most striking results in the theory of public goods. In its most elaborate form Shibata-Warr neutrality means that a redistribution of initial income will not change the Cournot-Nash equilibrium supply of the public good when neither the group of contributors nor their aggregate income is changed. One consequence of the result of this paper is that the range of redistribution for which this invariance theorem holds is very restricted: Starting from a certain given income distribution the formula contained in Proposition 2 can be used to describe which proportion the redistributions of income leading to Shibata-Warr neutrality have in all income distributions among current contributors. The Cobb-Douglas example then shows that this proportion will, under not very extraordinary assumptions, become very small even if the number of contributors is not very high. If we compare the proportion of income redistributions for which Shibata-Warr neutrality applies with all possible income distributions (including also some redistribution between contributors and non-contributors) even a much lower share will be obtained. This makes it evident that normally Shibata-Warr neutrality can only be expected when the redistribution of income is of an extremely limited extent.
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