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A two-period model of international trade and payments

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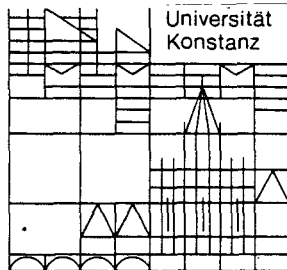
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**Fakultät für
Wirtschaftswissenschaften
und Statistik**

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**A Two-Period Model
of International
Trade and Payments**

Diskussionsbeiträge

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A TWO-PERIOD MODEL OF
INTERNATIONAL TRADE AND PAYMENTS

Serie A - Nr. 194

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an der Universität Konstanz

Serie A: Volkswirtschaftliche Beiträge
Serie B: Finanzwissenschaftliche Arbeitspapiere
Serie C: Betriebswirtschaftliche Beiträge

A Two-Period Model of International Trade and Payments

1. Introduction

The model presented here has been developed in order to provide a framework for analyzing international payments and debt problems from a classical point of view. The model starts from the very simplest case: two commodities (a consumer good and a capital good), two factors of production (labor and capital), two countries, and two periods. The framework can in principle be extended to any number of commodities, factors, countries, and periods; but in order to obtain qualitative results and to develop an intuitive grasp of the model, one must start from the very simplest case.

In conformity with the hypotheses of the Heckscher-Ohlin-Lerner-Samuelson theory, the following additional simplifying assumptions will be made:

1. Consumer goods (in this case, the single consumer good) are freely tradable with no transport costs.
2. Factors of production (in this case, labor and capital) are freely mobile between industries within countries, but completely immobile between countries. In particular, this implies that the capital good is nontradable.

3. Production functions are neoclassical (concave, homogeneous of degree 1, and strictly quasi-concave). In the main theorem it will also be assumed that they are identical between countries. A further assumption to be used is nonreversal of factor intensities.
4. Consumer preferences as between present and future consumer goods are identical and homothetic within and between countries.

The following notation will be used:

$l_i^k(t)$ = endowment of factor i ($i=1,2$) in country k ($k=1,2$) in period t ($t=0,1$). Factor 1 = labor, factor 2 = capital.

$v_{ij}^k(t)$ = allocation of factor i to industry j in country k in period t ($i,j,k=1,2; t=0,1$).

$y_j^k(t)$ = output of commodity j in country k in period t . Commodity 1 = consumer good; commodity 2 = capital good.

$x_j^k(t)$ = consumption (by households) of commodity j in country k in period t ($x_2^k(t) = 0$).

$p_j^k(t)$ = price of commodity j in country k at time t .

$w_i^k(t)$ = rental of factor i in country k at time t .

$z_1^k(t)$ = net import of consumer good by country k at time t
 $= x_1^k(t) - y_1^k(t)$.

Production is carried out by means of production functions

$$y_j^k(t) = f_j^k(v_{1j}^k(t), v_{2j}^k(t)) \quad j, k=1, 2; t=0, 1.$$

and subject to resource-allocation constraints

$$v_{i1}^k(t) + v_{i2}^k(t) \leq \ell_i^k(t) \quad i, k=1, 2; t=0, 1.$$

Endowments obey the following rule:

$$\ell_1^k(1) = \ell_1^k(0)$$

$$\ell_2^k(1) = \ell_2^k(0) + y_2^k(0).$$

That is, labor (population) is constant, and capital is augmented in period 1 by the output of the capital good in period 0.

The present value of the national product is defined as

$$p_1^k(0)y_1^k(0) + p_1^k(1)y_1^k(1) \quad k=1, 2.$$

The prices must here be interpreted as in Lerner's Economics of Control. The real interest rate may be defined as

$$r^k = p_1^k(0)/p_1^k(1) - 1.$$

Since by our assumption that the consumer good is freely traded with no transport costs, we have

$$p_1^1(t) = p_1^2(t) = p_1(t) \quad t=0, 1.$$

It follows that $r^1 = r^2 = r$, that is, the interest rate will be equal between countries (in equilibrium). The real interest factor may be defined as

$$R = p_1(0)/p_1(1) = 1 + r.$$

In our model, it plays the role of a temporal terms of trade.

Since newly-produced and old capital must both have the same rental in period 1, it follows that

$$w_2^k(1) = p_2^k(0).$$

The national accounting then proceeds as follows. In period 0, consumption and investment are given by

$$C^k(0) = p_1^k(0)y_1^k(0) = w_1^k(0)v_{11}^k(0) + w_2^k(0)v_{21}^k(0)$$

$$I^k(0) = p_2^k(0)y_2^k(0) = w_1^k(0)v_{12}^k(0) + w_2^k(0)v_{22}^k(0).$$

Assuming full employment, these sum to

$$Y^k(0) = C^k(0) + I^k(0) = w_1^k(0)\ell_1^k(0) + w_2^k(0)\ell_2^k(0).$$

In period 1 there is no production of capital goods, hence

$$\begin{aligned} C^k(1) &= p_1^k(1)y_1^k(1) = w_1^k(1)\ell_1^k(1) + w_2^k(1)\ell_2^k(1) \\ &= w_1^k(1)\ell_1^k(0) + w_2^k(1)\ell_2^k(0) + p_2^k(0)y_2^k(0). \end{aligned}$$

In order to avoid double-counting we may define

$$Y^k(1) = w_1^k(1)\ell_1^k(0) + w_2^k(1)\ell_2^k(0),$$

i.e., as the sum of factor rentals times initial endowments.

In the case of capital, this defines the return to capital in period 1 as the period-1 rental times the initial capital.

Then $I^k(1) = -I^k(0)$, and the present value of the national product can be expressed as

$$Y^k(0) + Y^k(1) = [w_1^k(0) + w_1^k(1)]\ell_1^k(0) + [w_2^k(0) + w_2^k(1)]\ell_2^k(0).$$

We may thus define for each factor the present value of its rentals:

$$W_i^k = w_i^k(0) + w_i^k(1) \quad i, k=1, 2.$$

Now let us define country k's inter-temporal production-possibility set as

$$\mathcal{Y}_1^k(\ell_1^k(0), \ell_2^k(0)) = \{(y_1^k(0), y_1^k(1)) : y_j^k(t) = f_j^k(v_{1j}^k(t), v_{2j}^k(t)), \\ v_{i1}^k(t) + v_{i2}^k(t) \leq \ell_i^k(t)\}.$$

Let us further define the inter-temporal national-product function by

$$\Pi^k(\ell_1^k(0), \ell_2^k(0), p_1^k(0), p_1^k(1)) = \max_{y_1^k \in \mathcal{Y}_1^k(\ell_1^k(0), \ell_2^k(0))} p_1^k(0)y_1^k(0) + p_1^k(1)y_1^k(1)$$

where y_1^k denotes the vector $(y_1^k(0), y_1^k(1))$.

The following may be shown:

$$\frac{\partial \Pi^k}{\partial \ell_i^k(0)} = W_i^k; \quad \frac{\partial \Pi^k}{\partial p_j^k(t)} = y_j^k(t).$$

These generalize Samuelson's reciprocity relations. In particular, the functions defined by the above are the generalizations of the Stolper-Samuelson and Rybczynski functions

$$W_i^k(\ell_1^k(0), \ell_2^k(0), p_1^k(0), p_1^k(1)); \quad \overline{y_j^k(t)}(\ell_1^k(0), \ell_2^k(0), p_1^k(0), p_1^k(1)).$$

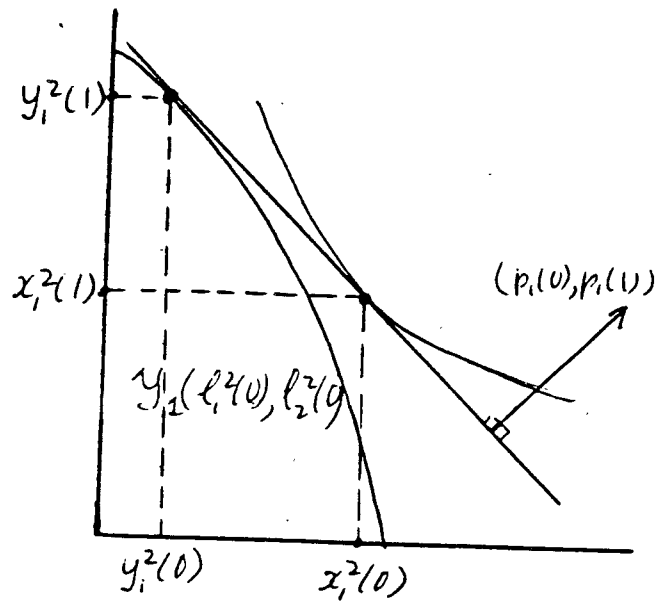
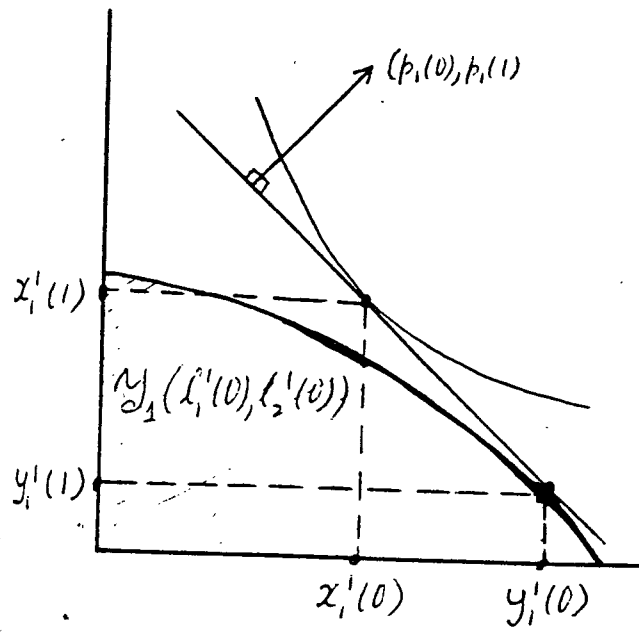


Figure 1

2. An Intertemporal Heckscher-Ohlin Theorem

Introducing the assumption of identical production functions between countries, the shapes of two countries' intertemporal production-possibility sets $\mathcal{Y}_1(\ell_1^k(0), \ell_2^k(0))$ depend entirely on the initial endowments $\ell_1^k(0), \ell_2^k(0)$. In fact, owing to the assumption of constant returns to scale, the slope depends entirely on the relative endowments $\ell_2^k(0)/\ell_1^k(0)$. Supposing that the slopes of country 1's and country 2's intertemporal production-possibility sets are as in the accompanying Figure 1, and supposing an equilibrium price-ratio $p_1(0)/p_1(1)$ to be established (given by the slopes of the two straight lines in the figure), with identical homothetic preferences as between present and future consumer goods, relative consumptions of these goods will be the same in the two countries (indicated by $x_1^k(0), x_1^k(1)$). However, if relative outputs $y_1^k(1)/y_1^k(0)$ are higher in country 1 than country 2, then country 1 will "export" an amount $-z_1^1(0) = y_1^1(0) - x_1^1(0)$ to country 2 and "import" an amount $z_1^1(1) = x_1^1(1) - y_1^1(1)$ from country 2. That is, country 1 will export present goods to country 2 in exchange for future goods. This means that country 1 will lend to country 2 in period 0, and country 2 will repay this debt to country 1 in period 1.

One thing that is immediately clear is that, in general, balanced trade (or balanced payments on current account) is

not optimal. The intertemporal counterpart to the proposition that "free trade is better than autarky" is that "international borrowing and lending is better than balanced trade."

What we would like to do is obtain enough information about the shapes of the intertemporal production-possibility sets, as a function of the relative initial capital endowments $\ell_2^k(0)/\ell_1^k(0)$, so that, with the assumption of identical homothetic preferences across and within countries, we could predict which country would lend and which would borrow.

One way in which this could be accomplished is the following. Suppose that, at whatever price ratio $p_1(0)/p_1(1)$ is chosen, the output ratio $y_1^k(1)/y_1^k(0)$ is a monotone function of the initial-endowment ratio $\ell_2^k(0)/\ell_1^k(0)$; and that if this function is monotone increasing (or decreasing) at one price ratio, it remains monotone increasing (resp. decreasing) at any other price ratio. Then since in particular this would hold at any equilibrium price ratio, the direction of borrowing or lending would be uniquely determined. Following this strategy, the object is to determine the sign of

$$\frac{\frac{\overline{y_1^k(1)}(p_1(0), p_1(1), \ell_1^k(0), \ell_2^k(0))}{\overline{y_1^k(0)}(p_1(0), p_1(1), \ell_1^k(0), \ell_2^k(0))}}{\partial \ell_i^k(0)}$$

$$= \frac{\frac{\overline{y_1^k(0)}}{y_1^k(0)} \frac{\partial \overline{y_1^k(1)}}{\partial \ell_i^k(0)} - \frac{\overline{y_1^k(1)}}{y_1^k(1)} \frac{\partial \overline{y_1^k(0)}}{\partial \ell_i^k(0)}}{\overline{y_1^k(0)}^2}, \quad \text{for } i = 1 \text{ or } 2.$$

In order to do this, let us set out the system of equations that determine equilibrium of a country at any given values of the variables $p_1(0)$, $p_1(1)$, $\ell_1^k(0)$, $\ell_2^k(0)$.

First, we define the national-cost function

$$\begin{aligned} \varphi^k(w_1^k(t), w_2^k(t), y_1^k(t), y_2^k(t)) = \\ g_1^k(w_1^k(t), w_2^k(t))y_1^k(t) + g_2^k(w_1^k(t), w_2^k(t))y_2^k(t), \end{aligned}$$

where

$$g_j^k(w_1^k(t), w_2^k(t))$$

is the Shephard minimum-unit-cost function dual to the production function $f_j^k(v_{1j}^k(t), v_{2j}^k(t))$. Note that φ^k is concave, homogeneous of degree 1, and strictly quasi-concave in $w_1^k(t), w_2^k(t)$, since this is true of each g_j^k . By Shephard's duality theorem

$$\frac{\partial g_j^k}{\partial w_i^k(t)} = b_{ij}^k(w_1^k(t), w_2^k(t)) = \frac{v_{ij}^k(t)}{y_j^k(t)},$$

the partial derivative of φ^k with respect to $w_i^k(t)$ is the demand for factor i :

$$\begin{aligned} \varphi_i^k(w_1^k(t), w_2^k(t), y_1^k(t), y_2^k(t)) &\equiv \frac{\partial \varphi^k(w_1^k(t), w_2^k(t), y_1^k(t), y_2^k(t))}{\partial w_i^k(t)} \\ &= b_{i1}^k(w_1^k(t), w_2^k(t))y_1^k(t) + b_{i2}^k(w_1^k(t), w_2^k(t))y_2^k(t). \end{aligned}$$

This function is homogeneous of degree 0 in $w_1^k(t), w_2^k(t)$, hence by Euler's theorem

$$\sum_{j=1}^2 \phi_{ij}^k [t] w_j^k(t) = 0,$$

where

$$\phi_{ij}^k [t] = \frac{\partial^2 \phi^k(w_1^k(t), w_2^k(t), y_1^k(t), y_2^k(t))}{\partial w_i^k(t) \partial w_j^k(t)}.$$

Defining the matrix

$$\phi^k [t] = [\phi_{ij}^k [t]],$$

it is symmetric (so long as each g_j^k is continuously twice differentiable) and negative semi-definite (from the concavity in $w_1^k(t), w_2^k(t)$ of ϕ^k), and since from above

$$\phi^k [t] w^k(t) = 0$$

(where $w^k(t) = (w_1^k(t), w_2^k(t))'$), it has rank at most 1. From strict quasi-concavity of g^k , it has rank 1 almost everywhere. We shall assume that $\phi_{ii}^k < 0$ everywhere.

We may now set out our system of seven equations in the seven unknowns $w_1^k(0), w_2^k(0), w_1^k(1), w_2^k(1), y_1^k(0), y_2^k(0), y_1^k(1)$ (recall that $y_2^k(1) = 0$).

$$g_1^k(w_1^k(0), w_2^k(0)) = p_1^k(0)$$

$$g_2^k(w_1^k(0), w_2^k(0)) - w_2^k(1) = 0$$

$$\varphi_1^k(w_1^k(0), w_2^k(0), y_1^k(0), y_2^k(0)) = \lambda_1^k(0)$$

$$\varphi_2^k(w_1^k(0), w_2^k(0), y_1^k(0), y_2^k(0)) = \lambda_2^k(0)$$

$$\varphi_1^k(w_1^k(1), w_2^k(1), y_1^k(1), 0) = \lambda_1^k(0)$$

$$\varphi_2^k(w_1^k(1), w_2^k(1), y_1^k(1), 0) - y_2^k(0) = \lambda_2^k(0)$$

$$g_1^k(w_1^k(1), w_2^k(1)) = p_1^k(1)$$

Taking differentials, we obtain the following system, where by definition

$$b_{ij}^k[t] = b_{ij}(w_1^k(t), w_2^k(t))$$

and

$$B^k[t] = [b_{ij}^k[t]].$$

$$\begin{bmatrix} b_{11}^k[0] & b_{21}^k[0] & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{12}^k[0] & b_{22}^k[0] & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ \varphi_{11}^k[0] & \varphi_{21}^k[0] & b_{11}^k[0] & b_{12}^k[0] & 0 & 0 & 0 & 0 & 0 \\ \varphi_{12}^k[0] & \varphi_{22}^k[0] & b_{21}^k[0] & b_{22}^k[0] & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varphi_{11}^k[1] & \varphi_{21}^k[1] & b_{11}^k[1] & 0 & 0 \\ 0 & 0 & 0 & -1 & \varphi_{12}^k[1] & \varphi_{22}^k[1] & b_{21}^k[1] & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{11}^k[1] & b_{21}^k[1] & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dw_1^k(0) \\ dw_2^k(0) \\ dy_1^k(0) \\ dy_2^k(0) \\ dw_1^k(1) \\ dw_2^k(1) \\ dy_1^k(1) \end{bmatrix} = \begin{bmatrix} dp_1^k(0) \\ 0 \\ d\lambda_1^k(0) \\ d\lambda_2^k(0) \\ d\lambda_1^k(0) \\ d\lambda_2^k(1) \\ dp_1^k(1) \end{bmatrix}$$

Denoting the above matrix by A, let us first evaluate its determinant, $|A|$. Partitioning A into its first four rows and columns and last three rows and columns, we have by Schur's formula

$$|A| = \begin{vmatrix} P & Q \\ R & S \end{vmatrix} = |P-QS^{-1}R| |S|,$$

where, denoting $g_{1,i}^k = \partial g_1^k / \partial w_i$ and $g_{1,ij}^k = \partial^2 g_1^k / \partial w_i \partial w_j$,

$$S = \begin{bmatrix} \phi_{11}^k[1] & \phi_{21}^k[1] & b_{11}^k[1] \\ \phi_{12}^k[1] & \phi_{22}^k[1] & b_{21}^k[1] \\ b_{11}^k[1] & b_{21}^k[1] & 0 \end{bmatrix} =$$

$$\begin{bmatrix} y_1^k(1) g_{1,11}^k[1] & y_1^k(1) g_{1,21}^k[1] & g_{1,1}^k[1] \\ y_1^k(1) g_{1,12}^k[1] & y_1^k(1) g_{1,22}^k[1] & g_{1,2}^k[1] \\ g_{1,1}^k[1] & g_{1,2}^k[1] & 0 \end{bmatrix}$$

To obtain the expression for S^{-1} we use the following

Lemma 1. Let $f(v) = f(v_1, v_2, \dots, v_n)$ be a concave, homogeneous-of-degree 1, and strictly quasi-concave production function, and let $g(w) = g(w_1, w_2, \dots, w_n)$ be its dual minimum-unit-cost function. Denote $f_i = \partial f / \partial v_i$, $f_{ij} = \partial^2 f / \partial v_i \partial v_j$, $g_i = \partial g / \partial w_i$, $g_{ij} = \partial^2 g / \partial w_i \partial w_j$. Assume $v_i, y > 0$ and $w_i, p > 0$,

where

$$\begin{aligned} f_i(v) - w_i/p &= 0; & g_i(w) - v_i/y &= 0; \\ f(v) &= y; & g(w) &= p. \end{aligned}$$

Then

$$\begin{bmatrix} p & \frac{\partial^2 f}{\partial v \partial v} & \frac{\partial f}{\partial v} \\ & \frac{\partial f}{\partial v} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} y & \frac{\partial^2 g}{\partial w \partial w} & \frac{\partial g}{\partial w} \\ & \frac{\partial g}{\partial w} & 0 \end{bmatrix}.$$

Proof. Differentiating the above two systems of equations, we obtain

$$\begin{aligned} \sum_{j=1}^n f_{ij}(v) dv_j + \frac{w_i}{p^2} dp &= dw_i/p; & \sum_{j=1}^n g_{ij}(w) dw_j + \frac{v_i}{y^2} dy &= dv_i/y; \\ \sum_{j=1}^n f_j(v) dv_j &= dy; & \sum_{j=1}^n g_j(w) dw_j &= dp. \end{aligned}$$

Using $f_i = w_i/p$, $g_i = v_i/y$ these may be written as

$$\begin{bmatrix} \frac{\partial^2 f}{\partial v \partial v} & p^{-1} \frac{\partial f}{\partial v} \\ p^{-1} \frac{\partial f}{\partial v} & 0 \end{bmatrix} \begin{bmatrix} dv \\ dp \end{bmatrix} = p^{-1} \begin{bmatrix} dw \\ dy \end{bmatrix}; \quad \begin{bmatrix} \frac{\partial^2 g}{\partial w \partial w} & y^{-1} \frac{\partial g}{\partial w} \\ y^{-1} \frac{\partial g}{\partial w} & 0 \end{bmatrix} \begin{bmatrix} dw \\ dy \end{bmatrix} = y^{-1} \begin{bmatrix} dv \\ dp \end{bmatrix}$$

Equating the two expressions we have

$$\begin{bmatrix} dw \\ dy \end{bmatrix} = \begin{bmatrix} p & \frac{\partial^2 f}{\partial v \partial v} & \frac{\partial f}{\partial v} \\ \frac{\partial f}{\partial v} & 0 \end{bmatrix} \begin{bmatrix} y & \frac{\partial^2 g}{\partial w \partial w} & \frac{\partial g}{\partial w} \\ \frac{\partial g}{\partial w} & 0 \end{bmatrix} \begin{bmatrix} dw \\ dy \end{bmatrix}$$

from which the conclusion follows. □

Thus, denoting $f_{1,i}^k[t] = \partial f_1^k(v_{11}^k(t), v_{21}^k(t)) / \partial v_{i1}^k(t)$ and $f_{1,ij}^k[t] = \partial^2 f_1^k(v_{11}^k(t), v_{21}^k(t)) / \partial v_{i1}^k(t) \partial v_{j1}^k(t)$, we have

$$S^{-1} = \begin{bmatrix} p_1^k(1) f_{1,11}^k[1] & p_1^k(1) f_{1,21}^k[1] & f_{1,1}^k[1] \\ p_1^k(1) f_{1,12}^k[1] & p_1^k(1) f_{1,22}^k[1] & f_{1,2}^k[1] \\ f_{1,1}^k[1] & f_{1,2}^k[1] & 0 \end{bmatrix}$$

Accordingly,

$$QS^{-1}R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_1^k(1) f_{1,22}^k[1] \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and thus

$$\begin{aligned}
 |P-QS^{-1}R| &= -p_1^k(1)f_{1,22}^k[1] \begin{vmatrix} b_{11}^k[0] & b_{21}^k[0] & 0 \\ \phi_{11}^k[0] & \phi_{21}^k[0] & b_{11}^k[0] \\ \phi_{12}^k[0] & \phi_{22}^k[0] & b_{21}^k[0] \end{vmatrix} + |P|. \\
 &= -p_1^k(1)f_{1,22}^k[1] \begin{vmatrix} \phi_{11}^k[0] & \phi_{21}^k[0] & b_{11}^k[0] \\ \phi_{12}^k[0] & \phi_{22}^k[0] & b_{21}^k[0] \\ b_{11}^k[0] & b_{21}^k[0] & 0 \end{vmatrix} + |P|.
 \end{aligned}$$

Note that

$$\begin{bmatrix} \phi_{11}^k[0] & \phi_{21}^k[0] & b_{11}^k[0] \\ \phi_{12}^k[0] & \phi_{22}^k[0] & b_{21}^k[0] \\ b_{11}^k[0] & b_{21}^k[0] & 0 \end{bmatrix} =$$

$$\begin{bmatrix} y_1^k(0)g_{1,11}^k[0] & y_1^k(0)g_{1,21}^k[0] & g_{1,1}^k[0] \\ y_1^k(0)g_{1,12}^k[0] & y_1^k(0)g_{1,22}^k[0] & g_{1,2}^k[0] \\ g_{1,1}^k[0] & g_{1,2}^k[0] & 0 \end{bmatrix} + \begin{bmatrix} y_2^k(0)g_{2,11}^k[0] & y_2^k(0)g_{2,21}^k[0] & 0 \\ y_2^k(0)g_{2,12}^k[0] & y_2^k(0)g_{2,22}^k[0] & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

hence

$$\begin{vmatrix} \phi_{11}^k[0] & \phi_{21}^k[0] & b_{11}^k[0] \\ \phi_{12}^k[0] & \phi_{22}^k[0] & b_{21}^k[0] \\ b_{11}^k[0] & b_{21}^k[0] & 0 \end{vmatrix} = \begin{vmatrix} y_1^k(0)g_{1,11}^k[0] & y_1^k(0)g_{1,21}^k[0] & g_{1,1}^k[0] \\ y_1^k(0)g_{1,12}^k[0] & y_1^k(0)g_{1,22}^k[0] & g_{1,2}^k[0] \\ g_{1,1}^k[0] & g_{1,2}^k[0] & 0 \end{vmatrix} > 0,$$

the sign following from strict quasi-concavity of g_1^k . Since $f_{1,22}^k[1] < 0$ from strict quasi-concavity of f_1^k , the first term in the above expression for $|P-QS^{-1}R|$ is positive. Since P is a block lower-triangular matrix, its determinant is $|B^k[0]|^2 > 0$. Thus, $|P-QS^{-1}R| > 0$. Since $|S| > 0$ from strict quasi-concavity of g_1^k , it follows that $|A| > 0$.

We now come to the main theorem:

Theorem (Inter-Temporal Heckscher-Ohlin Theorem). Let two countries each produce a tradable consumer good and a non-tradable capital good in each of two periods, with identical neo-classical production functions. Let the initial endowment of labor be constant over the two periods, and let the endowment of capital in period 1 be equal to the initial endowment of capital in period 0 plus the output of the capital good in period 0. Let preferences as between present and future consumer goods be identical and homothetic within and across countries. Let payments be balanced over the two periods.

Assume finally that, at all factor rentals, production of consumer goods uses a higher capital-labor ratio than production of capital goods. Then the country with a higher relative initial endowment of capital to labor in period 0 will lend in period 0 to the other country, and be repaid in period 1.

The main work of the theorem consists in proving the following

Lemma 2. Assume that, at all factor rentals w_1, w_2 ,

$$|B^k(w_1, w_2)| = b_{11}^k(w_1, w_2) b_{12}^k(w_1, w_2) \left[\frac{b_{22}^k(w_1, w_2)}{b_{12}^k(w_1, w_2)} - \frac{b_{21}^k(w_1, w_2)}{b_{11}^k(w_1, w_2)} \right] < 0.$$

Then

$$y_1^k(0) \frac{\overline{\partial y_1^k(1)}}{\partial \ell_1^k(0)} - y_1^k(1) \frac{\overline{\partial y_1^k(0)}}{\partial \ell_1^k(0)} > 0.$$

Proof. Let A be partitioned into its first two rows and columns and last five rows and columns, as

$$A = \begin{bmatrix} B' & C \\ D & E \end{bmatrix} \quad \text{where } B' = B^k[0]' .$$

Define E_{10} and E_{11} as the matrices such that the first and last columns of E are replaced, respectively, by the column $(1, 0, 1, 0, 0)'$. Then from the above matrix differential system it is clear that, applying Cramer's rule, and denoting

$$A_{10} = \begin{bmatrix} B' & C \\ D & E_{10} \end{bmatrix}, \quad A_{11} = \begin{bmatrix} B' & C \\ D & E_{11} \end{bmatrix},$$

we have

$$\frac{\overline{\partial y_1^k(0)}}{\partial \lambda_1^k(0)} = \frac{|A_{10}|}{|A|} \quad \text{and} \quad \frac{\overline{\partial y_1^k(1)}}{\partial \lambda_1^k(1)} = \frac{|A_{11}|}{|A|}.$$

Applying Schur's factorization

$$\begin{bmatrix} B' & C \\ D & E_{1t} \end{bmatrix} = \begin{bmatrix} B' & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ D & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & E_{1t} - DB'^{-1}C \end{bmatrix} \begin{bmatrix} I & B'^{-1}C \\ 0 & 0 \end{bmatrix}$$

we have immediately

$$|A_{1t}| = |B| |E_{1t} - DB'^{-1}C|.$$

We verify readily that

$$DB'^{-1}C = \begin{bmatrix} 0 & 0 & 0 & -\lambda_{21}^k[0] & 0 \\ 0 & 0 & 0 & -\lambda_{22}^k[0] & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the $\lambda_{ij}^k[0]$ are the elements of the matrix

$$\Lambda^k[0] = B^k[0]^{-1} \Phi^k[0],$$

i.e.,

$$\lambda_{21}^k[0] = \frac{-\phi_{11}^k[0]b_{21}^k[0] + \phi_{21}^k[0]b_{11}^k[0]}{|B^k[0]|}$$

$$\lambda_{22}^k[0] = \frac{-\phi_{12}^k[0]b_{21}^k[0] + \phi_{22}^k[0]b_{11}^k[0]}{|B^k[0]|}$$

Since $\phi^k[0]$ is negative semi-definite and symmetric and of rank 1, $\phi_{11}^k[0] < 0$, $\phi_{22}^k[0] < 0$ and $\phi_{12}^k[0] = \phi_{21}^k[0] > 0$. On the hypothesis $|B^k[0]| < 0$ we therefore have

$$\lambda_{21}^k[0] < 0 \quad \text{and} \quad \lambda_{22}^k[0] > 0.$$

Denoting $\bar{E}_{1t} = E_{1t} - DB'^{-1}C$ we find that

$$|\bar{E}_{10}| = b_{22}^k[0] \begin{vmatrix} \phi_{11}^k[1] & \phi_{21}^k[1] & b_{11}^k[1] \\ \phi_{12}^k[1] & \phi_{22}^k[1] & b_{21}^k[1] \\ b_{11}^k[1] & b_{21}^k[1] & 0 \end{vmatrix} +$$

$$b_{11}^k[1] \{-b_{21}^k[1]b_{22}^k[0]\lambda_{21}^k[0] + (b_{21}^k[1]b_{12}^k[0] - b_{11}^k[1])\lambda_{22}^k[0]\}.$$

The first term is positive, as is the term involving $\lambda_{21}^k[0]$.

However, the term involving $\lambda_{22}^k[0]$ is nonnegative only if

$$\frac{b_{21}^k[1]}{b_{11}^k[1]} \geq \frac{1}{b_{12}^k[0]}$$

i.e., if the capital-labor ratio in the consumer-good industry in period 1 exceeds or equals the output-labor ratio in the capital-good industry in period 0. This is a rather strong condition which cannot be expected to be realized in practice. Thus, we cannot in general conclude that $\overline{\partial y_1^k(0)/\partial \ell_1^k(0)} < 0$.

Proceeding to $\overline{\partial y_1^k(1)/\partial \ell_1^k(0)}$ we find that

$$\begin{aligned} |\bar{E}_{11}| &= b_{21}^k[0] \begin{vmatrix} \varphi_{11}^k[1] & \varphi_{21}^k[1] \\ b_{11}^k[1] & b_{21}^k[1] \end{vmatrix} + |B^k[0]| \begin{vmatrix} \varphi_{12}^k[1] & \varphi_{22}^k[1] \\ b_{11}^k[1] & b_{21}^k[1] \end{vmatrix} \\ &+ b_{11}^k[1] \{ b_{21}^k[0] \lambda_{21}^k[0] - b_{11}^k[0] \lambda_{22}^k[0] \}. \end{aligned}$$

The first two terms are unambiguously negative (given that $|B^k[0]| < 0$), and so are the two terms involving $\lambda_{21}^k[0]$ and $\lambda_{22}^k[0]$. Thus, we may conclude that $\overline{\partial y_1^k(1)/\partial \ell_1^k(0)} > 0$.

In order to obtain our desired result, we combine the terms in $|\bar{E}_{10}|$ and $|\bar{E}_{11}|$ involving $\lambda_{22}^k[0]$. That is, we compute the coefficient of $\lambda_{22}^k[0]$ in the expression

$$\begin{aligned} y_1^k(0) \frac{\overline{\partial y_1^k(1)}}{\partial \ell_1^k(0)} - y_1^k(1) \frac{\overline{\partial y_1^k(0)}}{\partial \ell_1^k(0)} \\ = \frac{|B^k[0]|}{|A|} \{ y_1^k(0) \bar{E}_{11} - y_1^k(1) \bar{E}_{10} \}. \end{aligned}$$

Omitting the common factor $b_{11}^k[1]$, the coefficient of the term in braces is easily seen to be, using our assumptions in the subsequent steps,

$$\begin{aligned}
 & -b_{11}^k[0]y_1^k(0) + b_{11}^k[1]y_1^k(1) - b_{12}^k[0]b_{21}^k[1]y_1^k(1) \\
 &= -v_{11}^k(0) + v_{11}^k(1) - b_{12}^k[0]v_{21}^k(1) \\
 &= -[\ell_1^k(0) - v_{12}^k(0)] + \ell_1^k(0) - b_{12}^k[0][\ell_2^k(0) + y_2^k(0)] \\
 &= v_{12}^k(0) - v_{12}^k(0) - b_{12}^k[0]\ell_2^k(0) \\
 &= -b_{12}^k[0]\ell_2^k(0) < 0.
 \end{aligned}$$

It follows that

$$y_k^1(0)\bar{E}_{11} - y_1^k(1)\bar{E}_{10} < 0,$$

and the lemma is proved. □

The proof of the main theorem now proceeds exactly as in the standard Heckscher-Ohlin theorem. Let us suppose that country 1 has a higher capital-labor endowment ratio than country 2, i.e.,

$$\frac{\ell_2^1(0)}{\ell_1^1(0)} > \frac{\ell_2^2(0)}{\ell_1^2(0)}.$$

Then from Lemma 2 it follows that

$$\frac{y_1^1(1)}{y_1^1(0)} < \frac{y_1^2(1)}{y_1^2(0)},$$

i.e., that country 1 will have a greater capability of producing present goods relative to future goods than country 2, as in Figure 1. On the other hand, the assumption of identical homothetical preferences implies that

$$\frac{x_1^1(1)}{x_1^1(0)} = \frac{x_1^2(1)}{x_1^2(0)} .$$

From the definition of net imports $z_1^k(t) = x_1^k(t) - y_1^k(t)$, and the material-balance condition

$$z_1^1(t) + z_1^2(t) = 0 \quad t=0,1$$

it follows that

$$\frac{y_1^1(1) + z_1^1(1)}{y_1^1(0) + z_1^1(0)} = \frac{y_1^2(1) + z_1^2(1)}{y_1^2(0) + z_1^2(0)} = \frac{y_1^2(1) - z_1^1(1)}{y_1^2(0) - z_1^1(0)} .$$

Cross-multiplying the two outside fractions we have, after cancellation,

$$y_1^1(1)y_1^2(0) - y_1^1(1)z_1^1(0) + y_1^2(0)z_1^1(1) = \\ y_1^1(0)y_1^2(1) - y_1^1(0)z_1^1(1) + y_1^2(1)z_1^1(0),$$

or

$$[y_1^1(0) + y_1^2(0)]z_1^1(1) - [y_1^1(1) + y_1^2(1)]z_1^1(0) = \\ y_1^1(0)y_1^2(1) - y_1^1(1)y_1^2(0) > 0.$$

Now, making use of the balanced-payments condition

$$p_1(0)z_1^k(0) + p_1(1)z_1^k(1) = 0,$$

or

$$Rz_1^k(0) + z_1^k(1) = 0$$

(the value of the repayment, $z_1^1(1)$, is equal to the loan, $-z_1^1(0)$, times the interest factor R), the above inequality becomes

$$0 < -z_1^1(0)\{R[y_1^1(0) + y_1^2(0)] + [y_1^1(1) + y_1^2(1)]\}$$

which implies

$$z_1^1(0) < 0,$$

i.e., that country 1 will export present goods to country 2, or lend to country 2 in period 0, and import future goods from country 2, or receive repayment from country 2 in period 1. \square