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Poddar, Satya N.; Genser, Bernd

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Fakultät für Wirtschaftswissenschaften und Statistik

Satya N. Poddar and Bernd Genser

Measurement of Effective Tax Progression

Diskussionsbeiträge

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MEASUREMENT OF EFFECTIVE TAX PROGRESSION

Satya N. Poddar* and Bernd Genser

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*) Department of Finance Canada, Ottawa

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Measurement of Effective Tax Progression

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Abstract

The paper first considers two properties of measures of effective progression, proportional yield neutrality and linear decomposability. It is shown that decomposability implies yield neutrality but not the converse. The concept of sensitivity of a measure to progression in various ranges of the income spectrum is then defined. This concept, when applied to the commonly used group of Lorenz curve based measure, shows them all to be sensitive only in the middle income range. A new class of measures is, therefore, introduced so that particular measures with any desired pattern of sensitivity can be constructed. The paper concludes with empirical illustrations for Canada and the U.S. The importance of separating measures of effective progression from measures of income redistribution is stressed, by showing that progressive taxes do not unambigously reduce the inequality of post tax incomes.

Zusammenfassung

Der Beitrag behandelt zwei Eigenschaften effektiver Progressionsmaße, die Invarianz gegenüber proportionalen Steuerbelastungsänderungen (proportional yield neutrality) und die lineare Dekomposition (linear decomposability). Es läßt sich zeigen, daß die Invarianzeigenschaft aus der Dekomposition folgt, jedoch nicht umgekehrt. Weiter wird das Konzept der Sensitivität eines Progressivitätsmaßes gegenüber Progressionseffekten in unterschiedlichen Intervallen der Einkommensskala formalisiert. Anhand dieses Konzeptes zeigt sich, daß die üblichen an Lorenzkurven anknüpfenden Progressivitätsmaße nur in einem mittleren Einkommensbereich sensitiv sind. Es wird daher eine neue Klasse von Progressivitätsmaßen vorgestellt, die es erlaubt, beliebige Sensitivitätsmuster entlang der Einkommensskala festzulegen. Abschließend werden einige empirische Resultate für Kanada und die USA angegeben. Dabei zeigt sich einmal mehr die Notwendigkeit, effektive Progressivitätsmaße von Maßen der Einkommensumverteilung zu unterscheiden, da progressive Steuerlastverteilungen nicht immer zu einer Verringerung der Disparität der Einkommen nach Steuern führen.

MEASUREMENT OF EFFECTIVE TAX PROGRESSION Satya N. Poddar^{*} and Bernd Genser^{**}

I. INTRODUCTION

While it is generally agreed that a tax system should be defined as progressive when the average rate of tax increases with income, comparisons of the extent or amount of progression are often made by politicians and economists with no precise meaning attached to them. A number of papers, however, have considered approaches to the measurement of progression and/ or the merits of specific alternative measures. These include the well known paper by Musgrave and Thin (1948) and more recent papers by Khetan and Poddar (1976), Kakwani (1977a), Jakobsson (1976), and Suits (1977). A comprehensive survey of different measures is Kiefer (1984). The various measures discussed in these papers can be divided into two broad categories: measures of structural progression and measures of effective progression.

Structural measures depend only upon the tax function, and are typically expressed as mathematical transformations of the tax function. Thus, they assign a specific value of tax liability to any income level y given the tax function t(y). The average-rate, marginal-rate, liability, and residual-income progression, defined by Musgrave and Thin and examined again and modified by Jakobsson (1976), Bös/Genser (1977), or Lambert (1985b) are well known examples of such measures. They require either graphical or tabular presentations (Genser 1980) or rather heroic aggregative views (Dilnot/Kay/Morris 1984) if empirical comparisons are to be made over the whole income scale.

^{*} Department of Finance Canada, Ottawa

Measures of effective progression, on the other hand, depend on both the tax function and the particular distribution of income to which they are being applied.

Although aggregate analogues of the structural measures, e.g. elasticities with respect to GDP, are used in macroeconomic analyses to identify progressivity on a national level the usual approach is to use concentration curves.

Our concern, in this paper, is with measures of effective progression based on concentration curves. We first briefly describe six measures of this type stated in the literature and analyse their basic characteristics. Two important properties of measures of effective progression have been pointed out in the literature: an invariance property and an additivity property:

- (1) If two tax schedules applied to the same income vector generate tax distributions which are strictly proportional, then both tax schedules ought be characterized by the same degree of progressivity. This invariance property will be called proportional yield neutrality (PYN).
- (2) If several taxes are considered simultaneously then the overall degree of progressivity ought be decomposable into the corresponding progressivity measures of the single taxes. This property will be formalized in a stricter version as linear decomposability (LD).

A strong interrelationship is shown to exist between these two seemingly unrelated properties.

As several of the progressivity measures suggested in the literature meet these properties but reflect differently tax shifts in various ranges of the income spectrum a further property is then formalized, the sensitivity to progression (SP) of a given progressivity measure in different income intervals.

- 2 -

Applying this property, it is shown that the measures used in the above mentioned analyses are sensitive only in the middle income ranges. This is obviously a serious limitation, as it involves the possibility that changes in tax progression elsewhere in the income spectrum will be missed.

To overcome this limitation, we define a new class of measures of effective progression and construct illustrative examples of progressivity measures that are sensitive in other income ranges. We then consider in some detail the conflict that exists between the notion of effective tax progression flowing through the concepts of Khetan and Poddar, Kakwani, and Suits (but not Musgrave and Thin), and the notion of income redistribution based on traditional measures of income inequality found in Dalton (1920) and Atkinson (1970). The paper concludes with some empirical illustrations based on Canadian and U.S. data.

- 3 -

II. LORENZ CURVE BASED MEASURES OF PROGRESSION

Perhaps the easiest way to see the relationship among the measures of effective progression cited above is by first indicating their "building blocks". Figure 1 presents four concentration curves, differing only in the specific variables on the vertical and horizontal axes. Labelling the areas under these respective concentration curves A through D the measures developed in the various papers are defined as follows:

Musgrave/Thin Measure ¹	MT	=	B/A
Khetan/Poddar's First Measure ²	KP1	=	A/C
Khetan/Poddar's Second Measure ²	KP2	=	1/2D
Reynolds/Smolensky Measure	RS	=	2(B-A)
Kakwani Measure	K	=	2(A-C)
Suits Measure	S	=	1-2D

It is clear that the Suits measure and Khetan and Poddar's second measure are simple transformations of one another. The conventional Gini coefficients of before- and after-tax income are simply 1-2A and 1-2B respectively.

If a tax is exactly proportional to income the concentration curves (ii) and (iii) coincide with the Lorenz curve (i) and (iv) runs along the diagonal. In this case the first three measures assign a value of one, while the other three measures assign a value of zero. A progressive tax defined by an increasing average tax rate and a marginal tax rate less than 100 % will unembigously be assigned a value greater than one or zero respectively (Jakobsson 1976, Kakwani 1977b, Eichhorn/Funke/Richter 1984).

We refer here to what Musgrave and Thin called their measure of effective progression. Kakwani (1977) incorrectly refers to MT as being 2(B-A), using our notation.

²Khetan and Poddar refer to their measures as YNCP1 and YNCP2 respectively, for "yield - neutral coefficient of progression".



5

(i)

(ii)



cumulative population



(iii)







TABLE 1 : FORMULAE FOR LORENZ CURVE ANALOGUE

Concen- tration Curve	Horizontal Axis	Vertical Axis	Shaded Area
i	cumulative population	cumulative before-tax income	$A = 1 + 1/2N - (\Sigma_{i=1}^{N} i \cdot y_{i}) / N^{2} \overline{y}$
ii	cumulative population	cumulative after-tax income	$B = 1 + 1/2N - \frac{\sum_{i=1}^{N} i(y_i - t(y_i))}{N^2(\overline{y} - \overline{t})}$
iii	cumulative population	cumulative tax	$C = 1 + 1/2N - (\Sigma_{i=1}^{N} i \cdot t(y_{i}))/N^{2}\overline{t}$
iv	cumulative before-tax income	cumulative tax	$D = 1 + \frac{\sum_{i=1}^{N} t(y_i) \cdot (y_i/2 - \sum_{j=1}^{L} y_j)}{N^2 \overline{y} \overline{t}}$

NOTATION

Y _i	=	ncome of i th individual ³ (i=1,,N)
У	=	(y_1, \dots, y_N) income distribution vector for N indivi- luals with $y_1 = y_2 = \dots = y_N$
t ^k i	-	$k^{k}(y_{i})$ amount of the k th tax payable ⁴ (k=1,,K) by individual i having income y_{i}
t ^k	=	$(t^{k}(y_{1}),,t^{k}(y_{N}))$ distribution vector for tax k
Y	=	$z_i y_i = N \cdot \overline{y}$ total income
r^k	= Σ	$t_i^k = N \cdot \overline{t}^k$ total revenue from tax k
т	=	$Z_k^{T^k} = N \cdot \bar{t}$ total tax revenue from all K taxes
P(t,	y)	measure of effective tax progression

³A discrete income distribution is used to simplify the presentation and some of the proofs. The analysis applies to income distribution cumula-tives generally.

⁴It is assumed throughout that $0 < t^{k}(y) < y$, and that $dt^{k}(y) / dy < 1$. This last relation assures that the tax function does not change the rank order of individuals along the income spectrum. Thus the concentration curve of post-tax income coincides with its Lorenz curve.

If on the other hand a tax schedule generates an aftertax distribution of income whose concentration curve is everywhere above that of another tax and the two taxes have identical yields, then the first can unambigously be shown as being the more progressive by standard structural progressivity measures (Jakobsson 1976, Hemming/Keen 1983).

These features of tax progression certainly correspond to widely held views and definitions. By themselves, however, dominance relations are only sufficient to generate a partial ordering of tax functions with regard to progression, while any specific measure of effective progression implies a mapping into the ray of real numbers and thus a complete ordering. The problem, then, is how to choose a specific set of measures to use for empirical work; or equivalently, what additional principles can be drawn upon to motivate a complete ordering of tax functions with respect to progressivity.

III. PROPORTIONAL YIELD NEUTRALITY AND LINEAR DECOMPOSABILITY

This section describes the two important properties of progressivity measures that have been emphasized in the literature and discusses their interrelationship.

A. Proportional Yield Neutrality (PYN)

. . . .

The condition of proportional yield neutrality (PYN) for measures of tax progression requires that measured progression be invariant to changes in total tax yield when they are the result of proportional changes in taxes. Formally, proportional yield neutrality is defined as homogeneity of degree zero of a progressivity measure in tax liabilities.

(PYN)

P(at,y) = P(t,y)

for any $\alpha > 0$

. .. .

Khetan and Poddar (1976) and Kakwani (1977a) have stressed the importance of this property which allows to filter out the influence of changes in the size of government budgets from that of changes in the relative distribution of tax liabilities. They both have expressed strong reservations about the use of the Musgrave-Thin coefficient as a measure of effective progression precisely because it is not yield neutral. On the other hand this invariance property has been critizised by authors, who aimed to arrive at reliable measures of effective redistribution and connect progressivity with increasing social welfare according to distributional equity judgements (Blackorby/Donaldson 1984, Kiefer 1984). Following their arguments against "tax-scale-invariance" they are unambigously right in stressing that scale variations in individual tax liabilities affect the distribution of post-tax income. But what they fail to recognize is that demanding proportional yield neutrality does not mean to deny the redistributive consequences of average tax variations but to separate tax level effects from tax variance effects.

Two examples of everyday tax administration may serve to illustrate the practical importance of progressivity measures fulfilling proportional yield neutrality.

In Canada, most provinces have entered into a tax collection agreement with the federal government whereby their personal income tax is expressed as a proportion of federal tax payable. It is widely felt that this method of determining provincial taxes does not affect progressivity. Furthermore, when changes in personal income taxes are being made or considered, it is often the case that evaluations of the progressivity of the tax changes are based on tables showing for various income levels the change in tax as a proportion of tax originally payable. Such tables only make

- 8 -

sense if they are based on an implicit assumption of yield neutrality. Examples include Sunley (1978), Bradford (1978) and the Department of Finance (1978).

For purposes of this paper, only measures satisfying the condition of proportional yield neutrality will be referred to as measures of effective progression. In contrast, measures based on comparisons of before- and after-tax income inequality, which in fact typically violate the condition of yield neutrality, will be referred to as measures of redistribution. Thus the Musgrave-Thin measure or the Reynolds/ Smolensky measure are measures of redistribution.

According to their definition there exist close relations between the measures of redistribution and the measures of effective progression mentioned above. Nevertheless it must be kept in mind that these two kinds of measures will not always agree in their ordering of tax functions as pointed out for US data by Formby/Seaks/Smith (1981). In other words, a more progressive tax is not always more redistributive⁵.

The basic conflict between measures of progression and redistribution can be highlighted by a simple numerical example, shown in Table 2. In this example, there are two individuals and two taxes, with Tax B raising twice as much revenue as Tax A.

Fable 2	:	Hypothetical	Example
	_		

Individual	Before Tax Income(\$)	Amount : of Tax ' (\$) (\$)		Shares of Total Tax (%)		Shares of Total After-Tax Income ((응)
		А	B	Tax A	Tax B	Tax A	Tax B	
1	5,000	0	500	0.0	5.0	20.0	22.5	
2	25,000 5	000	9500	100.0	95.0	80.0	77.5	

Musgrave/Thin and Kakwani also note this distinction between progression and redistribution.

5

Examination of the tax shares paid by each individual, which is equivalent to using measures of progression, suggests that Tax A is more progressive; a higher share of the tax burden falls on the high income individual under Tax A than under Tax B. However, the shares of aftertax income are more equally distributed under Tax B indicating that it is the more redistributive. If Tax B is viewed as the more "progressive", then a preference for a measure of redistribution stating some other kind of "progressivity" than effective progressivity is therefore implied.

B. <u>Linear decomposability (LD)</u>

Suits has pointed out that his index of tax progressivity S has the desirable property of being decomposable such that the overall progression of system of taxes is a weighted average of the progression of the various component taxes. Kakwani also makes use of the fact that his measure is decomposable. This property of decomposability can be formally stated as follows:

$$(LD) \qquad P(\sum_{K} t^{k}, y) = \sum_{K} w^{k}(\overline{t}^{k}, \overline{t}, \overline{y}) \cdot P(t^{k}, y) = \sum_{K} \frac{\overline{t}^{k}}{\overline{t}} P(t^{k}, y)$$

Thus, linear decomposability requires that the overall progressivity of a system of K taxes be a simple weighted average of the progressivities of all the different taxes.

While neither KP1 nor KP2 are decomposable as originally defined by Khetan and Poddar, it is the case that they both satisfy this property once transformed as below:

$$KP1' = 1/KP1$$

 $KP2' = 1 - (1/KP2) = S$

KP2' is identical to the Suits measure which he has already shown to be decomposable. That KP1'=C/A is decomposable follows from its definition as C is decomposable, a property which Kakwani has already used to proof decomposability of his measure. However, the Musgrave-Thin measure is not decomposable⁶.

C. Relation between yield neutrality and decomposability

One question is whether or not there is any relationship between the two apparently desirable properties for measures of effective tax progression, yield neutrality and decomposability. In fact there is, as the following theorems indicate.

- Theorem 1: Any continuos progressivity measure P(t,y) which is linear decomposable must also fulfill the condition of proportional yield neutrality; i.e. (LD) => (PYN)
- **<u>Proof</u>:** LD implies for an arbitrary replication of a tax structure t $P(n \cdot t, y) = \sum_{n} \frac{\overline{t}}{n \cdot t} P(t, y) = P(t, y)$ for any integer n>0 Substituting t^{*}= n · t and applying the definition of (LD) oncemore for an arbitrary integer m>0 leads to $P(t^*, y) = P(\frac{1}{n} t^*, y)$ $P(mt^*, y) = P(\frac{m}{n} t^*, y) = P(t^*, y)$ Thus we have for any rational $\alpha = \frac{m}{n} > 0$ $P(\alpha t, y) = P(t, y)$ which according to continuity of P holds for any real

 $\alpha > 0$ too.

It may also be noted that any linear transformation of a decomposable measure is also decomposable. MT has been shown to be not yield neutral (defined above) by Khetan and Poddar so that by Theorem 1 below it cannot be decomposable.

Theorem 2: Proportional yield neutrality does not imply linear decomposability of a progressivity measure P(t,y) i.e. (PYN ≠>LD)

Proof: Khetan/Poddar (1976) provide a counterexample. Their measure KP1, as well as its inverse [KP1]⁻¹ is yield neutral. But only one of these two measures, viz. [KP1]⁻¹, can be decomposable. So KP1 proves (PYN) =>(LD) to be wrong.

IV. SENSITIVITY TO PROGRESSION IN DIFFERENT INCOME RANGES

As Atkinson (1970) has shown different measures of income inequality may rank the same pair of distributions differently. Moreover, when the Lorenz curves of the two income distributions being compared cross then there can always be found two measures of income inequality based on inequality averse social welfare judgements of the Samuelson/Bergson type which rank the two distributions differently. This paradoxical result occurs as the sensitivity of inequality measures differs in different ranges of the income range. According to an approach by Love/Wolfson (1976)⁷ analytical methods can be used to determine the pattern of sensitivity of specific inequality measures along the income scale. Then in empirical work, a suitable number of such measures can be chosen in such a way as to assure that at least one of the measures is relatively sensitive in each of the relevant income ranges. For example, the Theil-Bernouilli coefficient, the Gini coefficient, and the coefficient of variation are most sensitive in the lower, middle, and upper income range respectively. When all three of these measures agree in the ranking of two income distributions, it is most likely that the corresponding Lorenz curves do not cross. However, when these three measures

- 12 -

^{&#}x27;Sensitivity of inequality measures is discussed informally in Sen (1973). The notion of sensitivity of measures of progression is also mentioned by Musgrave and Thin, and by Suits.

do not agree, the Lorenz curves do cross and one will have to justify any ranking by an additional explicit value judgement.

The same kind of situation can arise in the case of summary measures of effective tax progression. If only one measure is used, which is sensitive to tax progression in a certain range of the income spectrum, then it is quite possible that changes in tax progression elsewhere in the income spectrum will be missed. An obvious solution to this problem (other than a detailed examination of the underlying data) is to employ two or three measures of effective progression that embody different patterns of sensitivity to progression in various ranges of the income spectrum. This in turn requires a method for determining the pattern of sensitivity of a measure of effective progression.

The discussion of sensitivity to progression will proceed in three stages. In the first stage, the sensitivity of the measures already discussed will be determined by developing a formal definition of sensitivity to progression as a function of income⁸. Since all these measures turn out to be sensitive to progression in the middle ranges of the income spectrum, a new class of measures of progression is developed in the second stage. Then in the third stage, the notion of sensitivity to progression is applied to this new class of measures in order to obtain specific measures that are relatively more sensitive in other parts of the income spectrum.

- 13 -

^oEven though MT is not yield neutral, the analysis of sensitivity to progression that will be developed involves only comparisons of taxes having equal yields. Thus, for the moment, the distinction made above between measures of progression and measures of redistribution can be ignored.

In order to determine the sensitivity of specific summary measures of effective tax progression to progression in various ranges of the income spectrum, an approach analogous to the one developed by Love and Wolfson for summary measures of income inequality will be followed. This approach simply examines the change in the value of the measure of progression if tax is increased by an amount dt for an individual with income y_i while it is simultaneously decreased by the same amount dt for an individual with income y_j . Formally, a "sensitivity to progression" function for a measure of effective tax progression P can be defined as follows:

$$SP_{P} (Y_{i}, Y_{j}) = \partial P(t, y) / \partial t|_{Y_{i}} - \partial P(t, y) / \partial t|_{Y_{j}}$$

Since area A in Figure 1 is unaffected by tax changes, the SP functions for the Lorenz curve based measures will be proportional to the SP functions applied to the formulae for the three areas B, C, and D given in Table 1 above. These SP functions are:

 $SP_{B} = (i - j) / N^{2}(\overline{y} - \overline{t})$ $SP_{C} = (j - i) / N^{2}\overline{t} \text{ and}$ $SP_{D} = ((y_{i} - y_{j})/2 + \sum_{k=j+1}^{i-1} y_{k}) / N^{2}\overline{y}\overline{t}$

The SP functions for the Lorenz curve based measures can now be expressed as follows:

$$SP_{MT} = SP_B/A$$

 $SP_{KP1} = -A SP_C/C^2$
 $SP_{KP2} = -SP_D/2D^2$

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$$SP_{RS} = 2 SP_{B}$$

 $SP_{S} = -2 SP_{D}$
 $SP_{K} = -2 SP_{C}$

In all cases, these functions are positive whenever $y_i > y_j$. Thus, a tax increase for individual i with a corresponding tax cut for individual j increases progressivity, as it should.

The basic question for determining the pattern of sensitivity of the measures is for which y_i and y_j will the magnitude of the SP function be greatest? By inspection of the formulae, it is clear that SP_{MT} , SP_{KP1} , SP_{RS} , and SP_{K} depend on the "rank distance" between y_i and y_j while SP_{KP2} and SP_S depend on the "income distance" between y_i and y_j . Clearly the values of all six SP functions are positively related to the (rank or income) distance between y_i and y_i. This is simply a corollary of the fact that these are measures of progressivity. Thus, in order to assess their sensitivity, it is most appropriate to "standardize" the examination of the SP functions by keeping the difference between y, and y_i fixed at some arbitrary level h. Let $y = y_i$ and $y+h = y_i$. We can then ask for a given income difference h>O, at what income level y will the magnitude of the SP function be greatest? The answer for MT, KP1, RS, and K is identical to that for the Gini coefficient for sensitivity to inequality, namely at the modal income. In this region of the income spectrum, for a given width of the income interval h there are the greatest number of intervening incomes so that the rank distance will also be greatest. Since income distributions are typically uni-modal (except possibly for zero incomes) with the mode located in the lower middle income range, it follows that MT, KP1, RS, and K are most

sensitive to changes in the tax function in that income range.

In the case of KP2 and S, however, the range of greatest sensitivity will not be the income intervall with the highest density but the income intervall with the highest share of total income. Given the usual shape of income distribution this intervall with the highest first moment of cumulated densities will be above the modal income, its location will depend on the skewness of the distribution. Typically this will be in the upper middle range of the income spectrum.

The general conclusion is that all Lorenz curve based measures of tax progression are most sensitive in the middle income ranges.

B. A Broad Class of Measures of Effective Tax Progression

It is clear that in terms of their derivation as well as their patterns of sensitivity, the Musgrave/Thin, Suits, Kakwani, Reynolds/Smolensky and Khetan/Poddar measures are related to the Gini coefficient in the realm of measures of income inequality. Since many other measures of inequality exist, it seems intuitively plausible that a wide range of measures of effective tax progression should also exist, and that they would differ in, among other things, their sensitivity to tax progression at different points in the income spectrum. Since virtually all the income inequality measures⁹ in the literature with the exception of the Gini coefficient are additively separable¹⁰, it seems quite reasonable to define an analogous class of additively separable measures of tax progression,

$$P(t,y) = \sum_{i=1}^{N} w(y_i/\overline{y}) t(y_i)$$

where $w(y/\bar{y})$ is an arbitrary weighting function.¹¹ However, it is not yet clear that these measures will take on values that are easily interpreted, or that they are properly oriented. MT, KP1, and KP2 all take the value one when the tax is proportional, and are greater than one if the tax is progressive. To assure that the additively separable (AS) measures also have this property, they must be normalized as follows:

⁹Note that inequality measures, as we use the term, must satisfy the Pigou-Dalton condition of transfers, while inequality indicators need only satisfy the weaker requirement of not violating the Pigou-Dalton condition. This condition simply requires that if any two distributions of income differ only by a sequence of transfers, all from higher to lower income recipients with no one changing their rank order, then the after-transfer distribution must be measured as having less inequality. The condition is equivalent to a situation where the two Lorenz curves do not cross and the one closest to the diagonal is measured as having less inequality. The share of the top 5%, for example, is then an inequality indicator (it does not violate the Pigou-Dalton condition, but neither does it respond to transfers entirely within the top 5% or bottom 95%) while the variance of logs is-neither an inequality measure nor an inequality indicator (since it violates the Pigou-Dalton condition for transfers above about three times the mean).

¹⁰Additive separability simply requires that the measure can be written in an additive form. While its intuitive economic content is not clear, it is widely used because it greatly simplifies the mathematics. For a discussion in the context of inequality measures, see Sen (1973).

¹¹Note that the weighting function depends on y/\bar{y} , income relative to the mean. Such normalization corresponds to the widely accepted views that progressivity and inequality are both purely relative attributes. This view is implicit in all measures based on Lorenz curves.

(AS)
$$P(t,y) = H(y) \Sigma w(y/y) t(y_1)/T$$

where
$$H(y) = N / \Sigma_{i=1}^{N} y_{i} w(y_{i}/\overline{y})$$

Then if t(y) is a proportional tax, we have P(t,y) = P(xy,y)=1 for some constant $\alpha>0$. The following theorem is a straight-forward result:

Theorem 3: Any additively separable propressivity measure is linearly decomposable i.e. (AS) => (LD)

Proof: Defining an (AS) measure of tax k

$$P(t^{k}, y) = H(y) \cdot \sum_{i}^{\Sigma} w(y_{i}/\overline{y}) t^{k}(y_{i}) / T^{k}$$

and adding them up according to (LD) we arrive at

$$\begin{split} \frac{\Sigma \underline{T}^{k}}{k} P(t^{k}, y) &= \sum_{k} \underline{T}^{k} \cdot H(y) \cdot \sum_{i} w(y_{i}/\overline{y}) \cdot t^{k}(y_{i})/T^{k} = \\ &= H(y) \cdot \sum_{i} w(y_{i}/\overline{y}) \cdot \frac{\Sigma \underline{T}^{k}}{k} \cdot \frac{t^{k}(y_{i})}{T^{k}} = \\ &= H(y) \cdot \sum_{i} w(y_{i}/\overline{y}) \cdot \frac{\Sigma t^{k}(y_{i})}{T} = P(\Sigma t^{k}, y) \\ &= u(y) \cdot \sum_{i} w(y_{i}/\overline{y}) \cdot \frac{\Sigma t^{k}(y_{i})}{T} = P(\Sigma t^{k}, y) \\ &= u(y) \cdot \sum_{i} w(y_{i}/\overline{y}) \cdot \frac{\Sigma t^{k}(y_{i})}{T} = P(\Sigma t^{k}, y) \\ &= u(y) \cdot \sum_{i} w(y_{i}/\overline{y}) \cdot \frac{\Sigma t^{k}(y_{i})}{T} = P(\Sigma t^{k}, y) \\ &= u(y) \cdot \sum_{i} w(y_{i}/\overline{y}) \cdot \frac{\Sigma t^{k}(y_{i})}{T} = P(\Sigma t^{k}, y) \\ &= u(y) \cdot \frac{\Sigma t^{k}(y_{i})}{T} = u(y)$$

It then follows from Theorem 1 that (AS) = (PYN) so that any (AS) measure is both decomposable and yield neutral. It is still necessary to place some restrictions on the weighting function w(y) to make sure that the measure actually increases with increasing progressivity of t(y); however, this will be determined in the next section on sensitivity.

C. Sensitivity of (AS) Measures of Effective Progression

Exactly the same method that was used to determine the sensitivity to progression pattern of the Gini related measures can be applied to the additively separable class of measures. The result is the following SP function:

$$SP_{P}(y_{i}, y_{j}) = H(y) (w(y_{i}/\overline{y}) - w(y_{j}/\overline{y})/T)$$

In order to assure that P actually measures progressivity, it is necessary that $y_i > y_j$ implies $SP_P(y_i, y_j) > 0$. In turn, assuming that w(y) > 0 for all y assures that H(y) > 0. Thus, the condition $SP_P(y_i, y_j) > 0$ requires that w(y) be a strictly increasing function of y, in other words that w'(y) = dw(y)/dy > 0.

The argument above has established that any additive measure with both w(y) and w'(y) positive indeed measures effective tax progression. But the class of such measures is still very large. In order to choose among alternative measures, the SP functions' characteristics can be examined. Re-expressing the SP function in terms of incomes y and y+h, we have

$$SP_{p}(y,h) = H(y)(w((y+h)/\overline{y}) - w(y/\overline{y}))/T)$$

In this case, the SP function indicates the changes in progression that would result from an increase in tax for an individual with income y+h and a corresponding decrease in tax for an individual with income y.

¹²More generally, the requirement is that the signs of w(y) for all y and of w'(y) for all y should be the same. Note that this condition is different from the usual requirement in the context of inequality measures that w(y) be concave.

It is now possible to define generally increasing and decreasing sensitivity to progression as a function of income simply by taking the partial derivative of the SP function with respect to income:

$$\partial SP_{p}(y,h)/\partial y = w'((y+h)/\overline{y}) - w'(y/\overline{y})$$

It is now clear that if w(y) is convex (i.e. w'(y) is increasing) then a change in progression of the tax function at higher incomes will have a greater impact on measures progression than will an identical change at lower incomes. The converse applies when w(y) is concave.

Given this result, it is now possible to construct specific measures of effective tax progression with any pattern of sensitivity. For example, $w(y) = y^2$ implies a measure that will be relatively more sensitive to tax changes in the higher income ranges, while $w(y) = y^{0.5}$ implies a measure that is relatively more sensitive to changes at the lower end of the income spectrum. Correspondingly, for any specific additive measure of progression, it is possible to determine its pattern of sensitivity analytically simply by inspecting the implied SP function.

V. EMPIRICAL ILLUSTRATIONS

A fundamental question relating to the measures of effective tax progression described above is their correspondence with measured inequality. It may be recalled that in the discussion of yield neutrality above a distinction was drawn between measures of progression and measures of redistribution. The discussion up to this point has been almost entirely concerned with measures of effective tax progression that are yield neutral. We turn now to an analysis of the relationship between progression and redistribution, for the specific meanings that have been given to these terms. To be precise, a measure of redistribution will be defined as the ratio of the inequality of before-tax income to the inequality of after-tax income, where an "acceptable" inequality measure is being used. An inequality measure is "acceptable" if it ranks one income distribution as more equal than another whenever its Lorenz curve is closer to the diagonal and the two Lorenz curves do not cross.

Given this definition of a measure of redistribution, it is clear that it will take a value of one when a tax is proportional to income, just as does a measure of progression. When the tax reduces after-tax income inequality relative to before-tax income inequality, the measure of redistribution will take values greater than one, indicating a redistributive tax. The implication is then that the following three statements are equivalent:

the tax is progressive;

the tax is redistributive;

the tax reduces after-tax income inequality -

provided the relevant Lorenz curves do not cross. Finally, if two taxes have the same yield, one tax will be more progressive than the other if and only if it is more redistributive - again provided the Lorenz curves of tax, or equivalently after-tax income, do not cross¹³. Thus, if any conflict between progressivity and redistribution is to arise, it must be in cases where the total revenues (or equivalently average tax rates) of the taxes being compared are

¹³Tax A is more progressive than tax B of equal yield, going back to general and accepted principles, if and only if the share of tax A paid by low income groups is less than their share of tax B, or equivalently if the concentration curve of tax A is below the concentration curve of tax B (using curve iii in Figure 1 above) or equivalently if the concentration curve of after-tax A income is above the concentration curve of after-tax B income (using curve ii in Figure 1), or equivalently if the ratio of before tax inequality to after-tax inequality is greater for tax A than the corresponding ratio for tax B that is, if and only if tax A is measured as being more redistributive than tax B.

not the same. The question then is whether or not it is always the case that a tax change that both increases measured progression and changes the total yield will also increase measured redistribution, when the two Lorenz curves of tax do not cross. (If the Lorenz curves of tax do cross, no unambiguous ranking even among measures of progression is possible.) Unfortunately, the answer is not always in the affirmative.

In order to illustrate the response to tax changes of specific measures of progression and redistribution, several empirical examples are presented. The first set is based on a recent analysis of changes in the Canadian personal income tax system by Allen et. al. (1978). Table 3 displays the distributions for taxpayers in 1975 of (a) personal income tax actually paid, (b) the increase in personal tax liabilities had indexing of the tax system not been implemented, and (c) the effects of a hypothetical tax cut of \$100 per tax-payer with income below \$3.000, as well as a 5% of tax cut for taxpayers with incomes over \$40.000 (a cut that results in intersecting Lorenz curves). The effects of these tax changes on measured progression and measured redistribution are given in Table 4. Eight measures are considered:

- $\left. \begin{array}{c} \text{KP1} & \\ \text{KP2} & \end{array} \right\} \text{ as defined above} \\ \text{MT} & \end{array}$
- AMPL Additive Measure of Progression-Low, defined above with weighting function w(y) = y^{0,5}so as to be relatively more sensitive to progression in the lower range of the income spectrum
- AMPH Additive Measure of Progression-High, defined above with weighting function w(y) = y² so as to be relatively more sensitive to progression in the upper ranges of the income spectrum
- RGINI Gini-based measure of redistribution, ratio of Gini of before-tax income to Gini of after-tax income

Table 3:

Distributional Impact of Indexing and an Illustrative Tax Cut (Canada 1975)

J	r		(a)	(b)	(c)
Assessed Income Class (\$ 000's)	Distribution of Tax Filers (%)	Average Assessed Income (\$)	Average Federal Tax, 1975 (\$)	Actual Indexing Tax Cut (\$)	Illustrative Tax Cut (\$)
0 - 1	5.2	239	0	0	100
1 - 2	7.9	1,532	0	0	100
2 - 3	7.2	2,504	0	1	100
3 - 4	6.9	3,480	6	15	0
4 - 5	6.4	4,503	69	52	0
5 - 7	13.5	6,009	255	75	0
7 - 10	16.8	8,462	641	108	О
10 - 15	20.2	12,272	1,310	164	0
15 - 20	9.2	17,116	2,249	264	0
20 - 25	3.4	22,077	3,233	379	0
25 - 30	1.3	27,137	4,386	7	0
30 - 40	1.0	34,091	6,076	- 516	0
40 - 50	0.4	44,281	9,018	J	451
50 -100	0.5	54,663	15,989	988	899
2	0.1	165,143	50,490	1,453	2,525

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Table 4 :

Impact of Indexing and an Illustrative Tax Cut on Measures of Progression and Redistribution, (Canadian Federal Personal Income Tax. 1975).

	Numerical Values of the Measures								
,	Actual Tax Structure	Actual I ture Exc Indexing	ax Struc- clusive of	Actual Tax Structure with Illustrative Ta Cut [*]					
Measures of Progressivity									
KP1	1.675	1.604	(- 4.2%)	1.822	(+8.8%)				
KP2	1.405	1.364	(- 2.9%)	1.448	(+3.1%)				
AMPL	1.230	1.209	(- 1.7%)	1.242	(+1.0%)				
AMPH	2.273	2.122	(- 6.6%)	2.238	·(-1. 5%)				
Measures of Redistribution									
REXP	1.026	1.028	(+0.2%)	1.029	(+0.3%)				
RGINI	1.071	1.076	(+0.5%)	1.078	(+0.7%)				
MT	1.049	1.053	(+0.4%)	1.054	(+0 <u>.</u> 5%)				
RCV	1.319	1.329	(O.8%)	1.312	(-0.5%)				

*The numbers in parentheses are percentage changes in the values of the respective measures

Squared coefficient of variation (CV)-RCV based measure of redistribution, ratio of CV of before-tax income to CV of aftertax income (relatively more sensitive in the upper range of the income spectrum) Exponential (EXP)¹⁴ based measure of re-REXP distribution, ratio of EXP of before-tax income to EXP of after-tax income (relatively more sensitive in the lower range of the income spectrum)

Table 4 clearly indicates the differences between measures of progression and measures of redistribution in cases where the total tax yield does not remain constant. The tax changes resulting from the indexing of the Canadian personal income tax system were progressive but not redistributive. Furthermore, the "Illustrative Cut" clearly shows the importance of the patterns of sensitivity for both measures of progression and measures of redistribution. The combination of a lump sum tax cut in the lowest income ranges and a proportional tax cut in the highest income ranges shows up as an increase in both progression and redistribution for all the Gini related measures (KP1, KP2, MT, and RGINI), an even greater increase for measures specifically designed to be sensitive in the lower income ranges (AMPL and REXP), but as an actual decrease in both progression and redistribution for measures specifically designed to be sensitive in the upper income ranges (AMPH and RCV). These results provide emphasis for our general point that in empirical analyses of progression or redistribution, a set of measures should be used that have been explicitly chosen to have a range of patterns of sensitivity.

¹⁴The exponential measure was developed in Wolfson (1977) specifically in order to be relatively more sensitive to inequality in the lower wealth ranges and to be defined for non-positive wealth (unlike the Theil-Bernouilli measure used in Love and Wolfson, which has a similar pattern of sensitivity, but is undefined for non-positive incomes). It is defined as $EXP=p_iexp(-y_i/\bar{y})$ where p_i is the proportion of the population in the ith income (or wealth) interval, y_i is the average income in the interval, and \bar{y} is the mean income.

Table 5:PROGRESSIVITY AND REDISTRIBUTIVEIMPACT OF U.S. TAXES, 1966

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	Measures of Effective Progression Measures of Redistributi					tion		
Tax	KP1	KP2	AMPL AMPH		MI	RGINI	RCV	REXP
Personal Income Tax	.:1.331	1.199	1.279	1.101	1.023	1.032	1.084	1.013
Corporate Income Tax	1.780	1.559	1.795	1.237	1.018	1.025	1.094	1.011
Property Tax	1.368	1.296	1.519	1.151	1.008	1.011	1.045	1.005
Sales and Excise Taxes	.813	.862	.714	•901	.988	.983	•956	.993
Payroll Taxes	.830	.852	.628	.891	•991	.987	.954	.994
Taxes on Personal Property and Motor Vehicles	.852	.893	.779	.924	•999	.999	•998	1.000
All Federal Taxes	1.144	1.095	1.161	1.055	1.027	1.038	1.110	1.016
All State and Local Taxes	1.038	1.046	1.130	1.032	1.003	1.004	1.027	1.002
All Taxes	1.109	1.079	1.151	1.048	1.033	1.047	1.160	.1.020

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This point is further supported in Table 5, where the eight measures of progression and redistribution defined above have been computed from the data for the U.S. tax system in 1966 presented by Suits (1977). The most striking observation is that all four measures of effective progression rank the corporate income tax as most progressive, followed by property taxes and personal income taxes. On the other hand only the coefficient-of-variation-based measure of redistribution, which is most sensitive in the upper income range, ranks the corporate income tax as most redistributive. The other three measures of redistribution rank the personal income tax as the most "progressive" tax.

VI SUMMARY AND CONCLUSIONS

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This paper has considered various properties of measures of effective tax progression and redistribution, and interrelationships among them. The properties of proportional yield neutrality and linear decomposability were defined formally and it was shown that decomposability implies yield neutrality, but not the converse. The concept of sensitivity of measures of progression to changes in progression in various ranges of the income spectrum was developed. It was shown that the concentration curve measures most frequently being used are sensitive only in the middle income ranges so that changes in progressivity elsewhere in the income spectrum may have been missed. A new class of measures of progression has been defined in order to permit the construction of measures with any desired pattern of sensitivity. Finally, numerical examples have been presented to illustrate both the importance of the pattern of sensitivity and the conflict between measures of progression and measures of redistribution.

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