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Safety-First Portfolio Optimization:
Fixed versus Random Target

by

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Safety-First Portfolio Optimization: Fixed versus Random Target*

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Abstract

This paper analyzes the safety-first portfolio model under two different target assumptions, the fixed target, which is commonly assumed in the literature, and the random target, which has played only a minor role so far. As both targets can be easily motivated, the open question is, which target choice leads to a better performance? We answer this question by comparing optimal expected portfolio returns of the fixed and the random target strategy. Assuming multivariate normal returns the answer is: (1) The random target strategy outperforms the fixed target strategy if the portfolio return and the random target are positively correlated and riskless investing is prohibited, (2) the fixed target strategy outperforms the random target strategy if the portfolio return and the random target are not positively correlated and riskless investing is allowed. The first result is practically most relevant, in particular for institutional portfolio management and the skilled private investor, which is supported by an empirical analysis. Furthermore, we show that these results also hold when relaxing the normal assumption.

Keywords: safety-first portfolio model, chance constraint programming, portfolio optimization, random target, benchmarking

JEL Classification: G11

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1 Introduction

Portfolio optimization under the safety-first criterion is concerned with maximizing the expected portfolio return, while, simultaneously, the probability of failing to achieve a specified (fixed) target must fall below a critical level and has its origins in the early papers of Roy (1952), Telser (1955) and Kataoka (1963). The safety-first risk measure is commonly expressed by a probability statement as \( P(Z < T) \leq \alpha \) where \( Z \) is a random variable, e.g. portfolio return, \( T \) is a fixed target, e.g. a minimum desired portfolio return and \( \alpha \) is a critical level on the probability of failing to achieve the target.

A comparative advantage of the safety-first criterion over deviation risk measures, such as the variance, is its consistency with the way investors perceive risk (see for example Atwood et al., 1988; Harlow, 1991; Brogan and Stidham, 2005). Empirically, this is shown for example in Lopes (1987), Kroll et al. (1988), DeBondt (1998), Lopes and Oden (1999) and Neugebauer (2008). These behaviorally appealing feature have made the safety-first criterion attractive for behavioral portfolio theory, see for example Shefrin and Statman (2000) and the recent paper by Das et al. (2010), in which they transform the utility based interpretation of mean-variance portfolio theory to the more appealing target based interpretation of safety-first portfolio theory under the assumption of normal distributed asset returns. More generally, Kalin and Zagst (1999) show the equivalence of mean-variance and safety-first portfolio theory for a wide class of probability distributions.

What all the above cited papers about the safety-first model have in common is the assumption of a fixed target \( T \), which, however, leads to significant conceptual disadvantages: Suppose an investment fund which seeks to achieve a fixed return \( T \) for the next period. According to this target the fund manager purchases and sells assets. But, what happen when the market return within the next period is greater than \( T \)? The fund performed rather poorly. This situation could have been avoided if the manager had reallocated the assets according to the expected performance of the market, which is common practice in passive portfolio management. Or, suppose a fund which seeks to outperform the market (active portfolio management), i.e. the target for the next investment period is the sum of the expected market performance and some extra return.\(^1\) Or, from an individual perspective the target may not even be known. Many individuals have the target of “being successful”, but only a very few know precisely which selection of money, leisure time, culture etc. must be attained to achieve this target (Bordley and Licalzi, 2000). There are, thus, several situations in which an unknown or random target seems to be a more suitable choice. But, does in all these situations a random target lead to better results in terms of higher returns or are there situations in which a fixed target should be the preferred choice? As we pay particular attention to a portfolio optimization model, the question is, which target choice leads ceteris paribus to larger optimal expected portfolio returns? In this paper

\(^1\)For financial risk management with benchmarking see for example Basak et al. (2006), Browne (2000) and Gaivoronski et al. (2005).
we suggest a first answer to these questions.

In detail, assuming normal distributed asset returns, we know for example from Kalin and Zagst (1999) that the (probabilistic) safety-first risk measure can easily be transformed to a deterministic risk measure in terms of standard deviation. In section 2, we use this result to transform the safety-first portfolio model to an equivalent deterministic version, which is general enough to consider both, fixed and random targets. In section 3, which contains our main results, we compare optimal expected portfolio returns of the fixed and random target strategy and obtain following results: The random target strategy outperforms the fixed target strategy if the portfolio return and the random target are positively correlated and riskless investing is prohibited, whereas the fixed target strategy outperforms the random target strategy if the portfolio return and the random target are not positively correlated and riskless investing is allowed. By providing empirical evidence for the German stock market in section 4, we point out that the first case, in which the portfolio return and the random target are positively correlated, is practically most relevant. As the normal distribution is a good starting point analyzing the safety-first model (see Leibowitz and Henriksson, 1989; Leibowitz and Kogelman, 1991; Das et al., 2010), but typically violated in practice, we relax this assumption in section 5 and show, using a well-known approximation, that all results from section 3 remain the same. Section 6 offers a discussion and concludes the paper.

2 Safety-First Portfolio Optimization with Normal Distributed Asset Returns

Consider an investment universe of \( n \) different financial assets with \( \mathbf{R} := (R_1, \ldots, R_n)^T \) presenting the vector of random asset returns. A portfolio where short sales are prohibited is defined as a vector \( \mathbf{x} \in [0, 1]^n \) with \( x_i \) being the proportion invested in asset \( i \) and the proportions sum to one, which is also known under the “fully invested constraint”. Let the product \( \mathbf{x}^T \mathbf{R} \) be the random portfolio return, \( T \) be a fixed or random target and \( \alpha \) be a critical probability, then the safety-first portfolio model which maximizes the expected portfolio return subject to a safety-first constraint can be expressed as

\[
\text{Problem 1.} \quad \max_{\mathbf{x} \in [0,1]^n} \mu(\mathbf{x}^T \mathbf{R}) \quad \text{s.t.} \\
\quad P(\mathbf{x}^T \mathbf{R} < T) \leq \alpha, \\
\quad 1^T \mathbf{x} = 1.
\]

For a numerical treatment of problem 1 it is useful to provide a deterministic rather than a probabilistic expression of the safety-first constraint (1), which can be easily achieved under the normal assumption (see for example Kalin and Zagst, 1999). Many other papers studying the safety-first framework, among them Leibowitz and Henriksson (1989), Leibowitz and Kogelman (1991), Albrecht (1993) and Das et al. (2010), assume normal distributed asset returns. Define therefore \( Z := \mathbf{x}^T \mathbf{R} - T \) with
as a normal distributed random variable with expected value \( \mu(Z) \) and variance \( \sigma^2(Z) \). Then, employing the usual textbook transformation for the normal distribution, safety-first constraint (1) can be equivalently expressed as

\[
P(Z < 0) \leq \alpha \iff P \left( \frac{Z - \mu(Z)}{\sigma(Z)} < -\frac{\mu(Z)}{\sigma(Z)} \right) \leq \alpha \iff 
\Phi \left( -\frac{\mu(Z)}{\sigma(Z)} \right) \leq \alpha \iff 
\Phi^{-1}(\alpha)\sigma(Z) + \mu(Z) \geq 0, \tag{3}
\]

where \( \Phi^{-1}(\alpha) \) is the \( \alpha \)-quantile of the standard normal distribution and \( \sigma(Z) \) is the standard deviation obtained from drawing the positive square root of the variance, \( \sigma^2(Z) \). The following theorem, found in a slightly modified version in Kall and Mayer (2005, pp. 103-106) and Ruszczyński and Shapiro (2003, pp. 10), provides precise expressions for \( \mu(Z) \) and \( \sigma^2(Z) \) (and \( \sigma(Z) \)).

**Theorem 1.** Let \( R = (R_1, \ldots, R_n)^T \) be \( n \)-variate normal distributed with expected value vector \( \mu \) and covariance matrix \( \Sigma \), \( R \sim N(\mu, \Sigma) \), and \( T \) be univariate normal distributed with expected value \( \mu(T) \) and variance \( \sigma^2(T) \), \( T \sim N(\mu(T), \sigma^2(T)) \), then the random variable \( Z = \mathbf{x}^T R - T \) is univariate normal distributed with expected value \( \mu^T x - \mu(T) \) and variance \( \|L^T x - b\|^2 \), where \( L \in \mathbb{R}^{n \times r} \) and \( b \in \mathbb{R}^r \) are obtained from the factorization of the covariance matrix \( \text{cov}(R^T, T) \) and \( \|\cdot\| \) denotes the Euclidean norm.

**Proof:** A detailed proof, following Kall and Mayer (2005, pp. 103-106), and a remark on the numerical treatment of the factorization of the covariance matrix are provided in the separate appendix.

Note,

\[
\sigma^2(Z) = \|L^T x - b\|^2 = (L^T x - b)^T (L^T x - b) = \mathbf{x}^T L L^T x - 2(Lb)^T x + b^T b,
\]

where the first addend in the last equation of \( \sigma^2(Z) \) is the variance of \( \mathbf{x}^T R \). The middle addend is two times the covariance between \( \mathbf{x}^T R \) and \( T \) with \( Lb \in \mathbb{R}^n \) being the cross-covariance vector between \( R \) and \( T \). The third addend is the variance of \( T \).

Applying then the deterministic safety-first constraint (3) with \( \mu(Z) = \mu^T x - \mu(T) \) and \( \sigma(Z) = \|L^T x - b\| \), obtained from theorem 1, problem 1 can be reformulated as

**Problem 2.**

\[
\max_{\mathbf{x} \in [0,1]^n} \mathbf{x}^T \mu \quad s.t. \\
\Phi^{-1}(\alpha)\|L^T x - b\| + \mu^T x \geq \mu(T), \quad \text{subject to } 1^T x = 1.
\]
Problem 2 is the deterministic equivalent of problem 1 and can now be solved numerically. It is linear in its objective but quadratic in its constraints. It therefore relates to the class of quadratic optimization problems. If $\alpha \in (0, 1/2)$, the deterministic safety-first constraint (4) is concave, which was first shown by Kataoka (1963). Under this assumption problem 2 can easily be solved by concave optimization methods. As $\alpha$ represents the maximum probability of failing to achieve the target and is moreover specified by the investor herself, it is usually chosen to be low, e.g. 5% or 10%. Thus, the restriction of $\alpha$ not exceeding 50% does not limit the practical value of our results. We therefore stick throughout our analysis to this assumption.

Note, modeling a riskfree asset $j$ can be easily achieved by setting the $j$-th row of $L$ to the zero-vector. Then, we have $R_j = \mu_j$ with $\mu_j$ being the riskfree rate. Analogously, a fixed target instead of a normal distributed target can be achieved by setting $b$ to the zero-vector. Then, the correlation between the target and the asset returns is zero and $T$ collapses to a fixed target with value $\mu(T)$.

### 3 Fixed versus Random Target

This section concerns the comparison of the fixed target strategy ($S_1$) with the random target strategy ($S_2$). The fixed and the random target is denoted as $T_1$ and $T_2$, respectively. Both strategies face the same investment universe and the same estimates for the expected returns, $\mu = (\mu(R_1), \ldots, \mu(R_n))^T$, and covariances, $\Sigma$. Additionally, the crosswise covariations between the random target and the asset returns are denoted by the $n$-vector $(Lb) = \text{cov}(R, T_2)$. Note, the $n \times r$ matrix $L$ and the $r$-vector $b$ are obtained from the factorization of the covariance matrix $\text{cov}(R_1, \ldots, R_n, T_2)$ (see theorem 1).

As assumed in theorem 1, $T_2$ is normal distributed with expected value $\mu(T_2)$ and variance $\sigma^2(T_2)$. It appears reasonable to assume, $\mu(T_2) > T_1$, so that the expected target return of $S_2$ is greater than $T_1$. This is due to the uncertainty about $T_2$, making $S_2$ riskier than $S_1$. This higher risk must thus be compensated by a greater expected target return. However, to keep our results as simple as possible we assume throughout this section that $\mu(T_2) = T_1$. But, all results presented here can be straightforwardly modified such that $\mu(T_2) > T_1$, for example by defining $\mu(T_2) := T_1 + \varepsilon$, $\varepsilon > 0$ and adapting the calculations.

In the following two subsections, we evaluate the differences in the performance of $S_1$ and $S_2$ by comparing optimal expected portfolio returns. In subsection 3.1 the practical more interesting case, in which the random target and the asset returns are crosswise positively correlated, is discussed, whereas the opposite case is discussed in subsection 3.2.

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2 More precisely, (4) is a second-order cone constraint, also called ice-cream cone or Lorentz cone. For a detailed discussion on that, consult Kall and Mayer (2005, pp. 273-276) and the references therein.

3 If the feasible domain is concave and not empty, there exists a unique maximum. For an overview of convex optimization see Boyd and Vandenberghe (2007)
3.1 The Positive Correlated Case

Consider strategy $S_2$ in which a portfolio is managed subject to the performance of a benchmark, such as a stock index like the S&P500, without directly investing into the benchmark.\footnote{A direct investment in an index can be obtained by purchasing an exchange traded fund (ETF) on the index, which explicitly tracks the index and is, moreover, attractive because of low transactions costs and tax efficiency.} There are at least three situations where this appears reasonable: First, the portfolio manager seeks to outperform the benchmark, which is typically for actively traded funds. Second, the manager seeks to invest in a foreign market without suffering from foreign exchange risk. This can be realized by tracking or outperforming a foreign index, representing this market, by a domestic portfolio. Third, the latter situation can also be derived from an individual perspective, where a skilled private investor is attracted by the performance of a foreign market, but does not want to invest directly into the market. The individual investor rather seeks to track the performance of the foreign market by only investing in domestic stocks. This situation is derived from a behavioral phenomenon called home bias, which was first documented by French and Poterba (1991).

Reducing all these examples to the stock market, they have in common to track or outperform one stock market by investing in similar but different stocks from another market. We therefore assume that asset returns are positively correlated with the benchmark return, i.e. the cross-covariance vector between $R = (R_1, \ldots, R_n)^T$ and $T_2$ is

$$ (Lb) = \text{cov}(R, T_2) > 0. \quad (5) $$

This assumption is generally justified for most of the risky financial assets, in particular for stock markets. In section 4 we provide empirical evidence that even a stronger version of (5) is justified for the stock market. Remark that assumption (5) excludes riskless investing because it does not allow for a zero covariance. We have $S_1$ with fixed target, $T_1$, and $S_2$ with normal distributed target, $T_2 \sim \mathcal{N}(T_1, \sigma^2(T_2))$. Applying theorem 1 yields

$$ Z_1 = x^T R - T_1 \quad \text{with} \quad Z_1 \sim \mathcal{N}(\mu^T x - T_1, ||L^T x||^2), $$

$$ Z_2 = x^T R - T_2 \quad \text{with} \quad Z_2 \sim \mathcal{N}(\mu^T x - T_1, ||L^T x - b||^2). $$

Note, the expected values of $Z_1$ and $Z_2$ coincide, but the variances differ. Thus, in the normal distributed case, the question, whether $S_1$ outperforms $S_2$ or vice versa is simply the question of comparing variances. The following theorem shows that, under a weak additional assumption, the variance of $Z_2$ is smaller than the variance of $Z_1$ and therefore, $S_2$ outperforms $S_1$. To prove this theorem we make use of

**Lemma 1.** If

$$ \text{cov}(R_i, T_2) > \frac{1}{2} \sigma^2(T_2), \quad i = 1, \ldots, n, \quad (6) $$

holds, then for any critical probability $\alpha \in (0, 1/2)$ the following inequality is true:

$$ \Phi^{-1}(\alpha) \sigma(Z_2) + \mu(Z_2) > \Phi^{-1}(\alpha) \sigma(Z_1) + \mu(Z_1) $$
Proof: From the fully invested constraint (2) together with (6) follows
\[
\text{cov}(\mathbf{R}, \mathbf{T}_2)^T \mathbf{x} > \frac{1}{2} \sigma^2(T_2),
\]
which is equivalent to
\[
0 > -2\text{cov}(\mathbf{R}, \mathbf{T}_2)^T \mathbf{x} + \sigma^2(T_2) \iff \\
\|L^T \mathbf{x}\|^2 > \|L^T \mathbf{x}\|^2 - 2(L \mathbf{b})^T \mathbf{x} + \mathbf{b}^T \mathbf{b} \iff \\
\Phi^{-1}(\alpha)\|L^T \mathbf{x}\| + \mu^T \mathbf{x} - T_1 < \Phi^{-1}(\alpha)\|L^T \mathbf{x} - \mathbf{b}\| \iff \\
\Phi^{-1}(\alpha)\sigma(Z_1) + \mu(Z_1) < \Phi^{-1}(\alpha)\sigma(Z_2) + \mu(Z_2).
\]
Notice that the second last inequality reverses because \(\Phi^{-1}(\alpha) < 0 \ \forall \alpha \in (0, 1/2).\)

\[\square\]

**Theorem 2.** Provided (6) holds, for any critical probability \(\alpha \in (0, 1/2)\) the optimal expected portfolio return of \(S_2\) is larger or equal than the optimal expected portfolio return of \(S_1\).

Proof: It is sufficient to show that the set of feasible portfolios of \(S_1\), denoted as \(\mathcal{F}_1\), is a subset of the feasible domain of \(S_2\), denoted as \(\mathcal{F}_2\). If \(\mathcal{F}_1 = \emptyset\), then clearly \(\mathcal{F}_1 \subseteq \mathcal{F}_2\). All portfolios \(\mathbf{x} \in \mathcal{F}_1 \neq \emptyset\) satisfy
\[
P(Z_1 < 0) \leq \alpha \iff \Phi^{-1}(\alpha)\sigma(Z_1) + \mu(Z_1) \geq 0 \quad \text{lemma 1} \\
\Phi^{-1}(\alpha)\sigma(Z_2) + \mu(Z_2) > 0 \iff P(Z_2 < 0) < \alpha.
\]
Thus, \(\mathbf{x} \in \mathcal{F}_2\).

Remark, in the special case where the safety-first constraint for \(S_1\) is satisfied for all portfolios \(\mathbf{x}\), i.e. \(P(\mathbf{x}^T \mathbf{R} < T_1) < \alpha \ \forall \mathbf{x} \in [0, 1]^n\), the entire wealth is invested in the single asset with the highest expected return and thus the same is true for \(S_2\). In this case both investors obtain the same optimal expected portfolio return and only one asset is hold. Remark, if inequality (6) reverses, the result clearly reverses, i.e. \(S_1\) outperforms \(S_2\). But this is practically not the case as supported by the empirical analysis in section 4.

In figure 1, where the safety-first efficient frontiers\(^5\) for \(S_1\) and \(S_2\) are sketched, we illustrate the result: Suppose, both investors choose a critical probability of \(\alpha_1\), then the portfolio problem is neither feasible for \(S_1\) nor feasible for \(S_2\). Suppose, both choose \(\alpha_2\), then \(S_2\) outperforms \(S_1\) as \(\mu^T \mathbf{x}_2^* > \mu^T \mathbf{x}_1^*\). Finally, suppose that they choose \(\alpha_3\), then their optimal expected portfolio returns coincide and their entire wealth is invested in the single asset with the highest expected return.

\(^5\)Shefrin (2005) uses the term \(SP/A\) efficient frontier, which is the same as the safety-first efficient frontier, plotted in \((\mu^T \mathbf{x}^*, \alpha)\)-space. The safety-first efficient frontier is monotone non-decreasing as investors prefer higher portfolio returns \((\mu^T \mathbf{x}^*)\) but lower risk \((\alpha)\).
3.2 The Non-Positive Correlated Case

This case which assumes a non-positive correlation between the random target and the asset returns, i.e. the cross-covariance vector between $T_2$ and $R$ is non-positive,

$$(Lb) = \text{cov}(R, T_2) \leq 0,$$

(7)
can be motivated by at least three situations: First, from an individual perspective the target may not even be known. Many individuals have the target of “being successful”, but only a very few know precisely which selection of money, leisure time, culture etc. must be attained to achieve this target (Bordley and Licalzi, 2000). Second, individuals may follow a group target, because they are uncertain about their individual target. This can be interpreted as herd behavior (see for example Shiller, 2005, pp. 157-172). In these two examples the target is not clearly specified and can thus be interpreted as unknown or random and stochastically independent from the portfolio. Third, there exists a negative correlation between $T_2$ and $R$, which for example occurs when $T_2$ presents the return of a bond market whereas $R$ are stock returns.

As in the previous subsection, the corresponding random variables for the strategies $S_1$ and $S_2$ are

$$Z_1 = x^T R - T_1 \quad \text{with} \quad Z_1 \sim \mathcal{N}(\mu^T x - T_1, \|L^T x\|^2) \quad \text{and}$$

$$Z_2 = x^T R - T_2 \quad \text{with} \quad Z_2 \sim \mathcal{N}(\mu^T x - T_1, \|L^T x - b\|^2),$$
respectively, obtained from theorem 1. Notice, in the case where \( T_2 \) and \( R \) are stochastically independent, the variance of the sum is the sum of variances, that is
\[
\sigma^2(Z_2) = \sigma^2(x^T R - T_2) = \sigma^2(x^T R) + \sigma^2(T_2) = \|L^T x\|^2 + \sigma^2(T_2).
\]
From this, it follows immediately that \( S_2 \) is riskier than \( S_1 \) and therefore \( S_1 \) outperforms \( S_2 \). We now prove that this is also true, when asset returns and the random target are non-positively correlated. To prove this, we make use of

**Lemma 2.** Provided (7) holds, for any critical probability \( \alpha \in (0, 1/2) \) the following inequality is true:
\[
\Phi^{-1}(\alpha)\sigma(Z_2) + \mu(Z_2) < \Phi^{-1}(\alpha)\sigma(Z_1) + \mu(Z_1)
\]

**Proof:**
\[
0 < -2 \text{cov}(R, T_2)^T x + \sigma^2(T_2) \
\leq 0 
\Rightarrow 
\|L^T x\|^2 < \|L^T x\|^2 - 2(Lb)^T x + b^T b 
\Rightarrow 
\|L^T x\| < \|L^T x - b\| 
\Rightarrow 
\Phi^{-1}(\alpha)\|L^T x\| + \mu(T_1) > \Phi^{-1}(\alpha)\|L^T x - b\| + \mu(T_1) 
\Rightarrow 
\Phi^{-1}(\alpha)\sigma(Z_1) + \mu(Z_1) > \Phi^{-1}(\alpha)\sigma(Z_2) + \mu(Z_2)
\]
Notice that the second last inequality reverses because \( \Phi^{-1}(\alpha) < 0 \) \( \forall \alpha \in (0, 1/2) \).

\[\square\]

**Theorem 3.** Provided (7) holds, for any critical probability \( \alpha \in (0, 1/2) \) the optimal expected portfolio return of \( S_1 \) is larger or equal than the optimal expected portfolio return of \( S_2 \).

**Proof:** Employing lemma 2 instead of 1 and redoing the proof of theorem 2 yields the proposition.

\[\square\]

This result requires, compared to the positive correlated case discussed in subsection 3.1, no additional assumption. Furthermore, it allows riskless investing, which is prohibited in the previous case. As the fixed target strategy \( S_1 \) outperforms the random target strategy \( S_2 \), it reveals two issues: First, a random target which is stochastically independent from the portfolio return, induced by a general uncertainty about the target, leads to a poor performance and should be avoided. Second, an inappropriate target choice characterized by a negative covariation with the portfolio return leads also to a poor performance and should be avoided.
4 Empirical Evidence for Condition (6)

This section provides empirical evidence for condition (6) and therefore supports the practical value of the result from section 3.1. To do this, we estimate

\[ \text{cov}(R_i, T_2) - 1/2\sigma^2(T_2), \; i = 1, \ldots, n, \]  

in which \( R_i \) is the return of stock \( i \) and \( T_2 \) is the return of a stock index. We use German stock data from the DAX (blue chip stocks) and the MDAX (mid cap stocks). For \( T_2 \) we use data from important national and international indices. Condition (6) is satisfied if (8) is positive for all \( i \).

Table 1 reports the empirical estimator for (8), where \( R_i, \; i = 1, \ldots, 30 \), are the returns of all 30 DAX stocks and for \( T_2 \) we use the following stock indices: MSCI WORLD, DJ STOXX 50, DJ EURO STOXX 50, MSCI EUROPE, FTSE 100, S&P 500 and NYSE. We use a sample period for which monthly return data for all DAX stocks contained in the index in February 2010 is available, that is a period from February 2001 to February 2010. Table 1 shows that (8) is positive and thus (6) is true for all stocks in the DAX, except for three (BEIX: Beiersdorf, FMEX: Fresenius Medical Care and DTEX: Deutsche Telekom, for which only the condition using the MSCI EUROPE is not satisfied). Undocumented results for a larger sample period from September 1991 to February 2010, for which complete return data of 14 DAX stocks exist, reveal that (8) is entirely positive.

Table 2 reports the empirical estimator for (8) using the same index data, but stock data from the MDAX. Again, we use a sample period from February 2001 to February 2010, for which complete return data of 33 MDAX stocks is available. Table 2 shows that (8) is positive and thus (6) is true for all stocks in the MDAX, except for four (CLS1: Celesio, DEQ: Deutsche Euroshop, FIE: Fielmann and VOS: Vossloh).

The empirical results presented here indicate that condition (6) is mostly true, at least for the German stock market. This provides additional support that, in the context of section 3.1, the random target strategy \( S_2 \) should be preferred to the fixed target strategy \( S_1 \).

5 Return Distribution Is Unknown

It is shown empirically that return distributions are fat-tailed (see for example Adler, 1998, and the references therein) and skewed to the left, i.e. losses weigh heavier than gains, discussed for example in Harlow (1991). These findings indicate that the normal assumption does not necessarily hold in general. We therefore relax this assumption and assume only that good estimates for the first two moments of the return distributions exist. Then, we can use a well-known textbook inequality, the Chebyshev inequality. In the following we apply Chebyshev’s inequality to the safety-first constraint (1) and obtain a stronger but deterministic version of (1).\footnote{This idea has been first suggested by Roy (1952). For a detailed discussion on the application of Chebyshev’s inequality to the safety-first criterion consult Kall and Mayer (2005) or Birge and} Providing this, it is easy to verify
that under unknown return distributions the results from section 3 hold analogously.

Again, let $Z = x^T R - T$ be a random variable with $x^T R$ being the random portfolio return and $T$ being a target, either fixed or random. For the expected value and the variance of $Z$ we get

$$
\mu(Z) = \mu^T x - \mu(T),
$$

$$
\sigma^2(Z) = \|L^T x - b\|^2.
$$

The following inequalities provide an upper bound for the safety-first constraint (1):

$$
P(x^T R < T) = P(Z < 0)
\leq P(Z \leq 0) = P(Z - \mu(Z) \leq -\mu(Z)) = P(\mu(Z) - Z \geq \mu(Z))
\leq P(|\mu(Z) - Z| \geq \mu(Z))
\leq \frac{\sigma^2(Z)}{\mu(Z)^2} = \frac{\|L^T x - b\|^2}{(\mu^T x - \mu(T))^2},
$$

where the last inequality is obtained from Chebyshev’s rule.\(^7\) Instead of (1) the stronger inequality

$$
\frac{\|L^T x - b\|^2}{(\mu^T x - \mu(T))^2} \leq \alpha
$$

can be applied. Drawing the square root and rearranging yields

$$
-\alpha^{-\frac{1}{2}}\|L^T x - b\| + \mu^T x \geq \mu(T).
$$

(9)

Comparing (9) to the deterministic safety-first constraint (4) obtained for the multivariate normal case (see section 2), the sole difference is the multiplier for the term $\|L^T x - b\|$. In (4) the multiplier is $\Phi^{-1}(\alpha)$ whereas it is $-\alpha^{-\frac{1}{2}}$ for (9), both are negative for all $\alpha \in (0, 1/2)$. Thus, redoing the proofs from section 3 with (8) instead of (4) yields the same results. Moreover, as (9) is concave for all $\alpha$ the same efficient solving methods as for problem 2 can be used.

Note, applying the stronger inequality (9) instead of (1) reduces the number of feasible portfolios, i.e. the feasible domain of problem 1 with (9) instead of (1) is a subset of the feasible domain of the original problem 1. Thus, choosing $\alpha$ very small may lead to infeasibility of the safety-first problem under (9), but not necessarily under the true safety-first constraint (1). Nevertheless, this approach provides a useful and tractable alternative to the multivariate normal case.

### 6 Conclusion

This paper analyzes the safety-first portfolio model under two different target assumptions, the fixed target, which is commonly assumed in the literature, and the random

\(^{7}\)A detailed illustration for the univariate case is provided in Breuer et al. (2006, pp. 119-121). For the Chebyshev inequality in general consult a textbook on probability theory, such as Behnen and Neuhaus (1995).
target, which has played only a minor role in the existing literature on the safety-first model so far. As a random target can be easily motivated for this framework, the open question is, which target choice leads to a better performance? We answer this question by comparing optimal expected portfolio returns of the fixed and the random target strategy. Assuming multivariate normal returns the answer is: (1) The random target strategy outperforms the fixed target strategy if the portfolio return and the random target are positively correlated and riskless investing is prohibited, (2) the fixed target strategy outperforms the random target strategy if the portfolio return and the random target are not positively correlated and riskless investing is allowed. The first result is practically most relevant, in particular for institutional portfolio management and the skilled private investor, which is supported by the empirical analysis in section 4.

The second result suggests, general uncertainty about the target and an inappropriate target choice, characterized by a negative correlation between the portfolio return and the target, should be avoided. As the normal assumption is typically violated for real return data, we relax this assumption in section 5 and show that both results hold when approximating the safety-first statement by the well-known Chebyshev inequality.

The normal distribution and Chebyshev’s inequality are on the one hand very tractable and easy to implement, but on the other hand not very accurate. To overcome this limitation several extensions are possible: One, the normal assumption can be relaxed to the general distribution family depending on a shift and a scale parameter (see Kalin and Zagst, 1999). Two, the normal distribution can be generalized to the elliptical distribution. Third, a copula function, which provides a general technique for formulating a multivariate distribution, can be used. Four, considering higher order moments a more accurate probability inequality can be used. We recommend this issues for further research.

References


As $\mathbf{R}$ is $n$-variate and $T$ univariate normal distributed, the vector $(\mathbf{R}^T, T)^T$ is $n + 1$-variate normal distributed and can be written as

$$
\begin{pmatrix}
\mathbf{R} \\
T
\end{pmatrix} = 
\begin{pmatrix}
L \\
b^T
\end{pmatrix} \mathbf{Y} + 
\begin{pmatrix}
\mu \\
\mu(T)
\end{pmatrix},
$$

in which $L \in \mathbb{R}^{n \times r}$, $b \in \mathbb{R}^r$ and $\mathbf{Y} = (Y_1, \ldots, Y_r)^T$ has an $r$-variate normal distribution with mean $\mathbf{0}$ and identity matrix as covariance matrix (see for example Giri, 2004, pp. 14).
Simple algebra yields

\[ Z = x^T R - T = (R^T, T) \begin{pmatrix} x \\ -1 \end{pmatrix} = Y^T L^T x + \mu^T x - b^T Y - \mu(T) = Y^T (L^T x - b) + \mu^T x - \mu(T). \]

Hence, we get the expected value and the variance of \( Z \), respectively, as

\[
\mu(Z) = \mu^T x - \mu(T), \\
\sigma^2(Z) = (L^T x - b)^T (L^T x - b) = \| L^T x - b \|^2.
\]

\[ \square \]

**B A Note on the Numerical Treatment of the Covariance Matrix Factorization**

Kall and Mayer (2005) suggest the Cholesky-factorization

\[
cov(R^T, T) = cov(R_1, \ldots, R_n, T) = \begin{pmatrix} L \\ b^T \end{pmatrix} \begin{pmatrix} L \\ b^T \end{pmatrix}^T,
\]

with \( L \) being a lower triangular matrix, which, however, may lead to numerical problems. To compute the Cholesky-factorization the matrix must be positive definite, which may, due to stochastic independences between two or more assets, not be the case. In this case the covariance matrix is positive semidefinite and the Cholesky-factorization can not be computed. To correct this drawback we suggest a more general factorization for symmetric matrices, in which the covariance matrix is factorized as

\[
cov(R^T, T) = QDQ^T.
\]

The matrix \( D \) is a diagonal matrix and contains the eigenvalues of \( cov(R^T, T) \) and \( Q \) is an orthogonal matrix. Numerically, if the covariance matrix is positive semidefinite, small rounding errors may induce negative eigenvalues close to zero, which is theoretically impossible. This is corrected by hand by setting eigenvalues smaller than \( \varepsilon > 0 \) to zero. Doing this, \( \tilde{D} \) arise from \( D \) and the covariance matrix can finally be factorized as

\[
cov(R^T, T) = Q \sqrt{\tilde{D}} \sqrt{\tilde{D}} Q^T = \begin{pmatrix} \tilde{L} \\ \tilde{b}^T \end{pmatrix} \begin{pmatrix} \tilde{L} \\ \tilde{b}^T \end{pmatrix}^T,
\]

in which \( \tilde{L} \) is necessarily no triangular matrix.
Table 1: This table reports the empirical estimator for $\text{cov}(R_i, T_2) - 1/2\sigma_i^2$, where $R_i, i = 1, \ldots, 30$, are the returns of all 30 DAX (blue chip) stocks. For $T_2$ we use the following international stock indices: MSCI: MSCI WORLD, DJST: DJ STOXX 50, DJES: DJ EURO STOXX 50, MSCE: MSCI EUROPE, FTSE: FTSE 100, S&P: S&P 500 and NYSE. We use a sample period for which monthly return data for all DAX stocks contained in the index in February 2010 is available, that is a period from February 2001 to February 2010.

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Table 2: This table reports the empirical estimator for $COV(R_i, T_2) - 1/2\sigma^2_{T_2}$ where $R_i$, $i = 1, \ldots, 33$, are the returns of 33 MDAX (mid cap) stocks, for which complete return data for the sample period from February 2001 to February 2010 is available. For $T_2$ we use the following international stock indices: MSCI: MSCI WORLD, DJST: DJ STOXX 50, DJES: DJ EURO STOXX 50, MSCE: MSCI EUROPE, FTSE: FTSE 100, S&P: S&P 500 and NYSE.

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