

Singer, Nico

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Safety-First Portfolio Optimization:
Fixed versus Random Target

by

Nico Singer

Universität Rostock
Wirtschafts- und Sozialwissenschaftliche Fakultät
Institut für Volkswirtschaftslehre
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Safety-First Portfolio Optimization: Fixed versus Random Target*

Nico Singer[†]

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Abstract

This paper analyzes the safety-first portfolio model under two different target assumptions, the fixed target, which is commonly assumed in the literature, and the random target, which has played only a minor role so far. As both targets can be easily motivated, the open question is, which target choice leads to a better performance? We answer this question by comparing optimal expected portfolio returns of the fixed and the random target strategy. Assuming multivariate normal returns the answer is: (1) The random target strategy outperforms the fixed target strategy if the portfolio return and the random target are positively correlated and riskless investing is prohibited, (2) the fixed target strategy outperforms the random target strategy if the portfolio return and the random target are not positively correlated and riskless investing is allowed. The first result is practically most relevant, in particular for institutional portfolio management and the skilled private investor, which is supported by an empirical analysis. Furthermore, we show that these results also hold when relaxing the normal assumption.

Keywords: safety-first portfolio model, chance constraint programming, portfolio optimization, random target, benchmarking

JEL Classification: G11

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[†]Author's affiliation: University of Rostock, Department of Economics, Ulmenstraße 69, 18051 Rostock, Germany. Tel.: ++49 381 498 4390. Email: nico.singer@uni-rostock.de.

1 Introduction

Portfolio optimization under the safety-first criterion is concerned with maximizing the expected portfolio return, while, simultaneously, the probability of failing to achieve a specified (fixed) target must fall below a critical level and has its origins in the early papers of Roy (1952), Telser (1955) and Kataoka (1963). The safety-first risk measure is commonly expressed by a probability statement as $P(Z < T) \leq \alpha$ where Z is a random variable, e.g. portfolio return, T is a fixed target, e.g. a minimum desired portfolio return and α is a critical level on the probability of failing to achieve the target.

A comparative advantage of the safety-first criterion over deviation risk measures, such as the variance, is its consistency with the way investors perceive risk (see for example Atwood et al., 1988; Harlow, 1991; Brogan and Stidham, 2005). Empirically, this is shown for example in Lopes (1987), Kroll et al. (1988), DeBondt (1998), Lopes and Oden (1999) and Neugebauer (2008). These behaviorally appealing feature have made the safety-first criterion attractive for behavioral portfolio theory, see for example Shefrin and Statman (2000) and the recent paper by Das et al. (2010), in which they transform the utility based interpretation of mean-variance portfolio theory to the more appealing target based interpretation of safety-first portfolio theory under the assumption of normal distributed asset returns. More generally, Kalin and Zagst (1999) show the equivalence of mean-variance and safety-first portfolio theory for a wide class of probability distributions.

What all the above cited papers about the safety-first model have in common is the assumption of a fixed target T , which, however, leads to significant conceptual disadvantages: Suppose an investment fund which seeks to achieve a fixed return T for the next period. According to this target the fund manager purchases and sells assets. But, what happen when the market return within the next period is greater than T ? The fund performed rather poorly. This situation could have been avoided if the manager had reallocated the assets according to the expected performance of the market, which is common practice in passive portfolio management. Or, suppose a fund which seeks to outperform the market (active portfolio management), i.e. the target for the next investment period is the sum of the expected market performance and some extra return.¹ Or, from an individual perspective the target may not even be known. Many individuals have the target of “being successful”, but only a very few know precisely which selection of money, leisure time, culture etc. must be attained to achieve this target (Bordley and Licalzi, 2000). There are, thus, several situations in which an unknown or random target seems to be a more suitable choice. But, does in all these situations a random target lead to better results in terms of higher returns or are there situations in which a fixed target should be the preferred choice? As we pay particular attention to a portfolio optimization model, the question is, which target choice leads *ceteris paribus* to larger optimal expected portfolio returns? In this paper

¹For financial risk management with benchmarking see for example Basak et al. (2006), Browne (2000) and Gaivoronski et al. (2005).

we suggest a first answer to these questions.

In detail, assuming normal distributed asset returns, we know for example from Kalin and Zagst (1999) that the (probabilistic) safety-first risk measure can easily be transformed to a deterministic risk measure in terms of standard deviation. In section 2, we use this result to transform the safety-first portfolio model to an equivalent deterministic version, which is general enough to consider both, fixed and random targets. In section 3, which contains our main results, we compare optimal expected portfolio returns of the fixed and random target strategy and obtain following results: The random target strategy outperforms the fixed target strategy if the portfolio return and the random target are positively correlated and riskless investing is prohibited, whereas the fixed target strategy outperforms the random target strategy if the portfolio return and the random target are not positively correlated and riskless investing is allowed. By providing empirical evidence for the German stock market in section 4, we point out that the first case, in which the portfolio return and the random target are positively correlated, is practically most relevant. As the normal distribution is a good starting point analyzing the safety-first model (see Leibowitz and Henriksson, 1989; Leibowitz and Kogelman, 1991; Das et al., 2010), but typically violated in practice, we relax this assumption in section 5 and show, using a well-known approximation, that all results from section 3 remain the same. Section 6 offers a discussion and concludes the paper.

2 Safety-First Portfolio Optimization with Normal Distributed Asset Returns

Consider an investment universe of n different financial assets with $\mathbf{R} := (R_1, \dots, R_n)^T$ presenting the vector of random asset returns. A portfolio where short sales are prohibited is defined as a vector $\mathbf{x} \in [0, 1]^n$ with x_i being the proportion invested in asset i and the proportions sum to one, which is also known under the “fully invested constraint”. Let the product $\mathbf{x}^T \mathbf{R}$ be the random portfolio return, T be a fixed or random target and α be a critical probability, then the safety-first portfolio model which maximizes the expected portfolio return subject to a safety-first constraint can be expressed as

Problem 1.

$$\begin{aligned} \max_{\mathbf{x} \in [0,1]^n} \mu(\mathbf{x}^T \mathbf{R}) \quad & s.t. \\ P(\mathbf{x}^T \mathbf{R} < T) & \leq \alpha, \\ \mathbf{1}^T \mathbf{x} & = 1. \end{aligned} \tag{1}$$

For a numerical treatment of problem 1 it is useful to provide a deterministic rather than a probabilistic expression of the safety-first constraint (1), which can be easily achieved under the normal assumption (see for example Kalin and Zagst, 1999). Many other papers studying the safety-first framework, among them Leibowitz and Henriksson (1989), Leibowitz and Kogelman (1991), Albrecht (1993) and Das et al. (2010), assume normal distributed asset returns. Define therefore $Z := \mathbf{x}^T \mathbf{R} - T$ with

$Z \sim \mathcal{N}(\mu(Z), \sigma^2(Z))$ as a normal distributed random variable with expected value $\mu(Z)$ and variance $\sigma^2(Z)$. Then, employing the usual textbook transformation for the normal distribution, safety-first constraint (1) can be equivalently expressed as

$$\begin{aligned}
P(Z < 0) &\leq \alpha \Leftrightarrow \\
P\left(\frac{Z - \mu(Z)}{\sigma(Z)} < -\frac{\mu(Z)}{\sigma(Z)}\right) &\leq \alpha \Leftrightarrow \\
\Phi\left(-\frac{\mu(Z)}{\sigma(Z)}\right) &\leq \alpha \Leftrightarrow \\
\Phi^{-1}(\alpha)\sigma(Z) + \mu(Z) &\geq 0, \tag{3}
\end{aligned}$$

where $\Phi^{-1}(\alpha)$ is the α -quantile of the standard normal distribution and $\sigma(Z)$ is the standard deviation obtained from drawing the positive square root of the variance, $\sigma^2(Z)$. The following theorem, found in a slightly modified version in Kall and Mayer (2005, pp. 103-106) and Ruszczyński and Shapiro (2003, pp. 10), provides precise expressions for $\mu(Z)$ and $\sigma^2(Z)$ (and $\sigma(Z)$).

Theorem 1. *Let $\mathbf{R} = (R_1, \dots, R_n)^T$ be n -variate normal distributed with expected value vector $\boldsymbol{\mu}$ and covariance matrix Σ , $\mathbf{R} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$, and T be univariate normal distributed with expected value $\mu(T)$ and variance $\sigma^2(T)$, $T \sim \mathcal{N}(\mu(T), \sigma^2(T))$, then the random variable $Z = \mathbf{x}^T \mathbf{R} - T$ is univariate normal distributed with expected value $\boldsymbol{\mu}^T \mathbf{x} - \mu(T)$ and variance $\|L^T \mathbf{x} - \mathbf{b}\|^2$, where $L \in \mathbb{R}^{n \times r}$ and $\mathbf{b} \in \mathbb{R}^r$ are obtained from the factorization of the covariance matrix $\text{cov}(\mathbf{R}^T, T)$ and $\|\cdot\|$ denotes the Euclidean norm.*

Proof: A detailed proof, following Kall and Mayer (2005, pp. 103-106), and a remark on the numerical treatment of the factorization of the covariance matrix are provided in the separate appendix. □

Note,

$$\sigma^2(Z) = \|L^T \mathbf{x} - \mathbf{b}\|^2 = (L^T \mathbf{x} - \mathbf{b})^T (L^T \mathbf{x} - \mathbf{b}) = \mathbf{x}^T \underbrace{LL^T}_{=\Sigma} \mathbf{x} - 2(L\mathbf{b})^T \mathbf{x} + \mathbf{b}^T \mathbf{b},$$

where the first addend in the last equation of $\sigma^2(Z)$ is the variance of $\mathbf{x}^T \mathbf{R}$. The middle addend is two times the covariance between $\mathbf{x}^T \mathbf{R}$ and T with $L\mathbf{b} \in \mathbb{R}^n$ being the cross-covariance vector between \mathbf{R} and T . The third addend is the variance of T .

Applying then the deterministic safety-first constraint (3) with $\mu(Z) = \boldsymbol{\mu}^T \mathbf{x} - \mu(T)$ and $\sigma(Z) = \|L^T \mathbf{x} - \mathbf{b}\|$, obtained from theorem 1, problem 1 can be reformulated as

Problem 2.

$$\begin{aligned}
\max_{\mathbf{x} \in [0,1]^n} \mathbf{x}^T \boldsymbol{\mu} \quad & s.t. \\
\Phi^{-1}(\alpha) \|L^T \mathbf{x} - \mathbf{b}\| + \boldsymbol{\mu}^T \mathbf{x} &\geq \mu(T), \\
\mathbf{1}^T \mathbf{x} &= 1.
\end{aligned} \tag{4}$$

Problem 2 is the deterministic equivalent of problem 1 and can now be solved numerically. It is linear in its objective but quadratic in its constraints.² It therefore relates to the class of quadratic optimization problems. If $\alpha \in (0, 1/2)$, the deterministic safety-first constraint (4) is concave, which was first shown by Kataoka (1963). Under this assumption problem 2 can easily be solved by concave optimization methods.³ As α represents the maximum probability of failing to achieve the target and is moreover specified by the investor herself, it is usually chosen to be low, e.g. 5% or 10%. Thus, the restriction of α not exceeding 50% does not limit the practical value of our results. We therefore stick throughout our analysis to this assumption.

Note, modeling a riskfree asset j can be easily achieved by setting the j -th row of L to the zero-vector. Then, we have $R_j = \mu_j$ with μ_j being the riskfree rate. Analogously, a fixed target instead of a normal distributed target can be achieved by setting \mathbf{b} to the zero-vector. Then, the correlation between the target and the asset returns is zero and T collapses to a fixed target with value $\mu(T)$.

3 Fixed versus Random Target

This section concerns the comparison of the fixed target strategy (S_1) with the random target strategy (S_2). The fixed and the random target is denoted as T_1 and T_2 , respectively. Both strategies face the same investment universe and the same estimates for the expected returns, $\boldsymbol{\mu} = (\mu(R_1), \dots, \mu(R_n))^T$, and covariances, Σ . Additionally, the crosswise covariations between the random target and the asset returns are denoted by the n -vector $(L\mathbf{b}) = cov(\mathbf{R}, T_2)$. Note, the $n \times r$ matrix L and the r -vector \mathbf{b} are obtained from the factorization of the covariance matrix $cov(R_1, \dots, R_n, T_2)$ (see theorem 1).

As assumed in theorem 1, T_2 is normal distributed with expected value $\mu(T_2)$ and variance $\sigma^2(T_2)$. It appears reasonable to assume, $\mu(T_2) > T_1$, so that the expected target return of S_2 is greater than T_1 . This is due to the uncertainty about T_2 , making S_2 riskier than S_1 . This higher risk must thus be compensated by a greater expected target return. However, to keep our results as simple as possible we assume throughout this section that $\mu(T_2) = T_1$. But, all results presented here can be straightforwardly modified such that $\mu(T_2) > T_1$, for example by defining $\mu(T_2) := T_1 + \varepsilon$, $\varepsilon > 0$ and adapting the calculations.

In the following two subsections, we evaluate the differences in the performance of S_1 and S_2 by comparing optimal expected portfolio returns. In subsection 3.1 the practical more interesting case, in which the random target and the asset returns are crosswise positively correlated, is discussed, whereas the opposite case is discussed in subsection 3.2.

²More precisely, (4) is a second-order cone constraint, also called ice-cream cone or Lorentz cone. For a detailed discussion on that, consult Kall and Mayer (2005, pp. 273-276) and the references therein.

³If the feasible domain is concave and not empty, there exists a unique maximum. For an overview of convex optimization see Boyd and Vandenberghe (2007)

3.1 The Positive Correlated Case

Consider strategy S_2 in which a portfolio is managed subject to the performance of a benchmark, such as a stock index like the S&P500, without directly investing into the benchmark.⁴ There are at least three situations where this appears reasonable: First, the portfolio manager seeks to outperform the benchmark, which is typically for actively traded funds. Second, the manager seeks to invest in a foreign market without suffering from foreign exchange risk. This can be realized by tracking or outperforming a foreign index, representing this market, by a domestic portfolio. Third, the latter situation can also be derived from an individual perspective, where a skilled private investor is attracted by the performance of a foreign market, but does not want to invest directly into the market. The individual investor rather seeks to track the performance of the foreign market by only investing in domestic stocks. This situation is derived from a behavioral phenomenon called *home bias*, which was first documented by French and Poterba (1991).

Reducing all these examples to the stock market, they have in common to track or outperform one stock market by investing in similar but different stocks from another market. We therefore assume that asset returns are positively correlated with the benchmark return, i.e. the cross-covariance vector between $\mathbf{R} = (R_1, \dots, R_n)^T$ and T_2 is

$$(L\mathbf{b}) = \text{cov}(\mathbf{R}, T_2) > \mathbf{0}. \quad (5)$$

This assumption is generally justified for most of the risky financial assets, in particular for stock markets. In section 4 we provide empirical evidence that even a stronger version of (5) is justified for the stock market. Remark that assumption (5) excludes riskless investing because it does not allow for a zero covariance. We have S_1 with fixed target, T_1 , and S_2 with normal distributed target, $T_2 \sim \mathcal{N}(T_1, \sigma^2(T_2))$. Applying theorem 1 yields

$$\begin{aligned} Z_1 &= \mathbf{x}^T \mathbf{R} - T_1 \quad \text{with} \quad Z_1 \sim \mathcal{N}(\boldsymbol{\mu}^T \mathbf{x} - T_1, \|L^T \mathbf{x}\|^2), \\ Z_2 &= \mathbf{x}^T \mathbf{R} - T_2 \quad \text{with} \quad Z_2 \sim \mathcal{N}(\boldsymbol{\mu}^T \mathbf{x} - T_1, \|L^T \mathbf{x} - \mathbf{b}\|^2). \end{aligned}$$

Note, the expected values of Z_1 and Z_2 coincide, but the variances differ. Thus, in the normal distributed case, the question, whether S_1 outperforms S_2 or vice versa is simply the question of comparing variances. The following theorem shows that, under a weak additional assumption, the variance of Z_2 is smaller than the variance of Z_1 and therefore, S_2 outperforms S_1 . To prove this theorem we make use of

Lemma 1. *If*

$$\text{cov}(R_i, T_2) > \frac{1}{2} \sigma^2(T_2), \quad i = 1, \dots, n, \quad (6)$$

holds, then for any critical probability $\alpha \in (0, 1/2)$ the following inequality is true:

$$\Phi^{-1}(\alpha) \sigma(Z_2) + \mu(Z_2) > \Phi^{-1}(\alpha) \sigma(Z_1) + \mu(Z_1)$$

⁴A direct investment in an index can be obtained by purchasing an exchange traded fund (ETF) on the index, which explicitly tracks the index and is, moreover, attractive because of low transactions costs and tax efficiency.

Proof: From the fully invested constraint (2) together with (6) follows

$$\text{cov}(\mathbf{R}, T_2)^T \mathbf{x} > \frac{1}{2} \sigma^2(T_2),$$

which is equivalent to

$$\begin{aligned} 0 &> -2\text{cov}(\mathbf{R}, T_2)^T \mathbf{x} + \sigma^2(T_2) &&\Leftrightarrow \\ \|L^T \mathbf{x}\|^2 &> \|L^T \mathbf{x}\|^2 - 2(L\mathbf{b})^T \mathbf{x} + \mathbf{b}^T \mathbf{b} &&\Leftrightarrow \\ \|L^T \mathbf{x}\| &> \|L^T \mathbf{x} - \mathbf{b}\| &&\Leftrightarrow \\ \Phi^{-1}(\alpha) \|L^T \mathbf{x}\| + \boldsymbol{\mu}^T \mathbf{x} - T_1 &< \Phi^{-1}(\alpha) \|L^T \mathbf{x} - \mathbf{b}\| + \boldsymbol{\mu}^T \mathbf{x} - T_1 &&\Leftrightarrow \\ \Phi^{-1}(\alpha) \sigma(Z_1) + \mu(Z_1) &< \Phi^{-1}(\alpha) \sigma(Z_2) + \mu(Z_2). \end{aligned}$$

Notice that the second last inequality reverses because $\Phi^{-1}(\alpha) < 0 \forall \alpha \in (0, 1/2)$. \square

Theorem 2. *Provided (6) holds, for any critical probability $\alpha \in (0, 1/2)$ the optimal expected portfolio return of S_2 is larger or equal than the optimal expected portfolio return of S_1 .*

Proof: It is sufficient to show that the set of feasible portfolios of S_1 , denoted as \mathcal{F}_1 , is a subset of the feasible domain of S_2 , denoted as \mathcal{F}_2 . If $\mathcal{F}_1 = \emptyset$, then clearly $\mathcal{F}_1 \subseteq \mathcal{F}_2$. All portfolios $\mathbf{x} \in \mathcal{F}_1 \neq \emptyset$ satisfy

$$\begin{aligned} P(Z_1 < 0) \leq \alpha &\Leftrightarrow \Phi^{-1}(\alpha) \sigma(Z_1) + \mu(Z_1) \geq 0 \stackrel{\text{lemma 1}}{\Leftrightarrow} \\ \Phi^{-1}(\alpha) \sigma(Z_2) + \mu(Z_2) > 0 &\Leftrightarrow P(Z_2 < 0) < \alpha. \end{aligned}$$

Thus, $\mathbf{x} \in \mathcal{F}_2$. \square

Remark, in the special case where the safety-first constraint for S_1 is satisfied for all portfolios \mathbf{x} , i.e. $P(\mathbf{x}^T \mathbf{R} < T_1) < \alpha \forall \mathbf{x} \in [0, 1]^n$, the entire wealth is invested in the single asset with the highest expected return and thus the same is true for S_2 . In this case both investors obtain the same optimal expected portfolio return and only one asset is hold. Remark, if inequality (6) reverses, the result clearly reverses, i.e. S_1 outperforms S_2 . But this is practically not the case as supported by the empirical analysis in section 4.

In figure 1, where the safety-first efficient frontiers⁵ for S_1 and S_2 are sketched, we illustrate the result: Suppose, both investors choose a critical probability of α_1 , then the portfolio problem is neither feasible for S_1 nor feasible for S_2 . Suppose, both choose α_2 , then S_2 outperforms S_1 as $\boldsymbol{\mu}^T \mathbf{x}_2^* > \boldsymbol{\mu}^T \mathbf{x}_1^*$. Finally, suppose that they choose α_3 , then their optimal expected portfolio returns coincide and their entire wealth is invested in the single asset with the highest expected return.

⁵Shefrin (2005) uses the term *SP/A efficient frontier*, which is the same as the safety-first efficient frontier, plotted in $(\boldsymbol{\mu}^T \mathbf{x}^*, \alpha)$ -space. The safety-first efficient frontier is monotone non-decreasing as investors prefer higher portfolio returns ($\boldsymbol{\mu}^T \mathbf{x}^*$) but lower risk (α).

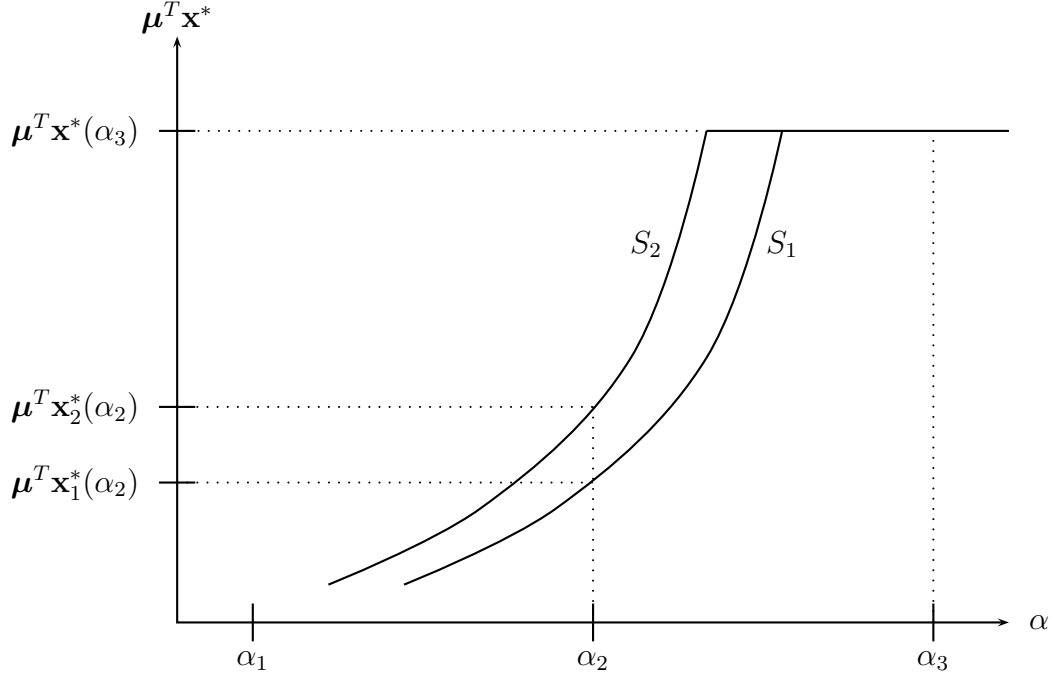


Figure 1: Safety-first efficient frontiers for S_1 and S_2 .

3.2 The Non-Positive Correlated Case

This case which assumes a non-positive correlation between the random target and the asset returns, i.e. the cross-covariance vector between T_2 and \mathbf{R} is non-positive,

$$(L\mathbf{b}) = \text{cov}(\mathbf{R}, T_2) \leq \mathbf{0}, \quad (7)$$

can be motivated by at least three situations: First, from an individual perspective the target may not even be known. Many individuals have the target of “being successful”, but only a very few know precisely which selection of money, leisure time, culture etc. must be attained to achieve this target (Bordley and Licalzi, 2000). Second, individuals may follow a group target, because they are uncertain about their individual target. This can be interpreted as *herd behavior* (see for example Shiller, 2005, pp. 157-172). In these two examples the target is not clearly specified and can thus be interpreted as unknown or random and stochastically independent from the portfolio. Third, there exists a negative correlation between T_2 and \mathbf{R} , which for example occurs when T_2 presents the return of a bond market whereas \mathbf{R} are stock returns.

As in the previous subsection, the corresponding random variables for the strategies S_1 and S_2 are

$$\begin{aligned} Z_1 &= \mathbf{x}^T \mathbf{R} - T_1 \quad \text{with} \quad Z_1 \sim \mathcal{N}(\mu^T \mathbf{x} - T_1, \|L^T \mathbf{x}\|^2) \quad \text{and} \\ Z_2 &= \mathbf{x}^T \mathbf{R} - T_2 \quad \text{with} \quad Z_2 \sim \mathcal{N}(\mu^T \mathbf{x} - T_1, \|L^T \mathbf{x} - \mathbf{b}\|^2), \end{aligned}$$

respectively, obtained from theorem 1. Notice, in the case where T_2 and \mathbf{R} are stochastically independent, the variance of the sum is the sum of variances, that is

$$\sigma^2(Z_2) = \sigma^2(\mathbf{x}^T \mathbf{R} - T_2) = \sigma^2(\mathbf{x}^T \mathbf{R}) + \sigma^2(T_2) = \|L^T \mathbf{x}\|^2 + \sigma^2(T_2).$$

From this, it follows immediately that S_2 is riskier than S_1 and therefore S_1 outperforms S_2 . We now prove that this is also true, when asset returns and the random target are non-positively correlated. To prove this, we make use of

Lemma 2. *Provided (7) holds, for any critical probability $\alpha \in (0, 1/2)$ the following inequality is true:*

$$\Phi^{-1}(\alpha)\sigma(Z_2) + \mu(Z_2) < \Phi^{-1}(\alpha)\sigma(Z_1) + \mu(Z_1)$$

Proof:

$$\begin{aligned} 0 &< \underbrace{-2 \operatorname{cov}(\mathbf{R}, T_2)^T \mathbf{x}}_{\leq 0} + \underbrace{\sigma^2(T_2)}_{> 0} && \Leftrightarrow \\ \|L^T \mathbf{x}\|^2 &< \|L^T \mathbf{x}\|^2 - 2(L\mathbf{b})^T \mathbf{x} + \mathbf{b}^T \mathbf{b} && \Leftrightarrow \\ \|L^T \mathbf{x}\| &< \|L^T \mathbf{x} - \mathbf{b}\| && \Leftrightarrow \\ \Phi^{-1}(\alpha)\|L^T \mathbf{x}\| + \boldsymbol{\mu}^T \mathbf{x} - T_1 &> \Phi^{-1}(\alpha)\|L^T \mathbf{x} - \mathbf{b}\| + \boldsymbol{\mu}^T \mathbf{x} - T_1 && \Leftrightarrow \\ \Phi^{-1}(\alpha)\sigma(Z_1) + \mu(Z_1) &> \Phi^{-1}(\alpha)\sigma(Z_2) + \mu(Z_2) \end{aligned}$$

Notice that the second last inequality reverses because $\Phi^{-1}(\alpha) < 0 \forall \alpha \in (0, 1/2)$. □

Theorem 3. *Provided (7) holds, for any critical probability $\alpha \in (0, 1/2)$ the optimal expected portfolio return of S_1 is larger or equal than the optimal expected portfolio return of S_2 .*

Proof: Employing lemma 2 instead of 1 and redoing the proof of theorem 2 yields the proposition. □

This result requires, compared to the positive correlated case discussed in subsection 3.1, no additional assumption. Furthermore, it allows riskless investing, which is prohibited in the previous case. As the fixed target strategy S_1 outperforms the random target strategy S_2 , it reveals two issues: First, a random target which is stochastically independent from the portfolio return, induced by a general uncertainty about the target, leads to a poor performance and should be avoided. Second, an inappropriate target choice characterized by a negative covariation with the portfolio return leads also to a poor performance and should be avoided.

4 Empirical Evidence for Condition (6)

This section provides empirical evidence for condition (6) and therefore supports the practical value of the result from section 3.1. To do this, we estimate

$$\text{cov}(R_i, T_2) - 1/2\sigma^2(T_2), \quad i = 1, \dots, n, \quad (8)$$

in which R_i is the return of stock i and T_2 is the return of a stock index. We use German stock data from the DAX (blue chip stocks) and the MDAX (mid cap stocks). For T_2 we use data from important national and international indices. Condition (6) is satisfied if (8) is positive for all i .

Table 1 reports the empirical estimator for (8), where R_i , $i = 1, \dots, 30$, are the returns of all 30 DAX stocks and for T_2 we use the following stock indices: MSCI WORLD, DJ STOXX 50, DJ EURO STOXX 50, MSCI EUROPE, FTSE 100, S&P 500 and NYSE. We use a sample period for which monthly return data for all DAX stocks contained in the index in February 2010 is available, that is a period from February 2001 to February 2010. Table 1 shows that (8) is positive and thus (6) is true for all stocks in the DAX, except for three (BEIX: Beiersdorf, FMEX: Fresenius Medical Care and DTEX: Deutsche Telekom, for which only the condition using the MSCI EUROPE is not satisfied). Undocumented results for a larger sample period from September 1991 to February 2010, for which complete return data of 14 DAX stocks exist, reveal that (8) is entirely positive.

Table 2 reports the empirical estimator for (8) using the same index data, but stock data from the MDAX. Again, we use a sample period from February 2001 to February 2010, for which complete return data of 33 MDAX stocks is available. Table 2 shows that (8) is positive and thus (6) is true for all stocks in the MDAX, except for four (CLS1: Celesio, DEQ: Deutsche Euroshop, FIE: Fielmann and VOS: Vossloh).

The empirical results presented here indicate that condition (6) is mostly true, at least for the German stock market. This provides additional support that, in the context of section 3.1, the random target strategy S_2 should be preferred to the fixed target strategy S_1 .

5 Return Distribution Is Unknown

It is shown empirically that return distributions are fat-tailed (see for example Adler, 1998, and the references therein) and skewed to the left, i.e. losses weigh heavier than gains, discussed for example in Harlow (1991). These findings indicate that the normal assumption does not necessarily hold in general. We therefore relax this assumption and assume only that good estimates for the first two moments of the return distributions exist. Then, we can use a well-known textbook inequality, the Chebyshev inequality. In the following we apply Chebyshev's inequality to the safety-first constraint (1) and obtain a stronger but deterministic version of (1).⁶ Providing this, it is easy to verify

⁶This idea has been first suggested by Roy (1952). For a detailed discussion on the application of Chebyshev's inequality to the safety-first criterion consult Kall and Mayer (2005) or Birge and

that under unknown return distributions the results from section 3 hold analogously.

Again, let $Z = \mathbf{x}^T \mathbf{R} - T$ be a random variable with $\mathbf{x}^T \mathbf{R}$ being the random portfolio return and T being a target, either fixed or random. For the expected value and the variance of Z we get

$$\begin{aligned}\mu(Z) &= \boldsymbol{\mu}^T \mathbf{x} - \mu(T), \\ \sigma^2(Z) &= \|L^T \mathbf{x} - \mathbf{b}\|^2.\end{aligned}$$

The following inequalities provide an upper bound for the safety-first constraint (1):

$$\begin{aligned}P(\mathbf{x}^T \mathbf{R} < T) &= P(Z < 0) \\ &\leq P(Z \leq 0) = P(Z - \mu(Z) \leq -\mu(Z)) = P(\mu(Z) - Z \geq \mu(Z)) \\ &\leq P(|\mu(Z) - Z| \geq \mu(Z)) \\ &\leq \frac{\sigma^2(Z)}{\mu(Z)^2} = \frac{\|L^T \mathbf{x} - \mathbf{b}\|^2}{(\boldsymbol{\mu}^T \mathbf{x} - \mu(T))^2},\end{aligned}$$

where the last inequality is obtained from Chebyshev's rule.⁷ Instead of (1) the stronger inequality

$$\frac{\|L^T \mathbf{x} - \mathbf{b}\|^2}{(\boldsymbol{\mu}^T \mathbf{x} - \mu(T))^2} \leq \alpha$$

can be applied. Drawing the square root and rearranging yields

$$-\alpha^{-\frac{1}{2}} \|L^T \mathbf{x} - \mathbf{b}\| + \boldsymbol{\mu}^T \mathbf{x} \geq \mu(T). \quad (9)$$

Comparing (9) to the deterministic safety-first constraint (4) obtained for the multivariate normal case (see section 2), the sole difference is the multiplier for the term $\|L^T \mathbf{x} - \mathbf{b}\|$. In (4) the multiplier is $\Phi^{-1}(\alpha)$ whereas it is $-\alpha^{-\frac{1}{2}}$ for (9), both are negative for all $\alpha \in (0, 1/2)$. Thus, redoing the proofs from section 3 with (8) instead of (4) yields the same results. Moreover, as (9) is concave for all α the same efficient solving methods as for problem 2 can be used.

Note, applying the stronger inequality (9) instead of (1) reduces the number of feasible portfolios, i.e. the feasible domain of problem 1 with (9) instead of (1) is a subset of the feasible domain of the original problem 1. Thus, choosing α very small may lead to infeasibility of the safety-first problem under (9), but not necessarily under the true safety-first constraint (1). Nevertheless, this approach provides a useful and tractable alternative to the multivariate normal case.

6 Conclusion

This paper analyzes the safety-first portfolio model under two different target assumptions, the fixed target, which is commonly assumed in the literature, and the random

Louveaux (1997).

⁷A detailed illustration for the univariate case is provided in Breuer et al. (2006, pp. 119-121). For the Chebyshev inequality in general consult a textbook on probability theory, such as Behnen and Neuhaus (1995).

target, which has played only a minor role in the existing literature on the safety-first model so far. As a random target can be easily motivated for this framework, the open question is, which target choice leads to a better performance? We answer this question by comparing optimal expected portfolio returns of the fixed and the random target strategy. Assuming multivariate normal returns the answer is: (1) The random target strategy outperforms the fixed target strategy if the portfolio return and the random target are positively correlated and riskless investing is prohibited, (2) the fixed target strategy outperforms the random target strategy if the portfolio return and the random target are not positively correlated and riskless investing is allowed. The first result is practically most relevant, in particular for institutional portfolio management and the skilled private investor, which is supported by the empirical analysis in section 4. The second result suggests, general uncertainty about the target and an inappropriate target choice, characterized by a negative correlation between the portfolio return and the target, should be avoided. As the normal assumption is typically violated for real return data, we relax this assumption in section 5 and show that both results hold when approximating the safety-first statement by the well-known Chebyshev inequality.

The normal distribution and Chebyshev's inequality are on the one hand very tractable and easy to implement, but on the other hand not very accurate. To overcome this limitation several extensions are possible: One, the normal assumption can be relaxed to the general distribution family depending on a shift and a scale parameter (see Kalin and Zagst, 1999). Two, the normal distribution can be generalized to the elliptical distribution. Third, a copula function, which provides a general technique for formulating a multivariate distribution, can be used. Four, considering higher order moments a more accurate probability inequality can be used. We recommend this issues for further research.

References

- R. J. Adler, editor. *A practical guide to heavy tails*. Birkhäuser, Boston, Mass., 1998.
- P. Albrecht. Shortfall returns and shortfall risk. Working Paper 59, University of Mannheim, 1993. Mannheimer Manuskripte zu Risikotheorie, Portfolio Management und Versicherungswirtschaft.
- J. A. Atwood, M. J. Watts, and G. A. Helmers. Chance-constrained financing as a response to financial risk. *American Journal of Agricultural Economics*, 70(1):79–89, 1988.
- S. Basak, A. Shapiro, and L. Teplá. Risk management with benchmarking. *Management Science*, 52(4):542–557, 2006.
- K. Behnen and G. Neuhaus. *Grundkurs Stochastik*. Teubner, Stuttgart, 1995.
- J. R. Birge and F. Louveaux. *Introduction to stochastic programming*. Springer, New York, NY, 1997.

- R. Bordley and M. Licalzi. Decision analysis using targets instead of utility functions. *Decisions in Economics and Finance*, 23(1):53–74, 2000.
- S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge Univ. Press, Cambridge, 2007.
- W. Breuer, M. Gürtler, and F. Schuhmacher. *Portfoliomanagement II*. Betriebswirtschaftlicher Verlag Dr. Th. Gabler — GWV Fachverlage GmbH Wiesbaden, Wiesbaden, 2006.
- A. J. Brogan and S. Stidham. A note on separation in mean-lower-partial-moment portfolio optimization with fixed and moving targets. *IIE Transactions*, 37(10):901–906, 2005.
- S. Browne. Risk-constrained dynamic active portfolio management. *Management Science*, 46(9):1188–1199, 2000.
- S. Das, H. Markowitz, J. Scheid, and M. Statman. Portfolio optimization with mental accounts. *Journal of Financial and Quantitative Analysis*, 45(2):311–334, 2010.
- W. F. M. DeBondt. A portrait of the individual investor. *European Economic Review*, 42(3):831–844, 1998.
- K. R. French and J. M. Poterba. Investor diversification and international equity markets. *The American Economic Review*, 81(2):222–226, 1991.
- A. A. Gaivoronski, S. Krylov, and N. van der Wijst. Optimal portfolio selection and dynamic benchmark tracking. *European Journal of Operational Research*, 163(1):115–131, 2005.
- N. C. Giri. *Multivariate Statistical Analysis*. Dekker, 2nd edition, 2004.
- W. V. Harlow. Asset allocation in a downside-risk framework. *Financial Analysts Journal*, September/October:28–40, 1991.
- D. Kalin and R. Zagst. Portfolio optimization: Volatility constraints versus shortfall constraints. *OR-Spektrum*, 21(1/2):97–122, 1999.
- P. Kall and J. Mayer. *Stochastic linear programming*. Springer, New York, NY, 1st edition, 2005.
- S. Kataoka. A stochastic programming model. *Econometrica*, 31(1/2):181–196, 1963.
- Y. Kroll, H. Levy, and A. Rapoport. Experimental tests of the mean-variance model for portfolio selection. *Organizational Behavior and Human Decision Processes*, 42(3):388–410, 1988.

- M. L. Leibowitz and R. D. Henriksson. Portfolio optimization with shortfall constraints: A confidence-limit approach to managing downside risk. *Financial Analysis Journal*, 45(2):34–41, 1989.
- M. L. Leibowitz and S. Kogelman. Asset allocation under shortfall constraints. *The Journal of Portfolio Management*, 17(2):18–23, 1991.
- L. L. Lopes. Between hope and fear: The psychology of risk. *Advances in Experimental Social Psychology*, (20):255–295, 1987.
- L. L. Lopes and G. C. Oden. The role of aspiration level in risky choice. *Journal of Mathematical Psychology*, 43(2):286–313, 1999.
- T. Neugebauer. Individual choice from a convex lottery set: Experimental evidence. In *Advances in Decision Making Under Risk and Uncertainty*, volume 42, pages 121–135. Abdellaoui, Hey (Hg.) 2008 – *Advances in Decision Making and Uncertainty*, Berlin, Heidelberg, 2008.
- A. D. Roy. Safety first and the holding of assets. *Econometrica*, 20(3):431–449, 1952.
- A. Ruszczyński and A. Shapiro, editors. *Stochastic programming*, volume 10. Elsevier, Amsterdam, 2003.
- H. Shefrin. *A Behavioral Approach to Asset Pricing*. Elsevier Science & Technology, 2005.
- H. Shefrin and M. Statman. Behavioral portfolio theory. *Journal of Financial and Quantitative Analysis*, 35(2):127–151, 2000.
- R. J. Shiller. *Irrational exuberance*. Currency Doubleday, New York, NY, 2005.
- L. G. Telser. Safety first and hedging. *The Review of Economic Studies*, 23(1):1–16, 1955.

A Proof of Theorem 1

As \mathbf{R} is n -variate and T univariate normal distributed, the vector $(\mathbf{R}^T, T)^T$ is $n + 1$ -variate normal distributed and can be written as

$$\begin{pmatrix} \mathbf{R} \\ T \end{pmatrix} = \begin{pmatrix} L \\ \mathbf{b}^T \end{pmatrix} \mathbf{Y} + \begin{pmatrix} \boldsymbol{\mu} \\ \mu(T) \end{pmatrix},$$

in which $L \in \mathbb{R}^{n \times r}$, $\mathbf{b} \in \mathbb{R}^r$ and $\mathbf{Y} = (Y_1, \dots, Y_r)^T$ has a r -variate normal distribution with mean $\mathbf{0}$ and identity matrix as covariance matrix (see for example Giri, 2004, pp.

81-82). Simple algebra yields

$$\begin{aligned}
Z &= \mathbf{x}^T \mathbf{R} - T = (\mathbf{R}^T, T) \begin{pmatrix} \mathbf{x} \\ -1 \end{pmatrix} \\
&= \mathbf{Y}^T L^T \mathbf{x} + \boldsymbol{\mu}^T \mathbf{x} - \mathbf{b}^T \mathbf{Y} - \mu(T) \\
&= \mathbf{Y}^T (L^T \mathbf{x} - \mathbf{b}) + \boldsymbol{\mu}^T \mathbf{x} - \mu(T).
\end{aligned}$$

Hence, we get the expected value and the variance of Z , respectively, as

$$\begin{aligned}
\mu(Z) &= \boldsymbol{\mu}^T \mathbf{x} - \mu(T), \\
\sigma^2(Z) &= (L^T \mathbf{x} - \mathbf{b})^T (L^T \mathbf{x} - \mathbf{b}) = \|L^T \mathbf{x} - \mathbf{b}\|^2.
\end{aligned}$$

□

B A Note on the Numerical Treatment of the Covariance Matrix Factorization

Kall and Mayer (2005) suggest the Cholesky-factorization

$$\text{cov}(\mathbf{R}^T, T) = \text{cov}(R_1, \dots, R_n, T) = \begin{pmatrix} L \\ \mathbf{b}^T \end{pmatrix} \begin{pmatrix} L \\ \mathbf{b}^T \end{pmatrix}^T,$$

with L being a lower triangular matrix, which, however, may lead to numerical problems. To compute the Cholesky-factorization the matrix must be positive definite, which may, due to stochastic independences between two or more assets, not be the case. In this case the covariance matrix is positive semidefinite and the Cholesky-factorization can not be computed. To correct this drawback we suggest a more general factorization for symmetric matrices, in which the covariance matrix is factorized as

$$\text{cov}(\mathbf{R}^T, T) = QDQ^T.$$

The matrix D is a diagonal matrix and contains the eigenvalues of $\text{cov}(\mathbf{R}^T, T)$ and Q is an orthogonal matrix. Numerically, if the covariance matrix is positive semidefinite, small rounding errors may induce negative eigenvalues close to zero, which is theoretically impossible. This is corrected by hand by setting eigenvalues smaller than $\varepsilon > 0$ to zero. Doing this, \tilde{D} arise from D and the covariance matrix can finally be factorized as

$$\text{cov}(\mathbf{R}^T, T) = Q\sqrt{\tilde{D}}\sqrt{\tilde{D}}Q^T = \begin{pmatrix} \tilde{L} \\ \tilde{\mathbf{b}}^T \end{pmatrix} \begin{pmatrix} \tilde{L} \\ \tilde{\mathbf{b}}^T \end{pmatrix}^T,$$

in which \tilde{L} is necessarily no triangular matrix.

Table 1: This table reports the empirical estimator for $cov(R_i, T_2) - 1/2\sigma_{T_2}^2$ where R_i , $i = 1, \dots, 30$, are the returns of all 30 DAX (blue chip) stocks. For T_2 we use the following international stock indices: MSCI: MSCI WORLD, DJST: DJ STOXX 50, DJES: DJ EURO STOXX 50, MSCE: MSCI EUROPE, FTSE: FTSE 100, S&P: S&P 500 and NYSE. We use a sample period for which monthly return data for all DAX stocks contained in the index in February 2010 is available, that is a period from February 2001 to February 2010.

Datastream code	MSCI	DJST	DJES	MSCE	FTSE	S&P	NYSE
ADSX	14.44	15.08	15.16	14.25	13.76	12.53	13.45
ALVX	39.31	41.08	46.68	40.00	35.37	39.63	39.00
BASX	22.65	22.63	24.30	22.93	21.00	20.57	22.53
BAYX	22.05	27.04	29.86	23.39	21.51	20.90	21.76
BEIX	-0.20	0.39	-3.34	-3.13	3.31	-1.18	-0.90
BMWX	16.45	20.41	20.58	14.25	18.49	16.07	16.53
CBKX	65.60	66.66	76.66	70.82	57.34	62.97	65.72
DAIX	31.76	32.08	34.93	30.78	29.45	30.56	31.87
DBKX	45.33	43.78	49.99	46.89	37.13	44.17	44.48
DB1X	27.61	26.80	28.50	27.31	24.02	26.10	27.39
LHAX	32.18	33.99	38.34	31.25	29.23	32.15	31.32
DPWX	28.87	27.00	29.15	29.76	26.06	28.46	29.59
DTEX	2.24	9.28	7.61	-1.60	4.37	3.88	0.92
EONX	10.02	11.04	9.45	9.54	10.96	7.69	10.16
FMEX	-1.99	-0.83	-3.92	-5.75	0.76	-1.76	-3.68
FR3X	12.78	12.86	10.77	10.79	14.21	11.55	10.76
HE3X	6.19	7.03	5.15	4.11	7.95	5.12	5.54
IFXX	92.54	94.40	106.32	97.91	75.35	87.42	90.11
SDFX	28.49	21.49	22.50	28.44	25.27	24.87	28.63
LINX	15.52	19.06	18.97	15.67	17.30	12.90	14.21
MANX	40.51	39.68	44.31	44.05	35.71	35.44	38.34
MRKX	4.16	4.78	2.38	2.50	5.41	3.45	3.52
MEOX	27.65	27.79	29.33	26.96	24.74	27.41	27.52

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	MSCI	DJST	DJES	MSCE	FTSE	S&P	NYSE
MU2X	19.46	23.84	27.02	17.88	19.18	20.06	18.25
RWEX	5.61	8.97	8.14	4.27	7.60	4.54	5.93
SZGX	30.44	26.91	27.81	28.36	29.78	29.04	32.05
SAPX	30.61	29.57	31.68	26.44	26.26	32.52	28.66
SIEX	38.23	40.86	45.52	38.91	35.73	37.17	36.20
TKAX	43.44	39.91	43.89	45.01	39.07	39.91	42.94
VO3X	26.67	22.06	25.04	26.77	19.64	24.52	26.72

Table 2: This table reports the empirical estimator for $COV(R_i, T_2) - 1/2\sigma_{T_2}^2$ where R_i , $i = 1, \dots, 33$, are the returns of 33 MDAX (mid cap) stocks, for which complete return data for the sample period from February 2001 to February 2010 is available. For T_2 we use the following international stock indices: MSCI: MSCI WORLD, DJST: DJ STOXX 50, DJES: DJ EURO STOXX 50, MSCE: MSCI EUROPE, FTSE: FTSE 100, S&P: S&P 500 and NYSE.

Datastream code	MSCI	DJST	DJES	MSCE	FTSE	S&P	NYSE
NDA	8.93	7.03	4.79	4.40	9.11	8.31	9.73
BYW6	25.17	18.38	18.15	27.75	21.58	20.71	24.68
GBF	30.56	26.51	29.02	30.74	29.80	27.83	30.16
BOS3	38.78	38.16	40.06	40.33	37.32	35.83	39.05
CLS1	-8.01	-4.11	-9.96	-12.27	-2.44	-7.60	-7.01
CON	41.96	36.63	39.87	44.77	37.72	38.90	42.12
DEQ	-2.68	-5.78	-9.52	-5.79	-0.51	-3.70	-2.36
DOU	3.91	4.84	3.09	1.24	7.19	2.69	3.51
EAD	27.13	32.06	34.40	26.37	28.79	25.60	26.65
ZIL2	30.67	24.83	25.85	28.54	25.50	29.57	31.17
FIE	-1.88	1.57	-0.96	-5.10	2.43	-2.23	-2.03
FPE3	21.53	15.38	14.73	23.08	17.90	17.82	21.49
G1A	40.42	35.13	37.35	41.38	37.10	37.63	40.13
GIL	41.06	38.18	40.60	47.72	36.16	34.86	40.80
HNR1	19.12	17.36	19.93	17.95	16.89	19.65	19.36
HDD	48.36	48.48	52.77	52.95	45.57	43.61	50.12
HEI	28.42	32.58	34.58	30.92	27.73	25.43	29.12
HOT	47.38	41.22	45.74	49.83	44.43	42.49	46.52
IVG	46.05	38.32	42.45	53.07	41.19	38.64	45.44
KRN	12.27	13.42	12.01	9.57	15.04	10.96	11.85
LEO	37.71	31.30	31.46	38.85	32.64	35.88	38.90
MLP	60.65	62.87	75.20	60.85	48.68	62.38	59.61
PFD4	47.96	43.30	48.20	51.23	39.89	44.97	48.77
PSM	83.52	81.32	91.31	88.84	69.61	78.76	81.95

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	MSCI	DJST	DJES	MSCE	FTSE	S&P	NYSE
PUM	25.11	22.51	22.50	25.40	24.67	23.00	24.71
RAA	17.86	16.47	16.84	18.11	18.14	14.99	16.42
RHM	12.25	11.20	8.60	11.95	14.90	9.81	12.63
RHK	4.70	7.55	4.61	2.74	9.69	3.34	4.26
SGL	52.18	39.24	46.29	52.16	40.80	50.59	50.24
SAZ	23.76	18.15	17.88	20.45	18.62	24.77	25.94
SZU	4.48	7.66	6.23	2.81	8.12	2.69	2.92
TUI1	51.02	52.12	59.03	55.17	46.99	48.36	52.19
VOS	-1.06	-3.47	-6.97	-4.25	2.55	-1.65	-0.62
