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Portfolio decisions on life annuities and financial assets with longevity and income uncertainty

by

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Abstract
There are two stylised facts, namely weak demand for life-annuities and flat age-wealth profile that contradict the life-cycle hypothesis. In this paper we design a theoretical framework, which combines plausible arguments, which have been put forward in the literature to reconcile theory with empirical evidence. Besides the existence of an annuity market and of a public pension system we assume risk-averse individuals who are uncertain about lifetime and disposable income and who have preferences for leaving bequests. It is shown that this framework can contribute to explain the observed portfolio decision in favour of financial assets relatively to annuities.

Keywords: savings, life annuities, bequests, uncertain lifetime, uncertain income, social security.

JEL codes: D81, D91, G22, H55.
1. Introduction

There are two empirical observations about the saving behaviour which recently have attracted much attention: First, demand for life annuities is weak. Second, age-wealth profiles show little if any tendency for elderly individuals to dissave in retirement. Both stylised facts contradict the life-cycle hypothesis, which state that the main reason for the individual saving behaviour is the desire to smooth consumption over one's lifetime appropriately.

Flat age-wealth profiles are only compatible with the standard life-cycle model given risk-averse individuals, uncertain about their life-expectancy, if annuity markets do not exist (see e.g. Davies, 1981). In this case individuals would save in order to self-insure against the risk of longevity and would leave accidental bequests in case that their life span turns out to be unexpectedly short. However, with access to life-annuities, individuals should choose the latter, since they can offer a higher rate on return than riskless bonds (see Yaari, 1965). Thus, the puzzle is why do individuals purchase so few annuities to provide for old-age and instead save in financial assets.

The incompatibility of the theory of consumption-savings behaviour with the empirical evidence is often subsumed under the catchword "retirement-savings-puzzle". It has been recognised as a major issue by the academics as well as by the politicians, since its resolution might have important implications for economic theory and for public policy, especially for the ongoing social security reforms which rely more strongly on private old-age provision.

Various explanations for a portfolio decision in favour of financial assets relatively to life annuities have been put forward in the literature, in order to reconcile theory and empirical evidence: First it might not be the life-cycle motive, which induces individuals to save in financial assets, but to pass wealth on to relatives (the bequest motive) and to self-insure against other risks than longevity like illness and unemployment (the precautionary motive). Next, the lack of participation in the annuity market can be explained by the existence of a generous public pension system. In case that individuals receive (more than) enough social security benefits to provide for old-age consumption, they will not buy life-annuities. Finally, low annuity demand can be due to high annuity prices above the "fair" one. Empirical studies find that annuity prices are about 20 – 40 % above the actuarially fair price corresponding to the average survival probability of the population (see e.g. Walliser, 2000; Mitchell et al., 1999; Friedman and Warshawsky, 1990). Part of the so-called "load factor" is attributable to overhead costs due to administration, taxes and monopoly profit; the other part is due to adverse selection among annuity purchasers with differing survival probabilities. This depresses the expected rate of return on annuities and makes them less attractive compared to other form of wealth holding.
While most studies concentrate on one of these possible explanations (see e. Davies, 1981; Friedman and Warshawsky, 1990, Rodepeter and Winter, 1998), the aim of this paper is to design a theoretical framework, which combines all possible arguments, and to study to which extent the combination of these possible explanations can contribute to the observed portfolio decision in favour of financial assets relatively to life annuities. By this, this paper tries to clarify whether (or at least to which extent) the retirement-savings puzzle can be resolved.

For this, we consider a two-period model which combines the existence of an annuity market and of a public pension system with risk-averse individuals who are uncertain about their life-expectancy and who have preferences for leaving bequests to their relatives. The study of the portfolio behaviour in this framework is done in the first part of the paper. The results obtained from this analysis will serve then in the second part of the paper as a benchmark to study the consequences, when the framework is extended by adding uncertainty about disposable income.

First, we consider solely longevity risk, but no income risk. Our main findings are that the empirical evidence of flat (or slightly increasing) age-wealth profiles in old-age which are mainly attributable to individuals at the upper end of the income distribution, while individuals at the lower end rather dissave, can be explained within this framework, when a positive influence of income on life-expectancy is supposed to exist. Further, we find that a generous public pension system combined with a high load factor of the annuity price and strong preferences of the individuals for leaving bequests can contribute to explain the observation of weak demand for life annuities.

However, it is also plausible that individuals avoid consuming out of wealth or buying annuities for fear of the consequences of a negative shock on disposable income due to unemployment, catastrophic illness or other unforeseen circumstances. Thus, the second and more complex issue, addressed in this paper, concerns the effects of income (expenditure, resp.) uncertainty on the portfolio decision of the individual. We extend the model by introducing the risk of a negative shock on disposable income, where we consider three cases, which differ in the timing, when the negative income shock may occur and when the uncertainty is resolved. First, we consider uncertainty about retirement income, which is either resolved at the beginning of the retirement period or continues to prevail in old-age. Further, we consider income uncertainty until the end of the working period. We find that income uncertainties indeed can contribute to explain higher (precautionary) savings and lower consumption levels over lifetime, however only labour income uncertainty can explain the low annuity demand, observed in real world.
This paper is as organized as follows. In Section 2 we develop the basic model under lifetime uncertainty, but without income risk, and study the portfolio decisions of an individual. In Section 3 it is analysed how the introduction of three different cases of income risks affects the portfolio decision. Section 4 summarises and concludes the paper.

2. Savings and annuity demand under lifetime uncertainty

Consider an individual who lives for a maximum of two periods $t = 0,1$ and has an initial wealth $M$ at the beginning of her life. In the working period $0$, she earns a fixed labour income $w$. At the end of the working period 0 the individual retires. Survival to the retirement period 1 is uncertain and occurs with probability $\pi$, $0 < \pi < 1$. Provision for future consumption is guaranteed by a social security system, organized according to the pay-as-you-go method. The individual pays a proportional social security tax rate $\tau$ on income and receives a benefit $S(w)$, which depends on income.

Preferences of the individual are represented by expected lifetime utility with a per-period utility function $u$ depending on consumption $c_t$, $t = 0,1$, and a per-period utility $v$ derived from leaving a bequest $B^i$, $i = sl, ll$. That is

$$U = u(c_0) + \pi u(c_1) + (1 - \pi) v(B^{sl}) + \pi v(B^{ll}) \tag{2.1},$$

where the superscript $sl$ and $ll$ represent short and long life, respectively. We assume that $u' > 0$, $u'' < 0$, $\lim_{c \to 0} u'(c) = \infty$ and $v' > 0$, $v'' < 0$.

To smooth consumption over the uncertain lifetime appropriately, the individual can make private old-age provision in addition to the social security system. The annuity market offers the individual a payoff $A$ in the retirement period 1 (conditional on the individual's survival), which she can purchase for a price $Q$ per unit of the payoff $A$. If the individual had no bequest motive, she would decide for life annuities against holding wealth in the form of bonds, since the former can offer a higher rate of return than the latter, i.e. $1/Q > R$ (see Yaari, 1965). However, the annuity pays nothing to her heirs, when the individual dies young. Thus due to her preferences for leaving bequests, she chooses also riskless bonds $E_0$, which guarantee a rate of return $R$ in the next period, regardless of whether she survives or not. The budget constraint in each period $t = 0, 1$ reads

$$c_0 = M + w(1 - \tau) - QA - E_0 \tag{2.2},$$
$$c_1 = S + A + RE_0 - B^{ll} \tag{2.3}.$$
At the beginning of the retirement period 1, lifetime uncertainty is resolved. If the individual does not survive to the retirement period 1, she leaves a bequest $B^s$.

$$B^s = RE_0.$$  \hspace{1cm} (2.4)

Otherwise she consumes $c_1$ in the retirement period and leaves a bequest $B^l$. By this, we make use of the standard assumption that the individual gives $B^s$ to her heir at the beginning of the retirement period (see e.g. Abel, 1986; Strawczynski, 1999). This allows us to define net savings $E_1$ (in form of bonds) in the retirement period as the difference between $B^l$ and $RE_0$, i.e.

$$E_1 = B^l - B^s$$  \hspace{1cm} (2.4')

Depending on whether this difference is positive, negative or zero, the individual has positive savings, dissaves or does not save at all in the retirement period. Further, we assume $A \geq 0$. By this, we rule out the possibility that the individual can sell annuities or raise a loan in the working period whose redemption is guaranteed through a life insurance. The individual decides on her consumption and bequest plan over the uncertain lifetime by maximizing (2.1) subject to (2.2), (2.3) and (2.4). Substituting (2.2), (2.3) and (2.4) into (2.1) and differentiating with respect to $A$, $E_0$ and $B^l$, we obtain the Kuhn-Tucker conditions of this maximization problem,

$$A > 0 \quad \text{and} \quad -Qu'(c_0) + \pi u'(c_1) = 0,$$

$$A = 0 \quad \text{and} \quad -Qu'(c_0) + \pi u'(c_1) \leq 0,$$  \hspace{1cm} (2.5a) \hspace{1cm} (2.5b)

$$E_0 > 0 \quad \text{and} \quad -u'(c_0) + R\left[\pi u'(c_1) + (1-\pi)v'(B^s)\right] = 0,$$

$$E_0 \leq 0 \quad \text{and} \quad -u'(c_0) + R\left[\pi u'(c_1) + (1-\pi)v'(B^s)\right] \leq 0,$$  \hspace{1cm} (2.6a) \hspace{1cm} (2.6b)

$$B^s > 0 \quad \text{and} \quad \pi u'(c_1) + v'(B^l) = 0,$$

$$B^s \leq 0 \quad \text{and} \quad \pi u'(c_1) + v'(B^l) \leq 0.$$  \hspace{1cm} (2.7a) \hspace{1cm} (2.7b)

Note that the interior solution (2.5a), (2.6a) and (2.7a) will hold, as long as the social security benefits $S$ are sufficiently small relatively to the survival probability $\pi$ (compare 2.5a) and as long as preferences for leaving bequests are sufficiently strong relatively to the preferences for consumption (compare 2.6a and 2.7a). Otherwise a boundary solution will hold: In case that the individual is over-annuitized due to high social security benefits, annuity demand is equal to zero. In case that the individual has a sufficiently low bequest motive, savings $E_0$ in the working period and/or bequests $B^s$ will be zero.
Next, we study the portfolio decision of the individual i.e. how much of the initial wealth and of the net income \( M + w(1 - \tau) \) is invested into annuities and financial assets in each of the two periods. It is shown that this decision depends on the relation between the rate of return \( R \) on riskless bonds and the expected rate of return on annuities\(^1\), which is \( \pi/Q \).\(^2\) If this relation is equal to one, annuities are said to be individually fair. This gives us the corresponding individually fair annuity price \( \hat{Q} = \pi/R \).

**Proposition 1:** Consider that \( A > 0, E_0 > 0, B_{ll} > 0 \) and the individually fair annuity price \( \hat{Q} = \pi/R \). If the annuity price is above (equal, below, resp.) the fair price, then an individual has negative (zero, positive, resp.) savings in financial assets in the retirement period. From this it follows that she consumes more than (exactly, less than, resp.) her retirement income, consisting of the social security benefits and the annuity payoffs, i.e. if \( Q > \hat{Q} \), then \( E_1 \leq 0 \) and \( c_1 \geq A + S \).

**Proof:** Consider the interior solution (2.5a), (2.6a) and (2.7a). Substituting the equation in (2.5a) and (2.7a) into the equation in (2.6a) yields

\[
\pi(1 - QR)v'(B_{ll}) = (1 - \pi)QRv'(B_{si}) \quad (2.8)
\]

If \( Q = \hat{Q} \), (2.8) reduces to \( v'(B_{ll}) = v'(B_{si}) \) and hence \( B_{ll} = B_{si} \). From this, together with (2.3) and (2.4) follows that \( c_1 = A + S \) and thus, zero-savings in the retirement period, i.e. \( E_1 = 0 \).

By the same argument, one can show that an individual will dissave in old-age, if \( Q > \hat{Q} \). In this case, \( \pi(1 - QR) < (1 - \pi)QR \), thus \( v'(B_{ll}) > v'(B_{si}) \), so that \( E_1 = B_{ll} - B_{si} < 0 \) and \( c_1 > A + S \). If \( Q < \hat{Q} \), all of the results are reversed. Q.E.D.

In case that annuities offer the individually fair price \( \hat{Q} \), the individual buys bonds in the working period to leave bequests \( B_{ll} = B_{si} \) and uses the social security benefits and the annuities to provide for consumption in the retirement. However, an individual will dissave in old-age, if \( Q > \hat{Q} \). In this case the expected rate of return on annuities is lower than the rate of return on bonds. Thus, it is an attractive strategy for the individual to provide for some of the consumption in the retirement period by financial assets. On the other hand, the individual has positive savings in the retirement, if confronted with an annuity price below her individually fair price \( \hat{Q} \). Hence, some of the retirement income, i.e. annuity payoffs and social security benefits, is not consumed but saved for bequests.

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\(^1\) See also Abel (1986).

\(^2\) Note that for each unit of expected annuity payoff \( \pi \) the individual pays a price \( Q \).
Next, consider a situation, where the individual has a sufficiently low bequest motive, so that (2.6b) or (2.7b) holds (besides 2.5a). From the above considerations, we can conclude that \( Q > \hat{Q} \) is a necessary, but not sufficient condition that (2.6a) together with (2.7b) holds. In this case, due to the unattractively high annuity price (compared to that of bonds), the individual will save in the working period for consumption in the retirement period, but does not leave any bequests \( B^{ll} \) after having lived her maximum lifespan. This is an attractive strategy, since \( E_0 \) can be used for two purposes, depending on whether or not she survives to the retirement period: old-age consumption and bequests \( B^{sl} \). However she does not leave any bequests \( B^{ll} \) after having lived her maximum lifespan, since preferences for giving bequests are not strong enough. By the same reasoning, \( Q < \hat{Q} \) is a necessary, but not sufficient condition that (2.6b) together with (2.7a) holds. In this case, \( B^{sl} = 0 \) and \( B^{ll} > 0 \). Since the expected rate of return on annuities is higher than the rate of return on bonds, the individual does not save in the working period and thus does not leave a bequest \( B^{ll} \), when she dies young. However she saves part of her annuity payoffs in the retirement period to leave a bequest \( B^{sl} \). To conclude: Although the bequest motive is relatively low, individuals may save financial assets, since the rate of return on bonds diverge from the expected rate of return on annuities.

Proceeding on the assumption that individuals differ in their life-expectancy and that there is asymmetric information in the private annuity, the individuals are confronted with a divergence of these two rates of return. Since annuity companies cannot distinguish individuals according to their life-expectancy, the first-best outcome, in which each individual receives her individually fair price according to her survival probability, cannot be realized. Instead the problem of adverse selection occurs. The fact that individuals with a long life-expectancy demand more annuities, leads to an over-representation of annuities bought by the high-risk individuals among aggregate annuity demand. As a consequence, insurance companies, in order to avoid losses, offer a price which is higher than the actuarially fair price based on the average survival probability of the population.\(^3\) From this we can conclude that individuals with a survival probability below and somewhat above the average survival probability are confronted with an annuity price above the individually fair price \( \hat{Q} \) and thus reduce their financial assets in old-age. On the other hand, individuals with a sufficiently high life-expectancy face a price below \( \hat{Q} \), and hence continue to accumulate financial assets even in the retirement period.

\(^3\) The adverse-selection problem in the annuity market was studied in various theoretical contributions, see e.g. see Abel, 1986; Eckstein, Eichenbaum and Peled (1985), Townley and Boadway (1988) and Brunner and Pech (2002, 2005), Pech (2004). Empirical studies for the well developed US annuity market give evidence that prices are about 7 – 15 % above the fair price due to adverse selection (Walliser, 2000; Mitchell et al., 1999; Friedman and Warshawsky, 1990). Finkelstein and Poterba (2002) find that adverse selection exists to some similar extent in the voluntary annuity market of the United Kingdom.
In the next proposition we investigate the saving behaviour of an individual, in case that she is over-annuitized due to high social security benefits, thus annuity demand is equal to zero.

**Proposition 2:** Assume that $A = 0$, $E_0 > 0$, $B^d > 0$. In this case, $B^d = B^0$ at a price $\overline{Q}$, which is above the individually fair price $\hat{Q}$. Only if $Q > \overline{Q}$, $B^d > B^0$ and the individual dissaves in the retirement period. If $Q < \overline{Q}$, then $B^d < B^0$ and the individual has positive net savings $E_1$ in the retirement period, i.e. if $Q < \overline{Q}$, then $E_1 > 0$ and $c_1 > \overline{S}$.

**Proof:** Consider (2.5b), (2.6a) und (2.7a). For an easy illustration of the result in this proposition, we rewrite the equation in (2.5b) and in (2.6a) as

$$u'(c_0) > \frac{\pi}{Q} u'(c_1) \quad (2.5b')$$

$$u'(c_0) = (1 - \pi)R v'(B^d) + \pi R u'(c_1) \quad (2.6a')$$

and we multiply the LHS of the equation in (2.7a) with $R(1 - \pi)/\pi$ and rearrange it to

$$R u'(c_1) = (1 - \pi)R v'(B^d) + \pi R u'(c_1). \quad (2.7a')$$

First consider that $Q = \hat{Q}$, $\hat{Q} = \pi/R$: In this case (2.5b') reduces to $u'(c_0) > R u'(c_1)$. From this it follows that the LHS of (2.6a') is greater than the LHS of (2.7a'). Consequently, the same holds for the RHS's of (2.6a') and (2.7a'), which implies that $v'(B^d) > v'(B^0)$, thus $B^d < B^0$.

From these considerations it follows that $R$ must be higher than $\pi/Q$, such that the two LHS's of (2.6a') and (2.7a') are equal, i.e. $u'(c_0) = R u'(c_1)$, as well as the inequality in (2.5b'), i.e. $u'(c_0) > (\pi/Q) u'(c_1)$, holds. We define $\overline{Q} = \hat{Q} > \pi/R$, as the price which fulfills these two conditions. Obviously, at $\overline{Q}$ $v'(B^d) = v'(B^0)$. By the same argument as in proposition 1, it follows that $v'(B^0) < v'(B^d)$, if $Q > \overline{Q}$ and $v'(B^0) > v'(B^d)$, if $Q < \overline{Q}$. Q.E.D

Consider an individual who does not purchase annuities, although they are offered to her at her individually fair price $\hat{Q}$, since she has (more than) enough retirement income in form of social security benefits. In this case, in order to leave bequests, the individual saves some of the labour income in the working period and saves some of the social security benefits in the retirement period. Thus, she leaves more bequests, when surviving to the retirement period. Only at a higher annuity price $\overline{Q}$ she would decide to dissave in old-age and hence leave bequests $B^d < B^0$. 

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This theoretical result may help to explain why in countries with a generous public pension system, one observes positive saving rates even in the periods of retirement, although lower than in the working periods. This empirical evidence was found especially for Germany; see e.g. Börsch-Supan et al. (1999) and Schnabel (1999). Lately attention has been paid to this empirical observation, considered as a contraction to the life-cycle hypothesis. However, when there is a strong presumption of a bequest motive combined with the existence of a generous social security system, the empirical evidence of low annuity demand and flat age-wealth profiles can be regarded to be in accordance with theoretical results.

Note, on the other hand, that the increasing wealth profile, which was found on the average, depends much on the income and wealth distribution of the population and can be mainly attributed to the upper end. At the lower end of the income distribution one observes a much flatter wealth accumulation during the working time and dissaving in the retirement time; see e.g. Schnabel (1999), Disney, Emmerson and Wakefield (2001). In the next proposition 3, we investigate the effect of income, initial wealth and of the survival probability on the portfolio behaviour. Empirical studies found evidence for a positive influence of income on life-expectancy; see e.g. Attanasio and Hoynes (2000), Lillard and Panis (1998), Lillard and Waite (1995), Menchik (1993), hence the combined effects can be considered to be of special relevance.

Proposition 3: Assume that \( A > 0, E_0 > 0, B^I > 0 \).

(i) An increase in the initial wealth \( M \) and in the income \( w \) has the following effects on the portfolio decision of an individual:

- Savings \( E_0 \) in the working period as well as the bequests \( B^u \) and \( B^l \) increase by an increase in \( M \). The same effect has an increase in income \( w \), if social security benefits do not decrease with income, i.e. if \( \partial S/\partial w \geq 0 \).

- Net savings \( E_1 \) in the retirement period do not change by an increase in \( M \) and \( w \), if annuities are offered at the fair price \( \hat{Q} \). Otherwise, the effect of an increase of these variables on \( E_1 \) depends on the third derivate of the utility function \( v(B^i) \) in the following way: Given that \( v'''' \geq 0 \), then \( E_1 \) increases, if \( Q > \hat{Q} \), while \( E_1 \) decreases, if \( Q < \hat{Q} \).\(^4\) Given that \( v'''' \leq 0 \) the effect is ambiguous.

- An increase in \( M \) induces a higher annuity demand \( A \), if savings \( E_1 \) in the retirement period do not decrease (too much) by an increase in \( M \). The same effect has an

\(^4\) The assumption of a positive third derivative denotes precautionary behaviour, as first explained by Leland (1968).
increase in income \( w \), if social security benefits do not increase with income, i.e. if \( \partial S / \partial w \leq 0 \). Otherwise the effect is ambiguous.

(ii) An increase in the survival probability \( \pi \) has the following effects on the portfolio decision of an individual: It induces a higher annuity demand \( A \), lower savings \( E_0 \) in the working period and higher net savings \( E_1 \) in the retirement period.

**Proof:** See Appendix.

Proposition 3 shows that individuals with higher initial wealth and income save more financial assets in the working period. Note that income has this effect only for certain, if social security benefits do not decrease with income, which is plausible probably for every existing social security system. But it is ambiguous whether higher income and higher initial wealth also means higher net savings \( E_1 \) in the retirement period. On the other hand, we find that a higher survival probability induces less savings \( E_0 \) in the working period, but higher net savings \( E_1 \) in the retirement period. Hence, when a positive influence of income and wealth on life-expectancy is supposed to exist, it is plausible to expect positive old-age savings of rich and long-living individuals, while poor and short-living people probably dissave in old-age. Further, the combined effects suggest that the latter have also a lower annuity demand than the former. Note however that this conclusion is only correct for initial wealth, but for income only, if social security benefits do not change with income.\(^5\) Again these theoretical findings point in the same direction as the empirical evidence, mentioned above, especially for countries with a generous public pension system combined with a high load factor of the annuity price: Aggregate annuity demand will be quite low and probably zero for individuals at the lower end of income and wealth distribution, while increasing age-wealth profiles will be observed at the upper end of the wealth distribution.\(^6\)

### 3. Savings and annuity demand under lifetime and income uncertainty

In this section we extend the model of section 2 by introducing income uncertainty to analyse how the portfolio decision of an individual is affected, when savings in form of bonds also have the purpose to self-insure against an income risk, which might arise e.g. due to an unemployment or ill-health risk. This seems a relevant issue since income risk might induce

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\(^5\) In many countries, regulations, which realise \( \partial S / \partial w = 0 \), at least below and above a certain threshold of income, are effective: On the one hand there are flat-rate pensions, which guarantee a minimum retirement income, on the other hand often an assessment ceiling for calculating benefits and contributions exists.

\(^6\) In this case, annuity payments - in empirical studies usually treated as savings in the working period - and annuity payoffs - treated as dissaving in retirement - are only a small fraction in total savings.
people to avoid consuming out of wealth or buying annuities. However, note that the prerequisite for the existence of this precautionary motive of savings are incomplete insurance markets, which means that individuals cannot insure against all risks they are confronted with.

In this section, we distinguish three cases, which differ in the timing, when the income risk is present and when it is resolved. In the first two cases we study the effects of uncertainty about old-age income. In case 1, it prevails until the individual dies, while in case 2 it is resolved already at the beginning of the retirement period. Note that in the first case, the risk of low income should be thought of as expenditures that due to unforeseen circumstances (such as large health expenditures) depress disposable income, which last until the end of life. On the other hand, the assumption of an old-age income risk that is resolved at the beginning of retirement seems more accurate, when considering individuals who learn about their health-status or get informed about their retirement income at the end of their working period, e.g. when it is closely related to the final earnings in the working time. The case 3 considers uncertainty about the labour income that prevails until the end of the working period. This can be regarded as a relevant situation, when the assumption is made that an annuity contract is concluded during the working period, when the income risk still prevails. One could think of annuities for which the premium is paid yearly during the working period. Such annual-payment annuity contracts are quite common in Germany and Austria and are typically those for which individuals receive subsidies by the state.

We introduce income (expenditure, resp.) risk into the model of section 2 in the simplest way: We assume an additive negative shock \(-\varepsilon\). This allows us to specify \(W_0\) and \(W_1\) as

\[
W_0 = \begin{cases} 
M + w(1 - \tau) - \varepsilon & \text{with probability } \beta \\
M + w(1 - \tau) & \text{with probability } 1 - \beta,
\end{cases} \tag{3.0}
\]

\[
W_1 = \begin{cases} 
S - \varepsilon & \text{with probability } \gamma \\
S & \text{with probability } 1 - \gamma,
\end{cases} \tag{3.1}
\]

where \(0 \leq \beta < 1\) and \(0 \leq \gamma < 1\). For the cases 1 and 2 with old-age income risk, we have \(\gamma > 0\) and \(\beta = 0\), while in case 3 with labour income risk, \(\gamma = 0\) and \(\beta > 0\). All results refer to the interior solution of the maximization problem, thus boundary solutions, as in section 2, are neglected.

**Case 1:** Old-age income uncertainty that prevails until the end of life (\(\gamma > 0\) and \(\beta = 0\))

Note that the individual is uncertain about her old-age income until the end of her life. Thus she will leave accidental bequests in case that the negative income shock is realized. Lifetime utility can be written as
\[
U = u(c_0) + \pi u(c_1) + (1 - \pi)v(B^{sl}) + \pi E[v(B^{ll})],
\]

where \(E\) is the expectation operator. Substituting (2.2) and (3.0) for \(\beta = 0\) into (2.4) yields

\[
B^{sl} = R(W_0 - QA - c_0).
\]

By use of (3.0) for \(\beta = 0\) and (3.1) for \(\gamma > 0\), (2.3) can be rearranged to

\[
B^{ll} = R(W_0 - QA - c_0) + W_1 + A - c_1.
\]

Substituting (3.3) and (3.4) into the lifetime utility function (3.2) and differentiating with respect to \(c_0, c_1\) and \(A\), respectively, yields the first-order conditions

\[
u'(c_0) = R\left((1 - \pi)v'(B^{sl}) + \pi E[v'(B^{ll})]\right),
\]

\[
u'(c_1) = E[v'(B^{ll})],
\]

\[(1 - \pi)QRv'(B^{sl}) = \pi(1 - QR)E[v'(B^{ll})].\]

**Proposition 4:** Old-age income uncertainty that prevails until the end of life (Case 1) has the following effect on the portfolio and consumption decision: Compared to a situation without the risk of a negative income shock in old-age, the individual consumes less in both periods of life. She demands more annuities, but buys less bonds in the working period and, hence, leaves less bequests \(B^{ll}\) in case that she does not survive to the retirement period. However, in case that the negative income shock is not realized, the individual will leave more ex-post bequests \(B^{ll}\), after having lived for two periods.

**Proof:** To show this result we take the interior solution without income uncertainty of section 2 as a benchmark and compare it with the interior solution (3.5) – (3.7) with old-age income uncertainty. First compare (3.6) with the equation in (2.7a*), which can be rewritten as,

\[
u'(c_1) = v'(B^{ll}).
\]

Since \(v'(B^{ll}) < E[v'(B^{ll})]\), the LHS in (3.6) is higher than the RHS in (2.7*). It follows that \(u'(c_1)\) has to increase in order to restore equality between both sides in (3.6). Hence, \(c_1\) decreases, which implies that the RHS of (3.6), \(E[v'(B^{ll})]\), decreases (see (3.4)). This in turn implies that \(B^{ll}\) is higher, if the negative income shock does not occur.
Next, we substitute (3.7) into (3.5) to obtain

\[ u'(c_0) = \frac{\pi}{Q} E[v'(B^{II})], \quad (3.8) \]

which we compare with

\[ u'(c_0) = \frac{\pi}{Q} v'(B^{II}), \quad (3.9) \]

which is obtained by substituting the equation in (2.7a) into the equation in (2.5a). By the same arguments as before, the LHS of (3.8) is higher than the LHS of (3.9). Thus, \( c_0 \) has to decrease (due to income uncertainty), since this increases the LHS of (3.8), \( u'(c_0) \), and decreases RHS of (3.9), \( E[v'(B^{II})] \).

Finally, we eliminate \( E[v'(B^{II})] \) by use of (3.7) into (3.5), which gives

\[ u'(c_0) = \frac{R(1-\pi)}{1-\pi R} v'(B^{III}). \quad (3.10) \]

Since \( u'(c_0) \) has increased, \( v'(B^{III}) \) has to increase too, to restore the equality on both sides of (3.10). From this follows that \( B^{III} \) and thus \( E_0 \) have to decrease. However, it turns out by use of budget equation (3.3) that this is only possible, if annuity demand increases. To be precisely the increase in the annuity expenditures +\( \Delta A \) has to be higher than the decrease in -\( \Delta c_0 \). Q.E.D.

The intuition for this result is as follows: Since annuities offer a higher rate of return than riskless bonds, annuities provide a better protection than savings against a negative income shock, which can occur only in the retirement period.

**Case 2:** Old-age income uncertainty that is resolved at the beginning of the retirement period \((\gamma > 0 \text{ and } \beta = 0)\)

Uncertainty about old-age income (expenditures, resp.) resolved at the beginning of the retirement is plausible, when it is assumed that individuals have the opportunity to get informed about their retirement income or to learn about their health-status. In this case the individual is confronted with a two-stage decision problem. In the working period 0, she chooses the following variables: Annuity demand \( A \) and her consumption level \( c_0 \) in the working period, and thus bequests \( B^{III} \) and her total uncertain "retirement-wealth" \( D = R(W_0 - QA - c_0) + W_1 + A \). For this decision she takes into account her optimal level of consumption \( c_1 \) and of bequests \( B^{III} \), which she will choose in the retirement period 1 after the resolution of both risks, i.e. knowing
about her retirement income and whether she has survived. Formally, this two-stage problem can be written as:

\[
\begin{align*}
    t = 0: & \quad \max_{c_0, A} u(c_0) + (1 - \pi)v(B^{sl}) + \pi E[\phi(R(W_0 - QA - c_0) + W_1 + A)], \\
    \text{s.t.} & \quad (3.2)
\end{align*}
\]

\[
\begin{align*}
    t = 1: & \quad \max_{c_1} u(c_1) + v(B^I) \\
    \text{s.t.} & \quad (3.3)
\end{align*}
\]

where \( \phi(D) = \max_{c_1} (u(c_1) + v(B^I)) \bigg| B^I = D - c_1 \).

By inserting (3.3) into (3.11) and differentiating with respect to \( c_0 \) and \( A \) as well as inserting (3.4) into (3.12) and differentiating with respect to \( c_1 \), we obtain the first-order conditions of this maximization problem:

\[
\begin{align*}
    u'(c_0) &= R \left( (1 - \pi)v'(B^{sl}) + \pi E \frac{\partial \phi(D)}{\partial D} \right), \\
    (1 - \pi)QRv'(B^{sl}) &= \pi(1 - QR)E \frac{\partial \phi(D)}{\partial D}, \\
    u'(c_1) &= v'(B^I),
\end{align*}
\]

where by application of the Envelope Theorem

\[ \frac{\partial \phi(D)}{\partial D} = v'(B^I). \]  

**Proposition 5:** Old-age income uncertainty that is resolved at the beginning of the retirement period (Case 2) has the following effects on the portfolio and consumption decision: Compared to a situation without an income risk, the individual consumes less in the working period, demands more annuities and less bonds in the working period (and hence leaves less bequests \( B^{sl} \) in case of death). If she survives to the retirement period, she allocates the – now certain - retirement-wealth \( W_1 \) on consumption \( c_1 \) and bequests \( B^I \). In case that the negative income shock does not occur, the individual has a higher consumption level \( c_1 \) and leaves more bequests \( B^I \), after having lived for two periods.

**Proof:** We consider the interior solution without income uncertainty of section 2 and compare it with the interior solution (3.13) – (3.15) with old-age income uncertainty. Inserting (3.16) and (3.14) into (3.13) yields (3.8); eliminating \( E[\partial \phi(D)/\partial D] \) by use of (3.13) and (3.14) gives (3.10). As shown in the proof of the foregoing Proposition 4, consumption \( c_0 \), bequests \( B^{sl} \) and savings
\( E_0 \) must be lower and annuity demand \( A \) must be higher compared to a situation without the risk of a negative income shock.

After the lifetime and income uncertainty has been resolved, the allocation of the realized "retirement-wealth" \( D \) (either with or without the negative income shock) on consumption \( c_1 \) and bequests \( B^l \) is made according to (3.15). We know that \( E[v'(B^l)] \) decreases (to restore equality on both sides of (3.8)). From this together with the comparison with (2.7a'), i.e. the F.O.C. without income uncertainty, it follows that \( B^l \) and \( c_1 \) are higher, if the negative income shock does not occur. Q.E.D.

Thus comparing Case 1 and Case 2, which differ only in the timing of the resolution of the old-age income uncertainty shows the following difference: In case 2, where the individual gets informed about her old-age income (or health-care expenditures) at the beginning of the retirement period, she splits it up on consumption and bequests. In case 1, however, where the individual fears old-age income risk (or the risk of the consequences of catastrophic illness) during the whole time of retirement, she has more precautionary savings in the retirement period, which leads to accidental bequests, in case that no negative income shock has occurred.

**Case 3: Labour income uncertainty that prevails until the end of the working period \((\gamma = 0 \text{ and } \beta > 0)\)**

Finally, we investigate the consequences of labour income uncertainty, e.g. due to the risk of unemployment in the framework employed throughout this paper. By this, the analysis is kept simple and comparable, but needs as a prerequisite that the annuity contract is concluded during the working period, when the income risk still prevails, such as a annuity for which the premium is paid yearly.

In this case again, the individual has a two-stage decision problem: In the working period \( 0 \), she chooses annuity demand \( A \) and her consumption level \( c_0 \) in the working period, and thus bequests \( B^s_l \) and her total uncertain retirement-wealth \( D \). For this decision she takes into account her optimal level of consumption \( c_1 \) and of bequests \( B^s_l \), which she will choose in the retirement period \( 1 \) after the resolution of both lifetime and labour income risk. Note that the difference to case 2, discussed above, is that the individual is uncertain about her income in the working period as well as about her retirement-wealth \( D \) at time \( t = 0 \). Thus, labour income risk produces accidental bequests \( B^s_l \), besides \( B^l \). Formally, this two-stage problem can be written as:
\begin{align*}
t = 0: \quad & \max_{c_0,A} \left( u(c_0) + (1 - \pi)E[v(B^{sl})] + \pi E[\varphi(R(W_0 - QA - c_0) + W_1 + A)] \right), \quad (3.17) \\
\text{s.t. } & (3.2) \\
t = 1: \quad & \max_{c_1} \left( u(c_1) + v(B^{ll}) \right) \quad \text{ (3.18)} \\
\text{s.t. } & (3.3) \\
\text{where } \varphi(D) = \max_{c_1} \left( u(c_1) + v(B^{ll}) \right) \Bigr| B^{ll} = D - c_1. \nonumber \\

\text{By inserting (3.3) into (3.17) and differentiating with respect to } c_0 \text{ and } A \text{ as well as inserting (3.4) into (3.18) and differentiating with respect to } c_1, \text{ we obtain the first-order conditions of this maximization problem:} \\
\begin{align*}
\partial \varphi/\partial \pi = & \quad u'(c_0) = R \left( (1 - \pi)E[v'(B^{sl})] + \pi E[\partial \varphi(D)/\partial D] \right), \quad (3.19) \\
(1 - \pi)QRE[v'(B^{sl})] = & \quad \pi(1 - QR)E[\partial \varphi(D)/\partial D], \quad (3.20) \\
u'(c_1) = & \quad v'(B^{ll}), \quad (3.21) \\
\text{where by application of the Envelope Theorem} \\
\frac{\partial \varphi(D)}{\partial D} = & \quad v'(B^{ll}). \quad (3.22) \\

\text{Proposition 6: Labour income uncertainty that prevails until the end of the working period (Case 3) has the following effect on the portfolio and consumption decision: Compared to a situation without an income risk, the individual consumes less in the working period, demands less annuities and more bonds in the working period. Hence, she leaves more bequests } B^{sl} \text{ in case of an early death, given that the negative income shock has not occurred. If she survives to the retirement period, she allocates the – now certain - retirement-wealth } D \text{ on consumption and bequests. In case that no negative income shock has occurred, she leaves more bequests } B^{ll} \text{ as well as has a higher consumption level } c_1. \\

\text{Proof: We consider the interior solution without income uncertainty of section 2 and compare it with the interior solution (3.19) – (3.21) with labour income uncertainty. Inserting (3.22) and (3.19) into (3.17) yields again (3.8). As shown in the proof of the Proposition 4, consumption } c_0 \text{ in the working period must be lower and } E[v'(B^{ll})] \text{ has to decrease. This in turn implies that } E[v'(B^{sl})] \text{ has to decrease too, which is straightforward to see by use of (3.20) and (3.22). From this it follows that in case that the negative income shock in the working period does not occur, the individual will leave higher bequests } B^{sl} \text{ in case of non-survival to the retirement period,}
compared to a situation without any income risk. Hence, the individual will have higher savings $E_0$ in the working period.

After lifetime and income uncertainty has been resolved, the allocation of the retirement-wealth $D$ (either with or without the negative income shock) on consumption $c_1$ and bequests $B^b$ is made according to (3.21), which is equal to (3.15). Thus, as shown in the proof of proposition 5, $B^b$ and $c_1$ are higher, if the negative income shock has not been realized. Q.E.D.

Thus, a comparison of the labour income uncertainty to old-age income uncertainty allows the following conclusions: Both labour and old-age income risk induces the individual to reduce consumption in the working period. However, the first risk increases savings $E_0$ in the working period, while the latter risk increases annuity demand. The intuition for this result is obvious: Only savings of financial assets are an appropriate strategy to self-insure against negative income shock in the working period. In case of uncertainty about income in the retirement period, both, savings and annuities can serve as a protection against a negative shock; however since annuities offer a higher rate of return, this strategy is more attractive.

4. Conclusions

There are two stylised facts, namely weak demand for life annuities and flat age-wealth profile in old-age, that contradict the life-cycle hypothesis. Many plausible arguments have been put forward in the literature to reconcile theory with empirical evidence. The aim of this paper is to design a theoretical framework, which combines these arguments, to study to which extent these can contribute to explain the observed portfolio decision in favour of financial assets relatively to annuities. To do so, we consider a two-period model which combines the existence of an annuity market and of a public pension system with risk-averse individuals who are uncertain about lifetime and disposable income and who have preferences for leaving bequests.

We found the following results given that there is a longevity risk, but no income risk: Individuals with higher wealth and income save more financial assets in the working period, but it is ambiguous whether this means also higher financial assets in the retirement period. However, when a positive influence of income and wealth on life-expectancy is supposed to exist, one can expect higher positive savings of rich (and long-living) individuals in old-age, while poor (and short-living) people probably dissave. These theoretical findings are in accordance with the empirical evidence, which show that the flat (or slightly increasing) age-wealth profiles in old-age, observed on the average, are mainly attributable to individuals at the upper end of the income distribution, while individuals at the lower end rather dissave. On the other hand, we
found that annuities are not an attractive strategy to provide for old-age in countries with a generous public pension system combined with a high load factor of the annuity price. In this case, aggregate annuity demand is quite low and probably zero for individuals at the lower end of income distribution.

Further, we have extended the model by introducing a negative income shock on disposable lifetime income, where we have distinguished between three cases, which differ in the timing, when and how long the income uncertainty prevails. We compare the results with those under income certainty: In case that there exists the risk of a negative shock on the disposable income in old-age, annuity demand increases and savings in riskless bonds as well as consumption in the working period decreases. If the individual gets informed about the level of her old-age income (or health-care expenditures) at the beginning of the retirement period, she splits it up between consumption and bequests. On the other hand, when the individual is uncertain about her disposable old-age income during the whole time of retirement, she saves more in the retirement period for precautionary motives, which leads to lower consumption and accidental bequests. In contrast, the risk of a negative shock on the labour income induces a lower annuity demand and precautionary savings in the working period and hence accidental bequests, if she dies young. When surviving to retirement, she splits up her whole retirement wealth, which she knows for certain at that time, on consumption and bequests. From these results we can conclude that income uncertainties can contribute to explain higher savings for precautionary motives and lower consumption levels over lifetime, however only labour-income uncertainty can explain the weakness of annuity demand.

**Appendix: Proof of Proposition 3**

We determine the effect of a marginal change of a parameter \( X = M, w, \pi \) on \( E_0, B^\Pi, E_1 \) and \( A \). For this we make use that the LHS of the equation in (2.5a), (2.6a) and (2.7a), resp., is the first derivate of the lifetime utility function (2.1) with respect to \( A, E_0 \) and \( B^\Pi \), resp.

Implicit differentiation of the interior solution (2.5a), (2.6a) and (2.7a) with respect to a parameter \( X = M, w, \pi \), gives

\[
\begin{align*}
\frac{\partial A}{\partial X} & = - \begin{pmatrix}
\frac{\partial^2 U}{\partial A^2} & \frac{\partial^2 U}{\partial A \partial E_0} & \frac{\partial^2 U}{\partial A \partial B^\Pi} \\
\frac{\partial^2 U}{\partial E_0 \partial A} & \frac{\partial^2 U}{\partial E_0^2} & \frac{\partial^2 U}{\partial E_0 \partial B^\Pi} \\
\frac{\partial^2 U}{\partial B^\Pi \partial A} & \frac{\partial^2 U}{\partial B^\Pi \partial E_0} & \frac{\partial^2 U}{\partial B^\Pi^2}
\end{pmatrix}^{-1} \begin{pmatrix}
\frac{\partial^2 U}{\partial A \partial X} \\
\frac{\partial^2 U}{\partial E_0 \partial X} \\
\frac{\partial^2 U}{\partial B^\Pi \partial X}
\end{pmatrix}
\end{align*}
\]

(A1)
where the first matrix of the LHS of (A1) is the (symmetric) Hessian Matrix. Its elements are

\[
\frac{\partial^2 U}{\partial A^2} = Q^2 u^*(c_0) + \pi u^*(c_1),
\]

\[
\frac{\partial^2 U}{\partial E_0^2} = u^*(c_0) + \pi R^2 u^*(c_1) + (1 - \pi) R^2 v^*(B^0),
\]

\[
\frac{\partial^2 U}{\partial B^{ii}} = \pi u^*(c_1) + \pi v^*(B^1),
\]

\[
\frac{\partial^2 U}{\partial A \partial E_0} = Qu^*(c_0) + \pi Ru^*(c_1),
\]

\[
\frac{\partial^2 U}{\partial A \partial B^{ii}} = -\pi u^*(c_1),
\]

\[
\frac{\partial^2 U}{\partial E_0 \partial B^{ii}} = -\pi Ru^*(c_1),
\]

Inverting the Hessian Matrix in (A1) and multiplying it with the vector on the LHS of (A1) gives:

\[
\frac{\partial A}{\partial X} = \theta \left[ \left( \frac{\partial^2 U}{\partial A \partial X} \left( \frac{\partial^2 U}{\partial E_0^2} \frac{\partial^2 U}{\partial B^{ii} \partial E_0} - 2 \frac{\partial^2 U}{\partial E_0 \partial B^{ii} \partial E_0} \right) \right) + \frac{\partial^2 U}{\partial E_0 \partial E_0} \frac{\partial^2 U}{\partial B^{ii} \partial B^{ii}} \right]
\]

\[
\frac{\partial E_0}{\partial X} = \theta \left[ \left( \frac{\partial^2 U}{\partial A \partial E_0} \left( -\frac{\partial^2 U}{\partial E_0^2} \frac{\partial^2 U}{\partial B^{ii} \partial E_0} + \frac{\partial^2 U}{\partial E_0 \partial B^{ii} \partial E_0} \right) \right) + \frac{\partial^2 U}{\partial E_0 \partial E_0} \frac{\partial^2 U}{\partial B^{ii} \partial B^{ii}} \right]
\]

\[
\frac{\partial B^{ii}}{\partial X} = \theta \left[ \left( \frac{\partial^2 U}{\partial A \partial B^{ii}} \left( \frac{\partial^2 U}{\partial E_0^2} \frac{\partial^2 U}{\partial B^{ii} \partial E_0} \right) + \frac{\partial^2 U}{\partial E_0 \partial E_0} \frac{\partial^2 U}{\partial B^{ii} \partial B^{ii}} \right) + \frac{\partial^2 U}{\partial E_0 \partial E_0} \frac{\partial^2 U}{\partial B^{ii} \partial B^{ii}} \right]
\]

where \( \theta = -1/|H| \) with \(|H|\) as the determinant of the Hessian Matrix, which is negative due to the second-order conditions of the maximisation problem. Thus, \( \theta > 0 \).

Finally, remember that \( E_1 = B^{ii} - RE_0 \); thus \( \frac{\partial E_1}{\partial X} \) can be determined by use of (A9) and (A10), i.e.

\[
\frac{\partial E_1}{\partial X} = \frac{\partial B^{ii}}{\partial X} - R \frac{\partial E_0}{\partial X}
\]
(i) First, we determine $\frac{\partial^2 U}{\partial A_0 \partial M}$ for $X = M, W, \pi$. Substituting (A2), (A4) – (A7) together with

\[ \frac{\partial^2 U}{\partial E_0 \partial M} = u^*(c_0), \quad (A12) \]
\[ \frac{\partial^2 U}{\partial B^i \partial M} = 0, \quad (A14) \]

into (A9) for $X = M$ yields

\[ \frac{\partial E_0}{\partial M} = -\theta(1 - QR)\pi^2 u^*(c_0)u^*(c_1)v^*(B^i), \quad (A15) \]

which is positive (remember that $\theta > 0$, $1/Q > R$ and strict concavity of $u(c_0)$ and $v(B^i)$). We calculate (A9) for $X = M$ by inserting

\[ \frac{\partial^2 U}{\partial A \partial W} = -Q(1 - \tau)u^*(c_0) + \pi u^*(c_1)\frac{\partial S}{\partial W}, \quad (A16) \]
\[ \frac{\partial^2 U}{\partial E_0 \partial W} = (1 - \tau)u^*(c_0) + \pi Ru^*(c_1)\frac{\partial S}{\partial W}, \quad (A17) \]
\[ \frac{\partial^2 U}{\partial B^i \partial W} = -\pi u^*\frac{\partial S}{\partial W}, \quad (A18) \]

and (A2), (A4) – (A7) into (A9) to obtain

\[ \frac{\partial E_0}{\partial W} = -\theta(1 - QR)(1 - \tau + Q \frac{\partial S}{\partial W})\pi^2 u^*(c_0)u^*(c_1)v^*(B^i). \quad (A19) \]

It follows that $\frac{\partial E_0}{\partial W} > 0$, if $\frac{\partial S}{\partial W} \geq 0$. To determine $\frac{\partial E_0}{\partial \pi}$, we calculate

\[ \frac{\partial^2 U}{\partial A \partial \pi} = u'(c_1), \quad (A20) \]
\[ \frac{\partial^2 U}{\partial E_0 \partial \pi} = Ru'(c_1) - Rv'(B^i), \quad (A21) \]
\[ \frac{\partial^2 U}{\partial B^i \partial \pi} = -u'(c_1) + v'(B^i), \]

According to the first-order-condition (2.7a) $u'(c_1) = v'(B^i)$, thus

\[ \frac{\partial^2 U}{\partial B^i \partial \pi} = 0. \quad (A22) \]

Inserting (A2), (A4) – (A7) and (A20) – (A22) into (A9) for $X = \pi$ gives
\[
\frac{\partial E_0}{\partial \pi} = -\theta \pi \left[ R \nu_u(B^u) \left( Q^2 \nu_u(c_o) \nu_v(B^d) + Q^2 \nu_u(c_o) \nu_v(B^d) + \pi \nu_u(c_o) \nu_v(B^d) + \nu(v(B^d) + \pi \nu_u(c_o) \nu_v(B^d)) \right] \right] \]

which is negative.

(ii) Next, we determine \( \frac{\partial B^v}{\partial X} \) for \( X = M, w, \pi \). We calculate (A10) for \( X = M \) by substituting (A2) – (A3), (A5) – (A7) and (A12) – (A14) into (A10) to obtain

\[
\frac{\partial B^v}{\partial M} = -\theta Q R (1 - \pi) \nu_u(c_o) \nu_v(B^d), \tag{A24}
\]

which is positive. In the same manner, we determine \( \frac{\partial B^v}{\partial w} \) by use of (A2) – (A3), (A5) – (A7), (A10) and (A16) – (A18), i.e.

\[
\frac{\partial B^v}{\partial w} = -\theta Q R (1 - \pi) (1 - \pi) \nu_u(c_o) \nu_v(B^d). \tag{A25}
\]

(A25) is positive, if \( \frac{\partial S}{\partial w} \geq 0 \). Finally, substituting (A2) – (A3), (A5) – (A7) and (A16) – (A18) into (A10) for \( X = \pi \) yields

\[
\frac{\partial B^v}{\partial \pi} = \theta \nu_u(c_o) \left[ (1 - QR) \nu_u(c_o) \nu_v(B^d) + (1 - QR) \nu_u(c_o) \nu_v(B^d) + R^2 (1 - \pi) \nu_u(c_o) \nu_v(B^d) \right], \tag{A26}
\]

which is positive.

(iii) Next, we determine \( \frac{\partial E_1}{\partial X} \) for \( X = M, w, \pi \). Substituting (A15) and (A24) into (A11) for \( X = M \) gives

\[
\frac{\partial E_1}{\partial M} = \theta Q R \left( 1 - \pi \right) \nu_u(c_o) \nu_v(B^d) \left[ \frac{(1 - QR) \pi}{QR(1 - \pi)} \nu_v(B^d) \right]. \tag{A27}
\]

Note that the sign of the RHS of (A27) is determined by the last term in the brackets on the RHS of (A27). Observe that if \( Q \geq \hat{Q} \), then \( \frac{1 - QR}{QR(1 - \pi)} \leq 1 \) and \( B^d \geq B^u \) (compare proposition 1). Obviously, \( \nu_v(B^u) / \nu_v(B^d) = 1 \), if \( B^d = B^u \). From these considerations follows that \( \frac{\partial E_1}{\partial M} = 0 \), if \( Q = \hat{Q} \).

However for \( B^d \geq B^u \), we have to make assumptions about the third derivative \( \nu_v(B^d) \) to assess the size of \( \nu_v(B^u) / \nu_v(B^d) \): (a) Given that \( \nu_v(B^d) = 0 \), then \( \nu_v(B^u) / \nu_v(B^d) = 1 \). (b) Given that \( \nu_v(B^d) > 0 \), \( \nu_v(B^u) / \nu_v(B^d) \geq 1 \). (c) Given that \( \nu_v(B^d) < 0 \), \( \nu_v(B^u) / \nu_v(B^d) \leq 1 \).
These considerations allows us to conclude: if \( Q \gtrless \hat{Q} \), then \( \frac{1 - QR}{QR(1 - \pi)} \frac{v^*(B^u)}{v(B^f)} \lesssim 0 \) and thus, \( \partial E_i / \partial M \gtrless 0 \), given that \( v^*(B') \gtrsim 0 \). However, given that \( v^*(B') \lesssim 0 \), the effect is ambiguous.

We determine \( \partial E_i / \partial w \) by use of (A11), (A19) and (A25), i.e.

\[
\frac{\partial E_1}{\partial w} = 0QR^2 \pi(1 - \pi)(1 - \tau + Q \frac{\partial S}{\partial w})u^*(c_0)u^*(c_i)v^*(B^u)\left\{ \frac{(1 - QR)}{QR(1 - \pi)} \frac{v^*(B^u)}{v(B^f)} \right\}.
\]

(A28)

Obviously, for (A28) the same considerations apply like for (A27), in case that \( \partial S / \partial w \gtrsim 0 \). If \( Q = \hat{Q} \), \( \partial E_i / \partial w = 0 \) (irrespective of the slope of \( v^*(B') \)). If \( Q \gtrsim \hat{Q} \), then \( \partial E_i / \partial w \gtrsim 0 \), given that \( v^*(B') \gtrsim 0 \). However, given that \( v^*(B') \lesssim 0 \), the effect is ambiguous.

Finally, we show that \( \partial E_i / \partial \pi > 0 \). This follows immediately from the fact that \( \partial B^f / \partial \pi > 0 \) and \( \partial E_0 / \partial \pi < 0 \) (see (A11), (A23) and (A26)).

(iv) Finally we determine \( \partial A / \partial X \) for \( X = M, w, \pi \). For \( X = M \), we substitute (A3) – (A7) and (A12) – (A14) into (A8). This gives

\[
\frac{\partial A}{\partial M} = -0QR^2 \pi u^*(c_i)v^*(B^u)v^*(B^f) + 0QR^2 \pi(1 - \pi)u^*(c_0)u^*(c_i)v^*(B^u)\left\{ \frac{(1 - QR)}{QR(1 - \pi)} \frac{v^*(B^u)}{v(B^f)} \right\},
\]

which can be rewritten by use of (A27) as

\[
\frac{\partial A}{\partial M} = -0QR^2 \pi u^*(c_i)v^*(B^u)v^*(B^f) + \frac{\partial E_1}{\partial M}.
\]

(A29)

From (A29) it is straightforward to see that \( \partial A / \partial M > 0 \), if \( \partial E_i / \partial M \gtrsim 0 \). Otherwise the effect is ambiguous. Next we calculate \( \partial A / \partial w \) by use of (A3) – (A8) and (A16) – (A18), i.e.

\[
\frac{\partial A}{\partial w} = 0\pi \left[ QR^2 \pi(1 - \tau)(1 - \pi)u^*(c_0)u^*(c_i)v^*(B^u)\left\{ \frac{(1 - QR)}{QR(1 - \pi)} \frac{v^*(B^u)}{v(B^f)} \right\} - QR^2 \pi(1 - \tau)(1 - \pi)u^*(c_0)v^*(B^u)v^*(B^f) + \pi u^*(c_i)v^*(B^f)\frac{\partial S}{\partial w}\left(1 - QR\right)u^*(c_0) + R^2 v^*(B^u) \right]\]

\]

(A30)

Note that the first term on the RHS of (A30) is positive (zero, negative, resp.), if (A28), i.e. \( \partial E_i / \partial w \), is positive (zero, negative, resp.). By use of this result, inspection of (A30) shows: If \( \partial S / \partial w \lesssim 0 \) and \( \partial E_i / \partial M \gtrsim 0 \), then \( \partial A / \partial w > 0 \). Otherwise, effect is ambiguous. Obviously,
the effect can reverse, i.e. $\partial A/\partial w$ may be non-positive, in case that $\partial S/\partial w > 0$ and $\partial E_\tau/\partial M < 0$.

Substituting (A3) – (A7) and (A20) – (A22) into (A8) for $X = \pi$ yields

$$
\frac{\partial A}{\partial \pi} = 0\pi\left[\begin{array}{c}
1 - QR\end{array}\right]u'(c_0)u'(c_1)\left[\begin{array}{c}
\nu'(B^1)\end{array}\right] + R^2\left(1 - \pi\right)u'(c_1)\nu'(B^1)\left[\begin{array}{c}
\nu'(B^1)\end{array}\right] + R\nu'(B^1)\left[\begin{array}{c}
u'(c_0)u'(c_1)\nu'(B^1) + Qu'(c_0)u'(c_1)\nu'(B^1) + R\pi u'(c_1)\nu'(B^1)\end{array}\right],
$$

(A31)

which is positive.

Q.E.D.

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