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On Ramsey's conjecture: Efficient allocations in the neoclassical growth model with private information

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Reihe Ökonomie
Economics Series

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May 2004

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Founded in 1963 by two prominent Austrians living in exile – the sociologist Paul F. Lazarsfeld and the economist Oskar Morgenstern – with the financial support from the Ford Foundation, the Austrian Federal Ministry of Education and the City of Vienna, the Institute for Advanced Studies (IHS) is the first institution for postgraduate education and research in economics and the social sciences in Austria. The **Economics Series** presents research done at the Department of Economics and Finance and aims to share “work in progress” in a timely way before formal publication. As usual, authors bear full responsibility for the content of their contributions.

Das Institut für Höhere Studien (IHS) wurde im Jahr 1963 von zwei prominenten Exilösterreichern – dem Soziologen Paul F. Lazarsfeld und dem Ökonomen Oskar Morgenstern – mit Hilfe der Ford-Stiftung, des Österreichischen Bundesministeriums für Unterricht und der Stadt Wien gegründet und ist somit die erste nachuniversitäre Lehr- und Forschungsstätte für die Sozial- und Wirtschaftswissenschaften in Österreich. Die **Reihe Ökonomie** bietet Einblick in die Forschungsarbeit der Abteilung für Ökonomie und Finanzwirtschaft und verfolgt das Ziel, abteilungsinterne Diskussionsbeiträge einer breiteren fachinternen Öffentlichkeit zugänglich zu machen. Die inhaltliche Verantwortung für die veröffentlichten Beiträge liegt bei den Autoren und Autorinnen.

Abstract

In his seminal paper of 1928, Ramsey conjectured that if agents discounted the future differently, in the long run all agents except the most patient would live at the subsistence level. The validity of this conjecture was investigated in different environments. In particular, it has been confirmed in the neoclassical growth model with dynamically complete markets. This paper studies this conjecture in a version of this model that includes private information and heterogeneous agents. A version of Bayesian Implementation is introduced and a recursive formulation of the original allocation problem is established. Efficient allocations are renegotiation-proof and the expected utility of any agent cannot go to zero with positive probability if the economy does not collapse. If the economy collapses all agents will get zero consumption forever. Thus, including any degree of private information in the neoclassical growth model will deny Ramsey's conjecture, if efficient allocations are considered.

Keywords

Dynamic contracts, capital accumulation, private information

JEL Classification

C610, D82, D90, D610, D310

Comments

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1 Introduction

Will modern societies display extreme levels of wealth concentration in the long-run if people discount the future differently? Ramsey [21] conjectured an affirmative answer to this question.¹ These extreme levels of concentration would be the natural outcome to the rational behavior of agents, as long as the market structure allowed for enough consumption smoothing through both time and states of nature. Thus, all consumers except the most patient would live at the subsistence level. The validity of this conjecture has been investigated in different environments. In fact, it turns out that in the standard stochastic neoclassical growth model with the limit assumption of dynamically complete markets, this conjecture is impressively accurate. That is, given an equilibrium interest rate sequence, impatient consumers will trade away their distant future wealth to consume as much as possible in the relatively near present.²

The literature has also been trying to analyze this conjecture under more realistic assumptions regarding the market structure. Would this conjecture be valid if some markets are missing? Important contributions have been made in this dimension as well. In general, these attempts have arbitrarily closed some markets to analyze how these predictions would change.³ Considering borrowing constraints, Becker [6] shows that Ramsey's conjecture holds for the stationary equilibria of one-sector economies.⁴

Becker and Zilcha [8] study the stochastic version of Becker [6] with similar credit

¹ “[E]quilibrium would be attained by a division into two classes, the thrifty enjoying bliss and the improvident at the subsistence level”. Ramsey [21], page 559.

² There are, however, variants of the neoclassical growth model in which the long-run distribution of wealth can be non-degenerate. Lucas & Stokey [17] show that if preferences are represented by recursive utility functionals satisfying certain assumptions, there exist stationary equilibria in which all households have positive wealth in the limit.

³ Of course, predictions will change in some other important dimensions. For example, Aiyagari [2] extends the standard neoclassical growth model to include uninsured idiosyncratic risk and borrowing constraints in an economy populated by a continuum of ex-ante identical agents. Compared with the complete markets economy, he shows that agents overaccumulate capital in order to smooth consumption in the face of idiosyncratic risk. See also Hugget [14] for a related result.

⁴ Becker & Foias [7] found sufficient conditions such that the equilibrium converges to the steady state.

market imperfections. They show that there exist stochastic stationary equilibria where Ramsey’s conjecture is not longer valid. See Ghigliino ([12], Section 4) for an excellent discussion of this literature.

But there is an another important issue involved here. Suppose that some markets are not present: can extreme levels of wealth concentration be still **efficient**? In general, in economies with incomplete markets it is possible to check if the resulting allocation satisfies some efficiency criteria. But after all, why are markets incomplete? One of the standard arguments to justify different incomplete market structures is the fact that there are informational frictions and therefore some markets will not be present (see, for example, Arrow [3]). Fundamental contributions were also made regarding this issue in a different branch of the literature. Consider first an endowment economy populated with a large number of ex-ante identical agents that are subject to privately observed idiosyncratic shocks every period. In this environment, Atkeson & Lucas [5] and Green [13] have shown that (constrained) efficient allocations, independently of the feasibility technologies, will display extreme levels of “immiserization”: the expected utility level of (almost) every agent in the economy converges to the lower bound with probability one. This result is also present in Thomas & Worrall [25]. Wang [27] shows that these results might not be robust to the assumption of considering a finite number of ex-ante identical agents.⁵

There were some extensions to this literature to allow for capital accumulation. Marcet & Marimon [18] study the stochastic growth model with incentive constraints. They characterize constrained efficient allocations in an economy with a risk-neutral principal and a risk-averse agent where investment is unobservable. In this partial equilibrium framework, they find that information constraints affect consumption

⁵However, Phelan [19] has shown that Wang’s results depends critically upon the assumptions on the utility function.

volatility while the Pareto efficient capital accumulation path can still be decentralized. In an important application of the techniques developed by Abreu, Pearce & Sttatcheti [1]’s seminal contribution, Atkeson [4] examines constrained efficient allocations between a risk-averse borrower and a sequence of risk-neutral lenders in an economy with both incomplete enforceability and moral hazard. In particular, investment is unobservable and capital fully depreciates. He shows that capital outflows could be optimal when a low realization of output is observed. In related work, Khan & Ravikumar [15],[16] introduce capital accumulation in Green’s model. There is a continuum of ex-ante identical agents endowed with linear technologies that are subject to privately observed idiosyncratic productivity shocks. For CRRA utility functions, they show that the optimal contract exhibits two-sided voluntary participation and numerical exercises show that both the expected valued and the dispersion of utility entitlements increase through time.

The interaction between these two branches of the literature has not been extensively studied. The purpose of this paper is to fill this gap. The existence of private information provides a basis for market incompleteness. Hence, the main goal of this paper is to characterize the set of efficient allocations in a particular informationally constrained version of the neoclassical growth model with many heterogeneous agents instead of specifying an arbitrary set of markets. That is, as motivated by Townsend [26]’s seminal contribution, I analyze Pareto optimal arrangements “to avoid the imposition of exogenous restrictions and so the nonexecution of some mutually perceived advantageous trade”. I will keep the model as simple as possible. The economy is populated by N heterogeneous, risk-averse, infinitely-lived agents. Each period agents are subject to idiosyncratic preference shocks. These shocks are assumed to be i.i.d. through time and independent across agents. At every date

t , the history of realizations are private information for each agent. Agents are endowed with one unit of time each period but they do not value leisure. A neoclassical technology is available at the aggregate level. Since there is private information, it is well-known since Townsend [26] that the relevant set of (constrained) efficient allocations can be history dependent. Hence, standard recursive methods with finite dimensional state spaces do not apply to characterize optimal allocations. This can be solved extending the set of state variables to include next period’s “expected discounted utility entitlements” as in the seminal contributions of Abreu et. al. [1], Spear & Srivastava [23] and Phelan & Townsend [20].

After introducing a version of Bayesian implementation for this particular dynamic environment, the existence of an efficient recursive formulation of the original allocation problem is established. These results might be of interest on their own. First, the restriction of implementing through Bayesian mechanisms differs from the rest of the literature described, in particular with Wang [27] and Phelan [19]. They introduce, at least implicitly, some hybrid between Dominant-Strategies and Bayesian implementation. There, agents are asked to report truthfully **independently** of the others’ reports today (related to Dominant-Strategies implementation), considering the **expected** discounted utility attained if all the other agents report truthfully from tomorrow on (related to Bayesian implementation). The set of incentive compatible allocations in this paper contains the set of incentive compatible allocations considered in those papers. With a continuum of agents, as in Atkeson & Lucas [5], both concepts collapse since agents are individually very “small” and then their own report cannot affect the entitlement to the others.

Second, I establish a version of the *Principle of Optimality* for this environment: future utilities lie on the frontier of the utility set and therefore ex-ante efficient

contracts are renegotiation-proof. This implies that there will never be an incentive to renegotiate the contract since any renegotiation would make at least one agent worse off. Here, the planner can use the free disposal technology available as a commitment device. This does not mean that any agent individually will not have incentives to renegotiate the contract. However, it is assumed that allocations are fully enforceable. In dynamic contracting, ex-ante Pareto efficient allocations need not be ex-post efficient since agents might find that modifications to the original long-term contract are mutually beneficial as future events unfold. This might happen because the planner needs to promise low future expected utility levels to the agents if some states are reported to provide incentives to truthfully report today. But if this happens and ex-ante efficient contracts are not renegotiation-proof, the assumption that contracts are costlessly enforceable might be stretched to the limit.

Some other important properties of the efficient allocation are investigated. In particular, it is shown that it is impossible for the level of expected utility of any agent to go to zero with positive probability in a non-collapsing economy. On the other hand, if the economy collapses the stock of capital drops to zero and thus **all** agents consume nothing forever. These results provide a remarkably different prediction with respect to a standard result in economies with full information. There, if the marginal utility of consumption goes to infinity when consumption goes to zero, impatient agents will end up consuming nothing in the limit. A novel property of the model presented here is that the introduction of **any** degree of private information (that is, even if probabilities differ from 1 by an arbitrary small number) will imply that this result will no longer hold when considering (constrained) Pareto optimal trading arrangements.

Finally, since risk-sharing is provided without restricting transfers additionally to

incentive compatibility and feasibility (in particular, an agent can receive a negative net transfer from the other agents in the economy), there might be nonexecuted mutually beneficial trade opportunities in a market economy where all insurance markets are arbitrarily closed (as, for example, in Hugget [14] and Becker & Zilcha [8]).

The rest of this paper is organized as follows. Section 2 introduces the original resource allocation problem with private information, and some basic properties are established. Section 3 proves the existence and describes some properties of an efficient allocation. Section 4 concludes. The Appendix contains all the proofs.

2 The Economy

The economy is populated by a finite number of infinitely-lived heterogeneous agents with names in the set $I = \{1, \dots, N\}$ (with typical element n). Time $t = 0, 1, 2, \dots$ is discrete. There is only one consumption good. Each agent is endowed with one unit of time each period. To simplify, let us assume that agents do not value leisure.

Production possibilities are represented by a standard neoclassical production function. That is, if at date t the stock of capital is K_t and the time used to produce is L_t , then the total output is given by $F(K_t, L_t)$. I assume that F is increasing in both arguments and thus, given that agents do not value leisure, $L_t = N$ each period. Let us denote $f(K) = F(K, N)$.

*Assumption 1: $f : R_+ \rightarrow R_+$ is strictly increasing, strictly concave and differentiable. Moreover, there exists $\bar{K} > 0$ such that $f(k) \geq k$ if and only if $k \leq \bar{K}$.*⁶

Each period agent n receives an idiosyncratic preference shock θ_n . I assume that θ_n takes value in the finite set $\Theta = \{\theta_1, \dots, \theta_J\}$ where $\theta_i > \theta_j$ if $i > j$. Preference

⁶Note that this representation is general and it may include capital depreciation.

shocks are assumed to be independent across agents and i.i.d. across time for each agent. That is, let $\pi_{nt}(\theta_j) = \pi_n(\theta_j) > 0$ be the probability at date t for agent n of having a preference shock θ_j . Since the number of agents is finite, there is aggregate uncertainty. Let $\theta = (\theta_1, \dots, \theta_N) \in \Theta^N$ denote the aggregate preference shock with probability $\pi(\theta) = \prod_{n \in N} \pi_n(\theta_n)$. Let μ^{t+1} be the probability distribution on the measurable space $(2^{\Theta^{N(t+1)}}, \Theta^{N(t+1)})$ induced by π . That is, $\mu^{t+1}(\theta^t)$ is the probability of the aggregate partial history up to date t , $\theta^t = (\theta_0, \dots, \theta_t) \in \Theta^{N(t+1)}$.

Let S denote the consumption set, which is defined in the following way:

$$S = \{ \{C_t\}_{t=0}^{\infty} : C_t : \Theta^{N(t+1)} \rightarrow R_+ \text{ and } \sup_{t, \theta^t} \{C_t(\theta^t)\} < \infty \}$$

Preferences over S are represented by a time-separable expected discounted utility function $U : S \rightarrow R_+$. More precisely, if $\mathbf{c} \in S$

$$U(\mathbf{c}) = E \left\{ \sum_{t=0}^{\infty} \beta_n^t \theta_{nt} u_n(C_t) \right\}$$

It is assumed that for all $n \in I$, $\beta_n \in (0, 1)$ and $u_n : R_+ \rightarrow R$ is strictly increasing, strictly concave and twice differentiable, where idiosyncratic preference shocks are multiplicative. Assume also that $\lim_{c_t \rightarrow 0} u'_n(c_t) = +\infty$ for all $n \in I$. Without loss of generality, assume $u_n(0) = 0$ and $\sum_{\theta} \pi_n(\theta) \theta = 1$ for all n . E represents the expectation operator.

Feasible Incentive Compatibility Allocations

Since it is assumed that θ_{nt} and θ_n^t are private information, agents will be asked to report their own preference shocks. I will assume that there is no way to audit or verify the answer that any agent chooses to give. I also assume that allocations are fully enforceable.

Given a privately observed partial history θ_n^t up to date t by agent n , he chooses to report $z_{nt}(\theta_n^t) \in \Theta$ at date t . Let $z_n = \{z_{nt}(\theta_n^t)\}_{t=0}^{\infty}$ represent agent n 's sequence of

reporting strategies where $z_{nt} : \Theta^{t+1} \rightarrow \Theta$ for all t . Denote $z = (z_1, \dots, z_N) = (z_n, z_{-n})$ the sequence of aggregate reporting strategies. Let z_n^* be the truthtelling reporting strategy for agent n where $z_{nt}^*(\theta_n^t) = \theta_{nt}$ for all t and $\theta_n^t \in \Theta^{t+1}$. Note that since each individual only observes his own preference shock, agent n 's reporting strategy depends only upon his own partial history.

Let $K' = \{K_{t+1}\}_{t=0}^\infty$ be an **investment rule** where $K_{t+1} : \Theta^{N(t+1)} \rightarrow R_+$ for all t . Similarly, let $C = \{C_t\}_{t=0}^\infty$ be a **consumption transfer** where $C_t : \Theta^{N(t+1)} \rightarrow R_+^N$ for all t . To interpret this, consider any aggregate realization θ^t up to date t and any aggregate reporting strategy z . Consumption for each agent n is given by $C_{nt}(z^t(\theta^t)) \geq 0$. Similarly, the stock of capital at period $t + 1$ will be given by $K_{t+1}(z^t(\theta^t)) \geq 0$. Any pair (C, K') satisfying these properties is called an **allocation**.

Definition 1 *Given k , an allocation (C, K') is **feasible** if for all aggregate reporting strategies z*

$$K_{t+1}(z^t(\theta^t)) + \sum_{n \in N} C_{nt}(z^t(\theta^t)) \leq f(K_t(z^{t-1}(\theta^{t-1}))) \quad (1)$$

for all t , all θ^t and $k_0 = k$.

The levels of capital will be also restricted to those levels that are sustainable. Suppose that $K_t(\theta^{t-1}) \leq \bar{K}$. Since consumption must be nonnegative, from feasibility and the definition of an allocation we have that for all t and for all reports θ^t

$$0 \leq K_{t+1}(\theta^t) \leq f(K_t(\theta^{t-1})) \leq \bar{K}$$

Denote $X \equiv [0, \bar{K}]$ as the set of sustainable capital levels. It will be assumed that $k_0 \in X$ and therefore any feasible allocation will necessarily satisfy that $K_{t+1}(\theta^t) \in X$ for all t and all θ^t .

Suppose that some arbitrary aggregate partial history θ^{t-1} has been reported. Let z' be an aggregate continuation reporting strategy from period t onwards. Given an

allocation (C, K') , define the level of expected discounted utility entitled to agent n at date t for each z' as follows:

$$\begin{aligned} U_{nt}(C, K', z' || \theta^{t-1}) &= \sum_{s=0}^{\infty} \beta_n^s \left\{ \int_{\Theta^{N(s+1)}} \theta_{ns} u_n(C_{nt}(\theta^{t-1}, z^s(\theta^s))) \right\} \mu^{t+s+1}(\theta^{t-1}, \theta^{s+1}) \\ &= \sum_{\theta \in \Theta^N} \pi(\theta) \{ \theta_n u_n(C_{nt}(\theta^{t-1}, z_0(\theta))) \\ &\quad + \beta_n U_{nt+1}(C, K', z'(\theta) || \theta^{t-1}, z_0(\theta)) \} \end{aligned}$$

Here $z'(\theta)$ is the continuation reporting strategy from period $t + 1$ onwards induced by z' when the first element θ is kept constant. When $t = 0$, we write for any z

$$U_n(C, K', z) = U_{n0}(C, K', z)$$

The following definition says that an allocation is *incentive compatible* if truthtelling is the best response for each agent whenever he considers that the other agents will truthfully report their own preference shocks not only today but also in the future.

Definition 2 *Given $k_0 \in X$, an allocation (C, K') is incentive compatible if for all agents, for all $t \geq 0$, all θ^{t-1} and all z'_n*

$$\begin{aligned} &\sum_{\theta_{-n}} \pi(\theta_{-n}) [\theta_n u_n(C_{nt}(\theta^{t-1}, \theta_n, \theta_{-n})) + \beta_n U_{nt+1}(C, K', z_n^*, z_{-n}^* || \theta^{t-1}, \theta_n, \theta_{-n})] \quad (2) \\ &\geq \sum_{\theta_{-n}} \pi(\theta_{-n}) [\theta_n u_n(C_{nt}(\theta^{t-1}, \tilde{\theta}_n, \theta_{-n})) + \beta_n U_{nt+1}(C, K', z'_n, z_{-n}^* || \theta^{t-1}, \tilde{\theta}_n, \theta_{-n})] \end{aligned}$$

for all $\tilde{\theta}_n$ and θ_n .

This can be interpreted as the natural extension of Bayesian implementation for this particular dynamic environment, which differs from the related literature as mentioned in the Introduction. Note that Definition 2 takes into account that agents can choose a continuation reporting strategy every period after they have observed their own preference shock histories. The restriction of analyzing incentive

compatible allocations is without loss of generality since it can be shown that the relevant version of the celebrated Revelation Principle holds.⁷ Roughly speaking, if there is any way in which some insurance can be provided through any allocation then there is an equivalent incentive compatible way in which agents report their true preference shocks.

The economy can be interpreted as both an N agent version of the economy studied by Atkeson & Lucas [5] with capital accumulation and a private information version of the model studied in, among many others, Becker [6] and Becker & Zilcha [8].

The notion of efficiency that will be analyzed throughout the paper can now be defined. Note that in this definition we are already using the fact that the allocation must be incentive compatible.

Definition 3 *An allocation (C^*, K'^*) is **efficient** at $(k_0, \{u_n\}_{n=2}^N)$ if*

$$(C^*, K'^*) \in \arg \max_{(C, K')} \{U_1(C, K', z^*) : (C, K') \text{ satisfies (1)-(2) and } U_n(C, K', z^*) = u_n \text{ for all } n = 2, \dots, N\}$$

Let $\Psi(k)$ be the utility possibility set for this economy when $k \in X$ is the initial stock of capital. That is,

$$\Psi(k) \equiv \{u \in R^N : \exists (C, K') \text{ satisfying (1)-(2) and } u_n = U_n(C, K', z^*) \forall n, k_0 = k\}$$

It is important to mention that if an efficient allocation exists, then it will also be (constrained) Pareto optimal. This follows from the fact that the utility frontier will in fact be strictly decreasing in this environment.

This correspondence Ψ , mapping X into R^N , has some properties that can be established immediately.

⁷More precisely, it is well-known that the revelation principle holds for any time horizon and any stochastic structure.

Remark 1 For all $k \in X$, $\Psi(k)$ is **nonempty**. Consider the allocation

$$\begin{aligned} C_{n0}(\theta) &= \frac{f(k)}{N} \text{ and } K_1(\theta) = 0 \text{ for all } \theta, n \\ C_{nt}(\theta^t) &= 0 \text{ and } K_{t+1}(\theta^t) = 0 \text{ for all } t \geq 1, n \text{ and } \theta^t \end{aligned}$$

If we define $u_n = \sum_{\theta_n \in \Theta} \pi_n(\theta_n) \theta_n u_n(\frac{f(k)}{N})$, then clearly $(u_n)_{n \in N} \in \Psi(k)$.

Remark 2 It is **uniformly bounded**. That is, for all $k \in X$ there exists a bounded subset of R^N , say H , such that $\Psi(k) \subset H$. To see this, note that for all agent n , $0 \leq c_{nt} \leq f(k_t) \leq \bar{K}$. Therefore, for any $k \in X$ if $u \in \Psi(k)$, then $0 \leq u_n \leq \frac{u_n(\bar{K})}{1-\beta_n}$.

Remark 3 There exists $u_1 > 0$ such that $u = (u_1, 0, \dots, 0) \in \Psi(k)$ for all $k > 0$. To see this, consider the following allocation: given $k > 0$, define for all for all $t \geq 0$ and all θ^t

$$\begin{aligned} C_{10}(\theta) &= f(k) \\ C_{nt}(\theta^t) &= 0 \text{ for all } n \geq 2 \\ K_{t+1}(\theta^t) &= 0 \text{ for } t \end{aligned}$$

Note that (C, K') is an incentive compatible, feasible allocation given the definitions of an allocation, reporting strategies and feasibility. Note also that $U_n(C, K') = 0$ for all $n \geq 2$ and $U_1(C, K') > 0$ given that $f(k) > 0$ for all $k > 0$.

The following Lemma will be useful to establish some results “in the limit”. This result allows the restriction to one-period deviation when one considers incentive compatible allocations.⁸

Lemma 1 Let (C, K') be any feasible allocation at $k \in X$. Consider any agent $n \in I$ and let z'_n, z_n^m be continuation reporting strategies where $z_{ns}^m = z_{ns}$ for all $s \leq m$ and

⁸See, for example, Atkeson & Lucas [5] for a related result.

$z_{ns}^m = z_{ns}^*$ thereafter. Then, for all $t \geq 0$ and any aggregate report θ^{t-1}

$$\limsup_{m \rightarrow \infty} [U_{nt}(C, K', z_n^m, z_{-n}^* | \theta^{t-1})] = U_{nt}(C, K', z_n^*, z_{-n}^* | \theta^{t-1})$$

2.1 The Full Information Case

I will briefly discuss Ramsey's conjecture in the model described above with full information. That is, I will consider the allocation problem just described without the incentive compatibility constraints. Since it is easy to establish that both the First and Second Welfare Theorems hold, the property described below will also hold in a economy with dynamically complete markets.

Consider the following problem that the planner has to solve:

$$\max_{(B, K')} \sum_{t=0}^{\infty} \beta_1^t \left\{ \int_{\Theta^{N(t+1)}} \theta_{1t} u_1(C_{1t}(\theta^t)) \mu^{t+1}(\theta^t) \right\}$$

subject to (1) and for all n

$$\sum_{t=0}^{\infty} \beta_n^t \left\{ \int_{\Theta^{N(t+1)}} \theta_{nt} u_n(C_{nt}(\theta^t)) \mu^{t+1}(\theta^t) \right\} = u_n$$

Note that nonnegativity of consumption is implicit in the definition of an allocation.

Suppose that $\beta_1 > \beta_n$ for all $n \neq 1$. Necessary first order conditions for the unique interior solution will imply that for all $n \in I$, for all t and for all θ^t

$$\alpha_n \beta_n^t \theta_{nt} u_n'(C_{nt}(\theta^t)) \mu^{t+1}(\theta^t) = \lambda_t(\theta^t)$$

Here, $\alpha_n > 0$ is the Lagrange multiplier corresponding to agent n (with $\alpha_1 = 1$) and $\lambda_t(\theta^t) > 0$ is the Lagrange multiplier corresponding to the feasibility constraint at period t if θ^t is the aggregate partial history. Thus, for all t and for all θ^t

$$\frac{\beta_1^t u_1'(C_{1t}(\theta^t))}{\alpha_n \beta_n^t u_n'(C_{nt}(\theta^t))} = 1 \tag{3}$$

Since $(\beta_1/\beta_n)^t \rightarrow +\infty$ as t goes to infinity, it follows from (3) that

$$\lim_{t \rightarrow \infty} [u_1'(C_{1t}(\theta^t))/u_n'(C_{nt}(\theta^t))] = 0$$

Given that consumption is uniformly bounded from above, it is clear then that $c_{nt}(\theta^t) \rightarrow 0$ **for all** $\{\theta_t\}_{t=0}^{\infty}$. Therefore, only the patient agent consumes in the limit. Impatient agents would want to trade future wealth for present consumption and therefore these individuals would have zero consumption **asymptotically**. These extreme levels of concentration were initially conjectured by Ramsey [21]. The validity of this conjecture for different environments was discussed in the Introduction. In what follows, one of the main results is that with the introduction of **any** degree of private information this result no longer holds.

3 Characterization

In this section I will first characterize the utility possibility correspondence defined by Ψ . After that, I will study some important properties of an efficient allocation.

Let $W : X \rightarrow R^N$ be a nonempty, uniformly bounded correspondence. Let (c, k') be a vector-valued function where $c : \Theta^N \rightarrow R_+^N$ and $k' : \Theta^N \rightarrow X$. Given any two functions (c, k') , we say that the function $w : \Theta^N \rightarrow R^N$ is a **continuation value function** with respect to W if $w(\theta) \in W(k'(\theta))$ for all $\theta \in \Theta^N$. Call (c, k', w) a **recursive allocation**.

Definition 4 *Given a correspondence W as before, a recursive allocation (c, k', w) is admissible with respect to W at $k \in X$ if*

(1) *w is a continuation value function with respect to W ;*

(2) *(c, k', w) satisfies*

(2.a) *For all $\theta \in \Theta^N$, $k'(\theta) + \sum_n c_n(\theta) \leq f(k)$*

(2.b) *Temporary Incentive Compatibility (t.i.c.):*

For all $n \in I$ and for all $\theta_n, \tilde{\theta}_n \in \Theta$

$$\begin{aligned} & \sum_{\theta_{-n}} \pi(\theta_{-n}) \{ \theta_n u_n(c_n(\theta_n, \theta_{-n})) + \beta_n w_n(\theta_n, \theta_{-n}) \} \\ & \geq \sum_{\theta_{-n}} \pi(\theta_{-n}) \{ \theta_n u_n(c_n(\tilde{\theta}_n, \theta_{-n})) + \beta_n w_n(\tilde{\theta}_n, \theta_{-n}) \} \end{aligned}$$

Let (c, k', w) be admissible with respect to W at $k \in X$ and define for all $n \in I$

$$e_n(c, k', w) = \sum_{\theta \in \Theta^N} \pi(\theta) \{ \theta_n u_n(c_n(\theta)) + \beta_n w_n(\theta) \}$$

Given $k \in X$, define the following operator:

$$\Phi(W)(k) \equiv \{ (e_n)_{n \in N} \in R^N : \exists (b, k', w) \text{ admissible w.r.t. } W \text{ at } k, e_n(b, k', w) = e_n \}$$

This operator maps the set of uniformly bounded correspondences into themselves.

The following definition extends to correspondences some definitions given by Abreu *et. al.* [1] for sets.⁹

Definition 5 A correspondence $W : X \rightarrow R^N$ is self-generating if it is nonempty and $W(k) \subset \Phi(W)(k)$ for all $k \in X$.

Proposition 2 Let W be a uniformly bounded and self-generating correspondence. Then, for all $k \in X$

$$\Phi(W)(k) \subset \Psi(k)$$

The intuition for this result can be interpreted as follows. If a correspondence W is self-generating, any value in $W(k)$ is also in its image $\Phi(W)(k)$. This allows one to choose any vector of utility levels of $\Phi(W)(k)$ and transform it period-by period recursively into a feasible incentive compatible allocation having the same utility levels.

The next result establishes that Ψ is self-generating itself and therefore is a fixed point of the operator Φ .

⁹A related extension was made by Atkeson [4].

Proposition 3 For all $k \in X$, $\Psi(k) = \Phi(\Psi)(k)$.

Thus, any utility level that can be attained with an allocation (C, K') can also be attained by delivering to each agent consumption for today, assigning capital for tomorrow and promising to each agent some contingent levels of expected utility from tomorrow on.

This characterization of the utility possibility correspondence turns out to be extremely important both to establish the existence of an efficient allocation and to investigate some of its properties. Below, I will discuss sufficient conditions such that a version of the *Principle of the Optimality* holds and thus the original problem can be restated as one genuinely recursive.

3.1 Existence of an Efficient Allocation

I will proceed to show that for every $k \in X$, there exists an efficient allocation as defined before. To do that, a few properties of the operator Φ need to be shown. Define the graph of a correspondence $W : X \rightarrow R^N$ by the following set:

$$\text{graph}(W) = \{(w, k) \in R^N \times X : w \in W(k)\}$$

The next Lemma shows that the operator Φ preserves compactness.

Lemma 4 If $\text{graph}(W)$ is compact, then $\text{graph}(\Phi(W))$ is also compact.

Observe that if $\text{graph}(W_1) \subset \text{graph}(W_2)$, then $\text{graph}(\Phi(W_1)) \subset \text{graph}(\Phi(W_2))$. This follows directly from the definition of the operator Φ . The last result we need to show the existence of an efficient allocation is the following.

Lemma 5 Ψ has a compact graph.

Given that Ψ has a compact graph, it follows that for all $k \in X$, $\Psi(k)$ is a compact subset of R^N . We need to introduce some notation. Let $\Psi_{-1}(k) \equiv \{u_{-1} = (u_n)_{n=2}^N \in R^{N-1} : \exists u_1 \text{ where } (u_1, u_{-1}) \in \Psi(k)\}$.

For any $k \in X$ and given $u_{-1} \in \Psi_{-1}(k)$, define $\Psi_1(k, u_{-1}) \equiv \{u_1 \in R : (u_1, u_{-1}) \in \Psi(k)\}$. It is clear that for all $k \in X$ and given $u_{-1} \in \Psi_{-1}(k)$, $\Psi_1(k, u_{-1})$ is a compact subset of R . Given k and u_{-1} , define the following function:

$$V(k, u_{-1}) = \max\{u_1 \in \Psi_1(k, u_{-1})\}$$

Therefore, it follows that for all $k \in X$ and given $u_{-1} \in \Psi_{-1}(k)$, there exists (C^*, K'^*) such that

$$V(k, u_{-1}) = U_1(C^*, K'^*, z^*) \quad \text{and} \quad u_{-1} = U_{-1}(C^*, K'^*, z^*)$$

which is, by definition, an efficient allocation at (k, u_{-1}) . It also follows by Proposition 3 that there exists an equivalent **efficient recursive allocation** (c^*, k'^*, w^*) (which is admissible with respect to Ψ at k) such that

$$V(k, u_{-1}) = e_1(c^*, k'^*, w^*) \quad \text{and} \quad u_{-1} = e_{-1}(c^*, k'^*, w^*) \quad (4)$$

It will be said that a recursive allocation (c, k', w) is **promise keeping at** u_{-1} if $u_n = e_n(c, k', w)$ for all $n \in \{2, \dots, N\}$.

The next property of the correspondence Ψ will be crucial to show the continuity of V .

Lemma 6 Ψ has a convex graph.

Hence, since Ψ is a compact-valued correspondence (Lemma 5) with a convex graph (Lemma 6), it follows that Ψ is continuous and compact-valued.¹⁰

¹⁰See Stokey, Lucas & Prescott [24], Theorem 3.4 & 3.5.

3.2 Some Properties of an Efficient Allocation

Some important properties of an efficient allocation (or its equivalent recursive representation) can now be investigated. Lemma 7 below shows that efficient trading arrangements will imply multiperiod relationships as in Townsend [26]. The nature of these multiperiod relationships comes from the incentive compatibility constraints to circumvent information difficulties. Efficient allocations are thus history dependent: each agent's report today affects not only his present consumption but also his consumption from tomorrow on. Given that shocks are i.i.d., this would not be the case with full information.

Lemma 7 *Let (c, k', w) be admissible with respect to Ψ at (k, u_{-1}) . For all $n \in I$, if $\theta_n > \tilde{\theta}_n$, then for all $n \in I$*

$$\begin{aligned} \sum_{\theta_{-n}} \pi(\theta_{-n}) u_n(c_n(\theta_n, \theta_{-n})) &\geq \sum_{\theta_{-n}} \pi(\theta_{-n}) u_n(c_n(\tilde{\theta}_n, \theta_{-n})) \\ \sum_{\theta_{-n}} \pi(\theta_{-n}) w_n(\theta_n, \theta_{-n}) &\leq \sum_{\theta_{-n}} \pi(\theta_{-n}) w_n(\tilde{\theta}_n, \theta_{-n}) \end{aligned}$$

The first of these inequalities has an implication that additionally distinguishes this environment from the rest of the literature. Either with a continuum of agents or with a finite number of agents with the implementation device as in Wang [27] and Phelan [19], consumption is increasing in the preference shock. That is, the more the agent values consumption today, the higher the transfer today, **independently of the others' reports**. Here, an agent will receive a random vector of consumption depending upon others' reports, which is increasing with respect to his preference shock in the sense of second order stochastic dominance. More precisely, the induced distribution of $c_n(\theta_j, \cdot)$ *second order stochastically dominates* the induced distribution of $c_n(\theta_i, \cdot)$ whenever $j > i$. Therefore, the conditional expectation of the momentary utility will be increasing in the reported preference shock. On the other hand, agents

reporting relatively lower preference shocks are rewarded with a relatively higher level of **conditional** expected utility from tomorrow on.

A further characterization of V is essential to show the next results. In particular, the potential nonconvexity imposed by the incentive compatibility constraints might make it difficult to apply standard arguments to show the continuity of V . In this sense, the fact that the preference shocks are multiplicative will play an important role in simplifying the analysis. The main properties are summarized in the following Lemma.

Lemma 8 *(i) V is strictly increasing in k and strictly decreasing in u_{-1} . (ii) V is a continuous function.*

The next result shows that some of the conditions in Lemma 7 will hold with strict inequality. Thus, Proposition 9 basically establishes that an allocation providing no insurance for some agent cannot be efficient.

Proposition 9 *Given any $k \in X$ and $u_{-1} \in \Psi_{-1}(k)$, consider an arbitrary recursive allocation $(\bar{c}, \bar{k}', \bar{w})$ admissible with respect to Ψ at k and promise keeping at u_{-1} . Consider any agent such that $u_n > 0$ and suppose that for all $\theta_n \in \Theta$*

$$\begin{aligned} \sum_{\theta_{-n}} \pi(\theta_{-n}) u(\bar{c}_n(\theta_n, \theta_{-n})) &= \bar{u}_n \\ \sum_{\theta_{-n}} \pi(\theta_{-n}) \bar{w}_n(\theta_n, \theta_{-n}) &= \bar{w}_n \end{aligned}$$

Then, $(\bar{c}, \bar{k}', \bar{w})$ cannot be efficient at (k, u_{-1}) .

The intuition of this result can be grasped as follows. Suppose that an agent's report does not affect his conditional expected utility from tomorrow on. The incentive compatibility constraints will imply that the conditional expectation of the

momentary utility will also be constant. This can be dominated by an allocation providing more consumption (in the sense discussed above) to agents with relatively high preference shocks and thus providing them some insurance.¹¹

The next result shows that the ex-ante efficient allocation is in fact renegotiation-proof. This is very important because otherwise **ex-post** mutually beneficial renegotiations must be assumed away.

Proposition 10 *Given any (k, u_{-1}) , if $(c^*, k'^*, w^*)(k, u_{-1})$ is an efficient recursive allocation, then it can be constructed such that $w_1^*(\theta) = V(k'^*(\theta), w_{-1}^*(\theta))$ for all θ .*

This result means that the ex-ante efficient allocation will be ex-post efficient.¹² That is, the continuation utility levels delivered by any efficient allocation will be on the utility possibility frontier. Hence, it will never be **mutually agreed** to renegotiate transfers implied by an efficient allocation. This does not mean that any agent **individually** will not have incentives to renegotiate the contract. However, it has been assumed that allocations are enforceable at the individual level. It is important to note that to get this result it is crucial that the planner can use the free disposal technology available as a commitment device. Here, the ability of the planner to manipulate the stock of capital is crucial to make “credible” that there will not be ex-post incentives to renegotiate the continuation of the original contract. In the Conclusion there is an additional discussion about the possibility of collapsing economies.

From now on, we say that the economy *collapses* if there exists some (k, u_{-1}) such that $k'^*(\theta)(k, u_{-1}) = 0$ for some θ . Note that when the economy collapses, no

¹¹Of course, if $u_n = 0$ then $\bar{u}_n = \bar{w}_n = 0$ is part of any solution.

¹²See Fudenberg, Holmstrom & Milgrom [11] and Wang [28] for related results. Also, a similar result is present in Wang [27] (Proposition 2, pg.582) but the proof presented there seems to be at least incomplete. More specifically, when showing this result, it does not consider the fact that the operator is not monotone and then his condition (9) is not necessarily satisfied by the recursive allocation being considered in the last part of the proof.

agent will consume from next period on. Very importantly, this is independent of any their particular characteristics, including their discount factors. But, in which situation might the economy collapse? While trying to provide incentives to report preference shocks truthfully, the planner might choose to “penalize” an aggregate report $(\bar{\theta}, \dots, \bar{\theta})$ by transferring higher levels of consumption today at the expense of low levels of consumption from tomorrow on. The extreme case would be zero consumption forever for all agents.

Now, I will show that in an economy not collapsing in the limit, no agent’s expected utility can converge to any number with positive probability. In particular, it cannot converge to the lower bound as initially conjectured by Ramsey [21].

Let $\{U_{nt}\}_{t=0}^{\infty}$ be the stochastic process representing agent n ’s expected utility entitlement given an efficient allocation. Call $\Omega = \{\{\theta_t\}_{t=0}^{\infty} : \theta_t \in \Theta^N \text{ for all } t\}$ and let $B(\Omega)$ be the Borel σ – *field* of Ω . Let μ be the unique probability measure on $(\Omega, B(\Omega))$ generated by the finite-dimensional distributions (μ^t) (as an application of the Kolmogorov’s Extension Theorem).

Proposition 11 $\mu\{\{\theta_t\}_{t=0}^{\infty} : \lim_{t \rightarrow \infty} K_{t+1}(\theta^t) = k > 0 \text{ and } \lim_{t \rightarrow \infty} U_{nt}(\theta^t) = \bar{U}_n \in \Psi_n(\bar{k}) \text{ for some } n\} = 0$.

Having this result, we can then conclude that the introduction of **any** degree of private information precludes the result described in Section 2.1. for economies with full information. There, the most patient agent consumed all the output in the limit. Why is this not the case if we consider **any** degree of private information? We have already discussed the implication of a collapsing economy. There, no agent consumes at all. If the economy does not collapse, the idea behind Proposition 11 is, roughly speaking, the following. Suppose that the expected utility level of some agent converges with positive probability when some efficient allocation is considered.

This will imply that there will be situations where allocations displaying no long-term relationships among all agents are efficient. But no allocation like that can be efficient (Proposition 9), even though agents are heterogeneous and can differ in their discount factor. The incentive compatibility constraints imply that, for paths with positive probability, any efficient allocation will need to spread out future expected utility according to their reports. If the expected utility level converges (as in Section 2.1), at some point it will be impossible to do that.

4 Conclusion

This paper has studied some properties of efficient allocations in a version of the stochastic neoclassical growth model with many heterogeneous agents and private information. The first step was to prove that the original allocation problem had a recursive formulation in the spirit of Abreu et. al [1] and Spear & Srivastava [23]. This work has also introduced a simple version of Bayesian implementation for this particular dynamic environment and has established that the **ex-ante** efficient allocations are **ex-post** efficient where future utility entitlements lie in the utility possibility frontier. Therefore, long-term efficient contracts are renegotiation-proof in this environment. To get this result it is important that the planner can manipulate the stock of capital for next period to make these allocations sequentially efficient. Then, two main properties of an efficient allocation are additionally discussed.

First, any efficient allocation should provide some insurance to the agents against idiosyncratic preference shocks. Unlike most of the literature considering an arbitrary set of incomplete markets, agents can make transfers contingent upon their own idiosyncratic productivity shocks. The type of analysis developed in this paper avoids the presence of some **ex-ante** mutually beneficial nonexecuted trade opportunities.

Secondly, I have shown that the level of expected discounted utility cannot con-

verge with positive probability to the lower bound in a non-collapsing economy. In a collapsing economy, the stock of capital drops to zero and all agents consume nothing. These results show that a standard property of efficient allocations in economies with full information, initially conjectured by Ramsey [21], does not longer hold when **any** degree of private information is considered. In those economies, and under standard assumptions, the impatient agents will end up consuming nothing in the limit and therefore the level of expected utility converges to the lower bound. The introduction of **any** degree of private information and the imposition of incentive compatibility constraints imply that, for paths with positive probability, any efficient allocation will need to spread out future expected utility according to the reports. If the expected utility level converges, this property cannot be satisfied. On the other hand, if the economy collapses all agents consume zero independently of their degree of patience.

Some potential extensions might be mentioned. In the first place, a natural theoretical extension would be to try to characterize in more detail both the dynamic and the limiting properties of this economy. At the level of generality presented in this paper, this might not be a standard task. However, an important issue must be mentioned. Nothing in this paper has ruled out the case where the economy collapses with probability one. In this case, the results of the paper are still important to answer the main question regarding Ramsey's conjecture, but are not very relevant in most of the other dimensions. Work in progress, however, shows that under certain standard assumptions, **collapsing economies are rare events with zero probability**.¹³

Secondly, an algorithm to compute efficient allocations in this environment might be developed.¹⁴ Numerical results could allow computing, for example, welfare losses

¹³A priori, one cannot consider this possibility completely unlikely. For example, for a dynamic agency problem with capital accumulation, Di Giannatale [10] numerically shows that the principal's stock of capital monotonically decreases over time. I would like to thank an anonymous associate editor for alerting me to this issue. Details are available upon request.

¹⁴Sleet & Yeltekin [22] is an important step to properly tackle this problem.

imposed by the information structure when compared with efficient allocations in economies with full information. Moreover, one could also compare the basic welfare properties of the economy described here with those emerging in economies where different arbitrary market structures are imposed. In general, one of the most relevant unanswered questions can then be summarized as follows: *does private information really matter?*¹⁵ These issues are left for future research.

5 Appendix

Proof of Lemma 1. Consider any $m \geq 0$. Note that since for all $t \geq 0$ and any aggregate report θ^{t-1}

$$\begin{aligned}
& |U_{nt}(C, K', z_n^m, z_{-n}^* \|\theta^{t-1}) - U_{nt}(C, K', z_n', z_{-n}^* \|\theta^{t-1})| \\
&= \beta_n \left| \sum_{\theta \in \Theta^N} \pi(\theta) \{ U_{nt+1}(C, K', z_n^{m-1}(\theta_n), z_{-n}^* \|\theta^{t-1}, z_{n0}(\theta_n), \theta_{-n}) \right. \\
&\quad \left. - U_{nt+1}(C, K', z_n'(\theta_n), z_{-n}^* \|\theta^{t-1}, z_{n0}(\theta_n), \theta_{-n}) \} \right| \\
&\leq \beta_n \sup_{(\theta_n, \theta_{-n}) \in \Theta^N} | U_{nt+1}(C, K', z_n^{m-1}(\theta_n), z_{-n}^* \|\theta^{t-1}, z_{n0}(\theta_n), \theta_{-n}) \\
&\quad - U_{nt+1}(C, K', z_n'(\theta_n), z_{-n}^* \|\theta^{t-1}, z_{n0}(\theta_n), \theta_{-n})
\end{aligned}$$

Using this inequality we can get that

$$\begin{aligned}
& |U_{nt}(C, K', z_n^m, z_{-n}^* \|\theta^{t-1}) - U_{nt}(C, K', z_n', z_{-n}^* \|\theta^{t-1})| \\
&\leq \beta_n^{m+1} \sup_{(\theta_n^m, \theta_{-n}^m) \in \Theta^{N \times m}} | U_{n,t+m+1}(C, K', z_n^*, z_{-n}^* \|\theta^{t-1}, z_{nm}(\theta_n^m), \theta_{-n}^m) \\
&\quad - U_{n,t+m+1}(C, K', z_n'(\theta_n^m), z_{-n}^* \|\theta^{t-1}, z_{nm}(\theta_n^m), \theta_{-n}^m) |
\end{aligned}$$

¹⁵Some contributions have been made in this direction. In particular, Khan & Ravikumar [16] show that the welfare costs and the growth effects of private information are typically small for their AK version of Green [13]'s model. See also Cole & Kocherlakota [9].

Since consumption must be uniformly bounded, the desired result is obtained taking $\limsup_{m \rightarrow \infty}$ in the previous expression ($\beta_n \in (0, 1) \forall n$). ■

Proof of Proposition 2. Let $w_0 \in \Phi(W, k)$ for some given $k \in X$. We need to show that there exists a feasible and incentive compatible allocation (C, K') such that for $n \in I$

$$U_n(C, K', z^*) = w_{n0}$$

Step 1. Since $w_0 \in \Phi(W, k)$, there exists $(c, k', w)(w_0)$ admissible with respect to W at k such that $e_n(b, k', w)(w_0) = w_0$. Then, it follows that for all $\theta \in \Theta^N$ $w(\theta)(w_0) \in W(k'(\theta)) \subset \Phi(W, k'(\theta))$ since W is assumed to be self-generating. It is then clear that we can recursively define, for all $t \geq 0$, for all θ^t and given w_0 ,

$$\begin{aligned} k_{t+1}(\theta^t) &= k'(\theta_t)(W_t(\theta^{t-1})) \\ C_t(\theta^t) &= c(\theta_t)(W_t(\theta^{t-1})) \\ W_{t+1}(\theta^t) &= w(\theta_t)(W_t(\theta^{t-1})) \end{aligned}$$

Note that in the construction of this candidate allocation (C, K') , we are only considering truthtelling continuation reporting strategies. We claim now that for all $n \in I$, for all $t \geq 0$ and for all θ^{t-1}

$$W_{nt}(\theta^{t-1}) = U_{nt}(C, K', z^* || \theta^{t-1}) \tag{5}$$

To see this, note that it follows from definition that

$$\begin{aligned} & |U_{nt}(C, K', z^* || \theta^{t-1}) - W_t(\theta^{t-1})| \\ &= \beta_n \left| \sum_{\theta \in \Theta^N} \pi(\theta) \{U_{nt+1}(C, K', z^* || \theta^{t-1}, \theta) - W_{t+1}(\theta^{t-1}, \theta)\} \right| \\ &\leq \beta_n \sup_{\theta} |U_{nt+1}(C, K', z^* || \theta^{t-1}, \theta) - W_{t+1}(\theta^{t-1}, \theta)| \\ &\leq \beta_n^s \sup_{\theta^s} |U_{nt+s}(C, K', z^* || \theta^{t-1}, \theta^s) - W_{t+s}(\theta^{t-1}, \theta^s)| \end{aligned}$$

Since W is a uniformly bounded correspondence and $\{C_t\}_{t=0}^\infty$ is uniformly bounded by construction, taking the *limsup* as $s \rightarrow \infty$ we get (5) as desired since $\beta_n \in (0, 1)$ for all $n \in I$.

Step 2. We need to show that (C, K') is a feasible and incentive compatible allocation.

(a) Feasibility follows because $(c(\theta_t), k'(\theta_t), w(\theta_t))(W_t(\theta^{t-1}))$ are admissible with respect to W at $K_t(\theta^{t-1}) \in X$ for all t and all θ^{t-1} .

(b) Incentive compatibility of our candidate allocation will be proved as usual. First, we will prove that it holds for strategies that have a finite number of deviation from truthtelling. Then it will follow then from Lemma 1 that it cannot be violated by any reporting strategy with infinitely many deviation from truthtelling.

It follows from admissibility, equality (5) and by construction of (C, K') that

$$\begin{aligned} & \sum_{\theta_{-n}} \pi(\theta_{-n}) \{ \theta_n u_n(C_{nt}(\theta^{t-1}, \theta_n, \theta_{-n})) + \beta_n U_{nt+1}(C, K', z_n^*, z_{-n}^* \| \theta^{t-1}, \theta_n, \theta_{-n}) \} \quad (6) \\ & \geq \sum_{\theta_{-n}} \pi(\theta_{-n}) \{ \theta_n u_n(C_{nt}(\theta^{t-1}, \tilde{\theta}_n, \theta_{-n})) + \beta_n U_{nt+1}(C, K', z_n^*, z_{-n}^* \| \theta^{t-1}, \tilde{\theta}_n, \theta_{-n}) \} \end{aligned}$$

for all $n \in I$, $t \geq 0$, θ^{t-1} , θ_n and $\tilde{\theta}_n$.

Since it has to hold for all z'_n , define z_n^m as in Lemma 1. We want to show that for all $m \geq 0$

$$\begin{aligned} & \sum_{\theta_{-n}} \pi(\theta_{-n}) \{ \theta_n u_n(C_{nt}(\theta^{t-1}, \theta_n, \theta_{-n})) + \beta_n U_{nt+1}(C, K', z_n^*, z_{-n}^* \| \theta^{t-1}, \theta_n, \theta_{-n}) \} \quad (7) \\ & \geq \sum_{\theta_{-n}} \pi(\theta_{-n}) \{ \theta_n u_n(C_{nt}(\theta^{t-1}, \tilde{\theta}_n, \theta_{-n})) + \beta_n U_{nt+1}(C, K', z_n^m, z_{-n}^* \| \theta^{t-1}, \tilde{\theta}_n, \theta_{-n}) \} \end{aligned}$$

for all $n \in I$, $t \geq 0$, θ^{t-1} , θ_n and $\tilde{\theta}_n$. Note that (7) holds for $m = 0$ since (6) holds.

Suppose that (7) holds for some m . Note that for all θ^t

$$\begin{aligned} U_{nt+1}(C, K', z_n^{m+1}, z_{-n}^* \| \theta^t) &= \sum_{\theta \in \Theta^N} \pi(\theta) \{ \theta_n u_n(C_{nt+1}(\theta^t, z_{n0}^{m+1}(\theta_n), \theta_{-n})) \\ & \quad + \beta_n U_{nt+2}(C, K', z_n^m(\theta), z_{-n}^* \| \theta^t, z_{n0}^{m+1}(\theta_n), \theta_{-n}) \} \end{aligned}$$

$$\begin{aligned}
&= \sum_{\theta_n} \pi(\theta_n) \left[\sum_{\theta_{-n}} \pi(\theta_{-n}) \{ \theta_n u_n(C_{nt+1}(\theta^t, z_{n0}^{m+1}(\theta_n), \theta_{-n}) \right. \\
&\quad \left. + \beta_n U_{nt+2}(C, K', z_n^m(\theta), z_{-n}^* \|\theta^t, z_{n0}^{m+1}(\theta_n), \theta_{-n}) \} \right] \\
&\leq \sum_{\theta_n} \pi(\theta_n) \left[\sum_{\theta_{-n}} \pi(\theta_{-n}) \{ \theta_n u_n(C_{nt+1}(\theta^t, \theta_n, \theta_{-n})) \right. \\
&\quad \left. + \beta_n U_{nt+2}(C, K', z^* \|\theta^t, \theta_n, \theta_{-n}) \} \right] \\
&= U_{nt+1}(C, K', z_n^*, z_{-n}^* \|\theta^t)
\end{aligned}$$

where the first inequality follows because (7) is supposed to hold for m . Hence, given this inequality and (6) we get

$$\begin{aligned}
&\sum_{\theta_{-n}} \pi(\theta_{-n}) \left\{ \theta_n u_n(C_{nt}(\theta^{t-1}, \tilde{\theta}_n, \theta_{-n}) + \beta_n U_{nt+1}(C, K', z_n^{m+1}, z_{-n}^* \|\theta^{t-1}, \tilde{\theta}_n, \theta_{-n}) \right\} \\
&\leq \sum_{\theta_{-n}} \pi(\theta_{-n}) \left\{ \theta_n u_n(C_{nt}(\theta^{t-1}, \tilde{\theta}_n, \theta_{-n}) + \beta_n U_{nt+1}(C, K', z^* \|\theta^{t-1}, \tilde{\theta}_n, \theta_{-n}) \right\} \\
&\leq \sum_{\theta_{-n}} \pi(\theta_{-n}) \left\{ \theta_n u_n(C_{nt}(\theta^{t-1}, \theta_n, \theta_{-n}) + \beta_n U_{nt+1}(B, K', z^* \|\theta^{t-1}, \theta_n, \theta_{-n}) \right\}
\end{aligned}$$

It follows by induction that (7) holds for all $m \geq 0$. Finally, consider any arbitrary reporting strategy z'_n (included those with infinitely many misreport). If (2) does not hold, then it follows by Lemma 1 that it should not hold for some large m . But that is a contradiction to (7). ■

Proof of Proposition 3. Given some arbitrary $k \in X$, let $\bar{u} \in \Psi(k)$. Then there exists a feasible and incentive compatible allocation (C, K') such that $\bar{u}_n = U_n(C, K', z^*)$ for all $n \in I$. Put for all $n \in I$ and all $\theta \in \Theta^N$

$$c_n(\theta) = C_{n0}(\theta), \quad k'_n(\theta) = K_{n1}(\theta) \quad \text{and} \quad w_n(\theta) = U_{n1}(C, K', z^* \|\theta)$$

Note that by construction, $\bar{u} = e_n(c, k', w)$. We need to check that (c, k', w) is admissible with respect to Ψ at k . To do so, we will first check that $w(\theta) \in \Psi(k'(\theta))$ for all

θ . Fix an arbitrary $\bar{\theta}$; put for all $t \geq 0$ and all θ^t

$$\bar{C}_t(\theta^t) = C_{t+1}(\bar{\theta}, \theta^t) \quad \text{and} \quad \bar{k}_{t+1}(\theta^t) = k_{t+2}(\bar{\theta}, \theta^t)$$

Clearly, (\bar{C}, \bar{K}) is feasible at $k_1(\bar{\theta})$ by construction. Also it is incentive compatible since (C, K) actually is (see condition (2)). Therefore, since $\bar{\theta}$ was arbitrary, we can conclude that Ψ is self-generating.

Since Ψ is uniformly bounded (see Remark (2) above), we can then conclude that $\Psi(k) = \Phi(\Psi, k)$ for all $k \in X$. ■

Proof of Lemma 4. Let $\{u^j, k^j\}_{j=0}^\infty$ be a sequence in $graph(\Phi(W))$. Then, for all j there exists (c^j, k'^j, w^j) admissible with respect to W at k^j where

$$u^j = e(c^j, k'^j, w^j) \quad \text{and} \quad w^j(\theta) \in \Psi(k'^j(\theta)) \quad \text{for all } \theta$$

Given that $\{k^j\}_{j=0}^\infty \subset X$, it has a convergent subsequence with limit $\bar{k} \in X$. But then since $\{w^j, k'^j\}_{j=0}^\infty$ is a sequence in $graph(\Psi)$, a compact set, it has a convergent subsequence as well. Also, by the definition of admissibility, $\{c^j\}$ is also in a compact set having then a convergent subsequence (and the limit satisfies all the conditions imposed by admissibility). Therefore, for each $\theta \in \Theta^N$, there exists $(\bar{c}(\theta), \bar{k}'(\theta), \bar{w}(\theta))$ being the limit point to this convergent subsequence. Clearly, given that momentary utility function are assumed to be continuous and weak inequalities are preserved in the limit, $(\bar{c}(\theta), \bar{k}'(\theta), \bar{w}(\theta))_{\theta \in \Theta^N}$ is admissible with respect to \bar{k} . Therefore,

$$\lim_{m \rightarrow \infty} \left(\sum_{\theta \in \Theta^N} \pi(\theta) \{ \theta_n u_n(c_n^{j_m}(\theta)) + \beta_n w_n^{j_m}(\theta) \} \right)_{n \in N} = (e_n(\bar{c}, \bar{k}', \bar{w}))_{n \in N} \in \Phi(\Psi, \bar{k})$$

which establishes that $graph(\Phi(W))$ is a compact set. ■

Proof of Lemma 5 . We already know that $graph(\Psi)$ is a bounded set. We need to show that it is also closed. Define the correspondence $\bar{\Psi}$ such that

$$graph(\bar{\Psi}) = closure(graph(\Psi))$$

Clearly, it follows by definition that $graph(\Psi) \subset graph(\bar{\Psi})$. By the previous remark, $graph(\Phi(\Psi)) \subset graph(\Phi(\bar{\Psi}))$. Since $\Psi = \Phi(\Psi)$, $graph(\Phi(\Psi)) = graph(\Psi)$ and $graph(\Psi) \subset graph(\Phi(\bar{\Psi}))$.

Since $graph(\bar{\Psi})$ is closed by definition, from Lemma 4 we have that $graph(\Phi(\bar{\Psi}))$ is also closed. Hence, $graph(\bar{\Psi}) = closure(graph(\Psi)) \subset closure(graph(\Phi(\bar{\Psi}))) = graph(\Phi(\bar{\Psi}))$ and therefore $\bar{\Psi}(k) \subset \Phi(\bar{\Psi})(k)$ for all $k \in X$. But then $\bar{\Psi}$ is self-generating and from Proposition 1 we know that $\bar{\Psi}(k) \subset \Psi(k)$ for all $k \in X$. This implies that $graph(\bar{\Psi}) \subset graph(\Psi)$ and thus $graph(\Psi)$ is closed. ■

Proof of Lemma 6. We complete this proof in two steps.

Step 1. We claim that if $graph(W)$ is convex, then $graph(\Phi(W))$ is also convex.

Let $(u, k), (\bar{u}, \bar{k}) \in graph(\Phi(W))$. We need to show that for all $\alpha \in [0, 1]$, $u^\alpha \equiv (\alpha u + (1 - \alpha)\bar{u}) \in \Phi(W)(\alpha k + (1 - \alpha)\bar{k})$. We know that there exist (c, k', w) and $(\bar{c}, \bar{k}', \bar{w})$ that are admissible with respect to W at k and \bar{k} , respectively, such that for all n

$$u_n = e_n(c, k', w) \text{ and } \bar{u}_n = e_n(\bar{c}, \bar{k}', \bar{w})$$

We need to show that there exists $(c^\alpha, k'^\alpha, w^\alpha)$ that is admissible with respect to W at $\alpha k + (1 - \alpha)\bar{k} \equiv k^\alpha$ such that $e_n(c^\alpha, k'^\alpha, w^\alpha) = u_n^\alpha$ for all n . Since u_n is continuous for all n , using the Intermediate Value Theorem, define $c_n^\alpha(\theta)$ such that $u_n(c_n^\alpha(\theta)) \equiv \alpha u_n(c_n(\theta)) + (1 - \alpha)u_n(\bar{c}_n(\theta))$ for all n and for all θ . Note that since u_n is concave, it follows that $c_n^\alpha(\theta) \leq \alpha c_n(\theta) + (1 - \alpha)\bar{c}_n(\theta)$ for all $\alpha \in [0, 1]$. Finally, define $w_n^\alpha(\theta) \equiv w_n(\theta) + (1 - \alpha)\bar{w}_n(\theta)$ and $k'^\alpha(\theta) \equiv k'(\theta) + (1 - \alpha)\bar{k}'(\theta)$ for all n and for all

θ . Note that since it is assumed that $\text{graph}(W)$ is convex, then $w_n^\alpha(\theta) \in W(k'^\alpha(\theta))$ for all θ . Also, the alternative allocation is clearly feasible and incentive compatible by construction since f is concave. Thus, $(c^\alpha, k'^\alpha, w^\alpha)$ is admissible with respect to W at k^α . Since by construction $e_n(c^\alpha, k'^\alpha, w^\alpha) = u_n^\alpha$ for all n , we are done.

Step 2. For any set $A \subseteq R^N$, let $\text{co}(A)$ be the convex hull of A . Define the correspondence $\tilde{\Psi}$ such that $\text{graph}(\tilde{\Psi}) = \text{co}(\text{graph}(\tilde{\Psi}))$. Clearly, $\text{graph}(\Psi) \subseteq \text{graph}(\tilde{\Psi})$ and therefore $\text{graph}(\Phi(\Psi)) \subseteq \text{graph}(\Phi(\tilde{\Psi}))$. Since $\Psi = \Phi(\Psi)$ by Proposition 3, $\text{graph}(\Psi) \subseteq \text{graph}(\Phi(\tilde{\Psi}))$. Since the $\text{graph}(\tilde{\Psi})$ is convex by definition, it follows by Step 1 that $\text{graph}(\Phi(\tilde{\Psi}))$ is also convex. Therefore,

$$\text{graph}(\tilde{\Psi}) = \text{co}(\text{graph}(\tilde{\Psi})) \subseteq \text{co}(\text{graph}(\Phi(\tilde{\Psi}))) = \text{graph}(\Phi(\tilde{\Psi}))$$

But then $\tilde{\Psi}(k) \subseteq \Phi(\tilde{\Psi})(k)$ for all $k \in X$ and therefore $\tilde{\Psi}$ is self-generating. By Proposition 3, it follows that $\tilde{\Psi}(k) \subseteq \Psi(k)$ for all $k \in X$ and then $\text{graph}(\Psi) \supseteq \text{graph}(\tilde{\Psi})$. Therefore, we can conclude that Ψ has a convex graph. ■

Proof of Lemma 7. Consider $\theta_n > \tilde{\theta}_n$ and note that

$$\begin{aligned} & \sum_{\theta_{-n}} \pi(\theta_{-n}) \{ \theta_n u_n(c_n(\theta_n, \theta_{-n})) + \beta_n w_n(\theta_n, \theta_{-n}) \} \\ & \geq \sum_{\theta_{-n}} \pi(\theta_{-n}) \{ \theta_n u_n(c_n(\tilde{\theta}_n, \theta_{-n})) + \beta_n w_n(\tilde{\theta}_n, \theta_{-n}) \} \end{aligned}$$

and

$$\begin{aligned} & \sum_{\theta_{-n}} \pi(\theta_{-n}) \{ \tilde{\theta}_n u_n(c_n(\tilde{\theta}_n, \theta_{-n})) + \beta_n w_n(\tilde{\theta}_n, \theta_{-n}) \} \\ & \geq \sum_{\theta_{-n}} \pi(\theta_{-n}) \{ \tilde{\theta}_n u_n(c_n(\theta_n, \theta_{-n})) + \beta_n w_n(\theta_n, \theta_{-n}) \} \end{aligned}$$

imply

$$(\theta_n - \tilde{\theta}_n) \sum_{\theta_{-n}} \pi(\theta_{-n}) [u_n(c_n(\theta_n, \theta_{-n})) - u_n(c_n(\tilde{\theta}_n, \theta_{-n}))] \geq 0$$

Also, from the previous inequalities, it follows that

$$\sum_{\theta_{-n}} \pi(\theta_{-n}) w_n(\theta_n, \theta_{-n}) \geq \sum_{\theta_{-n}} \pi(\theta_{-n}) w_n(\tilde{\theta}_n, \theta_{-n})$$

as desired. ■

Proof of Lemma 8. (i) Suppose first that $\bar{k} > k$. Suppose that (c^*, k'^*, w^*) is efficient at (k, u_{-1}) . Consider the following alternative allocation $(\bar{c}, \bar{k}', \bar{w})$. Let $\epsilon = f(\bar{k}) - f(k)$ and define $\bar{k}'(\theta) = k'^*(\theta)$. Let $\bar{c}_n(\theta) = c_n^*(\theta)$ for all $n \neq 1$ and for all θ . Also, put $\bar{w}_n(\theta) = w_n^*(\theta)$ for all θ and for all n . Define $\epsilon_j \in (0, \epsilon]$ such that $\bar{c}_n(\theta_j, \theta_{-n}) = c_n^*(\theta_j, \theta_{-n}) + \epsilon_j$ for all θ_{-n} . Finally, it is easy to see that $\{\epsilon_j\}_{j:1}^J$ can be chosen strictly positive such that incentive compatibility is satisfied. For example, if $\Theta = \{\underline{\theta}, \bar{\theta}\}$ where $\underline{\theta} < \bar{\theta}$, put $\bar{\epsilon}$ and $\underline{\epsilon}$ such that $\sum_{\theta_{-1}} \pi(\theta_{-1}) [u_1(c_1(\bar{\theta}, \theta_{-1}) + \bar{\epsilon}) - u_1(c_1(\bar{\theta}, \theta_{-1}))] = \sum_{\theta_{-1}} \pi(\theta_{-1}) [u_1(c_1(\underline{\theta}, \theta_{-1}) + \underline{\epsilon}) - u_1(c_1(\underline{\theta}, \theta_{-1}))]$ and $\max(\bar{\epsilon}, \underline{\epsilon}) \leq \epsilon$. This alternative allocation is admissible with respect to Ψ at \bar{k} and $V(k, u_{-1}) < e_1(\bar{c}, \bar{k}', \bar{w}) \leq V(\bar{k}, u_{-1})$.

A similar argument shows that V is strictly decreasing in u_{-1} but some additional comments might help to get some of the intuition. Suppose that for some agent n , his utility is reduced from u_n to \underline{u}_n . Any admissible and promise keeping allocation will assign a lower level of consumption either today or in the future (or both). Suppose that this happens at period t if θ^{t-1} has been reported. This will allow to increase the level of aggregate capital available at $t+1$ and thus one could increase consumption for agent 1 whenever θ^{t-1} has been reported and **for all** θ_t . Since $\mu(\theta^{t-1}) > 0$ for all θ^{t-1} , this will increase agent 1's expected utility.

(ii) Now we will show that V is a continuous function. First, observe that $e_1(c, k', w) = \sum_{\theta} \pi(\theta) \{u_1(c_1(\theta)) + \beta_1 w_1(\theta)\}$ is a continuous function with respect to (c, k', w) . Then, since Ψ is a correspondence mapping X into R^N with a compact

and convex graph, it follows that Ψ is a compact-valued and continuous correspondence (Stokey, Lucas and Prescott [24], Theorem 3.4 and 3.5). Let $A \equiv \{(k, u_{-1}) \in X \times R^{N-1} : u_{-1} \in \Psi_{-1}(k)\}$ and define for each $(k, u_{-1}) \in A$ the correspondence

$$\Gamma(k, u_{-1}) \equiv \{(c, k', w) : (c, k', w) \text{ is admissible with respect to } \Psi \text{ at } k \\ \text{and } u_n = e_n(c, k', w) \text{ for all } n \neq 1\}$$

Since Ψ is a continuous correspondence on a compact domain, it is easy to check that A is a compact set. Also, it is a standard exercise to check that Γ is a continuous, compact valued correspondence since Ψ is a continuous correspondence on a compact domain and $e_n(c, k', w)$ are continuous functions for all n .¹⁶

Therefore, it follows from the Theorem of the Maximum that V is a continuous function on A . ■

Proof of Proposition 9. Assume, on the contrary, that $(\bar{c}, \bar{k}', \bar{w})$ is efficient at (k, u_{-1}) , where $u_n = e_n(\bar{c}, \bar{k}', \bar{w})$. Without loss of generality, suppose that $n \neq 1$ and this imply that $V(k, u_{-1}) > 0$ (as it will be clear, the whole proof goes through when $n = 1$).¹⁷ Assume to simplify that $\Theta = \{\underline{\theta}, \bar{\theta}\}$ where $\underline{\theta} < \bar{\theta}$. First, we claim that if agent n is entitled to some positive expected utility level, then $c_n(\bar{\theta}, \theta_{-n}) > 0$ for all θ_{-n} . Let $\lambda_{\theta_i, \theta_j}^n$ be the Lagrange multiplier corresponding to ICC corresponding to agent n giving incentives to reveal the observed preference shock θ_i instead of θ_j . Similarly, let $\gamma(\theta_n, \theta_{-n})$ and η_n be the agent n 's Lagrange multipliers corresponding to the feasibility constraint and to the promise keeping constraint, respectively, when the aggregate state is (θ_n, θ_{-n}) ($\eta_1 = 1$). Consider now the necessary FOC with respect to $c_n(\bar{\theta}, \theta_{-n})$:

$$\pi_{-n}(\theta_{-n}) u'_n(c_n(\bar{\theta}, \theta_{-n})) [\eta_n \pi_n(\bar{\theta}) \bar{\theta} + \lambda_{\bar{\theta}, \underline{\theta}}^n \bar{\theta} - \lambda_{\underline{\theta}, \bar{\theta}}^n \underline{\theta}] \leq \gamma(\bar{\theta}, \theta_{-n})$$

¹⁶See Stokey, Lucas and Prescott [24], Exercises 3.12 and 3.13.

¹⁷If $V(k, u_{-1}) = 0$, then it follows that $\bar{c}_1(\theta) = \bar{w}_1(\theta) = 0$ for all θ .

with equality if $c_n(\bar{\theta}, \theta_{-n}) > 0$. Now, if $c_n(\bar{\theta}, \theta_{-n}) = 0$, then it follows that $\lambda_{\bar{\theta}, \underline{\theta}}^n \bar{\theta} - \lambda_{\underline{\theta}, \underline{\theta}}^n \underline{\theta} < 0$ ($\eta_n > 0$ if agent n is entitled with a positive utility level). Now consider the necessary FOC with respect to $c_n(\underline{\theta}, \theta_{-n})$, which can be written:

$$\pi_{-n}(\theta_{-n}) u'_n(c_n(\underline{\theta}, \theta_{-n})) [\eta_n \pi_n(\underline{\theta}) \underline{\theta} + \lambda_{\underline{\theta}, \underline{\theta}}^n \underline{\theta} - \lambda_{\bar{\theta}, \underline{\theta}}^n \bar{\theta}] \leq \gamma(\underline{\theta}, \theta_{-n})$$

If $\lambda_{\bar{\theta}, \underline{\theta}}^n \bar{\theta} - \lambda_{\underline{\theta}, \underline{\theta}}^n \underline{\theta} < 0$, then it follows that $c_n(\underline{\theta}, \theta_{-n}) > 0$ for all θ_{-n} . But this contradicts Lemma 7.

Since $c_n(\bar{\theta}, \theta_{-n}) > 0$ for all θ_{-n} , observe that if $\sum_{\theta_{-n}} \pi(\theta_{-n}) u(\bar{c}_n(\theta_n, \theta_{-n})) = \bar{u}_n$ for all $\theta_n \in \Theta$, it follows that $\bar{u}_n > 0$. Therefore, it must be that, for some $\tilde{\theta}_{-n}$, $c_n(\underline{\theta}, \tilde{\theta}_{-n}) > 0$. Observe also that $\pi(\underline{\theta}, \tilde{\theta}_{-n}) > 0$.

Let $\tilde{\theta} = (\underline{\theta}, \tilde{\theta}_{-n})$ and define an alternative recursive allocation as follows:

$$\tilde{c}_n(\tilde{\theta}) = \bar{c}_n(\tilde{\theta}) - \lambda, \quad \tilde{w}_n(\tilde{\theta}) = \bar{w}_n(\tilde{\theta}) + \delta_n, \quad \tilde{k}'(\tilde{\theta}) = \bar{k}'(\tilde{\theta}) + \lambda$$

For all $\theta \neq \tilde{\theta}$, put $\tilde{c}_n(\theta) = \bar{c}_n(\theta)$, $\tilde{w}_n(\theta) = \bar{w}_n(\theta)$ and $\tilde{k}'(\theta) = \bar{k}'(\theta)$. For all $i \neq n$, put simply $\tilde{c}_i(\theta) = \bar{c}_i(\theta)$ and $\tilde{w}_i(\theta) = \bar{w}_i(\theta)$ for all θ . We will restrict $(\lambda, \delta_n) \gg 0$ such that $(\tilde{c}, \tilde{k}', \tilde{w})$ is admissible with respect to Ψ at k and $e_n(\tilde{c}, \tilde{k}', \tilde{w}) > e_n(\bar{c}, \bar{k}', \bar{w})$.

Step 1. Note that by continuity of V , (λ, δ_n) can be chosen such that

$$\bar{w}_1(\tilde{\theta}) \leq V(\bar{k}'(\tilde{\theta}) + \lambda, (\bar{w}_n(\tilde{\theta}) + \delta_n, \{\bar{w}_i(\tilde{\theta})\}_{i \neq 1, n})) \quad (8)$$

since V is decreasing in w_{-1} and increasing in k' . Therefore, we can find $(\lambda, \delta_n) \gg 0$ such that $(\tilde{w}_1(\tilde{\theta}), \tilde{w}_n(\tilde{\theta}), \{\tilde{w}_i(\tilde{\theta})\}_{i \in I/\{1, n\}}) \in \Psi(\tilde{k}'(\tilde{\theta}))$.

Step 2. If $\lambda > 0$ and $c_n(\tilde{\theta}) - \lambda \geq 0$, then feasibility is satisfied by definition.

Step 3. Incentive Compatibility. Since the recursive allocation $(\bar{c}, \bar{k}', \bar{w})$ is assumed to be admissible with respect to Ψ at k , there is nothing to check for agent $i \neq n$. Consider agent n and for $c_n(\tilde{\theta}) - \lambda \geq 0$ define

$$g_n(\theta, \lambda) = \theta [u_n(\bar{c}_n(\tilde{\theta})) - u_n(\bar{c}_n(\tilde{\theta}) - \lambda)]$$

Note that $g_n(\theta, 0) = 0$ and it is strictly increasing in λ for $\lambda > 0$. Also if $\lambda > 0$, then $g_n(\underline{\theta}, \lambda) < g_n(\bar{\theta}, \lambda)$. Hence, since $(\bar{c}, \bar{k}', \bar{w})$ is incentive compatible, it is easy to check that

$$\begin{aligned} & \sum_{\theta_{-n}} \pi(\theta_{-n}) [\underline{\theta} u_n(\tilde{c}_n(\underline{\theta}, \theta_{-n})) + \beta_n \tilde{w}_n(\underline{\theta}, \theta_{-n})] \\ & \geq \sum_{\theta_{-n}} \pi(\theta_{-n}) [\underline{\theta} u_n(\tilde{c}_1(\bar{\theta}, \theta_{-n})) + \beta_n \tilde{w}_n(\bar{\theta}, \theta_{-n})] \end{aligned}$$

is satisfied if

$$g_n(\underline{\theta}, \lambda) \leq \beta_n \delta_n \tag{9}$$

Similarly,

$$\begin{aligned} & \sum_{\theta_{-n}} \pi(\theta_{-n}) [\bar{\theta} u_n(\tilde{c}_n(\bar{\theta}, \theta_{-n})) + \beta_n \tilde{w}_n(\bar{\theta}, \theta_{-n})] \\ & \geq \sum_{\theta_{-n}} \pi(\theta_{-n}) [\bar{\theta} u_n(\tilde{c}_n(\underline{\theta}, \theta_{-n})) + \beta_n \tilde{w}_n(\underline{\theta}, \theta_{-n})] \end{aligned}$$

is satisfied if

$$g_n(\bar{\theta}, \lambda) \geq \beta_n \delta_n \tag{10}$$

Put $g_n(\bar{\theta}, \lambda) = \beta_n \delta_n > g_n(\underline{\theta}, \lambda)$ and then conditions (8), (9) and (10) are satisfied for some $(\lambda, \delta_n) \gg 0$. For instance, put $\lambda = 1/k$ and define δ_n^k by letting $g_n(\bar{\theta}, 1/k) = \beta_n \delta_n^k > 0$. Note that $(1/k, \delta_n^k) \searrow (0, 0)$ as $k \rightarrow \infty$ and then condition (8) is satisfied by continuity.

Step 4. Finally, note that for all $i \neq n$, $e_i(\tilde{c}, \tilde{k}', \tilde{w}) = e_i(\bar{c}, \bar{k}', \bar{w})$ by construction.

Also, observe that

$$\begin{aligned} e_n(\tilde{c}, \tilde{k}', \tilde{w}) &= \sum_{\theta} \pi(\theta) \{u_n(\tilde{c}_n(\theta)) + \beta_n \tilde{w}_n(\theta)\} \\ &= e_n(\bar{c}, \bar{k}', \bar{w}) + \pi(\hat{\theta}) [\beta_n \delta_n - g_n(\underline{\theta}, \lambda)] = \tilde{u}_n \\ &> e_n(\bar{c}, \bar{k}', \bar{w}) = u_n \end{aligned}$$

Hence, $(\tilde{c}, \tilde{k}', \tilde{w})$ is a recursive allocation which admissible with respect to $\Psi(k)$ such that

$$e_1(\tilde{c}, \tilde{k}', \tilde{w}) = V(k, u_{-1}), \quad e_n(\tilde{c}, \tilde{k}', \tilde{w}) = \tilde{u}_n > u_n, \quad e_i(\tilde{c}, \tilde{k}', \tilde{w}) = u_i \text{ for all } i \in I/\{1, n\}$$

Let $\tilde{u}_{-1} = (\tilde{u}_n, \{u_n\}_{i \in I/\{1, n\}})$. Since $V(k, u_{-1}) > V(k, \tilde{u}_{-1}) \geq e_1(\tilde{c}, \tilde{k}', \tilde{w})$ by Lemma 8, we get the desired contradiction. ■

Proof of Proposition 10. Given any (k, u_{-1}) , let $(c^*, k'^*, w^*)(k, u_{-1})$ be the efficient recursive allocation. Suppose that there exists some $\tilde{\theta}$ such that

$$0 \leq w_1^*(\tilde{\theta}) < V(k'^*(\tilde{\theta}), w_{-1}^*(\tilde{\theta}))$$

Since $V(0, w_{-1}^*) = 0$ for any w_{-1}^* , this implies that $k'^*(\tilde{\theta}) > 0$. Since V is continuous, there exists $0 \leq \underline{k}'(\tilde{\theta}) < k'^*(\tilde{\theta})$ such that $V(\underline{k}'(\tilde{\theta}), w_{-1}^*(\tilde{\theta})) = 0$. Finally, since V is continuous and strictly decreasing in k , it follows by the mean value theorem that there exists some $\underline{k}'(\tilde{\theta}) \leq k'^{**}(\tilde{\theta}) < k'^*(\tilde{\theta})$ such that $w_1^*(\tilde{\theta}) = V(k'^{**}(\tilde{\theta}), w_{-1}^*(\tilde{\theta}))$. The alternative recursive allocation (c^*, k'^{**}, w^*) is incentive compatible, promise keeping and feasible by construction since a free disposal technology is available. It is immediate that in fact (c^*, k'^{**}, w^*) is also efficient at (k, u_{-1}) . ■

Proof of Proposition 11. Denote $\Delta(k, \bar{U}_n) = \{\{\theta_t\}_{t=0}^\infty : \lim_{t \rightarrow \infty} K_{t+1}(\theta^t) = k > 0 \text{ and } \lim_{t \rightarrow \infty} U_{nt}(\theta^t) = \bar{U}_n \in \Psi_n(\bar{k}) \text{ for some } n\}$. Given (k_0, U_0) , take any $\{\theta_t\}_{t=0}^\infty \in \Delta(\bar{k}, \bar{U}_n)$ and consider the path of the following stochastic vector

$$\{K_t, U_t, c^*(\theta)(K_t, U_t), k'^*(\theta)(K_t, U_t), w^*(\theta)(K_t, U_t)\}_{t=0}^\infty$$

where $w^*(\theta)(K_t, U_t) \in \Psi(k'^*(\theta)(K_t, U_t))$ for all t and all $\theta \in \Theta^N$. Also, for all t

$$\begin{aligned} U_{nt} &= \sum_{\theta \in \Theta^N} \pi(\theta) \{\theta_n u_n(c_n^*(\theta)(K_t, U_t)) + \beta_n w_n^*(\theta)(K_t, U_t)\} \\ U_{1t} &= V_1^*(K_t, U_{-1t}) \end{aligned}$$

Note that this is a sequence in a compact set and therefore it will have a convergent subsequence. Without loss of generality, assume that the relevant subsequence is the sequence itself. Denote the corresponding limit point by $\{\widehat{k}, \widehat{U}, \widehat{c}, \widehat{k}', \widehat{w}\}$. Note that $\widehat{k} = k$ and $\widehat{U}_n = \overline{U}_n$.

Step 1. We claim that $(\widehat{c}, \widehat{k}', \widehat{w})$ is admissible with respect to Ψ at \widehat{k} . Moreover, it is efficient at $(\widehat{k}, \widehat{U}_{-1})$.

To see this, note first that by definition $(w^*(\theta)(K_t, U_t), k'^*(\theta)(K_t, U_t)) \in \text{graph}(\Psi)$ for all t and all $\theta \in \Theta^N$. Since Ψ has a compact graph, it follows that for all $\theta \in \Theta^N$ $(\widehat{w}(\theta), \widehat{k}'(\theta)) \in \text{graph}(\Psi)$ and then $\widehat{w}(\theta) \in \Psi(\widehat{k}'(\theta))$ for all $\theta \in \Theta^N$.

Since weak inequalities are preserved in the limit and by continuity of f , it is also true that for all $\theta \in \Theta^N$

$$\sum_{n \in I} \widehat{c}_n(\theta) + \widehat{k}'(\theta) \leq f(\widehat{k}) \quad \text{and} \quad \widehat{c}_n(\theta) \geq 0$$

By continuity of u_n , it follows for all $n \in I$ that

$$\begin{aligned} \lim_{t \rightarrow \infty} U_{nt} &= \lim_{t \rightarrow \infty} \sum_{\theta \in \Theta^N} \pi(\theta) \{ \theta_n u_n(c_n^*(\theta)(K_t, U_t)) + \beta_n w_n^*(\theta)(K_t, U_t) \} \\ \widehat{U}_n &= \sum_{\theta \in \Theta^N} \pi(\theta) \{ \theta_n u_n(\widehat{c}_n(\theta)) + \beta_n \widehat{w}_n(\theta) \} \end{aligned}$$

Finally, note that since V is continuous it follows that

$$\lim_{t \rightarrow \infty} U_{1t} = \lim_{t \rightarrow \infty} V(K_t, U_{-1t}) = V(\widehat{k}, \widehat{U}_{-1})$$

That is, $(\widehat{c}, \widehat{k}', \widehat{w})$ is an efficient recursive allocation at $(\widehat{k}, \widehat{U}_{-1})$.

Step 2. Since in this case the economy does not collapse in the limit by assumption, consider any agent h such that $\widehat{U}_h > 0$. We claim that there exists $\widetilde{\theta}_{-h}$ such that either

$$(a) \quad \lim_{t \rightarrow \infty} w_h^*(\underline{\theta}, \widetilde{\theta}_{-h})(K_t, U_t) \neq \widehat{U}_h$$

or

$$(b) \quad \lim_{t \rightarrow \infty} w_h^*(\bar{\theta}, \tilde{\theta}_{-h})(K_t, U_t) \neq \widehat{U}_h$$

To see this, assume that it is not true. Then, it follows from Lemma 7 that for all θ_h

$$\begin{aligned} \widehat{U}_h &= \lim_{t \rightarrow \infty} \sum_{\theta_{-h}} \pi(\theta_{-h}) w_h^*(\underline{\theta}, \theta_{-h})(K_t, U_t) \geq \lim_{t \rightarrow \infty} \sum_{\theta_{-h}} \pi(\theta_{-h}) w_h^*(\theta_h, \theta_{-h})(K_t, U_t) \\ &\geq \lim_{\theta_{-h}} \sum_{\theta_{-h}} \pi(\theta_{-h}) w_h^*(\bar{\theta}, \tilde{\theta}_{-h})(K_t, U_t) = \widehat{U}_h \end{aligned}$$

Now observe that for all $\theta_h > \tilde{\theta}_h$, incentive compatibility and Lemma 7 imply that

$$\begin{aligned} &\beta_h \sum_{\theta_{-h}} \pi(\theta_{-h}) [w_h^*(\theta_h, \theta_{-h})(K_t, U_t) - w_h^*(\tilde{\theta}_h, \theta_{-h})(K_t, U_t)] \\ &\geq \theta_h \sum_{\theta_{-h}} \pi(\theta_{-h}) [u_h(c_h^*(\tilde{\theta}_h, \theta_{-h})(K_t, U_t)) - u_h(c_h^*(\theta_h, \theta_{-h})(K_t, U_t))] \\ &\geq \tilde{\theta}_h \sum_{\theta_{-h}} \pi(\theta_{-h}) [u_h(c_h^*(\tilde{\theta}_h, \theta_{-h})(K_t, U_t)) - u_h(c_h^*(\theta_h, \theta_{-h})(K_t, U_t))] \\ &\geq \beta_h \sum_{\theta_{-h}} \pi(\theta_{-h}) [w_h^*(\theta_h, \theta_{-h})(K_t, U_t) - w_h^*(\tilde{\theta}_h, \theta_{-h})(K_t, U_t)] \end{aligned}$$

Taking limits it follows that for all $(\theta_h, \tilde{\theta}_h)$

$$\lim_{t \rightarrow \infty} \sum_{\theta_{-h}} \pi(\theta_{-h}) [u_h(c_h^*(\theta_h, \theta_{-h})(K_t, U_t)) - u_h(c_h^*(\tilde{\theta}_h, \theta_{-h})(K_t, U_t))] = 0$$

Since u_h is assumed continuous and $(k_t, c_h^*(\theta)(K_t, W_t)) \rightarrow (\widehat{k}, \widehat{c}_h(\theta))$ for all θ , it follows that $\sum_{\theta_{-h}} \pi(\theta_{-h}) u_h(\widehat{c}_h(\theta_h, \theta_{-h})) = \bar{u}_h$ for all θ_h . But this is a contradiction to Proposition 9 since $\widehat{U}_h > 0$.

Step 3. Suppose that (a) holds and consider the sequence of q 's such that

$$U_{nt_q+1} = w_n^*(\underline{\theta}, \tilde{\theta}_{-n})(K_{t_q}, U_{t_q})$$

Since (a) holds, this equality can hold only for a finite number of q 's. Therefore,

$$\Delta(k, \bar{U}_n) \subset \{ \{ \theta_t \}_{t=0}^\infty \in \Omega : \theta_t = (\underline{\theta}, \tilde{\theta}_{-n}) \text{ finitely often} \}$$

must hold.¹⁸ But since all partial histories have strictly positive probabilities, this last set has probability zero (it is the countable union of zero probability sets). That is, $\mu\{\{\theta_t\}_{t=0}^\infty\} \in \Omega : \theta_t = (\underline{\theta}, \tilde{\theta}_{-n}) \text{ finitely often}\} = 0$. ■

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¹⁸Otherwise one can construct a convergent subsequence with an efficient limit point where condition (a) is violated. But that is a contradiction to the result in step 2.

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