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## Sector-specific human capital and the distribution of earnings

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Sector-Specific Human Capital  
and the Distribution of Earnings

Eric Smith

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## Sector-Specific Human Capital and the Distribution of Earnings

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**Abstract:** This paper incorporates assignment frictions and sector-specific training into the Roy model of occupational choice. Assignment frictions represent the extent of the market whereas differences in sector-specific training reflect worker specialization. This framework thus captures Adam Smith's idea that the extent of the market determines the division of labor. The paper demonstrates the way in which the relationship between assignment frictions and specialization affects the level and composition of human capital acquisition, aggregate output, and the distribution of income. Not surprisingly, economywide training, output, and specialization increase as the extent of the market increases. The distribution of these gains, however, is uneven. Within group or residual income, distribution does not converge monotonically as search frictions diminish. Comparisons across groups reveal that these effects can become more pronounced as average income increases.

JEL classification: E24, J24, D31

Key words: human capital, occupational choice, job assignment, income distribution

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"There are some sorts of industry, even of the lowest kind, which can be carried on no where but in a great town. ... In the lone houses and very small villages which are scattered about in so desert a country as the Highlands of Scotland, every farmer must be butcher, baker, brewer for his own family. ... The scattered families that live eight or ten miles distance from the nearest of them, must learn to perform themselves a great number of little pieces of work, for which, in more populous countries, they would call in the assistance of those workmen." (Adam Smith, The Wealth of Nations, p. 17)

Since Adam Smith if not before, the determination and distribution of labor market earnings have occupied the attention of economists. Recently, a vast literature has emerged establishing that earnings inequality in the US and elsewhere has grown dramatically over the last four decades. During the 1980's and 1990's inequality across groups deepened considerably while within group (or residual) inequality has also grown markedly since the 1970s. (See Katz and Autor, 1999, and Machin and van Reenen, 2007, for a survey and overview.) Although debate continues notably regarding the most recent trends (Card and DiNardo, 2002, Lemieux, 2006, Sattinger 2007), skilled biased technological change appears to be the most plausible cause for the rising spread in earnings across groups.

Unfortunately, skilled biased technological change, as well as other prominent explanations, offers little help in understanding the changes in residual wage disparity.<sup>1</sup> To address this shortcoming, this paper presents a model of earnings that incorporates labor market frictions and sector specific training into a Roy model of occupational choice.<sup>2</sup>

Suppose workers choose how much training to acquire for potential jobs before investigating and then choosing among available employment options in different

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<sup>1</sup> Unobserved differences among workers is perhaps the most compelling argument (Lemieux, 2006), but this explanation is by its very nature difficult to test. Theories that build upon market frictions and the sharing of quasi-rents offer an alternative approach. See Van Reenen (1996), Lloyd-Ellis (1999), Casselli (1999) and Kambourov and Manovskii (2004).

<sup>2</sup> Specialization in the Roy model without personal investments and without assignment frictions has been examined in detail. See Heckman and Honore (1990), Heckman and Sedlacek (1985), Willis and Rosen (1979) for expositions of the way in which self-selection affects the distribution of income and empirical evaluation of wage functions among individuals with heterogeneous abilities. Eleftheriou (2008) examines wage inequality in the Roy model with matching frictions but does not consider human capital investments or derive analytic results.

occupations. Following Adam Smith's lead, the likelihood with which an agent trades in a particular occupation, modeled here through assignment frictions, reflects the extent of the market for different occupations.<sup>3</sup> Agents take account of these frictions in their choices. The extent of the market determines not only training investments and the degree of skill specialization but also the level and distribution of earnings. Matching outcomes affect the division of labor.

This simple set-up links market frictions with different levels of human capital investments as well as uncertain income outcomes. Total training and aggregate income all rise as the extent of the market increases, that is, as the market thickens and assignment becomes more certain. This result is not surprising; however, the composition of human capital alters the distribution of these gains. As the extent of the market widens and workers become more able to locate the jobs they desire more, they specialize more in the acquisition of human capital. While greater specialization improves expected income, it exposes workers to a greater down side risk of not locating the more desired employment opportunity. More individuals match well but the unfortunate who do not locate jobs in their more desired sector have less training to fall back on and suffer lower incomes. As such, greater specialization increases the earnings divide between those who match well and those who do not. Less insurance through diversity is taken out against the less likely bad draw. Earnings within groups spread out and under some conditions leads to greater inequality.<sup>4</sup> Dispersion of income does not necessarily converge monotonically with a reduction in frictions.

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<sup>3</sup> Adam Smith emphasizes geographical dispersion in trading probabilities due to the extent of the market. Here the interpretation is on the improvement of matching over time in the frequency with which agents have the opportunity to trade. This specification abstracts from Adam Smith's second reason for the gain in productivity. There is no capital in the model so Smith's third rationale is also absent.

<sup>4</sup> Motivated by their empirical findings, Kambourov and Monovski (2004) also emphasize the importance of occupational human capital and matching frictions to resolve within group wage inequality variation. They argue, however, that increased variability of productivity shocks generate more occupational mobility. When coupled with experienced-related, occupation specific human capital, this shift accounts for a large share of overall wage inequality. Here, the simple two period model can be extended to allow for periodic transitions. Indeed it is possible to have mobility rise with matching rates. Such generality is left out in order to highlight the simplicity and intuition of proposed mechanism. Like Kambourov and Manovskii, occupation specific human capital is critical in this paper.

Although the focus of this study is on the way assignment frictions and sector specific human capital interact to alter the distribution of income for a particular ability group, it is possible to formulate comparisons across different groups by contrasting responses by ability. Comparisons of this sort line up with the growing disparity across groups and offer a complementary approach to skilled biased technological change for understanding the divide between groups. In particular, under standard conditions, the model also offers a potential explanation for the rise in inter-group heterogeneity documented in the literature.

## **1. Some Observations**

Does better occupational matching generate greater specialization and higher within group earnings dispersion? Evidence on specialization and occupational specific matching is difficult to come by but some observations from surveys of undergraduate students at US universities and colleges help motivate the proposed argument.

Table 1 documents the course credits for business, computer science and engineering students within and outside their field for three cohorts between 1972 and 1993. Despite some minor reductions in total credits, the overall pattern for these students is one in which they concentrate more in and around their chosen field while taking fewer outside options. For instance, engineers are receiving more hours of tuition in their discipline with fewer hours spent in disparate areas such as social science. The same general pattern holds in computer science and business majors albeit with slightly less clarity. The picture is none the less one in which students are less diversified. Based on this evidence bachelor degrees appear to be more concentrated in particular fields of study.

Table 2 documents the job placement (one year out) of graduates over roughly the same period. It is reasonable to expect improved matching given thicker markets and improved information systems. Over the last forty years in the US, this appears to be the case. Recent college graduates have experienced higher employment rates and have become more likely to find work in their field of study. Taken together, this evidence indicates that students are specializing more and are more likely to find their desired employment, often in jobs very closely related to their training.

**Table 1.**  
**Mean number of semester credits completed by bachelor's degree recipients, by course area and major:**  
**As percentage of total course hours, 1976, 1984, and 1993**

Selected college major	Total hours	<u>Course areas</u>								
		B'ness	Comp Sci	Egn'r-ing	Math	Nat sci	Total sci	Soc sci & psych	Other	
<b>1972 high school seniors who completed bachelor's degrees by 1976</b>										
Business and management	124.4	33.1	1.8	0.3	8.2	5.9	16.2	24.4	26.1	
Computer science	133.3	5.0	25.1	4.0	16.8	7.3	53.2	15.5	26.4	
Engineering	134.8	1.2	1.5	37.1	13.5	16.2	68.2	10.4	20.2	
<b>1980 high school seniors who completed bachelor's degrees by 1984</b>										
Business and management	122.8	33.6	3.7	0.9	7.2	5.0	16.8	22.4	27.1	
Computer science	129.3	9.1	21.6	3.6	16.5	8.0	49.7	14.7	26.5	
Engineering	132.3	0.8	1.7	39.7	12.2	16.1	69.8	9.3	20.2	
<b>1988-89 high school graduates who completed bachelor's degrees by 1993</b>										
Business and management	123.9	44.4	3.2	0.1	6.1	4.8	14.2	18.7	31.3	
Computer science	127.6	15.7	26.9	1.9	12.3	6.4	47.4	13.8	26.6	
Engineering	136.9	1.4	5.1	42.3	12.2	14.9	74.5	8.9	15.6	

NOTE: All majors total includes fields not shown separately. Because of rounding, details may not add to totals.  
 SOURCE: U.S. Department of Education, National Center for Education Statistics, National Longitudinal Study of 1972; High School and Beyond; and Baccalaureate and Beyond survey.

**Table 2.**  
**Employment Status of Bachelor Degree Recipients, 1 Year After Graduation**  
**1976 to 2001**

	1974-75 graduates in May 1976	1979-80 graduates in May 1981	1983-84 graduates in June 1985	1985-86 graduates in June 1987	1989-90 graduates in June 1991	1999-00 graduates in July 2001
Percent employed full-time	67	71	73	74	74	84
Percent employed full-time in a job closely related to field of study	35	38	38	38	39	52

NOTE: Data are from sample surveys of recent college graduates. Data exclude bachelor's recipients from U.S. Service Schools. Deceased graduates and graduates living at foreign addresses at the time of the survey are not included. Data not available for 2001 on percent of graduates employed full-time in nonprofessional jobs.

SOURCE: U.S. Department of Education, National Center for Education Statistics, "Recent College Graduates" surveys, 1976 through 1991; and 2000/01 Baccalaureate and Beyond Longitudinal Study

The findings here can alternatively be related to differences across geographic areas. For instance, the stylized picture of the US economy over the last thirty years is of an economy with flexible labor markets (low matching frictions and hence low unemployment rates) as well as high but widely dispersed income growth. On the other hand, several European countries are generally viewed as having more rigid labor markets along with lower per capita income that is however more evenly distributed. The basic mechanism outlined in this paper suggests that these observations are consistent with matching frictions affecting the types of skills workers acquire and the resulting distribution of labor market earnings.

## 2. The Economic Environment

This section presents a (homogeneous) Roy (1951) occupational choice model with human capital acquisition and uncertain assignment. The model is highly stylized in order to highlight the basic insight and the underlying mechanism behind the idea that reducing matching frictions will not only increase income, productivity and employment but can further make income more disperse by altering investments in different types of risky skills.



As such, market frictions and job search in this model have a broad interpretation, one that does not exclusively correspond to the usual notion of frictions in the job matching literature.<sup>5</sup> Assignment frictions in this model include standard job search as well as other experiences involved in finding the right job fit. Even after training has been completed, it can take a fair amount of time and effort for an individual to discover whether they have the aptitude, desire and inclination to make it in a given occupation. The premise is that these frictions have been declining over time.

Assignment frictions may alternatively be interpreted along the lines proposed by Adam Smith. Suppose matching occurred repeatedly over an individual's life so that the second and third life stages (described below) became repeated episodes of search and employment. Extending the model in this way does not alter the basic investment decision when workers are risk neutral as they are here. With no mobility costs across sectors, maximizing the discounted sum of income is equivalent to maximizing expected per period earnings. The probabilities of finding primary and secondary sector jobs can then be interpreted as the proportion of time spent in each activity.<sup>6</sup>

### Agents

A unit mass of risk neutral, income maximizing agents populate a Lucas island economy. Individuals are homogeneous. All agents possess the same innate and known abilities  $a_1 > 0$  and  $a_2 > 0$  for use in one of two skilled occupations or sectors. Ability  $a_i$ ,  $i = 1, 2$  is useful only in sector  $i$  and agents can work in only one sector. Agents who do not obtain work in either of these sectors can enter an unskilled sector where output is normalized to zero.

Individuals experience three distinct life phases. In the first phase, an individual invests in a portfolio of occupation specific human capital skills. In the second phase, the individual explores islands in the economy for employment opportunities in these two

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<sup>5</sup> The motivating evidence above focuses fits with familiar search due to data scarcity and difficulties finding more general measures of assignment frictions.

<sup>6</sup> Allowing for multiple periods of repeated search and employment raises issues related to the observational content of individual income levels over time. In general, the results hold for a cross section of the population but observations over time are less clear cut and depend on the frequency and duration of the search and employment phases.

occupations. In the third phase, the individual chooses from the available opportunities to produce and consume output.

Homogeneity focuses attention on within group differences. Given within group results, comparisons among heterogeneous groups follow from comparative statics with respect to ability. Note as well that the homogenous type specification adopted here is more narrow than grouping individuals by total accumulated education levels. Since there are a potentially large number of different  $(a_1, a_2)$  groups who acquire the same total (observed) training, using total education can muddle the picture of within group effects.

### Human capital acquisition

Innate ability requires training to become productive. Endowed talent in each skilled sector becomes effective only when combined with occupation or sector specific human capital  $(h_1, h_2)$  acquired through training in the first phase of life. Given that skills are sector specific, if a worker fails to receive an employment opportunity in a particular sector, any acquired human capital for that sector becomes redundant. Moreover, when choosing a  $(h_1, h_2)$  portfolio, individuals knowingly face the possibility that the second phase of search might not uncover opportunities to produce in one or both of the skilled sectors.

Suppose an agent who acquires  $h_i$  units at cost  $c$  per unit can potentially produce sector  $i=1,2$  output  $a_i\phi(h_i)$  where  $\phi(h) = h^\alpha/\alpha$  and  $\alpha \in (0,1)$ . Although some results – those related to human capital investments – can be established for general specifications of the production function  $\phi(\cdot)$ , the parameterization adopted here permits the finding of explicit expressions for individual earnings and hence economy-wide income distribution. Without an explicit expression for  $\phi(\cdot)$  these results would not be available. On the other hand, the broad results do not necessarily depend on the explicit functional form. Similar remarks apply for the adoption of linear utility.

### Matching

Agents search (with recall) over islands throughout their entire second life phase. Each island contains an opportunity in one of the two skilled sectors. Opportunities within each sector are the same. Let  $\rho \in (0,1)$  represent the proportion of islands with

opportunities in sector 1. For example, if (as in the original Roy model) the two sectors are hunting and (fresh water) fishing, each island contains either a lake or a forest but not both.

Normalize the duration of this second phase to a unit of continuous time and let  $\lambda$  be the rate at which agents find new islands. As the arrival rate captures market thickness, the comparative static implications of having  $\lambda$  increase will be the focus of what follows.

A constant arrival rate of islands over the second life-phase period implies that the probability of encountering  $N$  islands is Poisson and denoted here by

$$\pi_N = \frac{e^{-\lambda} \lambda^N}{N!}$$

Given the proportion of opportunities across islands, the probability of finding at least one island with an opportunity in sector 1 is then given by

$$p_1(\lambda) = \sum_{N=1}^{\infty} \pi_N [1 - (1 - \rho)^N] = 1 - e^{-\lambda\rho}$$

Similarly the likelihood of finding at least one sector 2 employment opportunity is denoted by

$$p_2(\lambda) = \sum_{N=1}^{\infty} \pi_N [1 - \rho^N] = 1 - e^{-\lambda(1-\rho)}$$

The probability of finding one or more opportunity but all of them in sector 1 is likewise calculated as

$$\sum_{N=1}^{\infty} \pi_N \rho^N = e^{-\lambda(1-\rho)} - e^{-\lambda} = p_1(\lambda)[1 - p_2(\lambda)]$$

whereas the probability of having one or more offer but all of them exclusively in sector 2 is

$$\sum_{N=1}^{\infty} \pi_N (1 - \rho^N) = e^{-\lambda\rho} - e^{-\lambda} = p_2(\lambda)[1 - p_1(\lambda)]$$

Less fortunate individuals do not find any islands and find themselves in the unskilled, low productivity sector. This outcome occurs with probability

$$p_0(\lambda) = \pi_0 = e^{-\lambda}$$

With probability

$$1 - e^{-\lambda\rho} - e^{-\lambda(1-\rho)} + e^{-\lambda} = p_1(\lambda)p_2(\lambda)$$

agents are fortunate and find islands with both types of employment.

### Production and consumption

In the third phase of life, individuals produce and consume. To keep things straightforward, assume that agents consume their own production. Production goods are perfect substitutes. The absence of meaningful exchange at this stage prevents prices from adjusting in response to different allocations of agents across islands. This restriction, however, is not critical. It is possible to generate the basic results found here with more general preferences and trade that enable prices to adjust as agents' opportunities vary over time. The mechanism is less transparent and not developed here.

Individuals who find only one type of skilled occupation or do not find any skilled production opportunities have simple choices. When only one skilled sector opportunity is available, the lone skilled opportunity will be taken, provided any skills have been developed. Individuals with both skilled opportunities available have a slightly more complicated problem. They choose their most profitable occupation to work in based on abilities and human capital investments.

## **3. Investment and Specialization**

Market thickness along with innate abilities determine human capital decisions and the resulting distribution of (within group) income. An individual, endowed with talent in two distinct areas, initially allocates resources to separately develop these raw abilities. For example, depending on talent and job prospects, a university student may diversify by choosing to major in chemistry and minor in music. On the other hand, if the employment outlook in music is bleak, the student may forgo diversity and specialize even more in chemistry. With non-scholastic training, some professionals may choose to

earn a number of highly specialized and closely related qualifications - say by obtaining particular technical qualifications. Once training is complete, the individual looks for offers. In the musician/chemist example, the individual may come across places to play music or to work at a lab. In some instances, both opportunities turn up and the worker can choose which to pursue. In other instances, no opportunities exist and the individual works in the unskilled sector.

An agent with a choice of occupations will opt for the one generating the greatest income. Given human capital training decisions, sector 1 is chosen if and only if

$$a_1\phi(h_1) \geq a_2\phi(h_2)$$

The nontrivial decision an individual faces is to choose sector specific investments  $h_1$  and  $h_2$  to maximize expected income :

$$\begin{aligned} \max_{h_1, h_2} & p_1(1-p_2)a_1\phi(h_1) + (1-p_1)p_2a_2\phi(h_2) \\ & + p_1p_2 \max\{a_1\phi(h_1), a_2\phi(h_2)\} - c(h_1 + h_2) \end{aligned}$$

where agents take the matching probabilities  $p_i = p_i(\lambda)$  and  $p_2 = p_2(\lambda)$  exogenously.

**Proposition:** Agents specialize according to innate abilities and matching frictions. If

$$\frac{a_1}{a_2} \geq \frac{p_2(\lambda)}{p_1(\lambda)} \left[ \frac{1 - (1 - p_1(\lambda))^{1/(1-\alpha)}}{1 - (1 - p_2(\lambda))^{1/(1-\alpha)}} \right]^{1-\alpha} \quad (1)$$

then

$$h_1 = \left( \frac{p_1(\lambda)a_1}{c} \right)^{1/(1-\alpha)} \quad (2)$$

$$h_2 = \left( \frac{p_2(\lambda)(1 - p_1(\lambda))a_2}{c} \right)^{1/(1-\alpha)}$$

else

$$h_1 = \left( \frac{p_1(\lambda)(1 - p_2(\lambda))a_1}{c} \right)^{1/(1-\alpha)} \quad (3)$$

$$h_2 = \left( \frac{p_2(\lambda)a_2}{c} \right)^{1/(1-\alpha)}$$

**Proof:** Let

$$EY_1 = p_1 a_1 \phi(h_1) + p_2(1-p_1)a_2 \phi(h_2) - c(h_1 + h_2)$$

and

$$EY_2 = p_1(1-p_2)a_1 \phi(h_1) + p_2 a_2 \phi(h_2) - c(h_1 + h_2)$$

Notice that

$$\left( \left( \frac{p_1 a_1}{c} \right)^{1/(1-\alpha)}, \left( \frac{p_2(1-p_1)a_2}{c} \right)^{1/(1-\alpha)} \right)$$

Maximizes  $EY_1$  and that

$$\left( \left( \frac{p_1(1-p_2)a_1}{c} \right)^{1/(1-\alpha)}, \left( \frac{p_2 a_2}{c} \right)^{1/(1-\alpha)} \right)$$

maximizes  $EY_2$ .

Given these solutions,  $EY_1 \geq EY_2$  if and only if condition (1) holds. Note as well that

$$a_1 \phi(h_1) \geq a_2 \phi(h_2) \Leftrightarrow \frac{a_1}{a_2} \geq \left( \frac{p_1}{p_2} (1-p_1) \right)^\alpha$$

a condition that holds if (1) holds. If (1) holds with equality, (3) is, of course, also a solution. ■

Condition (1) which can be re-written in terms of the underlying matching frictions

$$\frac{a_1}{a_2} \geq \frac{1 - e^{-\lambda(1-\rho)}}{1 - e^{-\lambda\rho}} \left[ \frac{1 - e^{-\lambda\rho/(1-\alpha)}}{1 - e^{-\lambda(1-\rho)/(1-\alpha)}} \right]^{1-\alpha} \quad (4)$$

ranks occupations. If this condition holds, sector 1 becomes the primary occupation or major field and sector 2 becomes the secondary occupation or minor field. Note that a preference for sector 1 does not imply  $h_1 \geq h_2$ , that is, observing  $h_1 > h_2$  does not imply that sector  $i$  is preferred. A low probability of matching coupled with sufficiently high

ability in a sector can induce less training for that sector but if the choice arises, the high ability induces the worker to prefer that sector.<sup>7</sup>

Depending on abilities, individuals may switch their favored sector as markets become thicker. The partition from (4) rotates as  $\lambda$  evolves. In the limit as markets become very thick,  $\lambda \rightarrow \infty$ , frictions disappear and this condition simplifies to  $a_1 \geq a_2$ . The primary occupation is simply the innately more productive one. On the other hand,

$$\lim_{\lambda \rightarrow 0} \frac{1 - e^{-\lambda(1-\rho)}}{1 - e^{-\lambda\rho}} \left[ \frac{1 - e^{-\lambda\rho/(1-\alpha)}}{1 - e^{-\lambda(1-\rho)/(1-\alpha)}} \right]^{1-\alpha} = \left[ \frac{1-\rho}{\rho} \right]^\alpha$$

In very thin markets with  $\rho < 1/2$ , agents prefer sector 1 provided their innate talent for that sector is sufficiently high –  $a_1/a_2 > [(1-\rho)/\rho]^\alpha$ . For intermediate cases where

$$1 < a_1/a_2 < [(1-\rho)/\rho]^\alpha,$$

and  $\lambda$  is small, agents prefer sector 2 but as markets thicken and  $\lambda$  increases they will at some point prefer sector 1. As shown in the appendix, (4) is monotonic in  $\lambda$  so that this switch can occur at most once.

Since individuals may switch sectors as markets expand, two steps are considered in order to gauge the effects of market thickness. The first step considers the impact of market growth for a given  $(a_1, a_2)$  within a preferred sector, that is without a change in the primary sector. As illustrated below, a rise (over time for instance) in matching probabilities represented by a rise in  $\lambda$  not only induces an increase in overall training, (a natural result), but also alters the composition of the human capital portfolio. Expected income rises in thicker markets but the dispersion changes as well. The second step weighs the impact on decisions and economic outcomes when individuals swap preferred sectors.

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<sup>7</sup> If

$$2e^{-\lambda\rho/(1-\alpha)} - e^{-\lambda/(1-\alpha)} > 1$$

then any individual who prefers sector 1, that is satisfies (1), will have  $h_1 < h_2$ , a condition that is independent of ability and holds for  $\lambda$  sufficiently small.

### Within sector impact

Suppose  $(a_1, a_2)$  and  $\lambda$  are such that  $p_1(\lambda)a_1$  is sufficiently large relative to  $p_2(\lambda)a_2$  to make  $a_1\phi(h_1) > a_2\phi(h_2)$ . Sector 1 is the primary sector. The impact of market thickness is straightforward. Primary sector human capital

$$h_1 = \left( \frac{p_1 a_1}{c} \right)^{1/(1-\alpha)} = \left( \frac{(1 - e^{-\lambda\rho}) a_1}{c} \right)^{1/(1-\alpha)}$$

rises

$$\frac{\partial h_1}{\partial \lambda} = \frac{\rho e^{-\lambda\rho} h_1}{(1-\alpha)(1 - e^{-\lambda\rho})} > 0$$

as markets become thicker. Improved employment probabilities induce workers to invest more in the primary sector.

On the other hand, the likelihood of ending up in the secondary sector depends on the probability of not finding a primary sector position as well as the probability of finding a secondary sector opportunity. As a result, investment in secondary occupation skills

$$h_2 = \left( \frac{p_2(1-p_1)a_2}{c} \right)^{1/(1-\alpha)} = \left( \frac{(e^{-\lambda\rho} - e^{-\lambda})a_2}{c} \right)^{1/(1-\alpha)}$$

will rise or fall depending on the way in which the overall likelihood of winding up in the secondary sector evolves. In particular,

$$\frac{\partial h_2}{\partial \lambda} = \frac{(e^{-\lambda} - \rho e^{-\lambda\rho})h_2}{(1-\alpha)(e^{-\lambda\rho} - e^{-\lambda})}$$

Secondary occupation training rises as markets thicken from relatively thin positions, that is, if and only if

$$\lambda < \frac{-\ln \rho}{(1-\rho)}$$

Although secondary training may either rise or fall, the increase in primary training is always sufficiently large so that both total investment in human capital,  $h_1+h_2$ , and the portfolio spread,  $h_1 - h_2$ , increase with  $\lambda$ . The impact of matching frictions on



specialization, defined here as the proportion of total human capital in the primary sector (even though it is feasible that  $h_1 < h_2$ ), follows accordingly.

$$\frac{\partial}{\partial \lambda} \left( \frac{h_1}{h_1 + h_2} \right) > 0$$

As primary sector matching becomes more likely, workers become more specialized. (See the appendix for details.)

This pattern of human capital portfolio decision making aligns with Adam Smith's idea that workers become more specialized as the markets become thicker in the sense that there are better trading opportunities, at least in the primary occupations of individuals.

"The division of labor ...occasions, in every art, a proportionable increase of the productive powers of labor. The separation of different trades and employments from one another, seems to have taken place, in consequence of this advantage. This separation too is generally carried furthest in those countries which enjoy the highest degree of industry " (p. 5)

### Sector shifting

As market thickness rotates (1), the preferred sector can change. Such a switch has no impact on the total amount of training acquired; however, the portfolio becomes more balanced.

When (1) holds with equality, individuals do not favor a particular sector; however, those who opt for making sector 1 their preferred choice acquire the same total level of training as those who opt for sector 2:

$$\left( \frac{p_1 a_1}{c} \right)^{\chi_{1-\alpha}} + \left( \frac{p_2 (1-p_1) a_2}{c} \right)^{\chi_{1-\alpha}} = \left( \frac{p_2 a_2}{c} \right)^{\chi_{1-\alpha}} + \left( \frac{p_1 (1-p_2) a_1}{c} \right)^{\chi_{1-\alpha}}$$

which holds if and only if

$$\frac{a_1}{a_2} = \frac{p_2}{p_1} \left[ \frac{1 - (1-p_1)^{1/(1-\alpha)}}{1 - (1-p_2)^{1/(1-\alpha)}} \right]^{1-\alpha}$$

At the occupational partition, total acquired training,  $h_1+h_2$ , is continuous not only in abilities but also in matching probabilities and hence market thickness  $\lambda$ .

Although switching sectors has no effect on the level of total training, discrete changes occur in sector specific investments, employment rates and specialization. In the newly preferred sector, investments and employment jump up whereas the old preferred sector experiences a discrete drop. The more interesting concern is whether the portfolio becomes more or less balanced.

Suppose  $h_2'$  satisfies (3) and thus denotes the choice of sector 2 training when sector 2 is the primary sector. If (1) holds with equality, then

$$\frac{h_1}{h_2'} = \left( \frac{(1-e^{-\lambda\rho})a_1}{(1-e^{-\lambda(1-\rho)})a_2} \right)^{1/\alpha} = \frac{1-e^{-\lambda\rho/(1-\alpha)}}{1-e^{-\lambda(1-\rho)/(1-\alpha)}}$$

Since total training remains constant, it immediately follows that the portfolio spread in training as well as specialization depend on whether  $p_1 > p_2$  at the switching point for  $(a_1, a_2)$ . Consider a switch from preferring sector 2 to preferring sector 1. It must be the case that

$$1 < a_1/a_2 < [(1-\rho)/\rho]^\alpha,$$

and  $\rho < 1/2$ . It then follows that when the switch occurs,  $h_1 < h_2'$ . In other words, after the change occurs, investment becomes more even across potential occupations. Individuals appear less specialized when the fallback option turns into the primary occupation.

## 4. Income

Human capital adjustments lead directly to changes in the expected value as well as the distribution of income. As noted above, total investment in human capital increases as the extent of the market, captured by  $\lambda$ , grows and the  $p_i$  rise. Suppose again that  $a_1 > a_2$  and  $\lambda$  are such that sector 1 is preferred. Higher investments yield higher mean or expected (gross) earnings

$$EY = p_1 a_1 \phi(h_1) + p_2 (1-p_1) a_2 \phi(h_2) = c(h_1 + h_2)/\alpha$$

as well as total higher employment in skilled occupations.

A more interesting pattern, one that reflects the changing job prospects of workers in the primary and secondary occupations, emerges when considering the impact of matching on the (within group) distribution of income. With better primary job prospects, investments become more specialized. Individuals who obtain positions in the (more likely) primary occupation have more skills and hence greater earnings. The less fortunate who wind up with their second choice have relatively less secondary training. Recall that  $\partial(h_1 - h_2)/\partial\lambda < 0$ . Indeed, it may turn out to be the case that these secondary sector workers acquire absolutely less training if  $\partial h_2/\partial\lambda < 0$ . In either case, the spread in skilled sector earnings,  $S = a_1\phi(h_1) - a_2\phi(h_2)$ , rises in thicker markets:

$$\frac{\partial S}{\partial\lambda} = \frac{\alpha}{1-\alpha} \left( \frac{a_1\phi(h_1)}{1-e^{-\lambda\rho}} \rho e^{-\lambda\rho} - \frac{a_2\phi(h_2)}{e^{\lambda\rho} - e^{-\lambda}} [e^{-\lambda} - \rho e^{-\lambda\rho}] \right)$$

(See appendix.)

Income inequality, measured by the Gini coefficient, can likewise rise or fall as the market thickens. Given  $(a_1, a_2)$ , the Gini for earned income is

$$G = \frac{\Delta_{12} + \Delta_{10} + \Delta_{20}}{EY}$$

where

$$\Delta_{12} = p_1(1-p_1)p_2[a_1\phi(h_1) - a_2\phi(h_2)]$$

$$\Delta_{10} = p_1(1-p_1)(1-p_2)a_1\phi(h_1)$$

$$\Delta_{20} = p_2(1-p_1)^2(1-p_2)a_2\phi(h_2)$$

represent the respective income differences between sectors 1 and 2, between sector 1 and the unskilled sector, and between sector 2 and the unskilled sector, each multiplied by the number of people involved.

Plugging in the human capital choices gives

$$\begin{aligned}
G &= 1 - p_1 - [p_1 + p_2(1 - p_1)] \frac{h_2}{h_1 + h_2} \\
&= e^{-\lambda\rho} - [1 - e^{-\lambda}] \frac{h_2}{h_1 + h_2}
\end{aligned}$$

The impact of market thickness on this Gini decomposes into the direct effects of matching probabilities  $(p_1, p_2)$  and the indirect changes to portfolio specialization. The direct component reduces inequality

$$\frac{\partial G}{\partial \lambda}_{|h_1, h_2} = -\rho e^{-\lambda\rho} - e^{-\lambda} \frac{h_2}{h_1 + h_2} < 0$$

Fewer individuals obtain low paying unskilled jobs and more earn primary sector skilled pay. These transitions lower inequality. If individuals maintained their initial investment levels, the Gini would decline as more individuals entered the primary sector.

The accompanying rise in specialization, on the other hand, causes greater income inequality. As primary sector matching improves, skilled income spreads creating greater inequality among the high and low earners, thereby offsetting the income equality gains that come directly from better matching prospects. Abusing notation somewhat, the effect through specialization on the Gini is given by :

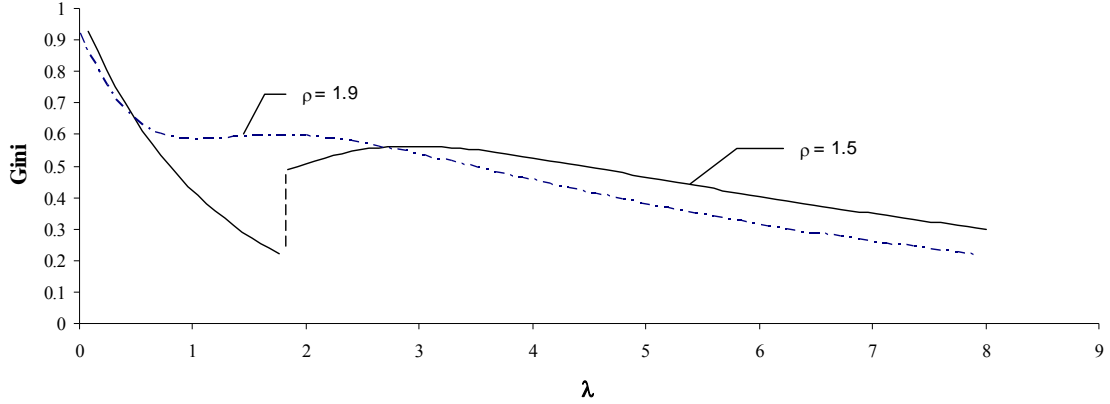
$$\frac{\partial G}{\partial \lambda}_{|p_1, p_2} = [1 - e^{-\lambda}] \frac{\partial [h_2 / (h_1 + h_2)]}{\partial \lambda} > 0$$

The net effect is ambiguous. For  $\lambda$  sufficiently large, this derivative is negative. Nearly all individuals are earning high, primary sector earnings. On the other hand, it is straightforward to find cases where earnings inequality increases with  $\lambda$ . For example, given  $(a_1, a_2, \alpha, \rho) = (12, 4, 0.75, 0.15)$ , the derivative of the Gini evaluated at  $\lambda = 1.80$  is roughly 0.127.

Figure 1 traces out the way in which earnings inequality in this example varies with  $\lambda$ . Following the sector switch that generates a discrete positive jump, the Gini exhibits an inverted U-shape progression as matching improves. This progression, however, is not unique. A variety of shapes emerge from varying the parameters. For instance, suppose in this example the proportion of sector 1 islands rises to  $\rho = 0.19$  which is sufficiently large

to make sector 1 the primary sector for all  $\lambda$ . In this new configuration, the Gini decreases for low and high  $\lambda$  but increases over an interior range of  $\lambda$ , thereby creating a local max. For  $\rho = 0.25$  in this example, the local max does not occur. Earnings inequality is monotonically decreasing.

**Figure 1:  
Income Inequality Gini**



These examples reveal that improved matching associated with thicker markets do not necessarily lead to predictions of either more or less earnings equality.

### Sector switching

By construction, agents have equal expected income when the evolution of  $\lambda$  induces a move from preferring one occupation to the other. Income dispersion, however, is altered when this transition occurs. Let  $h'_1, h'_2, S'$  denote the human capital acquisitions and the spread between primary and secondary income given sector 2 is favored. The impact of switching sectors on the spread of skilled earnings is then given by

$$\begin{aligned}
 S - S' &= a_1\phi(h_1) - a_2\phi(h_2) - a_2\phi(h'_2) + a_1\phi(h'_1) \\
 &= \frac{h_1}{p_1} [1 + (1 - p_2)^{\rho-1}] - \frac{h'_2}{p_2} [1 + (1 - p_1)^{\rho-1}] \\
 &= h'_2 \frac{[1 - (1 - p_1)^{\rho-1}]}{p_1 p_2} \left[ p_2 \frac{1 + (1 - p_2)^{\rho-1}}{1 - (1 - p_2)^{\rho-1}} - p_1 \frac{1 + (1 - p_1)^{\rho-1}}{1 - (1 - p_1)^{\rho-1}} \right]
 \end{aligned}$$

where use has been made of the equality of equation (1).

Recall that if an individual switches primary sectors from sector 2 to sector 1, it must be the case, among other things, that  $p_1 < p_2$ . Given this restriction, it is then straightforward to establish that  $S - S' > 0$  if and only if  $\alpha > 1/2$ .

Two factors underlie this outcome. As shown above, when individuals switch their preferred sector, they invest more evenly in occupational human capital. The more balanced portfolio reduces the income spread. On the other hand, after the switch occurs, individuals are now investing in their innately more gifted ability - a shift from sector 2 to sector 1 requires abilities such that  $1 < a_1/a_2 < [(1-\rho)/\rho]^\alpha$  where  $\rho < 1/2$ . Increased investment in higher innate ability spreads primary and secondary income. The net effect depends on the curvature of the human capital production function.

The Gini can likewise rise or fall at the occupational switch. As individuals swap sectors, a discrete jump occurs. Suppose  $G$  and  $G'$  represent the Gini when occupation 1 and occupation 2 are the preferred sector. Since total investments are equal at the switching threshold, the jump in inequality when (1) holds is given by

$$G - G' = \frac{h_2'}{h_1 + h_2} \left\{ -p_1 [1 + (1 - p_2)(1 - p_1)^{\lambda - \alpha}] + p_2 [1 + (1 - p_1)(1 - p_2)^{\lambda - \alpha}] \frac{1 - (1 - p_1)^{\lambda - \alpha}}{1 - (1 - p_2)^{\lambda - \alpha}} \right\}$$

It is not difficult to find examples in which the earnings inequality jump from this expression can discretely rise or fall when agents switch sectors.

## 5. Comparisons Across Types

To assess differences in earnings across groups, the paper turns now to the way in which changes (induced by market expansion) in expected gross earnings as well as changes in the distribution of earnings vary by ability.<sup>8</sup> Although definite predictions do not emerge for all  $\lambda$ , the analysis demonstrates the way in which thicker markets can provide greater benefits on average for the more well-off as well as create more

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<sup>8</sup>The difference between gross and net income is given by training costs. Since  $EY = c(h_1 + h_2)/\alpha$  this distinction involves a constant fraction independent of ability and matching rates. The results do not differ in a substantive way using either net or gross income.

pronounced dispersion within more affluent groups, results that accord with some recent observations mentioned above.<sup>9</sup>

Again assume that agents all prefer the same sector - sector 1 - and consider first the relationship between ability and the changes in expected income due to market expansion. The cross partials for the two abilities are:

$$\frac{\partial^2 EY}{\partial \lambda \partial a_1} = \frac{c \rho e^{-\lambda \rho} h_1}{\alpha (1 - \alpha)^2 (1 - e^{-\lambda \rho}) a_1} > 0$$

$$\frac{\partial^2 EY}{\partial \lambda \partial a_2} = \frac{c (e^{-\lambda} - \rho e^{-\lambda \rho}) h_2}{\alpha (1 - \alpha)^2 (e^{-\lambda \rho} - e^{-\lambda}) a_2}$$

If groups differ in primary sector abilities, the resulting difference in expected income becomes more pronounced as matching prospects improve. The more able have more to gain from better matching and this difference shows up directly in  $\partial^2 EY / \partial \lambda \partial a_1$ .

The picture is less transparent when the discrepancy in abilities lies in the secondary sector. The impact of market expansion mirrors the events occurring in the secondary sector human capital acquisition. In thin markets (where  $\lambda < -\ln \rho / (1 - \rho)$  and  $\partial h_2 / \partial \lambda > 0$ ), market expansion increases the overall likelihood of being in that sector. The more able secondary sector workers are positioned to take greater advantage of the expanding market. As a result, expected earnings differentials rise as market thickness expands:  $\partial^2 EY / \partial \lambda \partial a_2 > 0$ . In thicker markets where  $\lambda > -\ln \rho / (1 - \rho)$ , the converse holds. Individuals have lower expectations of winding up in the secondary sector. The change is more prominent for more able workers causing earnings disparity across groups to decline with market thickness.

Income dispersion across groups can be assessed in a similar way. The income spread among skilled workers  $S = a_1 \phi(h_1) - a_2 \phi(h_2)$  responds as investment and expected income respond to market thickness:

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<sup>9</sup> Due to the particular specification of  $\phi(h)$ , the return to education is  $EY / (h_1 + h_2) = c / \alpha$  is constant with respect to matching. A more general specification allows for an increasing or decreasing return.

$$\frac{\partial^2 S}{\partial \lambda \partial a_1} = \frac{\alpha \rho e^{-\lambda \rho} \phi(h_1)}{(1-\alpha)^2 (1-e^{-\lambda \rho})} > 0$$

$$\frac{\partial^2 EY}{\partial \lambda \partial a_2} = \frac{-\alpha (e^{-\lambda} - \rho e^{-\lambda \rho}) \phi(h_2)}{(1-\alpha)^2 (e^{-\lambda \rho} - e^{-\lambda})}$$

The spread in earnings becomes more pronounced between groups who differ in their primary sector ability. The skilled income spread becomes less pronounced with changes that raise secondary sector human capital acquisition. The mechanisms behind these results follow the logic behind the across group earning differentials.

The relationship between the changing spread and abilities does not entirely carry through to changes in earnings inequality. When matching frictions decline, the impact on the Gini change due to ability are given by :

$$\frac{\partial^2 G}{\partial \lambda \partial a_1} = \frac{-h_1 h_2}{a_1 (1-\alpha) (h_1 + h_2)^2} \Lambda$$

$$\frac{\partial^2 G}{\partial \lambda \partial a_2} = \frac{h_1 h_2}{a_2 (1-\alpha) (h_1 + h_2)^2} \Lambda$$

where

$$\Lambda = e^{-\lambda} + \frac{(1-e^{-\lambda})(e^{-\lambda} - \rho e^{-\lambda \rho})}{(1-e^{-\lambda \rho})(e^{-\lambda} - e^{-\lambda \rho})} \frac{(h_1 - h_2)}{(1-\alpha)(h_1 + h_2)}$$

These cross partials have similar forms that differ in sign and by  $a_i$ . The sign of  $\Lambda$  depends on both  $e^{-\lambda} - \rho e^{-\lambda \rho}$  and  $h_1 - h_2$ . For  $\lambda$  small,  $e^{-\lambda} - \rho e^{-\lambda \rho} > 0$  and  $h_1 - h_2 < 0$ . These factors switch sign at different values of  $\lambda$ , so comparisons across groups do not exhibit clear patterns. However,  $\Lambda < 0$  as markets become sufficiently thick, so that dispersion rises (falls) among groups that are ordered by increasing primary (secondary) ability. Although the Gini is a more involved measure of dispersion than the spread in earnings, the driving factors none the less can be seen at play in the Gini results.

## 6. Discussion

Over the last hundred years, income has grown steadily yet the distribution has exhibited an uneven progression, see for example Goldin and Katz, (1999) and Goldin



and Margo (1992). While a number of opposing factors may be at play, this paper argues that this pattern can follow from a gradual improvement of matching rates in different occupations. A smooth convergence or divergence should not necessarily be expected as the evolution of the distribution of income can exhibit arbitrary patterns.

The key factor is within group specialization in sector specific human capital. Improved matching in the primary sector generates greater specialization which in turn leads to greater income disparity between the well matched and the poorly matched. Increased residual disparity is more pronounced among the more able.

Other factors undeniably alter the distribution of income, both within and across groups. The mechanism outlined here complements these other factors by offering a simple explanation for the rise of within group earnings dispersion.

## References

- Card, D. and T. Lemieux, 2002, "Skilled-Biased Technological Change and Rising Wage Inequality: Some Problems and Puzzles," Journal of Labor Economics, 20:733-83.
- Eleftheriou, K., 2008, "Matching Specialties and Wage Inequality," Economics Bulletin, 11:1-12.
- Goldin, C. and L. Katz, 1999, "The Returns to Skill in the United States Across the Twentieth Century," NBER Working Paper No. 7126.
- Heckman, J. and B. Honore, 1990, "The Empirical Content of the Roy Model," Econometrica, 58:1121-1149.
- Heckman, J. and G. Sedlacek, 1985, "Heterogeneity, Aggregation and Market Wage Functions: An Empirical Model of Self-Selection in the Labor Market," Journal of Political Economy, 93:1077-1125.
- Kambourov, G. and I. Manovskii, 2004, "Occupational Mobility and Wage Inequality," PIER Working Paper, University of Pennsylvania.
- Katz, L. and D. Autor, 1999, "Changes in the Wage Structure and Earnings Inequality," in O. Ashenfelter and D. Card (eds.) Handbook of Labor Economics, Volume 3A, North-Holland.
- Lemieux, T., 2006, "Increased Residual Wage Inequality: Composition Effects, Noisy Data, or Rising Demand for Skill," American Economic Review, 96:461-498.
- Lloyd Ellis, H., 1999, "Endogenous Technological Change and Wage Inequality," American Economic Review, 89:47-77.
- Machin, S. and van Reenen, 2007, "Changes in Wage Inequality," CEP Special Paper No. 18, London School of Economics.
- Roy, A., 1951, "Some Thoughts on the Distribution of Earnings," Oxford Economic Papers (New Series) 3:135-146.
- Sattinger, M., 2007, "Assignment, Hierarchies and Upper Tails" mimeo, SUNY Albany
- Smith, A. 1776, The Wealth of Nations, Cannan edition, The Modern Library, New York.
- Van Reenen, J., 1996, "The Creation and Capture of Economic Rents: Wages and Innovation in a Panel of UK Companies," Quarterly Journal of Economics 111:195-226.
- Willis, R. and S. Rosen, 1979, "Education and Self-Selection," Journal of Political Economy, 87:S7-S36

## APPENDIX

This appendix presents the details for the comparative statics discussed in the text. To ease notation in what follows, let  $\theta = 1/(1-\alpha) > 1$

**Claim 1:** Equation (4) is monotonic in  $\lambda$ .

Proof : Two intermediate results are first established.

$$\text{Lemma 1: } \frac{\partial}{\partial \rho} \left[ \frac{\rho e^{-\lambda \rho \theta}}{1 - e^{-\lambda \rho \theta}} \right] = \frac{e^{-\lambda \rho \theta}}{[1 - e^{-\lambda \rho \theta}]^2} \{1 - e^{\lambda \rho \theta} - \lambda \rho \theta\} < 0$$

since  $1 - e^{-x} - x$  is decreasing in  $x$  and equal to zero at  $x = 0$ . ■

$$\text{Lemma 2: } \frac{\partial}{\partial \theta} \left[ \frac{\rho e^{-\lambda \rho \theta}}{1 - e^{-\lambda \rho \theta}} - \frac{(1-\rho)e^{-\lambda(1-\rho)\theta}}{1 - e^{-\lambda(1-\rho)\theta}} \right] = \frac{-\lambda \rho^2 e^{-\lambda \rho \theta}}{[1 - e^{-\lambda \rho \theta}]^2} + \frac{\lambda(1-\rho)^2 e^{-\lambda(1-\rho)\theta}}{[1 - e^{-\lambda(1-\rho)\theta}]^2} < 0 \quad \text{iff } \rho < 1/2$$

To establish this result, first note that

$$\frac{\partial}{\partial \theta} \left[ \frac{\rho e^{-\lambda \rho \theta}}{1 - e^{-\lambda \rho \theta}} - \frac{(1-\rho)e^{-\lambda(1-\rho)\theta}}{1 - e^{-\lambda(1-\rho)\theta}} \right]_{|\rho=1/2} = \frac{-\lambda \rho^2 e^{-\lambda \rho \theta}}{[1 - e^{-\lambda \rho \theta}]^2} + \frac{\lambda(1-\rho)^2 e^{-\lambda(1-\rho)\theta}}{[1 - e^{-\lambda(1-\rho)\theta}]^2} \Big|_{\rho=1/2} = 0$$

It is thus sufficient to show

$$\frac{\partial}{\partial \rho} \left[ \frac{\rho^2 e^{-\lambda \rho \theta}}{[1 - e^{-\lambda \rho \theta}]^2} \right] = \frac{\rho e^{-\lambda \rho \theta}}{[1 - e^{-\lambda \rho \theta}]^3} F(\lambda \rho \theta) < 0$$

where  $F(\lambda \rho \theta) \equiv 2 - 2e^{-\lambda \rho \theta} - \lambda \rho \theta - \lambda \rho \theta e^{-\lambda \rho \theta}$ .

$$\frac{\partial F(\lambda \rho \theta)}{\partial(\lambda \rho \theta)} = -1 + e^{-\lambda \rho \theta} + \lambda \rho \theta e^{-\lambda \rho \theta}$$

Since  $F(0) = 0$ ,  $\partial F(\lambda \rho \theta) / \partial(\lambda \rho \theta) |_{\lambda \rho \theta=0} = 0$  and

$$\frac{\partial^2 F(\lambda \rho \theta)}{\partial(\lambda \rho \theta)^2} = -\lambda \rho \theta e^{-\lambda \rho \theta} < 0$$

It follows that  $F(\lambda \rho \theta) < 0 \quad \forall \lambda \rho \theta > 0$  ■

To establish the claim, let

$$e^{M(\lambda)} = \left[ \frac{1 - e^{-\lambda(1-\rho)}}{1 - e^{-\lambda\rho}} \right]^\theta \left[ \frac{1 - e^{-\lambda\rho\theta}}{1 - e^{-\lambda(1-\rho)\theta}} \right]$$

Taking logs gives

$$M(\lambda) = \theta \ln[1 - e^{-\lambda(1-\rho)}] - \theta \ln[1 - e^{-\lambda\rho}] + \ln[1 - e^{-\lambda\rho\theta}] - \ln[1 - e^{-\lambda(1-\rho)\theta}]$$

$$M'(\lambda) = \theta[A - B + C - D]$$

where

$$A = \frac{(1-\rho)e^{-\lambda(1-\rho)}}{1 - e^{-\lambda(1-\rho)}}, B = \frac{\rho e^{-\lambda\rho}}{1 - e^{-\lambda\rho}}, C = \frac{\rho e^{-\lambda\rho\theta}}{1 - e^{-\lambda\rho\theta}} \text{ and } D = \frac{(1-\rho)e^{-\lambda(1-\rho)\theta}}{1 - e^{-\lambda(1-\rho)\theta}}$$

*Case 1:*  $\rho < 1/2$ . From Lemma 1,  $C - D > 0$ . From Lemma 2,

$$C - D > \frac{\rho e^{-\lambda\rho}}{1 - e^{-\lambda\rho}} - \frac{(1-\rho)e^{-\lambda(1-\rho)}}{1 - e^{-\lambda(1-\rho)}} = B - A$$

since  $\theta > 1$ . Therefore,  $A - B + C - D > 0$  for all  $\rho < 1/2$ .

*Case 2:*  $\rho > 1/2$ . From Lemma 1,  $A - B > 0$ . From Lemma 2,

$$D - C = \frac{(1-\rho)e^{-\lambda(1-\rho)\theta}}{1 - e^{-\lambda(1-\rho)\theta}} - \frac{\rho e^{-\lambda\rho\theta}}{1 - e^{-\lambda\rho\theta}} > \frac{(1-\rho)e^{-\lambda(1-\rho)}}{1 - e^{-\lambda(1-\rho)}} - \frac{\rho e^{-\lambda\rho}}{1 - e^{-\lambda\rho}} = B - A$$

hence  $A - B + C - D < 0$  for all  $\rho > 1/2$ . QED

**Claim 2:** Total human capital accumulation is increasing in market thickness:

$$\partial(h_1 + h_2) / \partial\lambda > 0$$

Proof:

$$\begin{aligned} \frac{\partial(h_1 + h_2)}{\partial\lambda} &= \theta \left\{ \frac{[(1 - e^{-\lambda\rho})a_1]^\theta}{1 - e^{-\lambda\rho}} \rho e^{-\lambda\rho} - \frac{e^{-\lambda\rho\theta} [(1 - e^{-\lambda(1-\rho)})a_2]^\theta}{e^{-\lambda\rho} - e^{-\lambda}} (\rho e^{-\lambda\rho} - e^{-\lambda}) \right\} \\ &= \frac{\theta \rho e^{-\lambda\rho} [1 - e^{-\lambda(1-\rho)}] a_2}{1 - e^{-\lambda\rho}} \left\{ \frac{[(1 - e^{-\lambda\rho})a_1]^\theta}{[(1 - e^{-\lambda(1-\rho)})a_2]^\theta} - \frac{e^{-\lambda\rho\theta} [(1 - e^{-\lambda\rho})(\rho - e^{-\lambda(1-\rho)})a_2]^\theta}{\rho e^{-\lambda\rho} [1 - e^{-\lambda(1-\rho)}]} \right\} \end{aligned}$$

$$\begin{aligned}
&> \frac{\theta \rho e^{-\lambda \rho} [(1 - e^{-\lambda(1-\rho)}) a_2]^\theta}{1 - e^{-\lambda \rho}} \left\{ \frac{1 - e^{-\lambda \rho \theta}}{1 - e^{-\lambda(1-\rho)\theta}} - \frac{e^{-\lambda \rho(\theta-1)} (1 - e^{-\lambda \rho}) (1 - \frac{1}{\rho} e^{-\lambda(1-\rho)})}{1 - e^{-\lambda(1-\rho)}} \right\} \\
&= \frac{\theta \rho e^{-\lambda \rho(\theta+1)} [(1 - e^{-\lambda(1-\rho)}) a_2]^\theta}{1 - e^{-\lambda \rho}} \left\{ \frac{e^{\lambda \rho \theta} - 1}{1 - e^{-\lambda(1-\rho)\theta}} - \frac{(e^{\lambda \rho} - 1) (1 - \frac{1}{\rho} e^{-\lambda(1-\rho)})}{1 - e^{-\lambda(1-\rho)}} \right\}
\end{aligned}$$

where the inequality above derives from (4). Define

$$G_1(\theta) \equiv \frac{e^{\lambda \rho \theta} - 1}{1 - e^{-\lambda(1-\rho)\theta}} - \frac{(e^{\lambda \rho} - 1) (1 - \frac{1}{\rho} e^{-\lambda(1-\rho)})}{1 - e^{-\lambda(1-\rho)}}$$

Note that  $\lim_{\theta \rightarrow \infty} G_1(\theta) = \infty$  and

$$\begin{aligned}
G_1(1) &\equiv \frac{(e^{\lambda \rho} - 1) e^{-\lambda(1-\rho)}}{\rho(1 - e^{-\lambda(1-\rho)})} > 0 \\
G_1'(\theta) &= \frac{1}{[1 - e^{-\lambda(1-\rho)\theta}]^2} \left\{ \rho \lambda e^{\lambda \rho \theta} [1 - e^{-\lambda(1-\rho)\theta}] - (e^{\lambda \rho \theta} - 1) \lambda (1 - \rho) e^{-\lambda(1-\rho)\theta} \right\} \\
&= \frac{\lambda e^{\lambda \rho \theta}}{[1 - e^{-\lambda(1-\rho)\theta}]^2} G_2(\rho)
\end{aligned}$$

where  $G_2(\rho) \equiv \rho(1 - e^{-\lambda \theta}) + e^{-\lambda \theta} - e^{-\lambda(1-\rho)\theta}$ . Since  $G_2(0) = G_2(1) = 0$  and

$$G_2''(\rho) = -(\lambda \theta) 2e^{-\lambda(1-\rho)\theta} < 0,$$

it follows that  $G_2(\rho) > 0$  for all  $\rho \in (0, 1)$ . It then follows that  $G_1(\theta) > 0$  which is sufficient to establish the claim QED.

**Claim 3:** Specialization is increasing in market thickness:

$$\frac{\partial [h_1 / (h_1 + h_2)]}{\partial \lambda} = \frac{h_2 \frac{\partial h_1}{\partial \lambda} - h_1 \frac{\partial h_2}{\partial \lambda}}{(h_1 + h_2)^2} > 0.$$

$$\begin{aligned}
\text{Proof: } \frac{h_2 \frac{\partial h_1}{\partial \lambda} - h_1 \frac{\partial h_2}{\partial \lambda}}{(h_1 + h_2)^2} &= \frac{\theta h_1 h_2}{(h_1 + h_2)^2} \left( \frac{\rho e^{-\lambda \rho}}{1 - e^{-\lambda \rho}} + \frac{\rho e^{-\lambda \rho} - e^{-\lambda}}{e^{-\lambda \rho} - e^{-\lambda}} \right) \\
&= \frac{\theta h_1 h_2 e^{-\lambda \rho}}{(h_1 + h_2)^2 (1 - e^{-\lambda \rho}) (e^{-\lambda \rho} - e^{-\lambda})} H(\rho)
\end{aligned}$$

where  $H(\rho) = \rho(1-e^{-\lambda}) + e^{-\lambda} - e^{-\lambda(1-\rho)}$ .  $H(0) = H(1) = 0$  and  $H''(\rho) = -\lambda^2 e^{-\lambda(1-\rho)} < 0$  imply that  $H(\rho) > 0$  for all  $\rho \in (0,1)$  QED.

**Claim 4:** The spread in skilled earnings increases with market thickness:  $\partial S / \partial \lambda > 0$ .

$$\begin{aligned}
\text{Proof: } \frac{\partial S}{\partial \lambda} &= \frac{\theta-1}{\alpha} \left\{ \frac{a_1^\theta (1-e^{-\lambda\rho})^\theta \rho e^{-\lambda\rho}}{[1-e^{-\lambda\rho}]^2} + \frac{a_2^\theta (e^{-\lambda\rho} - e^{-\lambda})^\theta (\rho e^{-\lambda\rho} - e^{-\lambda})}{[e^{-\lambda\rho} - e^{-\lambda}]^2} \right\} \\
&= \frac{(\theta-1)\rho e^{-\lambda\rho} [a_2(1-e^{-\lambda(1-\rho)})]^\theta}{\alpha [1-e^{-\lambda\rho}]^2} \left\{ \left[ \frac{a_1(1-e^{-\lambda\rho})}{a_2(1-e^{-\lambda(1-\rho)})} \right]^\theta + \frac{(1-e^{-\lambda\rho})^2 e^{-\lambda\rho\theta} (1-\frac{1}{\rho} e^{-\lambda(1-\rho)})}{[e^{-\lambda\rho} - e^{-\lambda}]^2} \right\} \\
&\geq \frac{(\theta-1)\rho e^{-\lambda\rho(\theta+1)} [a_2(1-e^{-\lambda(1-\rho)})]^\theta}{\alpha [1-e^{-\lambda\rho}]^2} \left\{ \frac{1-e^{-\lambda\rho\theta}}{1-e^{-\lambda(1-\rho)\theta}} e^{\lambda\rho\theta} + \left[ \frac{1-e^{-\lambda\rho}}{e^{-\lambda\rho} - e^{-\lambda}} \right]^2 (1-\frac{1}{\rho} e^{-\lambda(1-\rho)}) \right\}
\end{aligned}$$

given (4) holds. Since  $G_I(\theta)$  defined in Claim 2 is increasing in  $\theta$ , we need only establish that the term in brackets is positive for  $\theta = 1$ . Plugging in  $\theta = 1$  into the brackets yields

$$\frac{1-e^{-\lambda\rho}}{\rho(e^{-\lambda\rho} - e^{-\lambda})} \left\{ \rho + \frac{1-e^{-\lambda\rho}}{e^{-\lambda\rho} - e^{-\lambda}} (\rho - e^{-\lambda\rho}) \right\} = \frac{1-e^{-\lambda\rho}}{\rho(e^{-\lambda\rho} - e^{-\lambda})^2} K(\rho)$$

where  $K(\rho) = \rho(1-e^{-\lambda}) - e^{-\lambda(1-\rho)} + e^{-\lambda}$ . Since  $K(0) = K(1) = 0$  and  $K''(\rho) = \lambda^2 e^{-\lambda(1-\rho)} < 0$ ,  $K(\rho) > 0$  for all  $\rho \in (0,1)$ . QED