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# On optimal forest management: A bifurcation analysis

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# On Optimal Forest Management: A Bifurcation Analysis

by

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# On Optimal Forest Management: A Bifurcation Analysis*

Swapan Dasgupta† and Tapan Mitra ‡

**Abstract**

The paper examines the theory of optimal forest management with a view to describing its transition dynamics. In contrast to the literature's emphasis on long-run behavior of optimally managed forests, we focus on the nature of the optimal policy function, which describes the harvesting and replanting decisions that should be implemented currently, given an inherited forest. Bifurcation analysis is used to examine how the optimal policy function changes in response to variations in the parameters of the forestry model.

*Journal of Economic Literature* Classification Numbers: C61, D90, Q23.

*Keywords and Phrases*: Forest Management, Optimal Policy Function, Faustmann Threshold, Transition Dynamics, History Dependence, Bifurcation Analysis.

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# 1 Introduction

The theory of optimal forest management occupies an important place in the economic theory of natural resources, since forests constitute a major renewable resource. It is also important in one's understanding of the general theory of intertemporal allocation, as it constitutes one of the key examples of vintage capital theory.

This paper examines the theory of optimal forest management with a view to describing the forester's (optimal) *policy function*. In contrast to the literature's (almost exclusive) emphasis on long-run behavior of optimally managed forests, we focus on the optimal harvesting and replanting decisions that should be implemented currently, given any inherited forest.

For this purpose, we use an extremely simplified forestry model (introduced by Wan (1989)) involving a single species of trees on a unit piece of land, in which trees grow from newly planted saplings to young trees in one year, and at the end of two years achieve full maturity; after that the trees decay and become worthless. This allows us to describe the forest in terms of a single real variable, $x$, which represents the stock of (fraction of the land occupied by) mature trees. The growth of the timber content as young trees grow to maturity can be captured in terms of a real valued biological parameter, $a$. Welfare is derived from timber (by harvest of young or mature trees) using a (strictly concave) welfare function $w$, and future welfares are discounted by a discount factor $\delta$. The optimal forest management problem is to determine the harvesting (and replanting) decisions over time in order to maximize the discounted sum of welfares.

From the point of view of the general theory of intertemporal allocation, it is important to point out that a key feature of the model is the fact that its transition possibility set does not have free disposal of the initial stock. If one has a *higher* stock of mature trees today, then one necessarily has a lower stock of young trees today and therefore a *lower* stock of mature trees tomorrow. This "time to build" mature trees imposes a significant constraint on the forester, leading to important implications for the nature of the policy function.

We use standard dynamic programming methods to establish the existence of a (continuous) policy function on the set of stocks $I \equiv [0, 1]$. Thus the *transition dynamics* as well as *long-run behavior* of the optimally managed forest are fully described by the *dynamical system* $(I, h)$. In describing the nature of the optimal policy function the key concept is the *Faustmann threshold*.

Faustmann (1968) was concerned with the optimal forest rotation problem when (in our terminology) the welfare function $w$ is linear. In this case, it is optimal to cut all the mature trees and *only* the mature trees, regardless of the inherited forest. This policy can be called the *Faustmann policy*.[1] The Faustmann policy leads to persistent fluctuations in harvests, except when the initial forest is the stationary optimal forest.

[1] The formal demonstration of this, for a general forestry model, is contained in Mitra and Wan (1985, Theorem 4.2) for the discounted case, and in Mitra and Wan (1986, Theorem 5.2) for the undiscounted case..

When the welfare function $w$ is strictly concave, dynamic optimization has a tendency to smooth consumption over time. When future utilities are not discounted, this desire to smooth consumption takes over completely, leading to asymptotic convergence of optimal forests to the stationary optimal forest.[2] When future utilities are discounted, the consumption smoothing aspect is dampened by the fact that the benefits of such smoothing come in the future, which have less weight in the objective function compared with the present. This conflict is precisely captured by the existence of a stock of mature trees $x(\delta)$, such that for *all* initial stocks $x \in [x(\delta), 1]$, the Faustmann policy is optimal, while for *all* initial stocks $x \in [0, x(\delta))$, the Faustmann policy is *not* optimal. We refer to $x(\delta)$ as the *Faustmann threshold*.

For initial stocks $x \in [0, x(\delta))$ where the Faustmann policy is not optimal, an explicit solution of the policy function will depend on the actual welfare function used (apart from $a$ and $\delta$). However, the following qualitative description can be provided. The policy function is monotone non-decreasing in this range, and exhibits growth in the stock of mature trees.

This description of the policy function enables us to characterize the long-run behavior of the forest stock starting with any initial forest stock. The range of stocks $M \equiv [x(\delta), 1 - x(\delta)]$ is seen to be an invariant set of the dynamical system $(I, h)$; all other stocks are transitory, and $M$ is the global attractor of all trajectories generated by the optimally managed forest. But, the policy function also enables us to describe the optimal transition dynamics. For instance, when the initial stock $x$ of mature trees is small, the forester should cut down all the mature trees *and* some young trees as well, but take care to see that the remaining stock of young trees *exceeds* $x$, so that one ends up with a higher stock of mature trees tomorrow than one started out with today.

From the point of view of the general theory of intertemporal allocation, a particularly noteworthy aspect of the forestry model is that the *amplitude* of the period two Faustmann cycle that represents the long-run optimal forest depends on the inherited forest. The *history dependence* of long-run behavior suggests the intriguing possibility that starting with low initial forest stocks, which differ from each other only slightly, one might end up with significantly different long-run behavior in terms of the volatility of optimal harvests. This *sensitive dependence* of long-run behavior on initial conditions can be viewed as an "anti-turnpike" result.

In the final section, we provide an analysis of the change in long-run optimal behavior with respect to the parameters of the forestry model. Since long-run behavior is captured by the invariant set $[x(\delta), 1 - x(\delta)]$, we provide a formula characterizing the Faustmann threshold $x(\delta)$. Using this, we see that the average amplitude of fluctuations in the long-run increases with increases in the intertemporal elasticity of substitution and with the growth rate ($b = (1-a)/a$) of trees. The relationship is seen to be non-monotonic with respect to changes in the discount factor. We are able to identify a critical discount factor ($\hat{\delta} = (1/\sqrt{b})$) such that for all larger discount factors, the average amplitude

[2] This turnpike result was established in Mitra and Wan (1986, Theorem 6.1).

of long-run fluctuations decreases with increases in the discount factor; as one approaches the limiting case of perfect patience, the maximum amplitude of long-run fluctuations goes to zero, consistent with McKenzie's neighborhood turnpike theorem.

# 2 Preliminaries

## 2.1 A Model of Forest Management

The model of forest management we use is the simplest one whose primitives are described by a triplet $(a, w, \delta)$ where $a \in (0,1)$ is a biological parameter reflecting the growth possibilities of trees, $w$ is a real valued function on the non-negative reals which measures the welfare from timber harvest, and $\delta \in (0,1)$ is the discount factor, representing the forest manager's time preference.

Trees are assumed to grow from newly planted saplings to one year trees (young trees), and at the end of two years achieve full maturity, after which the trees decay and become worthless. It is assumed that no part of the land is left fallow so that at the end of each period, the forest is occupied by either mature trees or young trees. Denote by $x$ the fraction of land currently occupied by mature trees; then $1-x$ is the fraction of land occupied by young trees. The mature trees are necessarily going to be harvested because they decay after reaching maturity. However, for existing young trees, a decision has to be made as to what fraction $y$ of the land is to be to left with young trees, to mature during the coming period, while the remainder is harvested. Clearly, $0 \leq y \leq 1-x$. At the end of the period then, the fraction of land occupied by mature trees equals $y$. Assuming that the stock of each vintage of tree is proportional to the area occupied by that vintage and normalizing the area of land occupied by the forest to be one unit, we may identify the fractions $x$ and $y$ with the stocks of *mature trees* occupying the forest at the beginning and at the end of one period.

Choosing the timber content of a mature tree as the unit of measurement, we denote by $a$ the timber content of a young tree. The amount of timber harvested in the current period equals the timber content of the mature trees, $x$, together with the timber content of the young trees which are cut down. Since a fraction $1-x$ of the land is occupied by young trees, of which a part $y$ is left to mature, the fraction of land occupied by young trees which are harvested during the period equals $(1-x-y)$ and the timber content of this equals $a(1-x-y)$. The total timber harvested, therefore, equals $[x + a(1-x-y)]$.

Denoting the total timber harvested, by cutting trees of the two different maturities, during any period $t \geq 0$ by the number $c_t$, the welfare obtained from the timber harvested in that period is $w(c_t)$. The optimal forest management problem, starting from an initial stock of mature trees and young trees, is one of making a decision in each period $t = 0, 1, 2....$ as to how much to harvest $(c_t)$, so as to maximize the discounted sum of welfare obtained $\sum_0^\infty \delta^t w(c_t)$.

The problem of optimal forest management can be viewed as a special case of

discounted dynamic optimization in a standard *reduced form* model of optimal intertemporal allocation, described by the triplet $(\Omega, u, \delta)$, where $\Omega$ is a *transition possibility set*, defined by: $\Omega \equiv \{(x,y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1-x\}$, and $u : \Omega \to \mathbb{R}$ is a *utility function* defined by $u(x,y) \equiv w(x(1-a)+a(1-y))$. A *path* from $x \in I \equiv [0,1]$ is an infinite sequence $\langle x_t \rangle$ satisfying $x_0 = x$ and $(x_t, x_{t+1}) \in \Omega$ for all $t \geq 0$. A path $\langle x_t^* \rangle$ from $x$ is *optimal* if for every path $\langle x_t \rangle$ from $x$,

$$\sum_0^\infty \delta^t u(x_t^*, x_{t+1}^*) \geq \sum_0^\infty \delta^t u(x_t, x_{t+1})$$

The problem of optimal forest management is equivalent to that of finding an optimal path in this reduced form model.

## 2.2 Value and Policy Functions

The theory of dynamic programming is ideally suited to characterizing optimal behavior in our model. To apply this theory, we note some basic properties of the model.

The transition possibility set $\Omega$ is a compact, convex set which contains $(0,0)$. For every $x \in I$, there is $y \in I$ such that $(x,y) \in \Omega$. There is "free disposal" of the terminal stock: if $(x,y) \in \Omega$ and $0 \leq y' \leq y$ then $(x,y') \in \Omega$. However, free disposal of the initial stock is not possible; in fact, what is true is that if $(x,y) \in \Omega$ and $0 \leq x' \leq x$ then $(x',y) \in \Omega$. It is this last feature of $\Omega$ which distinguishes the forestry example from other examples of the general model of optimal intertemporal allocation that have been studied in detail in the literature.

The properties of the reduced form utility function, $u$, depend on properties of both $\Omega$ and $w$. In what follows, we assume that $w$ is continuous, concave and increasing on $\mathbb{R}_+$, twice continuously differentiable on $\mathbb{R}_{++}$ with $w'(c) > 0$ and $w''(c) < 0$ for all $c > 0$, and that it satisfies the end-point condition: $w'(c) \to \infty$ as $c \to 0$. Then it can be verified that $u$ is continuous on $\Omega$ and twice continuously differentiable on the interior of $\Omega$, with $u$ strictly increasing in the first argument and strictly decreasing in the second argument. Further, $u$ is concave on $\Omega$, with $u$ strictly concave in each argument separately.

These properties of the reduced form model $(\Omega, u, \delta)$ ensure that the discounted utility sum along any path $\langle x_t \rangle$ from $x \in I$, $\sum_0^\infty \delta^t u(x_t, x_{t+1})$, is absolutely convergent and that there is a unique optimal path from each initial stock $x \in I$. The *value function* associated with our dynamic optimization problem is a function $V : I \to \mathbb{R}$, defined by:

$$V(x) = \sum_0^\infty \delta^t u(x_t^*, x_{t+1}^*)$$

where $\langle x_t^* \rangle$ is the optimal path from $x \in I$. The standard theory can then be used to show that $V$ is a concave and continuous function of $x$ in $I$, and that it

satisfies the *principle of optimality*:

$$V(x) = \max_{y \in \Omega(x)} \{u(x,y) + \delta V(y)\} \text{ for each } x \in I$$

where $\Omega(x) = \{y : (x,y) \in \Omega\} = [0, 1-x]$. Further, a path $\langle x_t \rangle$ from $x \in I$ is optimal if and only if: $V(x_t) = u(x_t, x_{t+1}) + \delta V(x_{t+1})$ for all $t \geq 0$.

The *policy function* $h : I \to I$ is defined by:

$$h(x) = \arg \max_{y \in \Omega(x)} \{u(x,y) + \delta V(y)\} \text{ for each } x \in I$$

noting that for each $x \in I$, the maximization problem involved has a *unique* solution. The standard theory (see, for example, Mitra (2000)) can be used to verify that $h$ is continuous on $I$, and for all $(x,y) \in \Omega$, with $y \neq h(x)$, we have:

$$u(x,y) + \delta V(y) < V(x) = u(x, h(x)) + \delta V(h(x))$$

Further, a path $\langle x_t \rangle$ from $x \in I$ is optimal if and only if $x_{t+1} = h(x_t)$ for all $t \geq 0$. [3]

In our framework, given $x \in I$, it need not be the case that $(x, h(x))$ belongs to the interior of $\Omega$. However, *if* for some $x \in I$, we do have $(x, h(x))$ and $(h(x), h^2(x))$ in the interior of $\Omega$, then (using the differentiability property of $u$ in the interior of $\Omega$) the following *Ramsey-Euler equation* must hold:

$$u_2(x, h(x)) + \delta u_1(h(x), h^2(x)) = 0 \qquad \text{(RE)}$$

## 2.3 Stationary Optimal Stock

A *stationary optimal stock* (SOS) is any stock $k \in I$, satisfying $h(k) = k$. Since $h$ is a continuous function from $I$ to $I$, there exists a stationary optimal stock. The definition of $\Omega$ implies that $h(x) \leq (1-x)$ for all $x \geq 0$, and therefore for all $x \in (\hat{x}, 1]$, where $\hat{x} = (1/2)$, we must have $h(x) < x$. Thus, any stationary optimal stock $k$ must be in $[0, \hat{x}]$.

When a young tree grows to maturity, the the timber content grows from $a$ to 1, producing an increase in timber content of $(1-a)$; the *growth rate* associated with this process is therefore $b \equiv (1-a)/a$. In what follows we assume:

$$\delta b \equiv \delta(1-a)/a > 1 \qquad \text{(DP)}$$

Condition (DP) corresponds to what is called the $\delta-$*productivity* condition in the general theory of optimal intertemporal allocation. When this condition does not hold (that is, $\delta b \leq 1$), there is no incentive to increase the stock of mature trees even when one starts from a very low initial stock of mature trees, and in particular $h(0) = 0$.

[3] The policy function, $h$, of course, depends on the parameters of the model $(a, w, \delta)$. For later use, we note here that the standard theory (see Mitra (2000)) also establishes that the policy function varies continuously with $\delta$.

Under condition (DP), a stationary optimal stock $k$ cannot be in $[0, \hat{x})$. For, if an SOS $k \in [0, \hat{x})$, then $(k, k, k, k, ...)$ is optimal from $k \in [0, \hat{x})$. But a necessary condition for this is: $w'(c)(-a)+\delta w'(c)(1-a) \leq 0$, where $c = (k+a(1-2k)) > 0$, and this violates condition (DP). Thus, the only stationary optimal stock is $\hat{x} \equiv (1/2)$. Further, since $h(0) \in I$, the argument just given establishes that $h(0) > 0$, and the continuity of $h$ on $I$ also ensures that $h(x) > x$ for all $x \in [0, \hat{x})$.

# 3 The Nature of the Policy Function

A full qualitative description of the policy function enables one to describe the behavior of optimal paths completely. It not only provides information about long-run optimal composition of forests (which can be studied even without a full description of the policy function), but it also tells the forester what harvesting and replanting decisions to take *today* in order to attain optimality.

We have already established some useful properties of the policy function in the last two subsections of Section 2. In this section we describe two key qualitative features of the policy function, a *monotone property* and what we call the *Faustmann threshold*. After examining these features in the first two subsections, we proceed to indicate in the last two subsections how these properties enable us to describe the *transition dynamics* and *asymptotic behavior* of optimal paths.

## 3.1 A Monotone Property

As already noted, given $\bar{x} \in (0, 1)$, it need not be the case that $(\bar{x}, h(\bar{x}))$ belongs to the interior of $\Omega$. But, *if* $(\bar{x}, h(\bar{x})) \in int\ \Omega$, then one can establish the following (local) *monotone property*:

There is $\eta \in (0, \bar{x})$ such that $h$ is non-decreasing on $N(\bar{x}; \eta) = (\bar{x} - \eta, \bar{x} + \eta)$ (LM)

To see this, note that we can find $\varepsilon \in (0, \bar{x})$ such that the set $A$ defined by:

$$A \equiv \{(x, z) : (\bar{x} - \varepsilon, h(\bar{x}) - \varepsilon) \leq (x, z) \leq (\bar{x} + \varepsilon, h(\bar{x}) + \varepsilon)\}$$

is in the interior of $\Omega$. Since $h$ is continuous, we can find $\eta \in (0, \varepsilon)$ such that if $x \in N(\bar{x}, \eta)$ then $(x, h(x)) \in A$. Since $u$ is $C^2$ on $int\ \Omega$ and $u_{12}(x, y) = -a(1-a)w''(x+a(1-x-y)) > 0$ on $int\, \Omega$, it follows that $u(x, y)$ is supermodular on $A$, and so is $[u(x, y) + \delta V(y)]$. Now, the standard theory (see Topkis (1968), Mitra (2000)) of maximizers of supermodular functions ensures that $h$ is non-decreasing on $N(\bar{x}, \eta)$.

The monotone property in turn can be shown to yield the following useful result on optimal behavior at the *boundary*:

If $x' \in (0, 1]$ satisfies $h(x') = 1 - x'$, then $h(x) = 1 - x$ for all $x \in [x', 1]$ (B)

### 3.2 Existence of the Faustmann Threshold

Since the welfare function $w$ is strictly concave, dynamic optimization has a tendency to smooth consumption over time. Since future utilities are discounted, the consumption smoothing aspect is dampened by the fact that the benefits of such smoothing come in the future, which have less weight in the objective function compared with the present.

The conflicting effects of impatience and strict concavity of the welfare function lead to the following phenomenon in our example of the forestry model. There is an initial stock of mature trees $x(\delta) \in (0, \hat{x})$, such that for *all* initial stocks $x \in [x(\delta), 1]$, the Faustmann policy (of cutting down all the mature trees and only the mature trees, and replanting in the cleared area) is optimal, while for *all* initial stocks $x \in [0, x(\delta))$, the Faustmann policy is *not* optimal. We call $x(\delta)$ the *Faustmann threshold*. A key feature of the policy function is therefore to establish the existence of the Faustmann threshold, which is a *bifurcation value* of the initial stock.

To this end, the first observation we make is that for initial stocks $x \in [0, \hat{x})$ close to $\hat{x}$, it is *not optimal* to reach $\hat{x}$ immediately, even though it is feasible to do so. That is, there is $\varepsilon' \in (0, \hat{x})$, such that for all $\varepsilon \in (0, \varepsilon')$, we have $h(\hat{x} - \varepsilon) \neq \hat{x}$. To see this, let $\varepsilon \in (0, \hat{x})$, and note that if $h(\hat{x} - \varepsilon) = \hat{x}$, then $(\hat{x} - \varepsilon, \hat{x}, \hat{x}, ....)$ is optimal from $\hat{x} - \varepsilon$. However, it can be checked that the sequence $(\hat{x} - \varepsilon, \hat{x} + \varepsilon, \hat{x} - \varepsilon, \hat{x} + \varepsilon, ...)$ is a path from $\hat{x} - \varepsilon$, which gives a higher discounted sum of utilities for all $\varepsilon$ small enough, by using condition (DP).

This leads us to make the claim that there is some initial stock $x \in (\hat{x} - \varepsilon', \hat{x})$ for which $h(x) > \hat{x}$. For, if the claim is not true, then for *all* $x \in (\hat{x} - \varepsilon', \hat{x})$, it must be the case that $h(x) < \hat{x}$. But, since $h(x) > x$ for all $x \in [0, \hat{x})$, this implies that for all $x \in (\hat{x} - \varepsilon', \hat{x})$, we have $(x, h(x)) \in int\, \Omega$, and further $x < h(x) < \hat{x}$, so that $(h(x), h^2(x)) \in int\, \Omega$. This implies that the Ramsey-Euler equation:

$$u_2(x, h(x)) + \delta u_1(h(x), h^2(x)) = 0 \qquad \text{(RE)}$$

holds for all $x \in (\hat{x} - \varepsilon', \hat{x})$. But, it is easy to check that if (RE) holds for $x$ close to $\hat{x}$, then condition (DP) is violated. This establishes our claim that there is some $x^0 \in (\hat{x} - \varepsilon', \hat{x})$ such that $h(x^0) > \hat{x}$.

We define the Faustmann threshold as: $x(\delta) = \min\{x \in [0, \hat{x}] : h(x) = 1 - x\}$. This is well-defined and we have $h(x(\delta)) = 1 - x(\delta)$. We claim that $x(\delta) < \hat{x}$. For, if the claim is not true, then $h(x) < (1 - x)$ for all $x \in [0, \hat{x})$. And, since $h(x) > x$ for all $x \in [0, \hat{x})$, we have $(x, h(x)) \in int\, \Omega$ for all $x \in (0, \hat{x})$. Then, the local monotone property ensures that $D_+ h(x) \geq 0$ for all $x \in (0, \hat{x})$. But, this implies[4] $h(\hat{x}) \geq h(x^0) > \hat{x}$, a contradiction to the definition of a stationary optimal stock. This establishes our claim.

Next we claim that $x(\delta) > 0$. For, if the claim is not true, $x(\delta) = 0$, and so $h(0) = 1$. Since $\Omega(1) = \{0\}$, we also have $h(1) = 0$. Thus, $(0, 1, 0, 1, 0, ...)$ must be the optimal path from initial stock 0. But, for $\varepsilon \in (0, 1)$, the sequence $(0, 1 - \varepsilon, 0, 1, 0, ...)$ is a path from 0, which gives a higher discounted sum of

[4] Use Proposition 2 on page 99 of Royden (1988).

utilities for all $\varepsilon$ small enough, by using the fact that $w'(c) \to \infty$ as $c \to 0$. This contradiction establishes our claim, and we have $x(\delta) \in (0, \hat{x})$.

Notice that by definition of $x(\delta)$, we have $h(x) < (1-x)$ for all $x \in [0, x(\delta))$, and $h(x(\delta)) = 1 - x(\delta)$, by continuity of $h$ on $I$. Further, $h(x) > x$ for $x \in [0, x(\delta))$. Thus, for all $x \in (0, x(\delta))$, we have $(x, h(x)) \in int\,\Omega$, and so $D_+h(x) \geq 0$ for all $x \in (0, x(\delta))$.

We are now in a position to describe completely the nature of the policy function. On $[0, x(\delta)]$, we have $h$ non-decreasing with $(1-x) > h(x) > x$ for all $x \in [0, x(\delta))$. And, for $x \in [x(\delta), 1]$, we have $h(x) = 1 - x$ by property (B). This makes $x(\delta)$ a bifurcation value of the initial stock.

The policy function contains the answer to the question: "What is the forester's optimal harvesting and replanting decision today?" If the initial forest has mature tree stock $x \in [x(\delta), 1]$, the forester should follow the Faustmann policy of cutting down all the mature trees and *only* the mature trees (and replanting with seedlings in the cleared area). While the Faustmann threshold $x(\delta)$ depends on the form of the welfare function $w$, the policy itself (for this range) can be described independent of $w$. If, on the other hand, the initial forest has mature tree stock $x \in [0, x(\delta))$, the forester should cut down all the mature trees *and* some young trees as well (reflecting the fact that $h(x) < 1 - x$), but taking care to see that the remaining stock of young trees *exceeds* $x$ (reflecting the fact that $h(x) > x$), so that one ends up with a higher stock of mature trees tomorrow than one started out with today. The tension between consumption smoothing and impatience will determine the actual harvest of the young trees, so the description of the policy here necessarily depends on the actual welfare function $w$ used (apart from $a$ and $\delta$).

### 3.3 Transition Dynamics of Optimal Paths

Our qualitative description of the policy function allows us to fully characterize the transition dynamics of optimal paths. To this end, it is convenient to separate three ranges of stocks (of mature trees), which we might call low, medium and high. Let us define $L \equiv [0, x(\delta))$ to be the range of low stocks, $M \equiv [x(\delta), 1 - x(\delta)]$ to be the range of medium stocks, and $H \equiv (1 - x(\delta), 1]$ to be the range of high stocks.

We start with the range of medium stocks. If $x \in M$, then clearly $(x, 1 - x, x, 1 - x, ...)$ is the optimal path from $x$, exhibiting period two cycles.

Next, consider the range of low stocks. If $x \in L$, then the stock of mature trees will increase until it enters the range $M$ (in a finite number of periods), after which it will exhibit period two cycles as described in the previous paragraph.

Finally, consider the range of high stocks. If $x \in H$, then the stock of mature trees will enter the low range $L$ in one period, after which it will exhibit the transition dynamics described in the previous paragraph.

### 3.4 History Dependent Long Run Behavior

It is clear from the previous subsection that the middle range of stocks $M \equiv [x(\delta), 1-x(\delta)]$ is an invariant set of the dynamical system $(I, h)$. All other stocks are transitory, and $M$ is the global attractor of all trajectories generated by the system.[5] Thus, $M$ describes the long-run behavior of the forest stock (of mature trees), starting with any initial forest stock.

But, this description hides more than it reveals regarding the long-run behavior of the optimally managed forest. As mentioned before, in a variety of examples of intertemporal allocation models, discounting of future welfares can lead to persistent fluctuations, even when the welfare function is strictly concave, since discounting dampens the desire for consumption smoothing. [See Mitra and Nishimura (2001) for a detailed analysis of the well-known examples of Weitzman (described in Samuelson (1973)) and Sutherland (1968)]. However, in many of these examples, the unique golden-rule stock (which is the global attractor in the undiscounted case), is replaced by a unique cycle (which becomes the global attractor in the discounted case) as representative of long-run optimal behavior.

What distinguishes the forestry example is that the period two cycle that represents the long-run optimal forest depends on the forest one started with. This *history dependence* of long-run behavior can be seen by noting that for some initial forest stocks, one may end up at the stationary optimal stock after a finite number of periods (thereby exhibiting no fluctuations in harvests in the long-run) while from other initial forests, one may reach the Faustmann threshold $x(\delta)$ in a finite number of periods (thereby exhibiting harvests fluctuating between $x(\delta)$ and $(1-x(\delta))$ in the long-run). And, from still other initial forests, one may end up at $x \in (x(\delta), \hat{x})$ after a finite number of periods, thereby exhibiting persistent fluctuations in the long-run but of *smaller amplitude* than in the previously mentioned case.

The history dependence of long-run behavior suggests the intriguing possibility that (when $h(0)$ is small) starting with low initial forest stocks, which differ from each other only slightly, one might end up with significantly different long-run behavior in terms of the volatility of optimal harvests. This *sensitive dependence* of long-run behavior on initial conditions can be viewed as an "anti-turnpike" result.

The nature of the dependence of long-run behavior on history appears to be quite complex. If one starts with a completely even distribution of young and mature trees (the initial stock is $x = \hat{x} = (1/2)$), then of course the long-run behavior exhibits no fluctuations in the harvest. But, it is *not* the case that if one starts with two distributions of young and mature trees, the first more uneven than the second, one necessarily ends up in the long-run with higher amplitude fluctuations in the first case compared with the second.

This can be seen most transparently as follows. Keeping $(a, w)$ fixed, write the policy function as $h(x; \delta)$, to explicitly recognize the dependence of the

[5] That is, if $\langle x_t \rangle$ is optimal from any $x \in I$, then $d(x_t, M) \to 0$ as $t \to \infty$, where $d(x, M)$ is defined as $\inf\{d(x, z) : z \in M\}$ for all $x \in I$.

policy on the stock ($x$) and the discount factor ($\delta$). Recall from Section 2 that $h(x,\delta)$ is continuous in $\delta$, and that $h(0;(1/b))=0$. Now, consider $\delta>(1/b)$, so that (DP) is satisfied, with $\delta$ close to $(1/b)$. Then, $h(0,\delta)$ will be close to 0, and consequently, there must be $\tilde{x}\in(0,x(\delta))$, such that $h(\tilde{x};\delta)=\hat{x}$. Thus, starting from the forest consisting of $\tilde{x}$ mature trees and $(1-\tilde{x})$ young trees, one ends up at $\hat{x}$ in one period, leading to no fluctuations in harvest in the long-run. On the other hand, from the more evenly distributed initial forest, consisting of $x(\delta)\in(0,\hat{x})$ mature trees and $(1-x(\delta))$ young trees, the optimal harvest fluctuates between $x(\delta)$ and $(1-x(\delta))$ in the long-run.

# 4 Bifurcation Analysis

In this section, we provide an analysis of the change in the policy function with respect to the parameters of the forestry model. Since the Faustmann threshold is a key feature of the policy function, our primary task is to provide a formula to characterize this threshold in terms of the parameters of the model.

## 4.1 A Formula for the Faustmann Threshold

We first show that the value function is continuously differentiable on $(0,1-x(\delta))$. Let us define a function, $W:I\to\mathbb{R}$ as follows:

$$W(x)=[w(x)+\delta w(1-x)]/(1-\delta^2)$$

Note that from any $x\in I$, the sequence $(x,1-x,x,1-x,...)$ is a (feasible) program from the initial condition, $x$. Then, $W(x)$ is the discounted utility sum obtained by following this program. Clearly, $W$ is concave and continuous on $I$, and twice continuously differentiable on $J\equiv(0,1)$, with $W''(x)<0$ for $x\in J$.

For $x\in(0,x(\delta))$, we have, of course $V(x)>W(x)$, while at $x=x(\delta)$, we have $V(x)=W(x)$. Thus, we have:

$$V(x(\delta))-V(x)<V(x(\delta))-W(x)=W(x(\delta))-W(x) \text{ for } x\in(0,x(\delta))$$

This yields $V'^{-}(x(\delta))\leq W'(x(\delta))$.For $x\in M\equiv[x(\delta),1-x(\delta)]$, we have $V(x)=W(x)$. This yields $V'^{+}(x(\delta))\leq W'(x(\delta))$.Combining the two inequalities we get $V'^{-}(x(\delta))\leq V'^{+}(x(\delta))$. But, by concavity of $V$, we also have $V'^{-}(x(\delta))\geq V'^{+}(x(\delta))$. Thus, $V'^{-}(x(\delta))=V'^{+}(x(\delta))$ and $V$ is differentiable at $x(\delta)$, with $V'(x(\delta))=W'(x(\delta))$.

For $x\in M\equiv[x(\delta),1-x(\delta)]$, we have $V(x)=W(x)$, and so $V$ is continuously differentiable on $(x(\delta),1-x(\delta))$. For $x\in(0,x(\delta))$, we have $(x,h(x))\in int\,\Omega$ and so, by the standard theory (see Benveniste and Scheinkman (1979)), we have $V$ continuously differentiable on $(0,x(\delta))$. We have now demonstrated that $V$ is differentiable for all $x\in(0,1-x(\delta))$. Since $V$ is concave, $V$ is continuously differentiable on $(0,\,1-x(\delta))$.[6]

[6]Use Theorem B on page 7 of Roberts and Varberg (1973).

By the envelope theorem, we have $V'(x) = u_1(x, h(x)) = w'(x + a(1 - x - h(x)))(1 - a)$ for all $x \in (0, x(\delta))$. By the continuity of $V'$ at $x(\delta)$, we get $V'(x(\delta)) = w'(x(\delta))(1 - a)$. But, since $V'(x(\delta)) = W'(x(\delta))$, we also have:

$$V'(x(\delta)) = [w'(x(\delta)) - \delta w'(1 - x(\delta))]/(1 - \delta^2)$$

And this yields the formula for the Faustmann threshold:

$$w'(x(\delta))/w'(1 - x(\delta)) = \delta/[1 - (1 - \delta^2)(1 - a)] \qquad \text{(FT)}$$

## 4.2 Sensitivity of Long-Run Behavior

We concentrate on studying the effect of a change in the parameters of the model $(a, w, \delta)$ on long-run behavior. Since long-run behavior is captured by the invariant set $[x(\delta), 1 - x(\delta)]$, our focus is on how the Faustmann threshold changes when the parameters change.

### 4.2.1 Change in the Growth Rate of Trees

A decrease in $a$ can be interpreted as an *increase* in the growth rate $b \equiv [(1 - a)/a]$ achieved as young trees become mature. Note that, in order to satisfy condition (DP), we must have $a \in (0, \delta/(1+\delta))$. As $a$ increases (the growth rate $b$ falls), the right-hand side of (FT) decreases and accordingly the Faustmann threshold $x(\delta)$ *increases monotonically*, leading to *lower amplitude* long-run fluctuations (on the average).

As $a \uparrow \delta/(1+\delta)$, it can be checked that the right hand side of (FT) decreases to 1, and so $x(\delta)$ increases to $\hat{x} = (1/2)$. Thus, in this limiting case, all long-run fluctuations are eliminated. On the other hand, as $a \downarrow 0$, the right-hand side of (FT) increases to $(1/\delta) > 1$, and so $x(\delta)$ decreases to a lower limit $\underline{x} \in (0, \hat{x})$. Since $\underline{x} > 0$ the optimal asymptotic composition of the forest is one of part young and part mature trees.

### 4.2.2 Change in Impatience

An increase in the discount factor $(\delta)$ is to be interpreted as a *decrease* in impatience. Note that, in order to satisfy condition (DP), we must have $\delta \in (1/b, 1) \equiv (a/(1 - a), 1)$. As $\delta$ approaches the value 1 (the forester becomes very patient) the right-hand side of (FT) converges to 1 and so the Faustmann threshold $x(\delta)$ converges to $\hat{x} = (1/2)$. Thus, the maximum amplitude of long-run fluctuations goes to zero, consistent with the neighborhood turnpike theorem of McKenzie (1982).

However, as the discount factor decreases from 1 to $(1/b)$, we observe that the right-hand side expression behaves *non-monotonically*, first increasing and then decreasing. As $\delta \to (1/b)$, the right hand side of (FT) converges to 1, and again long-run fluctuations are eliminated in the limiting case (this case being exactly the limiting case of the previous subsection when $a$ converges to $\delta/(1 + \delta)$).

The expression on the right-hand side of (FT) attains a maximum at the critical discount factor, $\hat{\delta} = 1/\sqrt{b}$. Thus, for this discount factor, the corresponding Faustmann threshold $x(\delta)$ reaches its minimum, and $(1 - x(\delta))$ its maximum, producing the highest-amplitude two-period optimal cycle among all specifications of discount factors.

### 4.2.3 Change in the Intertemporal Substitution Elasticity

In our framework, the intertemporal elasticity of substitution is the inverse of the elasticity of the marginal welfare (for the welfare function $w$). For example, for $w(c) = c^{1-\alpha}/(1-\alpha)$, the elasticity of the marginal welfare is:

$$-\frac{cw''(c)}{w'(c)} = \alpha$$

and the intertemporal elasticity of substitution is $(1/\alpha)$.

Intuitively, the higher the intertemporal substitution elasticity, the more the agent can tolerate fluctuations, and therefore the less the agent's need for consumption smoothing. This leads to a lower Faustmann threshold and therefore higher amplitude long-run fluctuations (on the average). As the intertemporal substitution elasticity approaches infinity, the welfare function approaches the linear case considered by Faustmann, and the Faustmann threshold goes to zero, indicating that in this limiting case, the Faustmann policy becomes optimal to follow from all initial stocks.

To confirm this intuition, consider welfare functions $w$ and $v$ with the properties described in Section 2. Define:

$$g_w(x) = \frac{w'(x)}{w'(1-x)} \; ; \; g_v(x) = \frac{v'(x)}{v'(1-x)} \;\; \text{for all } x \in (0,1)$$

and denote by $R_w(c) \equiv -cw''(c)/w'(c)$ and $R_v(c) = -cv''(c)/v'(c)$ the elasticities of the marginal welfare for the two functions for all $c \in (0,1)$. Assume that for all $c \in (0,1)$, we have $R_w(c) > R_v(c)$, so that $w$ has the *lower* intertemporal substitution elasticity for all $c \in (0,1)$.

It is straightforward to verify that:

$$\frac{g_w'(x)}{g_w(x)} = -\left[\frac{R_w(x)}{x} + \frac{R_w(1-x)}{1-x}\right]$$

and a similar formula holds for the function $v$. Defining $G_w(x) = \ln g_w(x)$ and $G_v(x) = \ln g_v(x)$ for all $x \in (0,1)$, we then obtain for all $x \in (0, \hat{x})$,

$$-\int_x^{\hat{x}} \left[\frac{R_w(s)}{s} + \frac{R_w(1-s)}{1-s}\right] ds = \int_x^{\hat{x}} G_w'(s) ds = G_w(\hat{x}) - G_w(x) = -G_w(x)$$

This establishes that $G_w(x) > G_v(x)$ and therefore $g_w(x) > g_v(x)$ for all $x \in (0, \hat{x})$, using $R_w(c) > R_v(c)$ for all $c \in (0,1)$.

Given $a$ and $\delta$, the right hand side of (FT) is independent of the welfare function. Thus, we must have the Faustmann threshold *higher* for the welfare function $w$, leading to *smaller amplitude* long-run fluctuations (on the average), compared to the welfare function $v$.

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