

Streufert, Peter A.

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# A PLEASANT HOMEOMORPHISM FOR CONDITIONAL PROBABILITY SYSTEMS

PETER A. STREUFERT

ABSTRACT. Suppose that there are  $n$  states, each denoted by  $i \in \{1, 2, \dots, n\}$ . This paper shows that the function  $[p_{i|E}]_{i \in E} \mapsto [\sum_{j \neq i} p_{i\{i,j\}}]_i$  is a homeomorphism from the set of conditional probability systems onto the convex hull of all permutations of the  $n$ -dimensional vector  $(0, 1, 2, \dots, n-1)$ .

Suppose  $n$  is a positive integer. A *conditional probability system* over  $\{1, 2, \dots, n\}$  is a function  $p : \{(i, E) | i \in E \subseteq \{1, 2, \dots, n\}\} \rightarrow [0, 1]$  such that

- (a)  $(\forall E) \sum_{i \in E} p_{i|E} = 1$  and
- (b)  $(\forall i \in E' \subseteq E) p_{i|E'} = \frac{p_{i|E}}{\sum_{j \in E'} p_{j|E}}$  when  $\sum_{j \in E'} p_{j|E} > 0$ .

(Elements of  $\{1, 2, \dots, n\}$  are called states, subsets of  $\{1, 2, \dots, n\}$  are called events, condition (a) requires that each  $p_{\cdot|E}$  is a probability distribution over the states in  $E$ , and condition (b) requires that these probability distributions are consistent with one another.) Section 2 of Hammond (1994) notes that such conditional probability systems have interested philosophers, mathematicians, and game theorists.

Let  $P$  denote the set of all conditional probability systems (for some fixed  $n$ ). McLennan (1989b), Monderer, Samet, and Shapley (1992), and Vieille (1996) have each contributed homeomorphisms which map  $P$  onto a convex compact subset of  $\mathbb{R}^n$ . This note contributes a particularly pleasant homeomorphism.

THEOREM. *The function*

$$F(p) = [\sum_{j \neq i} p_{i\{i,j\}}]_i$$

*is a homeomorphism from  $P$  onto the convex hull of the set of all permutations of the  $n$ -dimensional vector  $(0, 1, 2, \dots, n-1)$ .*

PROOF. Monderer, Samet, and Shapley (1992, p. 33) note that a conditional probability system  $p \in P$  over states and events is identical to a weight system  $w \in \mathcal{W}$  over players and coalitions, and that  $\mathcal{W}$  is

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homeomorphic to the core of any strictly convex cooperative game  $v$  defined over the same players. The contribution of this paper is to notice a particularly pleasant game:  $v = \Sigma\{u_T | \#T = 2\}$ , which is the sum of the set of binary unanimity games.

Since  $v$  is defined as the sum of all binary unanimity games, the weighted Shapley value of  $v$ , denoted  $\phi^v$ , is the sum of the weighted Shapley values of all binary unanimity games, each denoted  $\phi^{u_T}$  (Monderer, Samet, and Shapley (1992, p. 32)). Hence the weighted Shapley value  $\phi^v$  equals the function  $F$  defined above:

$$\begin{aligned} (\forall p) \phi^v(p) &= [\phi_i^v(p)]_i = [\Sigma\{\phi_i^{u_T}(p) | \#T = 2\}]_i \\ &= [\Sigma\{p_{i|T} | \#T = 2 \text{ and } i \in T\}]_i = [\Sigma_{j \neq i} p_{j\{i,j\}}]_i = F(p). \end{aligned}$$

The definition of  $v$  also implies that

$$\begin{aligned} (1) \quad (\forall S \subseteq \{1, 2, \dots, n\}) \quad v(S) &= \Sigma\{u_T(S) | \#T = 2\} \\ &= \#\{T | \#T = 2 \text{ and } T \subseteq S\} = \binom{\#S}{2} = \#S(\#S - 1)/2 = \Sigma_{m=0}^{\#S-1} m. \end{aligned}$$

Hence

$$\begin{aligned} (\forall T \subset S) (\forall i \notin S) \quad v(T \cup \{i\}) - v(T) &= \Sigma_{m=0}^{\#T} m - \Sigma_{m=0}^{\#T-1} m = \#T \\ &< \#S = \Sigma_{m=0}^{\#S} m - \Sigma_{m=0}^{\#S-1} m = v(S \cup \{i\}) - v(S), \end{aligned}$$

and consequently,  $v$  is strictly convex by Shapley (1971, Equation (6)).

The equality  $\phi^v = F$ , the strict convexity of  $v$ , and Monderer, Samet, and Shapley (1992, Theorem C) together yield that  $F$  is a homeomorphism from  $P$  onto the core of  $v$ . Furthermore, the convexity of  $v$  and Shapley (1971, Theorems 3 and 5) together yield that the core of  $v$  is equal to the convex hull of the set of contribution vectors of  $v$ . By (1), the set of contribution vectors equals the set of all permutations of the vector  $(0, 1, 2, \dots, n-1)$ .  $\square$

Figure 1 depicts the images of the four conditional probability systems listed below. A conditional probability system is written as a matrix with states labelling its rows and events labelling its columns (hyphens occur where the state is not in the event). The value of  $F$  is determined by summing the second, third, and fourth columns of the matrix.

$$\begin{aligned} F \left( \begin{bmatrix} 1 & 1 & 1 & - & 1 & - & - \\ 0 & 0 & - & 2/3 & - & 1 & - \\ 0 & - & 0 & 1/3 & - & - & 1 \end{bmatrix} \right) &= \begin{bmatrix} 2 \\ 2/3 \\ 1/3 \end{bmatrix} \\ F \left( \begin{bmatrix} 1 & 1 & 1 & - & 1 & - & - \\ 0 & 0 & - & 1 & - & 1 & - \\ 0 & - & 0 & 0 & - & - & 1 \end{bmatrix} \right) &= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

$$F \left( \begin{bmatrix} 4/7 & 2/3 & 4/5 & - & 1 & - & - \\ 2/7 & 1/3 & - & 2/3 & - & 1 & - \\ 1/7 & - & 1/5 & 1/3 & - & - & 1 \end{bmatrix} \right) = \begin{bmatrix} 22/15 \\ 1 \\ 8/15 \end{bmatrix}$$

$$F \left( \begin{bmatrix} 1/3 & 1/2 & 1/2 & - & 1 & - & - \\ 1/3 & 1/2 & - & 1/2 & - & 1 & - \\ 1/3 & - & 1/2 & 1/2 & - & - & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

It follows immediately from the theorem and the functional form of  $F$  that  $P$  is homeomorphic to its own projection on  $\{(i, E) | \#E = 2\}$  (the same observation was derived differently by McLennan (1989a, p. 145–146) and Hammond (1994, p. 36–41)). Thus  $P$  can be characterized by binary probabilities alone.

This observation can be developed geometrically by noticing that the  $\binom{n}{2}$  binary probabilities constitute a coordinate system for  $F(P)$ . This coordinate system is illustrated for the case  $n = 3$  by the three sets of gridlines which criss-cross the hexagon  $F(P)$  in Figure 1. The many dark, somewhat vertical gridlines are the images of many subsets of  $P$  which each share the same binary probability  $p_{2|\{2,3\}}$ . Similarly, the many lighter, somewhat diagonal gridlines are the images of many subsets of  $P$  which each share either the same  $p_{1|\{1,2\}}$  or the same  $p_{1|\{1,3\}}$ .

It is well-known that every *positive* element of  $P$  can be characterized by only  $n - 1$  binary probabilities. This might be considered a “double redundancy:” the first redundancy being from all of  $P$  to only  $\binom{n}{2}$  binary probabilities, and the second redundancy being from all binary probabilities to only  $n - 1$  of them. The second redundancy is illustrated by the three sets of gridlines (strictly) inside Figure 1’s two-dimensional hexagon: in that region any two gridlines determine the third. The corresponding algebra is easiest if the gridlines are labelled with relative probabilities of the form  $p_{i|\{i,j\}}/p_{j|\{i,j\}}$ . In the third example, the relative-probability labels indicate that 1 is twice as likely as 2, that 2 is twice as likely as 3, and that consequently, 1 is four times as likely as 3.

It is also well-known that the second redundancy does not hold everywhere. In particular, each binary probability is indispensable when the two states it concerns are both zero-probability, or both unit-probability, with respect to all other states. In the first and second examples, the binary probability  $p_{2|\{2,3\}}$  is indispensable because 2 and 3 are zero-probability with respect to the only other state. More generally, there is an indispensable binary probability anywhere along the six

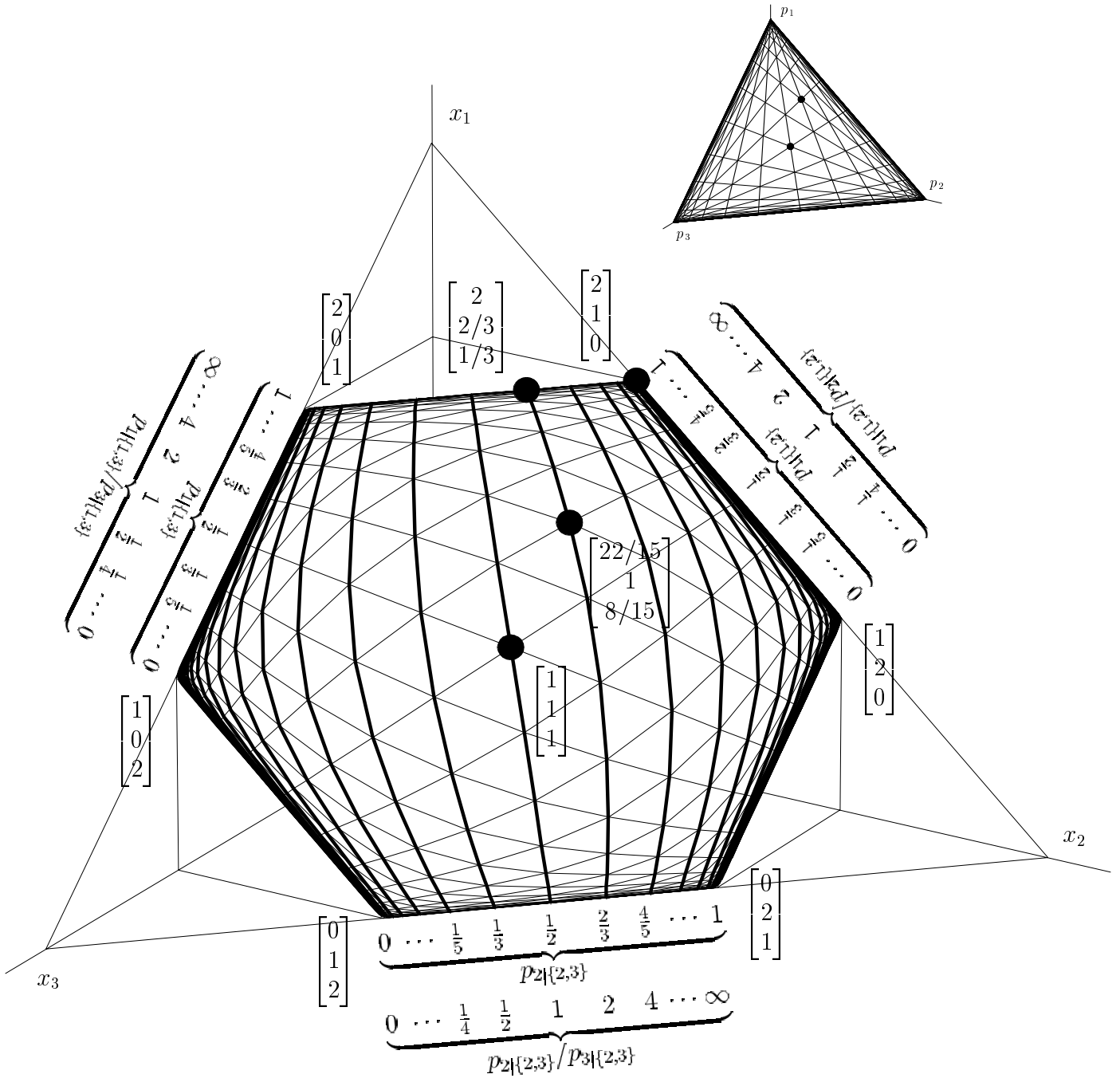


FIGURE 1. The hexagon is  $F(P)$ , the image of  $P$  under the homeomorphism  $F$ . The three sets of gridlines are determined by binary probabilities.

edges of the Figure 1's hexagon: the indispensable gridline is the one labelling the opposite edge (the zero-probability case) or the edge itself (the unit-probability case). Geometrically, this must happen because the other two gridlines lie on top of each other.

Finally, note that analogous gridlines are displayed on the standard simplex of Figure 1's upper-right corner. The simplex cannot depict the first two examples because a standard probability distribution cannot specify  $p_{2\{2,3\}}$  while assigning both 2 and 3 probability zero. The hexagon provides a means of depicting these two examples by flattening the point at the top of the triangle. The gridlines become bow-shaped when the other two corners are symmetrically flattened.

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ECONOMICS DEPARTMENT, UNIVERSITY OF WESTERN ONTARIO, LONDON  
ONTARIO N6A 5C2 CANADA

*E-mail address:* pstreuf@uwo.ca