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## A general equilibrium world with price and quantity signals: A duality approach

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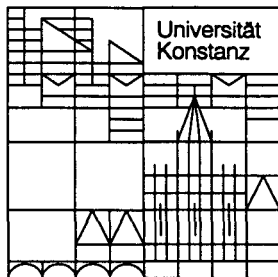
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**A General Equilibrium World With Price and Quantity Signals:  
A Duality Approach**

**Bodo Hilgers and Ulrich Wacker**

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# A General Equilibrium World With Price and Quantity Signals: A Duality Approach

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## Abstract

Our contribution offers a methodological exhibition of general equilibrium models with price and quantity signals. By the application of optimal value functions the household as well as the firm behaviour is modelled. The concepts of virtual prices and virtual quantities allow us to express constrained by unconstrained functions. Starting with these optimal value functions, behavioral functions describing the market coordination of supply and demand by price and quantity signals can be derived. We will show that by means of our method we can derive any general equilibrium model characterized by price and quantity signals with an arbitrary number of goods and factor markets. Furthermore, our method allows to construct a fully integrated general equilibrium framework not only with a finite number of goods and factor markets characterized by quantity constraints but also with unconstrained goods and factor markets.

## 1 Introduction

The Walras and Arrow-Debreu general equilibrium theory leads to the conclusion that under certain assumptions a system of actors pursuing only their self-interest

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ends up not in chaos but in a Pareto-efficient allocation in a world of scarce goods and factors. Unfortunately the price vector which clears all markets simultaneously and instantaneously only exists under very restrictive assumptions being far from reality. Equilibrium is determined by the notion that the market clearing prices are the endogenous solution of excess demand functions equaling zero. The Walras and Arrow-Debreu general equilibrium theory does not explain where these market clearing prices do originate. In terms of the definition of the equilibrium concept, this means that all prices must be treated parametrically by the actors. Therefore the definition of the equilibrium concept in a Walras and Arrow-Debreu world excludes the explanation of existing unexploited arbitrage opportunities like involuntary unemployment. All quantity signals are reducible to price signals. By contrast, in Keynesian economics income signals cannot be reduced to price signals.

Clower (1965) and Barro and Grossman (1971, 1976) developed the so-called fix-price rationing concept. By this equilibrium concept, quantity signals are explicitly taken into account. However, the assumption of exogenously given prices differing from Walrasian market clearing prices, with market imbalance and quantity signals as a consequence, has been the object of severe criticism because unexploited arbitrage opportunities are not explained by individual rational behaviour but postulated by exogenous prices. Another problem remains unsolved as well: why are there no forces leading to price changes if both sides of the market could benefit like in the Keynesian regime? I.e., fix-price rationing models do not explain who has the power to prevent actors from changing prices and why she should do so. We do not contradict all this criticism, but we do not conclude that fix-price rationing models are useless. On the contrary, fix-price rationing models provide a sound basis for a general equilibrium concept with price and quantity signals where Non-Walrasian price determination is endogenous. Benassy (1993) integrated imperfect competition into a Fix-price Rationing framework to explain sticky prices and quan-

tity signals.<sup>1</sup> In our contribution we will formulate, by means of duality theory<sup>2</sup>, a general equilibrium concept with an arbitrary number of flexible goods and factor prices, exogenously given goods as well as factor prices and quantity constraints on goods and factor markets.

The remainder is organized as follows. In section 2 we will describe household behaviour on goods and factor markets with and without quantity rationing. In section 3 firm behaviour will be modelled on both goods and factor markets with and without quantity rationing. Then, in section 4 we will bring together the representative household and the representative firm within a general equilibrium framework and we will set out specific general equilibrium scenarios. Finally, we will conclude with our main results.

## 2 The Representative Household

### 2.1 The Transfer Function

First we define our notation:<sup>3</sup>

$\mathbf{v} = [v_i]$  vector of unrationed factors with flex-price vector  $\mathbf{z} = [z_i]$ ,  $i = 1, \dots, k$

$\bar{\mathbf{l}} = [\bar{l}_r]$  vector of rationed factors with fix-price vector  $\bar{\mathbf{w}} = [\bar{w}_r]$ ,  $r = 1, \dots, m$

$\hat{\mathbf{w}} = [\hat{w}_r]$  vector of virtual factor prices,  $r = 1, \dots, m$

$\check{\mathbf{l}} = [\check{l}_r]$  vector of virtual factor quantities,  $r = 1, \dots, m$

$\mathbf{x} = [x_j]$  vector of unrationed goods with flex-price vector  $\mathbf{p} = [p_j]$ ,  $j = 1, \dots, s$

$\bar{\mathbf{y}} = [\bar{y}_t]$  vector of rationed goods with fix-price vector  $\bar{\mathbf{q}} = [\bar{q}_t]$ ,  $t = 1, \dots, n$

$\hat{\mathbf{q}} = [\hat{q}_t]$  vector of virtual goods prices,  $t = 1, \dots, n$

$\check{\mathbf{y}} = [\check{y}_t]$  vector of virtual goods quantities,  $t = 1, \dots, n$

<sup>1</sup>For an excellent survey of imperfect competition and general equilibrium see Hart (1985).

<sup>2</sup>For an excellent exposition of duality theory see Cornes (1992).

<sup>3</sup>Bold variables symbolise vectors. Let the quantity variables be column vectors and the price vectors be line vectors. Thus the inner product of quantity and price vectors becomes a scalar.

$u$  utility

For the case of the household being neither constrained on goods nor on factor markets, we get the following definition of the unconstrained transfer function:

$$T(\mathbf{p}, \mathbf{q}, \mathbf{z}, \mathbf{w}, u) \equiv \min_{\mathbf{x}, \mathbf{y}, \mathbf{v}, l} \{ \mathbf{p}\mathbf{x} + \mathbf{q}\mathbf{y} - \mathbf{z}\mathbf{v} - \mathbf{w}l \mid U(\mathbf{x}, \mathbf{y}, \mathbf{v}, l) \geq u \}. \quad (1)$$

Like the expenditure function, the transfer function is an optimal value function that characterizes household behaviour in perfect competitive goods and factor markets. The transfer function shows the minimal transfer income necessary for the household to finance utility level  $u$  at given goods and factor prices. So the transfer function can either be positive or negative. In contrast to the concept of the expenditure function, variable factor supplies can explicitly be taken into account. This will be of special importance in the context of the virtual wage concept. If the transfer function describes household behaviour solely on perfect competitive markets, it becomes zero.

**Definition 1** *The unconstrained transfer function gives the minimum transfer income necessary for the household to finance utility level  $u$  at given goods and factor prices.*

Now we pass over to a situation where the representative household is up against quantity constraints due to given goods and factor prices. So we can define the constrained transfer function

$$\tilde{T}(\bar{\mathbf{y}}, \bar{\mathbf{l}}, \mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}, u) \equiv \min_{\mathbf{x}, \mathbf{y}, \mathbf{v}, l} \{ \mathbf{p}\mathbf{x} + \bar{\mathbf{q}}\mathbf{y} - \mathbf{z}\mathbf{v} - \bar{\mathbf{w}}l \mid U(\mathbf{x}, \mathbf{y}, \mathbf{v}, l) \geq u, l \leq \bar{\mathbf{l}}, \mathbf{y} \leq \bar{\mathbf{y}} \}. \quad (2)$$

**Definition 2** *The constrained transfer function gives the minimum transfer income necessary for the household to finance utility level  $u$  when treating prices as well as quantity constraints on goods and factor markets parametrically.*

In what follows we will assume a utility function being additively separable in consumption and production quantities. Hence changes in quantity constraints only lead to income effects on factor markets and the household's consumption decision will not be influenced by substitution effects. Compared to definition (1), definition (2) contains binding quantity constraints. By contrast, goods prices  $\mathbf{p}$  and factor prices  $\mathbf{z}$  still remain perfectly flexible.

Unfortunately we do not know the properties of the constrained transfer function because of the quantity constraints  $\bar{\mathbf{I}}$  and  $\bar{\mathbf{y}}$ . Therefore we express the constrained transfer function by an unconstrained or virtual transfer function whose properties are known, as this function only depends on price signals. To derive a relationship between the constrained and unconstrained transfer function we make use of a characteristic feature of all market coordination mechanisms with price distortions: if quantity constraints are binding, there exists more than one price system. Market prices and shadow prices are falling apart. This is the reason for the existence of unexploited arbitrage possibilities. Neary and Roberts (1980) have made use of the the concept of virtual prices in order to analyse household behaviour in the case of quantity constraints. Their contribution offers the methodological background of our analysis.

## 2.2 The Concept of Virtual Prices

The concept of virtual prices allows us to derive a relationship between constrained and unconstrained transfer functions. This relationship allows us to model the behaviour of a quantity constrained household by a transfer function which does not depend on quantity signals:

**Definition 3** *At the virtual price the household supplies and demands voluntarily the constrained quantity. The virtual price cannot be observed on distorted markets.*



We start with extending the constrained transfer function by  $\hat{w}\bar{l} - \hat{w}\bar{l}$  and  $\hat{q}\bar{y} - \hat{q}\bar{y}$ .

So we get

$$\begin{aligned} \tilde{T}(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u) &\equiv \mathbf{p}\mathbf{x}(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u) + \bar{q}\bar{y} + \hat{q}\bar{y} - \hat{q}\bar{y} \\ &\quad - \mathbf{z}\mathbf{v}(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u) - \bar{w}\bar{l} + \hat{w}\bar{l} - \hat{w}\bar{l} \\ &\equiv \hat{T}(\mathbf{p}, \hat{q}, \mathbf{z}, \hat{w}, u) - (\hat{q} - \bar{q})\bar{y} + (\hat{w} - \bar{w})\bar{l}. \end{aligned} \quad (3)$$

In the second line we apply the concept of virtual prices:<sup>4</sup>

$$\bar{y} = \mathbf{y}(\mathbf{p}, \hat{q}, \mathbf{z}, \hat{w}, u) \quad \text{and} \quad \bar{l} = \mathbf{l}(\mathbf{p}, \hat{q}, \mathbf{z}, \hat{w}, u). \quad (4)$$

The virtual prices  $\hat{q}$  and  $\hat{w}$  are those where  $\bar{y}$  and  $\bar{l}$  are demanded and supplied voluntarily. And from equation (4) we see

$$\hat{q} = \mathbf{q}(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u) \quad \text{and} \quad \hat{w} = \mathbf{w}(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u). \quad (5)$$

**Definition 4** *The transfer function  $\hat{T}(\mathbf{p}, \hat{q}, \mathbf{z}, \hat{w}, u)$  is called virtual price transfer function. As an unconstrained function, the virtual price transfer function does not depend on quantity constraints.*

Apart from utility level  $u$  the virtual price transfer function depends on flexible as well as virtual goods and factor prices and can be defined by means of a minimization problem:

$$\hat{T}(\mathbf{p}, \hat{q}, \mathbf{z}, \hat{w}, u) \equiv \min_{\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{l}} \{ \mathbf{p}\mathbf{x} + \hat{q}\mathbf{y} - \mathbf{z}\mathbf{v} - \hat{w}\mathbf{l} \mid U(\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{l}) \geq u \}. \quad (6)$$

The last two terms in the third line of equation (3) are nothing else but the household's evaluation of being quantity constrained. While the second term  $-(\hat{q} - \bar{q})\bar{y}$

<sup>4</sup>In (3) we have made use of the following relationships:

$$\begin{aligned} \mathbf{l}(\mathbf{p}, \hat{q}, \mathbf{z}, \hat{w}, u) &= \bar{l} \quad \text{and} \quad \mathbf{y}(\mathbf{p}, \hat{q}, \mathbf{z}, \hat{w}, u) = \bar{y}, \\ \mathbf{v}(\mathbf{p}, \hat{q}, \mathbf{z}, \hat{w}, u) &= \mathbf{v}(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u) \quad \text{and} \quad \mathbf{x}(\mathbf{p}, \hat{q}, \mathbf{z}, \hat{w}, u) = \mathbf{x}(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u). \end{aligned}$$

on the right-hand side of equation (3) is describing the rationing of the household on goods markets, the third term  $(\hat{w} - \bar{w})\bar{l}$  stands for the rationing of the household on factor markets. Neary (1981) calls the difference between the virtual and actual prices PATINKIN GAP. According to Laroque (1981), the PATINKIN GAP is a measure for the power of market distortions and therefore a determining force of changes in goods and factor prices in an intertemporal model.

Deriving identity (3) with respect to  $\bar{y}$  and  $\bar{l}$  shows that the terms  $-(\hat{q} - \bar{q})$  and  $(\hat{w} - \bar{w})$  can be interpreted as vectors of shadow prices:

$$\begin{aligned}\frac{\partial \tilde{T}}{\partial \bar{y}} &= \left( \frac{\partial \hat{T}}{\partial \hat{q}} - \bar{y} \right) \frac{\partial \hat{q}}{\partial \bar{y}} - (\hat{q} - \bar{q}) + \left( \frac{\partial \hat{T}}{\partial \hat{w}} + \bar{l} \right) \frac{\partial \hat{w}}{\partial \bar{y}} = -(\hat{q} - \bar{q}) < 0 \\ \frac{\partial \tilde{T}}{\partial \bar{l}} &= \left( \frac{\partial \hat{T}}{\partial \hat{w}} + \bar{l} \right) \frac{\partial \hat{w}}{\partial \bar{l}} + (\hat{w} - \bar{w}) + \left( \frac{\partial \hat{T}}{\partial \hat{q}} - \bar{y} \right) \frac{\partial \hat{w}}{\partial \bar{l}} = \hat{w} - \bar{w} < 0.\end{aligned}\quad (7)$$

In (7) we take into account that virtual goods and factor prices are functions of the quantity constraints  $\bar{y}$  and  $\bar{l}$ . This follows from (5). An infinitesimal relaxation of goods or factor market constraints leads to a reduction of the minimal transfer income amounting to  $-(\hat{q} - \bar{q})$  and  $(\hat{w} - \bar{w})$  respectively, necessary to finance utility level  $u$ .

These differences as part of the products representing costs of the household due to being rationed on goods and factor markets is important for our analysis. Naturally there is also an economic interpretation of identity (3): let us assume that the household needs a certain transfer income to reach utility level  $u$  at prevailing goods and factor prices and at a given quantity constraint  $\bar{l}_r$  on, say, the market for low-skilled labour. By way of contrast, let there be no quantity constraints on goods markets. In case of involuntary unemployment the actual wage rate  $\bar{w}_r$  exceeds the virtual wage rate  $\hat{w}_r$ . Then  $(\hat{w}_r - \bar{w}_r)\bar{l}_r$  becomes negative. Therefore the value of the virtual transfer function has to exceed the value of the constrained transfer function. The virtual wage rate is lower than the actual wage rate at an unchanged labour input. Thus, the virtual labour income is smaller than the actual labour income at

given employment  $\bar{l}_r$ . This loss in labour income just amounts to  $(\hat{w}_r - \bar{w}_r)\bar{l}_r$ .

In addition, identity (3) refers to a further important point: if one or two quantity constraints are binding and  $-(\hat{q} - \bar{q})\bar{y}$  and  $(\hat{w} - \bar{w})\bar{l}$  respectively differ and are unequal to zero in consequence, the constrained function and the virtual price transfer function cannot be of the same value. So at least one of the two functions is unequal to zero.

**Proposition 1** *The household's behaviour when treating goods and factor prices as well as quantity constraints parametrically can either be described by means of the constrained transfer function or, alternatively, by the unconstrained or virtual price transfer function and the linear terms  $-(\hat{q} - \bar{q})\bar{y}$  and  $(\hat{w} - \bar{w})\bar{l}$ . This follows from identity (3).*

### 2.3 The Concept of Virtual Quantities

The disadvantage of the concept of virtual prices lies in virtual prices not being observable. We can overcome this problem by making use of the concept of virtual quantities.

**Definition 5** *The virtual quantity results from the demand or supply of the household at the exogenously given goods and factor price respectively. However, this quantity cannot be transacted because of the quantity constraint.*

The idea inherent in this concept is quite simple. We no longer ask for the virtual price voluntarily bringing about the quantity constraint but for the quantity which is voluntarily supplied or demanded at the exogenously given price.

Once again we start with the constrained transfer function (2) which is now extended by  $\bar{w}\bar{l} - \bar{w}\check{l}$  and  $\bar{q}\bar{y} - \bar{q}\check{y}$ . So we get

$$\begin{aligned} \tilde{T}(\bar{y}, \bar{l}, p, \bar{q}, z, \bar{w}, u) &\equiv px(\bar{y}, \bar{l}, p, \bar{q}, z, \bar{w}, u) + \bar{q}\bar{y} + \bar{q}\check{y} - \bar{q}\check{y} \\ &\quad -zv(\bar{y}, \bar{l}, p, \bar{q}, z, \bar{w}, u) - \bar{w}\bar{l} + \bar{w}\check{l} - \bar{w}\check{l} \end{aligned}$$

$$\equiv \check{T}(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}, u) - (\check{\mathbf{y}} - \bar{\mathbf{y}})\bar{\mathbf{q}} + (\check{\mathbf{l}} - \bar{\mathbf{l}})\bar{\mathbf{w}}.^5 \quad (8)$$

This leads us to the following definition:

**Definition 6** *The transfer function  $\check{T}(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}, u)$  is called virtual quantity transfer function.*

Apart from utility level  $u$  the virtual quantity transfer function depends on flexible as well as exogenously given goods and factor prices and can be defined as a minimization problem:

$$\check{T}(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}, u) \equiv \min_{\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{l}} \{ \mathbf{p}\mathbf{x} + \bar{\mathbf{q}}\mathbf{y} - \mathbf{z}\mathbf{v} - \bar{\mathbf{w}}\mathbf{l} \mid U(\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{l}) \geq u \}. \quad (9)$$

The product  $(\check{\mathbf{l}} - \bar{\mathbf{l}})\bar{\mathbf{w}}$  represents the household's evaluation of being rationed on factor markets at actual factor rewards. So this product is equivalent to the costs of involuntary unemployment with all variables being observable. By contrast, the product  $-(\check{\mathbf{y}} - \bar{\mathbf{y}})\bar{\mathbf{q}}$  represents the household's evaluation of being rationed on goods markets at actual goods prices. So this product is equivalent to the household's renouncement of consumption.

**Proposition 2** *The household's behaviour when treating prices as well as quantity constraints on goods and factor markets parametrically can either be described by means of the constrained transfer function or, alternatively, by the unconstrained or virtual quantity transfer function and the linear terms  $-(\check{\mathbf{y}} - \bar{\mathbf{y}})\bar{\mathbf{q}}$  and  $(\check{\mathbf{l}} - \bar{\mathbf{l}})\bar{\mathbf{w}}$ . This follows from identity (8).*

The interested reader might ask how far the interpretation of identity (3) and (8) are related to each other:

**Proposition 3** *Identity (8) differs from identity (3) in two respects:*

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<sup>5</sup>In the second line of equation (8) we apply the concept of virtual quantities, i.e.

$$\check{\mathbf{y}} = \mathbf{y}(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}, u) \quad \text{and} \quad \check{\mathbf{l}} = \mathbf{l}(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}, u).$$

1. *If there is no rationing on goods markets, the value of the virtual quantity transfer function does not become larger but smaller than the value of the constrained transfer function by the evaluation of the rationing on factor markets.*
2. *If there is no rationing on factor markets, the value of the virtual quantity transfer function does not become smaller but larger than the value of the constrained transfer function by the evaluation of the rationing on goods markets.*

The reason for the relationship stated under 1 lies in virtual factor supplies  $\check{I}$  that could not be realized. In case of involuntary unemployment these exceed the actual employment  $\bar{I}$ . In consequence, a higher virtual factor income, ceteris paribus, necessitates a lower transfer income to finance utility level  $u$ . The difference between the desired factor income  $\check{w}\check{I}$  at exogenously given factor rewards  $\bar{w}$  and the actually realized factor income  $\bar{w}\bar{I}$  is described by the third term on the right-hand side of identity (8).

The reason for the relationship stated under 2 lies in the virtual goods demand  $\check{y}$  that could not be realized either. This exceeds the actual goods transaction quantity  $\bar{y}$ . In consequence, a higher consumption expenditure necessitates, ceteris paribus, a higher virtual transfer income. The difference between the desired consumption expenditure  $\check{q}\check{y}$  at exogenously given goods prices  $\bar{q}$  and the actually realized consumption expenditure  $\bar{q}\bar{y}$  is described by the second term on the right-hand side of identity (8).

The virtual quantity transfer function has the following properties:  $\check{T}(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}, u)$  is non-decreasing in goods prices, non-increasing in factor prices, linear homogeneous and quasi concave in goods and factor prices, non-increasing in factor supplies and finally non-decreasing in utility. Under the assumption of the existence of the first- and second-order derivatives with respect to all goods and factor prices we can derive goods demand and factor supply functions by SHEPHARD'S LEMMA:

$$\frac{\partial \check{T}}{\partial \mathbf{p}} = \check{\mathbf{x}}^d(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}, u) \quad \text{and} \quad \frac{\partial \check{T}}{\partial \bar{\mathbf{q}}} = \check{\mathbf{y}}^d(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}, u). \quad (10)$$

The unconstrained or virtual goods demand functions are homogeneous of degree zero in goods and factor prices. All goods are normal goods. Therefore a price increase leads to a decrease in demand for the corresponding good. In addition, the goods are treated as substitutes by the household. The unconstrained or virtual factor supply functions become

$$\frac{\partial \check{T}}{\partial \bar{z}} = -\check{v}^s(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}, u) \quad \text{and} \quad \frac{\partial \check{T}}{\partial \bar{\mathbf{w}}} = -\check{l}^s(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}, u). \quad (11)$$

The virtual quantity transfer function delivers virtual goods demand and factor supply functions solely depending on actual price signals. Therefore these functions are called unconstrained or desired goods demand and factor supply functions. The constrained or effective demand and supply functions can be derived from the constrained transfer function.

Apart from identity (8) there is the following relationship between the constrained and unconstrained transfer function:<sup>6</sup>

$$\check{T}(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}, u) = \min_{\bar{\mathbf{y}}, \bar{\mathbf{l}}} \tilde{T}(\bar{\mathbf{y}}, \bar{\mathbf{l}}, \mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}, u). \quad (12)$$

**Proposition 4** *The virtual quantity transfer function is the envelope of the constrained transfer function. This follows from equation (12).*

As according to identity (8) the constrained and unconstrained transfer function only differ in two linear terms, the virtual transfer function's properties can be transferred to the constrained transfer function. We cannot derive effective goods demand and factor supply functions from the virtual but from the constrained transfer function. On this occasion the demanded effective supply and demand functions can either be derived directly from the constrained transfer function or, by means of identity (8), be expressed by unconstrained functions. In what follows we will take both alternatives into consideration.

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<sup>6</sup>See Deaton and Muellbauer (1980).

By SHEPARD'S LEMMA effective goods demand functions can be derived from identity (8), i.e.:

$$\frac{\partial \tilde{T}}{\partial \mathbf{p}} \equiv \frac{\partial \check{T}}{\partial \mathbf{p}} - \bar{q} \frac{\partial \check{y}^d}{\partial \mathbf{p}} + \bar{w} \frac{\partial \check{l}^s}{\partial \mathbf{p}}$$

and as a result

$$\check{x}^d(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u) \equiv \check{x}^d(\mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u) - \bar{q} \frac{\partial \check{y}^d}{\partial \mathbf{p}} + \bar{w} \frac{\partial \check{l}^s}{\partial \mathbf{p}}. \quad (13)$$

And

$$\frac{\partial \tilde{T}}{\partial \bar{q}} \equiv \frac{\partial \check{T}}{\partial \bar{q}} - (\check{y}^d - \bar{y}) - \bar{q} \frac{\partial \check{y}^d}{\partial \bar{q}} + \bar{w} \frac{\partial \check{l}^s}{\partial \bar{q}}$$

and thus

$$0 \equiv -\bar{q} \frac{\partial \check{y}^d}{\partial \bar{q}} + \bar{w} \frac{\partial \check{l}^s}{\partial \bar{q}}. \quad (14)$$

In the same way we get effective factor supply functions

$$\frac{\partial \tilde{T}}{\partial \mathbf{z}} \equiv \frac{\partial \check{T}}{\partial \mathbf{z}} - \bar{q} \frac{\partial \check{y}^d}{\partial \mathbf{z}} + \bar{w} \frac{\partial \check{l}^s}{\partial \mathbf{z}}$$

and thus

$$\check{v}^s(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u) \equiv \check{v}^s(\mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u) + \bar{q} \frac{\partial \check{y}^d}{\partial \mathbf{z}} - \bar{w} \frac{\partial \check{l}^s}{\partial \mathbf{z}}. \quad (15)$$

And

$$\frac{\partial \tilde{T}}{\partial \bar{w}} \equiv \frac{\partial \check{T}}{\partial \bar{w}} + (\check{l}^s - \bar{l}) - \bar{q} \frac{\partial \check{y}^d}{\partial \bar{w}} + \bar{w} \frac{\partial \check{l}^s}{\partial \bar{w}}$$

and as a result

$$0 \equiv -\bar{q} \frac{\partial \check{y}^d}{\partial \bar{w}} + \bar{w} \frac{\partial \check{l}^s}{\partial \bar{w}}. \quad (16)$$

## 2.4 The Concepts of Virtual Prices and Virtual Quantities Combined

There may exist situations where it makes sense to apply the concept of virtual prices in one market and the concept of virtual quantities in another market. From an empirical point of view, it seems fruitful to describe quantity constraints on factor markets by means of the concept of virtual quantities and quantity constraints on goods markets by means of the concept of virtual prices. Let us assume for example that all factor market constraints are expressed by the concept of virtual quantities and all goods market constraints by the concept of virtual prices. Nevertheless, it is important to point out that these two concepts can be mixed up arbitrarily not only between goods and factor markets, but also within goods markets and within factor markets. Now, our objective is to derive a relationship analogous to identity (3) and (8). Once again we start with the definition of the constrained transfer function (2). We expand the right-hand side by  $\hat{q}\bar{y} - \hat{q}\bar{y}$  and  $\bar{w}\bar{l} - \bar{w}\bar{l}$ :

$$\begin{aligned}
 \tilde{T}(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u) &\equiv \mathbf{p}\mathbf{x}(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u) + \bar{q}\bar{y} + \hat{q}\bar{y} - \hat{q}\bar{y} \\
 &\quad - \mathbf{z}\mathbf{v}(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u) - \bar{w}\bar{l} - \bar{w}\bar{l} + \bar{w}\bar{l} \\
 &\equiv \tilde{T}(\mathbf{p}, \hat{q}, \mathbf{z}, \bar{w}, u) - (\hat{q} - \bar{q})\bar{y} + (\bar{l} - \bar{l})\bar{w}. \quad (17)
 \end{aligned}$$

**Definition 7** *The transfer function  $\tilde{T}(\mathbf{p}, \hat{q}, \mathbf{z}, \bar{w}, u)$  is called virtual price-quantity transfer function.*

Apart from utility level  $u$  the virtual price-quantity transfer function depends on flexible goods and factor prices, on virtual goods prices and, finally, on the exogenously given factor prices. The virtual price-quantity transfer function can be defined as a minimization problem:

$$\tilde{T}(\mathbf{p}, \hat{q}, \mathbf{z}, \bar{w}, u) \equiv \min_{\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{l}} \{ \mathbf{p}\mathbf{x} + \hat{q}\mathbf{y} - \mathbf{z}\mathbf{v} - \bar{w}\mathbf{l} \mid U(\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{l}) \geq u \}. \quad (18)$$



Through identity (17) we can now derive virtual or unconstrained and constrained or effective goods demand and factor supply functions as we did in section 2.2 and 2.3. This is left to the reader as a little exercise. Our main point is to make clear that there arise no problems when combining both concepts which are perfectly symmetric.

### 3 The Firm

#### 3.1 The Profit Function

The production side can be modelled by unconstrained and constrained profit functions. From profit functions unconstrained and constrained goods supply and factor demand functions can be derived for an arbitrary number of rationing scenarios on goods and factor markets. Hence, these functions describe a representative firm's behaviour on goods and factor markets. The strong quasi-concave production functions

$$\begin{aligned}x_j &= f^{x_j}(\mathbf{v}, \mathbf{l}) & \text{with } j = 1, \dots, s \\y_t &= f^{y_t}(\mathbf{v}, \mathbf{l}) & \text{with } t = 1, \dots, n\end{aligned}\tag{19}$$

are constant returns to scale.

The representative firm behaves on goods as well as on factor markets as a price taker, being quantity constrained neither on factor nor on goods markets. Taking these conditions as a basis, we can define the aggregate unconstrained profit function:<sup>7</sup>

$$\pi(\mathbf{p}, \mathbf{q}, \mathbf{z}, \mathbf{w}) \equiv \max_{\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{l}} \{\mathbf{p}\mathbf{x} + \mathbf{q}\mathbf{y} - \mathbf{z}\mathbf{v} - \mathbf{w}\mathbf{l} \mid g(\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{l}) \leq 0\}.\tag{20}$$

<sup>7</sup> $g(\cdot)$  represents the convex production possibility set. In an infinitesimal neighbourhood the existence of an aggregate profit function was proved by Schweinberger (1995). The global proof has been furnished by Woodland (1982).

**Definition 8** *The aggregate profit function gives the maximum profit income of the representative firm under efficient production.*

The aggregate profit function can be interpreted as an objective function of a representative multi-product firm that behaves as a price-taker both on goods and on factor markets. That means that all prices are treated parametrically. The firm maximizes its profit by choosing profit maximizing goods supplies and factor demands. In case of perfect competitive goods and factor markets, the profit function becomes zero because of constant returns to scale. So EULER's theorem holds.

Now a constrained profit function can be defined describing the behaviour of the representative firm in case of quantity constraints on goods and factor markets:

$$\tilde{\pi}(\bar{y}, \bar{l}, p, \bar{q}, z, \bar{w}) \equiv \max_{x, y, v, l} \{px + \bar{q}y - zv - \bar{w}l \mid g(x, y, v, l) \leq 0, l \leq \bar{l}, y \leq \bar{y}\}. \quad (21)$$

**Definition 9** *The constrained profit function gives the maximum profit income of the firm when treating prices as well as quantity constraints on goods and factor markets parametrically.*

Compared to (20) definition (21) contains binding quantity constraints.

### 3.2 The Concept of Virtual Prices

Not only in case of the transfer function but also in case of the profit function we are able to derive a relationship between unconstrained and constrained profit functions by virtual prices. Thus we define:

**Definition 10** *The virtual price is the price inducing the firm to voluntarily supply and demand the rationing quantity.*

Extension of the profit function by  $\hat{q}\bar{y} - \bar{q}\bar{y}$  and  $\hat{w}\bar{l} - \bar{w}\bar{l}$  leads to the following relationship:

$$\begin{aligned}\tilde{\pi}(\bar{y}, \bar{l}, \mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}) &\equiv \mathbf{p}\mathbf{x}(\bar{y}, \bar{l}, \mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}) + \bar{\mathbf{q}}\bar{y} + \hat{\mathbf{q}}\bar{y} - \bar{\mathbf{q}}\bar{y} \\ &\quad - \mathbf{z}\mathbf{v}(\bar{y}, \bar{l}, \mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}) - \bar{\mathbf{w}}\bar{l} + \hat{\mathbf{w}}\bar{l} - \bar{\mathbf{w}}\bar{l} \\ &\equiv \hat{\pi}(\mathbf{p}, \hat{\mathbf{q}}, \mathbf{z}, \hat{\mathbf{w}}) - (\hat{\mathbf{q}} - \bar{\mathbf{q}})\bar{y} + (\hat{\mathbf{w}} - \bar{\mathbf{w}})\bar{l}.\end{aligned}\quad (22)$$

The second line embodies the concept of virtual prices.

**Definition 11** *The profit function  $\hat{\pi}(\mathbf{p}, \hat{\mathbf{q}}, \mathbf{z}, \hat{\mathbf{w}})$  is referred to as virtual price profit function.*

The virtual price profit function depends on flexible as well as on virtual goods and factor prices and can be defined as a maximization problem:

$$\hat{\pi}(\mathbf{p}, \hat{\mathbf{q}}, \mathbf{z}, \hat{\mathbf{w}}) \equiv \max_{\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{l}} \{ \mathbf{p}\mathbf{x} + \hat{\mathbf{q}}\mathbf{y} - \mathbf{z}\mathbf{v} - \hat{\mathbf{w}}\mathbf{l} \mid g(\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{l}) \leq 0 \}.\quad (23)$$

Whereas the last two terms of identity (22) evaluate the firm's binding rationing constraints on goods and factor markets, the first term is an unconstrained function. So we can make use of the advantage that the derivation properties of this unconstrained function are known.

Deriving identity (22) with respect to  $\bar{y}$  and  $\bar{l}$  shows that the terms  $-(\hat{\mathbf{q}} - \bar{\mathbf{q}})$  and  $(\hat{\mathbf{w}} - \bar{\mathbf{w}})$  can be interpreted as vectors of shadow prices:

$$\begin{aligned}\frac{\partial \tilde{\pi}}{\partial \bar{y}} &= \left( \frac{\partial \hat{\pi}}{\partial \hat{\mathbf{q}}} - \bar{y} \right) \frac{\partial \hat{\mathbf{q}}}{\partial \bar{y}} - (\hat{\mathbf{q}} - \bar{\mathbf{q}}) + \left( \frac{\partial \hat{\pi}}{\partial \hat{\mathbf{w}}} + \bar{l} \right) \frac{\partial \hat{\mathbf{w}}}{\partial \bar{y}} = -(\hat{\mathbf{q}} - \bar{\mathbf{q}}) > 0 \\ \frac{\partial \tilde{\pi}}{\partial \bar{l}} &= \left( \frac{\partial \hat{\pi}}{\partial \hat{\mathbf{w}}} + \bar{l} \right) \frac{\partial \hat{\mathbf{w}}}{\partial \bar{l}} + (\hat{\mathbf{w}} - \bar{\mathbf{w}}) + \left( \frac{\partial \hat{\pi}}{\partial \hat{\mathbf{q}}} - \bar{y} \right) \frac{\partial \hat{\mathbf{q}}}{\partial \bar{l}} = \hat{\mathbf{w}} - \bar{\mathbf{w}} > 0.\end{aligned}\quad (24)$$

In (24) we take into account that virtual goods and factor prices are functions of the quantity constraints  $\bar{y}$  and  $\bar{l}$ . An infinitesimal relaxation of goods and factor market constraints respectively leads to an increase in profit measured by  $-(\hat{\mathbf{q}} - \bar{\mathbf{q}})$  and  $(\hat{\mathbf{w}} - \bar{\mathbf{w}})$  respectively.

Identity (22) hints to a further important point: if one or two quantity constraints are binding and  $-(\hat{q} - \bar{q})\bar{y}$  and  $(\hat{w} - \bar{w})\bar{l}$  respectively differ and thus are unequal to zero, the constrained and the virtual price profit function cannot be of the same value. So at least one of the two functions is unequal to zero.

**Proposition 5** *The behaviour of the firm treating goods and factor prices as well as quantity constraints parametrically can either be described by the constrained profit function or alternatively by the unconstrained or virtual price profit function and the linear terms  $-(\hat{q} - \bar{q})\bar{y}$  and  $(\hat{w} - \bar{w})\bar{l}$ . This follows from identity (22).*

### 3.3 The Concept of Virtual Quantities

Again the concept of virtual prices suffers from the disadvantage of nonobservable virtual goods and factor prices. Therefore, in what follows a virtual quantity profit function will be derived. Now the constrained profit function (21) is extended by  $\bar{q}\check{y} - \bar{q}\check{y}$  and  $\bar{w}\check{l} - \bar{w}\check{l}$ :

$$\begin{aligned} \tilde{\pi}(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}) &\equiv \mathbf{p}\mathbf{x}(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}) + \bar{q}\bar{y} + \bar{q}\check{y} - \bar{q}\check{y} \\ &\quad - \mathbf{z}\mathbf{v}(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}) - \bar{w}\bar{l} + \bar{w}\check{l} - \bar{w}\check{l} \\ &\equiv \tilde{\pi}(\mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}) - (\check{y} - \bar{y})\bar{q} + (\check{l} - \bar{l})\bar{w}. \end{aligned} \quad (25)$$

The second line embodies the concept of virtual quantities. The identification of  $\check{y}$  and  $\check{l}$  as virtual quantities makes clear that these are desired goods supply and factor demands at flexible and exogenously given goods and factor prices.

**Definition 12** *The profit function  $\tilde{\pi}(\mathbf{p}, \bar{q}, \mathbf{z}, \bar{w})$  is referred to as virtual quantity profit function.*

The virtual quantity profit function depends on flexible as well as on exogenously given goods and factor prices and can be defined as maximization problem:

$$\tilde{\pi}(\mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}) \equiv \max_{\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{l}} \{ \mathbf{p}\mathbf{x} + \bar{q}\mathbf{y} - \mathbf{z}\mathbf{v} - \bar{w}\mathbf{l} \mid g(\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{l}) \leq 0 \}. \quad (26)$$

**Proposition 6** *The behaviour of the firm treating goods and factor prices as well as quantity constraints parametrically can either be described by the constrained profit function or alternatively by the unconstrained or virtual quantity profit function and the linear terms  $-(\check{y} - \bar{y})\bar{q}$  and  $(\check{l} - \bar{l})\bar{w}$ . This follows from identity (25).*

The virtual quantity profit function exhibits the following properties:  $\check{\pi}(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}})$  is non-decreasing in goods prices, non-increasing in factor prices and linear homogeneous as well as convex in goods and factor prices. Under the assumption of the existence of the first- and second-order derivatives with respect to all goods and factor prices, HOTELLING'S LEMMA yields the following virtual goods supply functions:

$$\frac{\partial \check{\pi}}{\partial \mathbf{p}} = \check{\mathbf{x}}^s(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}) \quad \text{and} \quad \frac{\partial \check{\pi}}{\partial \bar{\mathbf{q}}} = \check{\mathbf{y}}^s(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}). \quad (27)$$

The unconstrained or virtual goods supply functions are homogeneous of degree zero in goods and factor prices. All goods are normal goods. Hence a price increase causes an increase in supply. Besides, for the representative firm the factors are substitutes. The unconstrained or virtual factor demand functions are

$$\frac{\partial \check{\pi}}{\partial \bar{\mathbf{z}}} = -\check{\mathbf{v}}^d(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}) \quad \text{and} \quad \frac{\partial \check{\pi}}{\partial \bar{\mathbf{w}}} = -\check{\mathbf{l}}^d(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}). \quad (28)$$

Apart from identity (25) there exists another relationship between the constrained and unconstrained profit function:

$$\check{\pi}(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}) = \min_{\bar{\mathbf{y}}, \bar{\mathbf{l}}} \bar{\pi}(\bar{\mathbf{y}}, \bar{\mathbf{l}}, \mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{\mathbf{w}}). \quad (29)$$

**Proposition 7** *The virtual quantity profit function is the envelope function of the constrained profit function. This follows from equation (29).*

We do not derive effective goods supply and factor demand functions from the virtual but from the constrained profit function. In the light of identity (25), effective demand and supply functions can either be derived directly from the constrained profit function or be expressed by an unconstrained profit function and the two

linear terms as shown in (25). In what follows both alternatives will be taken into consideration.

Through HOTELLING's LEMMA, effective goods supply and factor demand functions can be derived from identity (25):

$$\frac{\partial \tilde{\pi}}{\partial \mathbf{p}} \equiv \frac{\partial \tilde{\pi}}{\partial \mathbf{p}} - \bar{q} \frac{\partial \check{y}^s}{\partial \mathbf{p}} + \bar{w} \frac{\partial \check{l}^d}{\partial \mathbf{p}}$$

and thus

$$\bar{x}^s(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}) \equiv \check{x}^s(\mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}) - \bar{q} \frac{\partial \check{y}^s}{\partial \mathbf{p}} + \bar{w} \frac{\partial \check{l}^d}{\partial \mathbf{p}}. \quad (30)$$

Furthermore

$$\frac{\partial \tilde{\pi}}{\partial \bar{q}} \equiv \frac{\partial \tilde{\pi}}{\partial \bar{q}} - (\check{y}^s - \bar{y}) - \bar{q} \frac{\partial \check{y}^s}{\partial \bar{q}} + \bar{w} \frac{\partial \check{l}^d}{\partial \bar{q}}$$

and thus

$$0 \equiv -\bar{q} \frac{\partial \check{y}^s}{\partial \bar{q}} + \bar{w} \frac{\partial \check{l}^d}{\partial \bar{q}}. \quad (31)$$

In the same way we get the effective factor demand functions:

$$\frac{\partial \tilde{\pi}}{\partial \mathbf{z}} \equiv \frac{\partial \tilde{\pi}}{\partial \mathbf{z}} - \bar{q} \frac{\partial \check{y}^s}{\partial \mathbf{z}} + \bar{w} \frac{\partial \check{l}^d}{\partial \mathbf{z}}$$

and as a result

$$\bar{v}^d(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}) \equiv \check{v}^d(\mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}) + \bar{q} \frac{\partial \check{y}^s}{\partial \mathbf{z}} - \bar{w} \frac{\partial \check{l}^d}{\partial \mathbf{z}}. \quad (32)$$

Furthermore

$$\frac{\partial \tilde{\pi}}{\partial \bar{w}} \equiv \frac{\partial \tilde{\pi}}{\partial \bar{w}} + (\check{l}^d - \bar{l}) - \bar{q} \frac{\partial \check{y}^s}{\partial \bar{w}} + \bar{w} \frac{\partial \check{l}^d}{\partial \bar{w}}$$

and as a result

$$0 \equiv -\bar{q} \frac{\partial \check{y}^s}{\partial \bar{w}} + \bar{w} \frac{\partial \check{l}^d}{\partial \bar{w}}. \quad (33)$$

### 3.4 The Concepts of Virtual Prices and Virtual Quantities Combined

Similarly to the model of household behaviour, again we are able to combine the concepts of virtual prices and virtual quantities. Let us assume that the concept of virtual quantities is appropriate to describe factor market rationing, and the concept of virtual prices is appropriate to describe goods market rationing. We start with expanding the right-hand side of (22) by  $\hat{q}\bar{y} - \hat{q}\bar{y}$  and  $\bar{w}\bar{l} - \bar{w}\bar{l}$ :

$$\begin{aligned}\tilde{\pi}(\bar{y}, \bar{l}, \mathbf{p}, \hat{q}, \mathbf{z}, \bar{w}) &\equiv \mathbf{p}\mathbf{x}(\bar{y}, \bar{l}, \mathbf{p}, \hat{q}, \mathbf{z}, \bar{w}) + \bar{q}\bar{y} + \hat{q}\bar{y} - \hat{q}\bar{y} \\ &\quad - \mathbf{z}\mathbf{v}(\bar{y}, \bar{l}, \mathbf{p}, \hat{q}, \mathbf{z}, \bar{w}) - \bar{w}\bar{l} - \bar{w}\bar{l} + \bar{w}\bar{l} \\ &\equiv \tilde{\pi}(\mathbf{p}, \hat{q}, \mathbf{z}, \bar{w}) - (\hat{q} - \bar{q})\bar{y} + (\bar{l} - \bar{l})\bar{w}.\end{aligned}\quad (34)$$

**Definition 13** *The profit function  $\tilde{\pi}(\mathbf{p}, \hat{q}, \mathbf{z}, \bar{w})$  is called virtual price-quantity profit function.*

The virtual price-quantity profit function depends on flexible goods and factor prices, on virtual goods prices and the exogenously given factor prices. It can be defined as a maximization problem:

$$\tilde{\pi}(\mathbf{p}, \hat{q}, \mathbf{z}, \bar{w}, u) \equiv \max_{\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{l}} \{ \mathbf{p}\mathbf{x} + \hat{q}\mathbf{y} - \mathbf{z}\mathbf{v} - \bar{w}\mathbf{l} \mid g(\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{l}) \leq 0 \}.\quad (35)$$

Correspondingly, we are now able to derive appropriate goods supply and factor demand functions. Again this is left to the reader as a little exercise.

## 4 General Equilibrium

Starting with our methodological formulation of quantity constraints in a general equilibrium model we are interested in two theoretically well established equilibrium scenarios. These scenarios are first the Keynesian one and second the Classical one.

The following general equilibria fulfill four important equilibrium conditions: first, all quantity constraints are treated parametrically by each actor. Second, the short side of the market determines the quantity transacted. This is called the short side or HAHN-NEGISHI rule. Third, the two sides cannot be rationed simultaneously on the same market. Fourth, every transaction remains voluntarily.

In what follows we will apply the concept of virtual quantities on factor markets and the concept of virtual prices on goods markets. This approach seems to be appropriate because of the intuition given by empirical considerations. At least for constrained labour markets this line of reasoning is straightforward in the case of binding exogenously given minimum wages due to institutional settings.

#### 4.1 Keynesian General Equilibrium

While the representative household is facing constraints on  $m$  factor markets, the representative firm is constrained on  $n$  goods markets. Hence the vector  $\bar{y}$  represents the entire quantity constraints the representative firm faces on goods markets and the vector  $\bar{l}$  contains all quantity constraints faced by the representative household on factor markets.

First, the goods market equilibria with flexible prices are

$$\check{x}^d(\mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u) + \frac{\partial \bar{l}}{\partial \mathbf{p}} \bar{w} = \hat{x}^s(\mathbf{p}, \hat{q}, \mathbf{z}, \bar{w}) - \frac{\partial \hat{q}}{\partial \mathbf{p}} \bar{y}. \quad (36)$$

Second, the goods market equilibria with exogenously given prices are

$$\check{y}^d(\mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u) + \frac{\partial \bar{l}}{\partial \bar{q}} \bar{w} = \left(1 - \frac{\partial \hat{q}}{\partial \bar{q}}\right) \bar{y}. \quad (37)$$

Third, the factor market equilibria with flexible prices can be written as follows:

$$\check{v}^s(\mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u) - \frac{\partial \bar{l}}{\partial \mathbf{z}} \bar{w} = \hat{v}^d(\mathbf{p}, \hat{q}, \mathbf{z}, \bar{w}) + \frac{\partial \hat{q}}{\partial \mathbf{z}} \bar{y}. \quad (38)$$



The fourth component which our Keynesian general equilibrium consists of are the equilibria on factor markets where prices are exogenously given:

$$\bar{I} - \frac{\partial \bar{I}}{\partial \bar{w}} \bar{w} = \hat{I}^d(\mathbf{p}, \hat{\mathbf{q}}, \mathbf{z}, \bar{w}, u) + \frac{\partial \hat{\mathbf{q}}}{\partial \bar{w}} \bar{y}. \quad (39)$$

Finally, the Keynesian general equilibrium is closed by the economy's budget constraint

$$\tilde{T}(\bar{I}, \mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{w}, u) = \tilde{\pi}(\bar{y}, \mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{w}) \quad (40)$$

and

$$\tilde{T}(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{w}, u) + (\bar{I} - \bar{I}) \bar{w} = \tilde{\pi}(\mathbf{p}, \hat{\mathbf{q}}, \mathbf{z}, \bar{w}) - (\hat{\mathbf{q}} - \bar{\mathbf{q}}) \bar{y} \quad (41)$$

respectively.

The equilibrium conditions (36), (37), (38), (39) and (41) constitute a general equilibrium model with  $s$  competitive goods markets,  $n$  goods markets on which the representative firm is quantity constrained,  $k$  competitive factor markets and, finally,  $m$  factor markets on which the representative household is quantity constrained. Now we are able to analyse all kinds of spillover effects between constrained and unconstrained goods and factor markets. One example is the analysis of exogenous shocks or distortions on capital markets and their effects on employment and utility by explicitly taking into account involuntary unemployment in a Keynesian regime.

## 4.2 Classical General Equilibrium

In contrast to the Keynesian equilibrium, in a classical regime the representative household not only faces constraints on  $m$  factor markets but is also constrained on  $n$  goods markets. Therefore the Classical equilibrium is characterized by excess demands on goods markets and excess supplies on factor markets. First, the goods market equilibria with flexible prices are

$$\bar{\mathbf{x}}^d(\mathbf{p}, \hat{\mathbf{q}}, \mathbf{z}, \bar{w}, u) - \frac{\partial \hat{\mathbf{q}}}{\partial \mathbf{p}} \bar{y} + \frac{\partial \bar{I}}{\partial \mathbf{p}} \bar{w} = \mathbf{x}^s(\mathbf{p}, \bar{\mathbf{q}}, \mathbf{z}, \bar{w}). \quad (42)$$

Second, the goods market equilibria with exogenously given prices are

$$\bar{y} - \frac{\partial \hat{q}}{\partial \bar{q}} \bar{y} + \frac{\partial \check{l}}{\partial \bar{q}} \bar{w} = y^s(\mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}). \quad (43)$$

Third, the factor market equilibria with flexible prices can be written as follows:

$$\check{v}^s(\mathbf{p}, \hat{q}, \mathbf{z}, \bar{w}, u) + \frac{\partial \hat{q}}{\partial \mathbf{z}} \bar{y} - \frac{\partial \check{l}}{\partial \mathbf{z}} \bar{w} = \mathbf{v}^d(\mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}). \quad (44)$$

The fourth component which the Classical general equilibrium consists of are the equilibria on factor markets with exogenously given prices:

$$\bar{l} + \frac{\partial \hat{q}}{\partial \bar{w}} \bar{y} - \frac{\partial \check{l}}{\partial \bar{w}} \bar{w} = l^d(\mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}). \quad (45)$$

Finally, the Classical general equilibrium is also closed by the economy's budget constraint

$$\check{T}(\bar{y}, \bar{l}, \mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}, u) = \pi(\mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}) \quad (46)$$

and

$$\check{T}(\mathbf{p}, \hat{q}, \mathbf{z}, \bar{w}, u) - (\hat{q} - \bar{q}) \bar{y} + (\check{l} - \bar{l}) \bar{w} = \pi(\mathbf{p}, \bar{q}, \mathbf{z}, \bar{w}) \quad (47)$$

respectively.

The equilibrium conditions (42), (43), (44), (45) and (47) constitute a general equilibrium model with  $s$  competitive goods markets as well as  $k$  competitive factor markets and  $n$  goods markets as well as  $m$  factor markets on which the household is quantity constrained. Again, like in the Keynesian regime, we are able to analyse spillover effects due to various exogenous shocks.

## 5 Conclusions

In our contribution we have integrated quantity constraints both into factor markets and into goods markets in a general equilibrium manner. Thus our model pays

attention to the interaction of quantity constraints in different markets. Hence important spillover effects can be analyzed. The implications of price distortions like efficiency wages can only be well understood if all spillover effects on other goods and factor markets are taken into consideration. Our general equilibrium model with quantity constrained goods and factor markets offers a perspective for such an investigation.

Due to unknown properties of the constrained transfer and profit functions we made use of the concept of virtual prices following Neary and Roberts (1980). As virtual prices cannot be observed and because a virtual factor supply does not necessarily exist in the case of price inelastic factor supply for each given Walrasian factor price, we have developed the concept of virtual quantities: constrained functions cannot only be translated into unconstrained functions and linear terms evaluating constraints by virtual prices. They also can be translated by the concept of virtual quantities. So virtual transfer and profit functions could be derived where flexible and exogenously given goods and factor prices served as arguments. The inner product of the vector of virtual and constrained quantities' differences and the vector of exogenously given prices showed us the evaluation of quantity constraints both by the representative household and by the representative firm. Incidentally, the linear terms which evaluate the quantity constraints offer an important advantage: all variables of these terms can be observed. The virtual transfer and profit functions derived possess the well-known envelope properties that allow to derive demand and supply functions. Equating demand and supply on quantity constrained and unconstrained markets and writing down the economy's budget constraint delivers the general equilibrium system. In summary, the core of our method is imitating a perfect competitive Walrasian equilibrium system by a system of virtual prices and of virtual quantities in order to integrate quantity constraints without losing important properties of the reference system.

Whereas other models, especially macro-models, analysing quantity constraints

only refer to one goods and one factor market, our model exhibits an economy with an arbitrary number of quantity constrained and unconstrained goods and factor markets. So we reach consistence in our model, in contrast to the well-known standard rationing models whose consistence has to be enabled by introducing a second period with a future good. However, this is only the attempt to make a static model consistent by a dynamic interpretation. The problem introduced by the necessity of a future market in a pure static model can be solved in a more sophisticated way by a model which really considers many quantity constraints in several markets. Up to now such a model was missing. Our model closes this gap by offering a more general framework with many goods and factors for a consistent comparative static analysis. Besides, our framework offers a perspective for the further investigation of spillover effects between quantity constrained markets as well as between quantity constrained and unconstrained markets.

Above all, endogenising exogenously given goods and factor prices without losing the properties of optimal value functions is the direction for future research. This can be done especially for the labour market by means of an efficiency wage hypothesis. Furthermore, necessary and sufficient global conditions for gains from trade can be derived. But these are only few applications of our duality approach that intends to contribute to a deeper understanding of market interactions with quantity and price signals. Future research will lead the way.

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