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# **Stability Problems in Hyperinflation Models**

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# STABILITY PROBLEMS IN HYPERINFLATION MODELS

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## Abstract

Traditional monetary models of hyperinflation suffer from severe stability problems: equilibria with realistic comparative-static properties are unstable while those with unrealistic comparative statics are stable. In the present paper I develop a model of hyperinflation which is based on an analysis of firms' payment behavior and which uses a weaker concept of stability. The "stability" properties of this model are consistent with the empirical evidence.

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## STABILITY PROBLEMS IN HYPERINFLATION MODELS

### 1. Introduction

Traditional monetary models of hyperinflation are criticized for severe stability problems: Equilibria with the "right" comparative static are unstable and those with the "wrong" comparative static are stable. "Right" comparative static refers to the hypothesis that an increase of money financed deficit should increase inflation. This problem has been discussed by Auernheimer (1973), Evans and Yarrow (1981), Sargent and Wallace (1987), Bruno and Fischer (1990), Bernholz and Jaksch (1989), Bernholz and Gersbach (1991) and others. Bruno and Fischer consider a Cagan-type model, Cagan (1956), in which a simple money market equilibrium condition is supplemented by a seigniorage driven money supply growth rule. They show, basically, that there exist two equilibria with constant rates of inflation and that with high enough adaptation speed the high inflation equilibrium is stable while the low inflation equilibrium is unstable. When fiscal deficit is increased then at the high inflation equilibrium inflation decreases while at the other equilibrium inflation increases.<sup>1 2 3 4</sup>

As Bernholz and Gersbach (1992) and others have discussed, the above comparative static behavior of the stable high inflation equilibrium contradicts with empirical evidence. Advanced inflations and hyperinflations have never emerged without sizable government budget deficits. Countries entering hyperinflation were always substantially increasing their real government budget deficit. Successful reforms ending hyperinflations had always proceeded with strong reductions of these fiscal deficits.

In this paper I develop a model for advanced inflation and hyperinflation. Money demand is derived from firms' payment behavior. In this model for small enough size of the deficit there always exists a unique rational expectations equilibrium and inflation increases with increasing fiscal deficits. At higher deficits there are two rational expectations equilibria. When there are two equilibria the high inflation equilibrium has the wrong comparative static. If the deficit exceeds a certain ignition point then inflation explodes to infinity.<sup>5</sup>

To analyze the stability properties of the equilibria I consider alternative adaptation hypotheses. Conventional methods of linear approximations are applied. If firms adjust their payment frequencies unrestrictedly after each payment date then, in general, both equilibria are unstable. However, if the magnitude of the payment frequency adjustment is subject to any finite flexibility bound then the stability properties conform

with the empirical evidence again. The "right" low inflation equilibrium becomes stable in the following sense: There exists a region around the low inflation equilibrium such that after any deviation the adaptation process arrives at this region after a finite number of payment dates and thereafter it remains within this region forever.<sup>6</sup> The size of this attracting region is proportional to the flexibility bound. These properties hold both for firms having adaptive or rational expectations during adaptation. If no linear approximation is applied in the stability analysis, though, this equilibrium may become unstable, provided the flexibility bound or government expenditure are too large. The "wrong" high inflation equilibrium, on the other hand, remains unstable for any flexibility bound, both whether linear approximations are applied or not.

Conventionally the transactions motive for money demand is modeled in a framework where private households receive their endowment or factor payment according to an exogenously given payment scheme. Usually households receive cash once per period and then money demand is determined by their choice of an asset portfolio over time. For examples see Baumol (1952), Tobin (1956) and Niehans (1978). Barro (1970) derives money demand from firms optimizing their payment frequencies.<sup>7</sup> In the present paper I also concentrate on an analysis of the payment scheme which firms choose to pay factor owners. In crucial extension of Barro's model, though, I take into account incentive problems associated with low payment frequencies. Portfolio decisions of private households on the other hand are simplified and adapted to a situation with high inflation. Each household uses all its cash receipts to buy goods as soon as he receives payment. This assumption reflects the empirical fact that during hyperinflations the expected rate of inflation mostly exceeds the nominal rate of interest thus impairing any opportunity for households and firms to hedge against inflationary devaluations of monetary assets (compare Meerwarth (1922), Soecknick (1927) and Bresciani-Turroni (1937)).<sup>8</sup>

The paper is organized according to the following plan. Section 2 introduces the general assumptions of the model for exogenous payment frequencies and analyzes in this framework how fiscal deficits affect inflation. Section 3 extends these assumptions to allow for an endogenization of payment schemes, it determines optimal payment frequencies and the effect of fiscal deficits on inflation. Section 4 analyzes stability properties of high and low inflation equilibria under three alternative adaptation hypotheses. It also discusses the effects of linearization. Conclusions and Appendices complete the paper.

## 2. The effect of seigniorage financed fiscal deficit on inflation under exogenous payment schemes

### Assumptions

We consider an economy with exogenous real national product of  $y$  per period of length one and real government expenditure of  $g$  which is financed by seigniorage obtained through monetary expansion. For simplicity we assume that output consists of a homogeneous good produced by  $n$  firms each of which produces  $y/n$  units. In intervals of  $\tau$  time units each firm transfers its revenue to factor owners, that is to households. Hence, in each period there are  $\nu=1/\tau$  payment dates common to all firms where integer problems with respect to  $\nu$  are left aside here for simplicity. These dates are denoted by  $t_1, t_2, t_3, \dots$  with  $t_1=0$  and  $t_{i+1}-t_i=\tau$  for  $i=1, 2, \dots$ , and the interval  $[t_i, t_{i+1})$  is denoted as subperiod  $i$ .

Since we have in mind a situation with high inflation, we assume simplifyingly that immediately after each payment date each customer spends all its monetary assets for purchasing goods. Hence, during each payment interval  $[t_i, t_{i+1})$  the market opens once only at the beginning of this interval after payment has been disbursed at time  $t_i$ . In this market  $i$ , nominal demand of the private sector,  $D_i^P$ , is equal to aggregate factor payments disbursed to private households at date  $t_i$ . Nominal demand of the government,  $D_i^G$ , on the other hand, depends on that price  $P_i$  which clears market  $i$ :  $D_i^G$  is so large that government can satisfy its real demand of  $g\tau$  units,  $D_i^G = g\tau P_i$ . Thus, the underlying assumption is that government smooths out its demand uniformly over all subperiods. The revenue which firms receive in market  $i$  is disbursed to factor owners only at the beginning of the next subperiod at time  $t_{i+1}$ . Thus, at any payment date  $t_i$  after disbursement but before market  $i$  opens all money assets  $M_{i-1}$  are in the hands of private households. After market  $i$  has closed all money assets  $M_i$  are in the hands of firms. Since government expenditure is financed by seigniorage, during any subperiod  $i$  money supply increases by  $D_i^G$ . Hence at the end of each subperiod  $i$ , money supply is given by,  $M_i = M_{i-1} + D_i^G$ .

### Calculation of the inflation rate

The key to the calculation of the inflation rate is provided by the determination of price levels  $P_i$  for each subperiod  $i$ .  $P_i$  follows from the market clearing condition: nominal aggregate supply  $y\tau P_i$  has to be equal to nominal aggregate demand  $M_i = M_{i-1} + P_i g\tau$ ,

$$(1) \quad y\tau P_i = M_i = M_{i-1} + P_i g\tau.$$

This implies

$$(2) \quad P_i = M_{i-1} / [(y-g)\tau].$$

Since the market clearing condition is also valid in period  $i-1$ ,  $y\tau P_{i-1} = M_{i-1}$ , we can substitute  $M_{i-1}$  in (2) by  $y\tau P_{i-1}$  to get

$$(3) \quad P_i / P_{i-1} = y / (y-g).$$

Hence, the inflation rate  $\pi$  with respect to the whole period is given by

$$(4) \quad 1+\pi = P_{i+v} / P_i = [y / (y-g)]^\nu, \quad \nu = 1/\tau.$$

We denote  $\rho := \ln(1+\pi)$  as inflation index and  $\gamma := \ln[y / (y-g)]$  as deficit index. Both  $\rho$  and  $\gamma$  represent monotone transformations of  $\pi$  and  $g$ , respectively. With these abbreviations equation (4) yields

$$(5) \quad \rho = \nu\gamma.$$

Hence the inflation rate  $\pi$  is positively related to government expenditure  $g$ . This relation is later described in Figure 7 as inflation ray.<sup>9</sup>

### 3. The effect of fiscal deficit on inflation under endogenous payment scheme Assumptions

In this section we extend the above analysis to take into account how inflation affects payment frequency. At any date  $i$ , after closing of the market, each firm can vary its payment frequency. We assume that for each firm each payment scheme  $\nu$  entails real overhead cost  $X$  per period composed of money holding cost  $A$ , disbursement cost  $B$ , and monitoring cost  $C$ ,  $X = A + B + C$ . Since at any date all firms face the same overhead cost function all firms will choose the same payment frequency.

Money holding cost  $A$  arise from inflationary value reduction of money balances held by firms. We assume that each firm knows the inflation index and takes it as given. If a firm chooses to pay in intervals of  $\tau$  time units then, during any such interval, her real balances reduce from  $\tau(y/n)$  to  $\tau(y/n)(P_{i-1}/P_i)$ . Hence, since there are  $1/\tau$  payment intervals per period for this firm, her money holding cost amount to

$$(6) \quad A(\nu|\rho) = (y/n)[1 - (1+\pi)^{-1/\nu}] = (y/n)[1 - e^{-\rho/\nu}], \quad \rho := \ln(1+\pi).$$

Disbursement cost  $B(\nu)$  arise from payment handling and thus they increase with increasing payment frequency while marginal disbursement cost decrease. For computational simplicity we assume

$$(7) \quad B(\nu) = \mu[1 - e^{-\alpha\nu}], \quad \alpha > 0, \mu > 0.$$

Monitoring cost  $C(\nu)$  arise from incentive problems which grow with

increasing length of payment intervals. To provide for a simple notion suppose factor payment consist of wage and profit payment only. Then either a firm pays its workers in advance (for more than half of the coming payment interval) or workers deliver labor in advance of payment (for more than half of the payment interval). In the first case workers face moral hazard since shirking of workers can no longer be punished by rather immediate wage reduction. Thus monitoring of workers has to increase with payment interval length. This argument is explained in more detail in Appendix 1. In the second case workers need to finance their living from accumulated wealth which becomes impossible as the payment interval has increased high enough. For computational simplicity we thus assume

$$(8) \quad C(\nu) = \lambda e^{-\beta\nu}, \quad \beta > 0, \lambda > 0, \beta > \alpha, \beta\lambda > \alpha\mu,$$

where the conditions on  $\beta$  and  $\lambda$  ensure that the sum of disbursement and monitoring cost  $B(\nu) + C(\nu)$  has a unique minimum for  $\nu \in (0, \infty)$ . These conditions are fulfilled if  $\beta$  is large enough in relation to the other parameters, that is if monitoring costs are significant only for small payment frequencies.<sup>10</sup>

While money holding cost  $A$  directly represent a value reduction of firm profits we assume that disbursement and monitoring cost represent disutility of firm management measured in output units and that variations of these cost do not affect output. Hence, for any given inflation index  $\rho$  each firm chooses that payment frequency  $\nu$  which minimizes her overhead cost  $X = A(\nu|\rho) + B(\nu) + C(\nu)$ . Money holding cost, disbursement cost and monitoring cost are described in Figure 1.

### Optimal payment frequencies

In the following we investigate the overhead cost function  $X(\nu|\rho)$ . A rigorous analysis is provided in Appendix 2. As can be seen from Figure 1, at zero payment frequency  $\nu$  payment cost  $X$  start at a level  $X(0|\rho) = \lambda + (y/n)$ , independently of the inflation index  $\rho$ . Initially, for small enough  $\nu$ , it decreases in  $\nu$  and for  $\nu \rightarrow \infty$  it converges to  $\mu$ . With increasing inflation index  $\rho$ ,  $X$  increases monotonously. There are unique inflation indices  $\rho_{glob}, \rho^*, \rho^{**}$  with  $0 < \rho_{glob} < \rho^* < \rho^{**} < \infty$  and the following properties:

- (i) For  $\rho \in (0, \rho^*)$ ,  $X(\nu|\rho)$  has a unique minimum over  $\nu$ , denoted as  $\nu_1^*(\rho)$ , and a unique maximum over  $\nu$  at some  $\nu_2^*(\rho) > \nu_1^*(\rho)$ .
- (ii) For  $\rho \in (\rho^*, \rho^{**})$ ,  $X(\nu|\rho)$  decreases monotonously in  $\nu$ .
- (iii) For  $\rho \in (\rho^{**}, \infty)$ ,  $X(\nu|\rho)$  has a unique minimum over  $\nu$ , denoted as  $\nu_1^{**}(\rho)$ , and a unique maximum over  $\nu$  at some  $\nu_2^{**}(\rho) > \nu_1^{**}(\rho)$ .
- (iv) For  $\rho < \rho_{glob}$ , the minimum is global and for  $\rho > \rho_{glob}$  the minimum is local.<sup>11</sup>

FIGURE 1: Money holding, disbursement and monitoring cost curves  
 FIGURE 2: Frequency curves and inflation rays

### Stationary equilibria

To analyze the effect of the deficit index  $\gamma$  on the inflation index  $\rho$  while the payment frequency is endogenous we consider stationary equilibria. Stationary equilibria are described by intersections of the inflation ray  $\nu=(1/\gamma)\rho$  with the frequency curves  $\nu_1^*(\rho)$  and  $\nu_1^{**}(\rho)$  in Figure 2. Since any such point is located on the frequency curve  $\nu_1^*$  or on  $\nu_1^{**}$ , no firm can gain from a small variation of her payment frequency. However, these equilibria are based on local optimization. Only for  $\rho \in (0, \rho_{\text{glob}})$  the equilibria are consistent with global optimization. Since the equilibrium is located on the inflation ray, output markets are in equilibrium while government expenditure is financed by money creation. The slope of the inflation ray,  $1/\gamma$ , is decreasing in the deficit  $g$  since  $\gamma = \ln[y/(y-g)]$ . For  $g=0$  the slope is infinite, for  $g=y$  it is zero. According to Lemma 4 of Appendix 2 described above, there are unique deficit indices  $\gamma_Q, \gamma_S, \gamma_L, \gamma_Z$  with  $0 < \gamma_Q < \gamma_S < \gamma_L < \gamma_Z < \infty$  such that for each  $\gamma \in (0, \gamma_Q)$  there is a unique intersection with  $\nu_1^*$ , for each  $\gamma \in (\gamma_Q, \gamma_S)$  there are two intersections with  $\nu_1^*$ , for  $\gamma \in (\gamma_S, \gamma_L)$  there is no intersection at all, for each  $\gamma \in (\gamma_L, \gamma_Z)$  there is an intersection with  $\nu_2^{**}$ , and for each  $\gamma \in (\gamma_Z, \infty)$  there is a unique intersection with  $\nu_1^{**}$ .

### Stability properties for slow frequency adaptations

Next we prepare for an investigation of the stability properties of the above equilibria. First we consider the case of slow frequency adaptations. The case of flexible payment frequencies is discussed in Section 4. With slow adaptations, at any time during the adaptation process the inflation index will approximately be given by (5),  $\rho = \nu\gamma$ , since in equations (1) to (4) we may assume approximately that during any period all payment intervals are of the same length  $\tau$ . To infer how firms respond with their payment frequency in any given situation  $Q=(\nu, \rho)$  we assume firms to base their decision on local considerations. If, in  $Q$ , the overhead cost  $X(\nu|\rho)$  increases in  $\nu$  then firms reduce  $\nu$ , if the overhead cost function  $X(\nu|\rho)$  decrease in  $\nu$  then firms increase  $\nu$ . This assumption implies that (a) firms reduce their payment frequency in any point located to the left of the frequency curves  $(\nu_1^*, \nu_2^*)$  and in any point located to the right of the frequency curves  $(\nu_1^{**}, \nu_2^{**})$ , and (b) firms increase their payment frequencies in any other point off the frequency curves. While frequencies are assumed to adapt slowly, our market clearing condition (1) implies that inflation rates adapt with infinite speed. Hence, for constant fiscal deficit indices  $\gamma$ , all movements of the economy during

adaptation take place along the relevant inflation ray and with a finite speed which is determined by the adaptation speed of payment frequencies. Such movements are depicted by small arrows in Figure 2. If firms optimized globally then all firms would increase their payment frequency to infinity once the deficit index is so high that the inflation index is above  $\rho_{glob}$ .

Now we can determine the stability properties of the above equilibria from Figure 2. For  $\gamma \in (0, \gamma_Q)$  and  $\gamma \in (\gamma_Z, \infty)$  equilibria are unique and stable. For  $\gamma \in (\gamma_Q, \gamma_S)$  that equilibrium with the lower inflation rate is stable, that with the higher inflation rate is unstable. For  $\gamma \in (\gamma_L, \gamma_Z)$  there are intersections with  $\nu_2^{**}$ . Such intersections are no equilibria since - for given inflation index - any single firm would gain from small deviations of its payment frequency. However, these constellations are nevertheless stable and, thus, they can be considered as a description of reality. For  $\gamma \in (\gamma_S, \gamma_L)$ , finally, the economy moves to ever increasing inflation rates. This deficit index interval we denote as hyperinflation breach.

### Hyperinflation

Finally we discuss the comparative static properties of the above equilibria and how hyperinflation can come about in the present model. If the deficit index  $\gamma$  increases within the interval  $(0, \gamma_S)$  then the inflation ray becomes flatter and, hence, the intersection moves to the right along the frequency curve  $\nu_1^*$ . Thus the equilibrium inflation rate increases. With fixed payment frequency,  $\rho$  would grow in proportion to the increase of  $\gamma$  as described in (5). We denote this direct effect of  $\gamma$  on  $\rho$  as pure seigniorage effect. With increased inflation, however, all firms will increase their payment frequency. Hence the proportion between the inflation index  $\rho$  and the deficit index  $\gamma$  rises too. Thus  $\rho$  rises more than proportionally in  $\gamma$ . We denote this additional effect of  $\gamma$  on  $\rho$  as payment frequency effect.

With a deficit index of  $\gamma_S$  the inflation ray is tangential to the payment frequency curve  $\nu_1^*$ . At deficit indices  $\gamma$  below  $\gamma_S$ ,  $\gamma$  had a more than proportional but finite effect on the inflation index  $\rho$ . As  $\gamma$  approaches  $\gamma_S$  the payment frequency effect accelerates so much that the marginal effect of  $\gamma$  on  $\rho$  increases to infinity.

As  $\gamma$  increases above  $\gamma_S$  the economy moves into the hyperinflation breach. There does no longer exist any intersection of the inflation ray with frequency curves. The economy moves to the right along the relevant inflation ray with increasing inflation rates since each firm tries to reduce its overhead cost  $X$  by increasing its payment frequency. Thus inflation increases

to infinity with a speed determined by the adaptation of the payment frequency. We denote the crucial deficit index  $\gamma_S$  as the hyperinflation ignition point.

The above case of accelerating hyperinflation occurs if the deficit increases slowly in relation to the adaptation speed of the payment frequency. However, the development changes if the deficit index increases relatively quickly. In the extreme case, suppose frequency  $\nu$  remains fixed while the deficit index  $\gamma$  increases somewhere into the interval  $(\gamma_L, \gamma_\infty)$ . Then, according to Figure 2, the economy adapts to some point on the frequency curves  $\nu_1^*$  or  $\nu_2^*$ . In this case hyperinflation would not emerge but the economy would remain in a situation of extreme inflation,  $\rho > \rho^{**}$ . For  $\gamma > \gamma_Z$  inflation increases in the deficit index. A reduction of inflation, though, is difficult in this situation since, as  $\gamma$  is reduced below  $\gamma_Z$ , inflation will increase with further reductions of  $\gamma$ .<sup>12</sup>

#### 4. Stability properties for bounded flexibility of payment frequencies

In this section the stability properties of the above stationary equilibria are analyzed more rigorously. In particular, we allow payment frequencies to adapt quickly. Speed limits are introduced later in this section. We consider only those equilibria which arise for deficit indices  $\gamma \in (0, \gamma_S)$ , that is from intersections of the inflation ray with the frequency curve  $\nu_1^*$ . For simplicity we abbreviate  $\nu_1^*$  as  $\nu^*$ . Since for high enough deficit indices  $\gamma$  there exist two equilibria, we denote as low inflation equilibria those where the frequency curve  $\nu^*$  is flatter than the inflation ray, and as high inflation equilibrium those where  $\nu^*$  is steeper.

For the analysis of adaptation processes we have to consider how pricing is affected by variations of payment interval lengths. Let  $\tau_i = t_i - t_{i-1}$  be the length of subperiod  $i$ . Then in any market  $i$ , the market clearing condition is now given by

$$(1') \quad y\tau_i P_i = M_i = M_{i-1} + P_i g\tau_i,$$

and, hence, the intertemporal price ratio by

$$(3') \quad P_i / P_{i-1} = [y / (y - g)] (\tau_{i-1} / \tau_i).$$

With the inflation rate  $\pi_i$  defined by

$$(4') \quad 1 + \pi_i = [P_i / P_{i-1}]^{1/\tau_i},$$

and  $\gamma = \ln[y / (y - g)]$ , the inflation index is now

$$(5') \quad \rho_i = (1/\tau_i) [\gamma + \ln(\tau_{i-1}/\tau_i)] = \nu_i [\gamma + \ln(\nu_i/\nu_{i-1})].$$

In case of constant payment frequencies, (5') coincides with (5).

In the following we investigate the stability properties of stationary equilibria. Correspondingly, in the next three subsections we consider three alternative assumptions about the adaptation of payment frequencies  $\nu_i$ . While for stationary equilibria we assume firms to have rational expectations, for the adaptation process we first assume firms to choose their payment frequencies adaptively. Rational expectations are introduced to the adaptation process later in Section 4c. In all these cases we apply suitable linearization procedures. Only in Section 4d we discuss how the system evolves without linearization.

#### (a) Stepwise adaptation with unbounded flexibility of payment frequencies

First we consider the case where payment frequencies are unrestrictedly adapted on the basis of previous inflation indices. Let  $(\nu^+, \rho^+)$  be a stationary equilibrium of either the low or the high inflation type. Without linearization we would assume that the economy evolves according to  $\nu_i = \nu^*(\rho_{i-1})$  and (5').<sup>13</sup> We transform this nonlinear system into the following system of linear difference equations:

$$(9a) \quad \nu_i = a + b\rho_{i-1},$$

$$(9b) \quad \rho_i = (1+\gamma)\nu_i - \nu_{i-1}$$

where the coefficients  $a$  and  $b$  are such that  $\nu = a + b\rho$  is a linear approximation of  $\nu^*(\rho)$  at the stationary inflation index  $\rho^+$ . We note that (9b) is a linear approximation of (5') at  $(\nu_i, \rho_i) = (\nu_{i-1}, \rho_{i-1})$ . If  $(\nu^+, \rho^+)$  is a low inflation stationary equilibrium then at  $(\nu^+, \rho^+)$  the payment frequency curve  $\nu^*$  has a smaller slope than the inflation ray and hence  $a > 0$  and  $0 < b < 1/\gamma$ . If  $(\nu^+, \rho^+)$  is a high inflation equilibrium then at  $(\nu^+, \rho^+)$  the payment frequency curve  $\nu^*$  is steeper than the inflation ray and hence  $a < 0$  and  $b > 1/\gamma$ .

For the case of unrestricted adaptation we only investigate the initial adaptation steps by means of a graphical example.<sup>14</sup> In Figures 3 and 4 equation (9a) is represented as frequency line  $\Phi$  and, for any previous payment frequency  $\nu_{i-1}$ , equation (9b) is represented as inflation line  $\Psi_i$ ,  $\nu_i = (\nu_{i-1} + \rho_i)/(1+\gamma)$ .<sup>15</sup> For the low inflation equilibrium, the frequency line  $\Phi$  is flatter than the inflation ray  $\Gamma$  and for the high inflation equilibrium  $\Phi$  is steeper than  $\Gamma$ . Figure 3 shows (for the case of  $\Psi_i$  being flatter than  $\Phi$ ) that at the low inflation equilibrium the economy first moves into a direction such that the initial deviation is reduced. However it overshoots. The adaptation of payment frequencies adds so much inertia to the initial

correction that the system is eventually destabilized (given that  $\Psi_i$  is flatter than  $\Phi$ ). For the high inflation equilibrium, Figure 4 shows that the economy moves away from the equilibrium at all times.

FIGURE 3: Early stage adaptation with unbounded flexibility.

Case: Low inflation equilibrium,  $b < 1/\gamma$

FIGURE 4: Early stage adaptation with unbounded flexibility.

Case: High inflation equilibrium,  $b > 1/\gamma$

#### (b) Stepwise adaptation with bounded flexibility of payment frequencies

The above investigation has shown that the low inflation equilibrium triggers strong correction forces after disturbances but that these forces may bring forth so large variations of payment frequencies that the system is eventually destabilized. In the following we investigate the behavior of the economy for the case that variations of the payment frequency (between successive subperiods) have some exogenous bound  $S$ ,  $|v_i - v_{i-1}| \leq S$  for any  $i$ . We will show that any such bound  $S$  prevents the low inflation equilibrium from exploding. Instead, after any local disturbance of this equilibrium the economy will move back into a neighborhood of the equilibrium and, forever, it will remain within this neighborhood the size of which is proportional to the flexibility bound  $S$ . The high inflation equilibrium, though, remains unstable for any flexibility bound.

Let  $(v^+, \rho^+)$  be a stationary equilibrium of either the low or the high inflation type and  $P_0 = (v_0, \rho_0)$  any starting point. For any flexibility bound  $S$  the dynamics of the system is given by the frequency adaptation equation

$$(10) \quad v_i = \begin{cases} v_{i-1} + S \cdot \text{sign}[(a + b\rho_{i-1}) - v_{i-1}] & \text{if } |(a + b\rho_{i-1}) - v_{i-1}| \geq S \\ a + b\rho_{i-1} & \text{if } |(a + b\rho_{i-1}) - v_{i-1}| \leq S, \end{cases}$$

where we define  $\text{sign}[x]$  as 1, 0, -1 according to  $x$  being positive, zero, or negative, and by the inflation line equation (9b),  $\rho_i = (1 + \gamma)v_i - v_{i-1}$ .

We will develop the stability properties of the equilibrium  $(v^+, \rho^+)$  in three stages. First we show that the economy will always remain near the inflation ray  $\Gamma$ ,  $v = (1/\gamma)\rho$ .

PROPOSITION 1: For any starting point  $P_0 = (v_0, \rho_0)$  and any date  $i \geq 1$ ,  $P_i = (v_i, \rho_i)$  will be within the set  $G_S := \{(v, \rho) : |\rho - \gamma v| \leq S\}$ , denoted as inflation pipe.  $\square$

PROOF: This property can be seen from Figure 5. For any previous frequency  $v_{i-1}$ , the actual frequency  $v_i$  fulfills  $v_{i-1} - S \leq v_i \leq v_{i-1} + S$ . Further, the actual

inflation index  $\rho_i$  is on the inflation line  $\Psi$ ,  $\rho=(1+\gamma)\nu-\nu_{i-1}$ . Hence  $P_i$  is a point in the interval

$$(11) \quad \bar{\Psi}_i = \bar{\Psi}_i(\nu_{i-1}, S) := \{(\nu, \rho) : \rho = (1+\gamma)\nu - \nu_{i-1} \text{ and } |\nu - \nu_{i-1}| \leq S\}.$$

$\bar{\Psi}_i$  passes through  $P'_{i-1} = (\nu_{i-1}, \gamma\nu_{i-1})$  and has slope  $1/(1+\gamma)$ . We denote  $\bar{\Psi}_i$  as inflation possibility set. Thus in Figure 5,  $P_i$  is on the diagonal AB and  $|\rho_i - \gamma\nu_i| \leq AB/2$ . Due to the slope properties  $1/(1+\gamma) = BC/AC$ ,  $1/\gamma = BC/DC$  and the identities  $AD = AC - DC$  and  $BC = 2S$  we have  $AD = 2S$  which proves the proposition.  $\square$

FIGURE 5: The inflation pipe  $G_S$

Next we investigate how the economy moves within the inflation pipe  $G_S$ . We show that in case of  $(\nu^+, \rho^+)$  being a low inflation equilibrium,  $P_i$  moves "towards" the stationary equilibrium. In case of  $(\nu^+, \rho^+)$  being a high inflation equilibrium,  $P_i$  moves away from  $(\nu^+, \rho^+)$ . These dynamics can also be seen from Figures 6 and 7.

PROPOSITION 2: Let  $(\nu^+, \rho^+)$  be a stationary equilibrium of either the low or the high inflation type. For some  $i-1 \geq 0$  let the economy be in some point  $P_{i-1}$  in the inflation pipe  $G_S$  but below (above) the frequency line  $\Phi$ , i.e.  $|\rho_{i-1} - \gamma\nu_{i-1}| \leq S$  and  $\nu_{i-1} (\lessgtr) a + b\rho_{i-1}$ . Then the frequency increases (decreases),  $\nu_i (\gtrless) \nu_{i-1}$ . In case of a low inflation equilibrium this movement is directed towards equilibrium even though overshooting may occur. In case of a high inflation equilibrium the movement is directed away from equilibrium and  $P_i$  stays below (above) the frequency line  $\Phi$ .  $\square$

PROOF: As can be seen from Figures 6 and 7 for both types of equilibrium the proposition follows directly from the frequency adaptation equation (10) and the frequency line having a positive slope: At date  $i$  the economy will be in some point  $P_i$  of the inflation possibility set  $\bar{\Psi}_i(\nu_{i-1}, S)$ . If  $P_{i-1}$  is below (above) the frequency line  $\Phi$  then  $P_i$  is on the upper (lower) half of  $\bar{\Psi}_i$ . Hence  $\nu_i (\gtrless) \nu_{i-1}$ . In case of a high inflation equilibrium  $P_i$  is at the upper (lower) endpoint of  $\bar{\Psi}_i$  and this implies additionally that  $P_i$  stays below (above) the frequency line  $\Phi$ .  $\square$

FIGURE 6: Movement in inflation pipe, low inflation equilibrium case

FIGURE 7: Movement in inflation pipe, high inflation equilibrium case

Propositions 1 and 2 show that the high inflation equilibrium is unstable for any flexibility bound  $S$ . Further, based on Propositions 1 and 2, the

following Proposition 3 proves the stability of the low inflation equilibrium.

PROPOSITION 3: Let  $(v^+, \rho^+)$  be a low inflation equilibrium and  $P_0$  any starting point. Then the economy will reach the attractor

$$(12) \quad R_S := \{(v, \rho) \in G_S : |(v - v^+)| \leq rS\}, \quad r := [b/(1 - \gamma b)] > 0,$$

after a finite number  $j$  of subperiods and it will remain there forever,  $j := \min\{k \geq 0 : P_k \in R_S\} < \infty$  and  $P_i \in R_S$  for all  $i \geq j$ .  $\square$

FIGURE 8: Attractor  $R_S$

PROOF: First we note that  $r > 0$  follows from  $\Phi$  being flatter than  $\Gamma$ ,  $b < 1/\gamma$ . Next, the finite arrival time property  $j < \infty$  follows directly from Propositions 1 and 2 and the frequency adaptation equation (14): as long as the economy has not arrived  $R_S$  yet,  $v_i$  approaches  $v^+$  in steps of length  $S$ .

Finally, for some  $i-1 \geq 0$  let  $P_{i-1}$  be in  $R_S$ . It suffices to prove that  $P_i$  fulfills the vertical bound property  $|v_i - v^+| \leq rS$ . As is depicted in Figure 8 let  $(v_{So}, \rho_{So})$  and  $(v_{Su}, \rho_{Su})$  be the intersection points of the frequency line  $\Phi$  with the right edge and the left edge, respectively, of the inflation pipe,  $a + b\rho_{So} = (1/\gamma)(\rho_{So} - S) = v_{So}$  and  $a + b\rho_{Su} = (1/\gamma)(\rho_{Su} + S) = v_{Su}$ . Then  $v_{So} - v^+ = v^+ - v_{Su} = rS$  follows from the slope relations  $b = BC/AC$ ,  $1/\gamma = BC/DC$  and  $DC = AC - S$ . It suffices to prove that  $v_i \in [v_{Su}, v_{So}]$ . First we show that  $v_i$  remains below  $v_{So}$ . Since  $P_{i-1} \in R_S$  we have  $v_{i-1} \leq v_{So}$ . Then the frequency adaptation equation (10) implies that  $v_i \leq \max(v_{i-1}, a + b\rho_{i-1})$ : in case of  $v_{i-1} > a + b\rho_{i-1}$  we have  $v_i < v_{i-1}$ , in case of  $v_{i-1} = a + b\rho_{i-1}$  we have  $v_{i-1} = v_i$ , and in case of  $v_{i-1} < a + b\rho_{i-1}$  we have  $v_i > v_{i-1}$  but  $v_i \leq a + b\rho_{i-1}$ . Thus,  $v_i \leq v_{So}$ . Analogously, we have  $v_{i-1} \geq v_{Su}$ . Then (10) implies that  $v_i \geq \min(v_{i-1}, a + b\rho_{i-1})$ : in case of  $v_{i-1} < a + b\rho_{i-1}$  we have  $v_i > v_{i-1}$ , in case of  $v_{i-1} = a + b\rho_{i-1}$  we have  $v_{i-1} = v_i$ , and in case of  $v_{i-1} > a + b\rho_{i-1}$  we have  $v_i < v_{i-1}$  but  $v_i \geq a + b\rho_{i-1}$ . Thus,  $v_i \geq v_{Su}$ .  $\square$

### c) Stepwise adaptation with bounded flexibility of payment frequencies and conditionally rational expectations

Stepwise adaptation may involve overshooting of the frequency line  $\Phi$  which is not consistent with rational behavior: in Figure 9 firms with rational expectations would have chosen for date  $i$  a lower payment frequency than  $v_i$ . To make the adaptation process consistent with rational behavior, in the following we introduce the notion of conditional rational expectations. We show that with such expectations for any flexibility bound  $S$  an adaptation process is induced under which low inflation equilibria are stable in the sense of Proposition 3 and high inflation equilibria are unstable if deviations are large enough.

FIGURE 9: Stepwise adaptation without rational expectations

Let  $(\nu^+, \rho^+)$  be a stationary equilibrium of either the low or the high inflation type and let  $P_0 = (\nu_0, \rho_0)$  be any starting point. Suppose at some date  $i-1 \geq 0$  the economy is in some point  $P_{i-1} = (\nu_{i-1}, \rho_{i-1})$  in  $G_S$ , e.g. as in Figure 10. Then, at date  $i$  the economy  $P_i$  will be in some point of the inflation possibility set  $\bar{\Psi}_i(\nu_{i-1}, S) := \{(\nu, \rho) : \rho = (1+\gamma)\nu - \nu_{i-1} \text{ and } |\nu - \nu_{i-1}| \leq S\}$ . Which frequency  $\nu_i$  will individual firms choose out of  $[\nu_{i-1} - S, \nu_{i-1} + S]$ ? The answer will depend on the a priori expectation  $\hat{P}_i = (\hat{\nu}_i, \hat{\rho}_i)$  which emerges in the economy with respect to the economy's position in  $(\nu, \rho)$ -space. We denote an a priori expectation  $\hat{P}_i$  as feasible if  $\hat{P}_i \in \bar{\Psi}_i(\nu_{i-1}, S)$ . If firms expected the economy at date  $i$  to be in some point  $\hat{P}_i = (\hat{\nu}_i, \hat{\rho}_i)$  of  $\bar{\Psi}_i$  then each firm would have an incentive to increase (decrease) her payment frequency in relation to  $\hat{\nu}_i$  if and only if  $\hat{P}_i$  is below (above) the frequency line  $\Phi$ , i.e. iff  $\hat{\nu}_i (\lessgtr) a + b\hat{\rho}_i$ . Hence firms would keep increasing (decreasing) their payment frequency  $\nu_i$  in relation to  $\hat{\nu}_i$  as long as both inequalities,  $|\nu_i - \nu_{i-1}| \leq S$  and  $\nu_i (\lessgtr) a + b\hat{\rho}_i$ , remain valid where  $\rho_i$  evolves according to the inflation equation (9b)  $\rho_i = (1+\gamma)\nu_i - \nu_{i-1}$ . Let  $P_i = \xi(\hat{P}_i)$  be that point which emerges in this way in response to feasible a priori expectations  $\hat{P}_i$ . We denote  $P_i$  as conditional rational with respect to a priori expectations  $\hat{P}_i$ .<sup>16</sup> We note that for given  $P_{i-1}$  we may get two different conditional rational positions for  $P_i$  depending on the a priori expectation  $\hat{P}_i$ . Hence the idea of conditional rational expectations is to determine rational expectations explicitly as conditioned by exogenous a priori expectations.

FIGURE 10: A priori expectations and conditional rational expectations equilibria, case:  $b > 1/(1+\gamma)$

Under conditional rational expectations all stability results remain essentially invariant: (i) the economy will always remain within the inflation pipe, (ii) in case of a low inflation equilibrium the economy will move in the direction of the equilibrium, after a finite number of dates it will reach the attractor the size of which is proportional to the flexibility bound  $S$ , and it will remain within the attractor forever, (iii) in case of a high inflation equilibrium the economy will move away from the equilibrium provided the initial deviation is large enough. For precise statements the reader is referred to Appendix 3.

#### (d) Stability analysis without linearization procedures

Without linearization the dynamics of the economy is given by equations

$v_i = v^*(\rho_{i-1})$  and (5')  $\rho_i = v_i[\gamma + \ln(v_i/v_{i-1})]$ , modified in analogy to the preceding analysis by the flexibility restriction  $|v_i - v_{i-1}| \leq S$  and optionally by the assumption of conditional rational expectations.<sup>17</sup> The inflation pipe is given by  $\tilde{G}_S = \{(v, \rho) : v[\gamma + \ln(v/(v+S))] \leq \rho \leq v[\gamma + \ln(v/(v-S))]\}$ , the two sides of which are of different size now. As frequency set we now define  $\tilde{\Phi} = \{(v, \rho) : v = v^*(\rho) \text{ and } \rho < \rho_{glob}^*\}$  where  $\rho_{glob}^* = \rho^*$  if firms minimize overhead cost locally, and  $\rho_{glob}^* = \rho_{glob}$  if firms minimize overhead cost globally. We denote  $v^*(\rho_{glob}^*)$  as the maximal finite equilibrium payment frequency. Further, for any previous frequency the inflation possibility set is given by  $\tilde{\Psi}_i = \{(v, \rho) \in \tilde{G}_S : \rho = v[\gamma + \ln(v/v_{i-1})]\}$ . It is intuitively clear that both with and without conditional rational expectations all stability properties of the preceding analysis remain valid in the present framework, provided the flexibility bound  $S$  is so small that (a) the frequency set  $\tilde{\Phi}$  intersects with the right edge of the inflation pipe  $\tilde{G}_S$  and (b) for any  $v_{i-1}$   $\tilde{\Phi}$  intersects with  $\tilde{\Psi}_i$  at most once. Hence, in this case the low inflation equilibrium remains stable and the high inflation equilibrium unstable. We note that the above condition on  $S$  requires the flexibility bound to be the smaller the higher government expenditure  $g$ .

Now we show that larger flexibility bounds can induce instability of the low inflation equilibrium. It suffices to give an example. As in Figure 11 let  $(v^+, \rho^+)$  be the low inflation equilibrium and let  $S$  be so large that for some  $v_{i-1} < v^+$  we have: (a)  $v_{i-1} < v^+ < v^*(\rho_{glob}^*) < v_{i-1} + S$  and (b) the frequency set  $\tilde{\Phi}$  is above the inflation possibility set  $\tilde{\Psi}_i$ . Then, without conditional rational expectations, clearly  $v_i$  and  $\rho_i$  increase forever. With conditional rational expectations, any sequence of feasible a priori expectations  $(\hat{v}_j, \hat{\rho}_j)_{j \geq i}$  with  $\hat{v}_j \geq v_{j-1}$  induces ever increasing inflation.

FIGURE 11: Instability of low inflation equilibria with high  $S$  and without linearization

On the other hand, as can be seen from Figure 7, the instability of high inflation equilibria remains valid for any flexibility bound: any deviation of the economy to the right of a high inflation equilibrium will induce ever increasing inflation.

## 5. Conclusions

The present paper reconsiders the alerting instability results of traditional monetary inflation models as described e.g. in Bruno and Fischer (1990). Bruno and Fischer showed that in these models with high enough adaptation speed the "wrong" high inflation equilibrium (with the unrealistic comparative static

property) is stable while the "right" low inflation equilibrium (with the realistic comparative statics) is unstable. With conventional linearization procedures applied to the model of the present paper, this result is reversed: for any finite "adaptation speed" the "right" low inflation equilibrium is "stable" while the "wrong" high inflation equilibrium is unstable.

The model deviates from the theoretical framework of the above authors in several respects. First, money demand is derived from an analysis of firms' payment behavior.<sup>18</sup> Second, the present model uses another concept of adaptation speed: It is the flexibility bound, i.e. the maximal variation of payment frequency each firm may adapt with after any payment date which is interpreted as adaptation speed. Third, it uses another stability concept: An equilibrium is stable if, after any local disturbance the process is attracted into a region around the equilibrium while the size of this region is proportional to the above adaptation speed. Fourth, the model has a rather special time structure restricting the actions of firms and households: All firms choose their payment frequency simultaneously. Households spend their money income on goods immediately after payment dates. Inflation is determined only after payment schemes have been chosen but during the adaptation process firms may have conditional rational expectations while in equilibrium firms always have rational expectations.<sup>19</sup>

The stability analysis is based on linear approximations of the frequency curve and the dynamic inflation curves. Without these linearizations the stability properties of the present model are different. While the high inflation equilibrium remains unstable in any case, the low inflation equilibrium may now become unstable too. Instability occurs at the low inflation equilibrium if the flexibility bound is so large that within a single payment frequency adaptation step the economy can move beyond the hyperinflation ignition point.

The model exhibits all the stylized facts of hyperinflation as far as a simple closed economy model can do (compare Bernholz and Gersbach): real money supply decreases with increasing inflation, nominal money growth is positively correlated with inflation, growing fiscal budget deficits increase inflation and may trigger hyperinflation, successful fights of hyperinflation may require extreme reductions of fiscal deficits.

The model points to the crucial role of firms' payment schemes. To reverse hyperinflation, payment frequencies have to be brought down. Empirical research on the role of payment frequencies during hyperinflations seems to be quite neglected.

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#### APPENDIX 1: An explanation for monitoring cost

In this appendix we present an explanation for the monitoring cost function  $C$  without, however, deriving the special shape of  $C$ . We have already argued that with long enough payment intervals workers must be paid in advance since otherwise they cannot finance their living. Hence we assume that each worker is paid in advance at payment dates.

We consider a worker who expects to get paid at each payment date  $t_i$  with a real wage rate  $w_i$ , who expects to work with effort  $e_i$  during the subsequent payment intervals  $(t_i, t_i + \tau_i)$ , and who has a time preference of  $\theta$ . We assume that at time 0 this worker has expected utility of

$$(A1-1) \quad EU = \sum_{i=0}^{\infty} (1-\theta)^{t_i} (w_i - e_i) \tau_i.$$

Effort can either be  $\bar{e}$  or  $e_{\min} < \bar{e}$ . A worker is asked by the firm to give effort  $\bar{e}$  and he is shirking if he reduces his effort to  $e_{\min}$ .  $e_{\min}$  is a lower bound for effort which can be enforced by the firm through monitoring. Monitoring cost increase with  $e_{\min}$ . Shirking gets detected with intensity  $\delta$ , that is, in any time interval of length  $\tau_i$  a given shirking worker gets detected with probability  $1 - (1-\delta)^{\tau_i}$ . For simplicity we assume that a worker is dismissed if he is detected of shirking. Since he has been paid in advance, however, dismissal can only become effective at the next payment date. Firms minimize monitoring cost under the restriction that workers choose effort  $\bar{e}$ .

Now we can calculate for alternative levels of  $e_{\min}$  the expected utility a worker receives in case of shirking and non-shirking,  $EU_s$  and  $EU_{ns}$ . We assume that  $w$  and  $\tau$  are constant over time. For each  $i$  we then have  $t_i = \tau i$  and

$$(A1-2) \quad EU_{ns} = \sum_{i=0}^{\infty} (1-\theta)^{\tau i} (w - \bar{e}) \tau = (w - \bar{e}) \tau / [1 - (1-\theta)^{\tau}],$$

$$(A1-3) \quad EU_s = (w - e_{\min}) \tau + (1-\delta)^{\tau} (1-\theta)^{\tau} EU_s.$$

Now straightforward calculation shows that

$$(A1-4) \quad \frac{EU_s}{EU_{ns}} = \frac{w-e_{\min}}{w-\bar{e}} \Psi(\tau)$$

where

$$(A1-5) \quad \Psi(\tau) := \frac{1-(1-\theta)^\tau}{1-(1-\delta)^\tau(1-\theta)^\tau} > 0, \quad \frac{d\Psi}{d\tau} > 0, \quad \lim_{\tau \rightarrow 0} \Psi \in (0,1) \quad \text{and} \quad \lim_{\tau \rightarrow \infty} \Psi = 1.$$

Hence the minimal  $e_{\min}$  which a firm can choose under the incentive compatibility constraint  $EU_s \leq EU_{ns}$  is given by

$$(A1-6) \quad e_{\min}^* = w - (w-\bar{e})/\Psi.$$

Thus, with increasing payment interval length  $\tau$  the firm has to increase this lower effort bound, if  $\tau$  increases to  $\infty$  then  $e_{\min}^*$  increases to  $\bar{e}$ . Monitoring cost thus increase in  $\tau$ .

## APPENDIX 2: Calculation of the optimal payment frequency

As a function of the inflation index  $\rho = \ln(1+\pi)$  we investigate that payment frequency  $\nu(\rho)$  which minimizes payment cost  $X=A(\nu|\rho)+B(\nu)+C(\nu)$ . For notational simplicity we denote the sum of disbursement cost and monitoring cost as disbursement and monitoring cost  $D$ ,  $B+C=D$ . First we analyze the extrema of  $X$ , i.e. those frequencies at which the marginal disbursement and monitoring cost function  $\dot{D}(\nu) := dD(\nu)/d\nu$  intersects with the negative of the marginal money holding cost function  $-\dot{A}(\nu|\rho) := -dA(\nu|\rho)/d\nu$ . We note that  $\dot{D}(\nu) = \mu\alpha e^{-\alpha\nu} - \lambda\beta e^{-\beta\nu}$  and  $-\dot{A}(\nu|\rho) = (y/n)e^{-\rho/\nu} \rho/\nu^2$ . As a preparation for the analysis of these intersection points, in the next three Lemmas we analyze  $\dot{D}$  and  $\dot{A}$  as functions of  $\nu$ ,  $y/n$ , and  $\rho$ ; all these properties follow from straightforward calculation.

LEMMA 1: Marginal disbursement and monitoring cost  $\dot{D}(\nu)$  exhibit the following properties:

$\dot{D}(\nu)$  is independent of  $\pi$

$$\dot{D}(0) = \mu\alpha - \lambda\beta < 0$$

$\dot{D}(\nu)$  has a unique root at  $\nu_{do} = [\ln(\lambda\beta/(\mu\alpha))]/(\beta-\alpha) > 0$

$\dot{D}(\nu)$  has a unique maximum at  $\nu_{dmax} = [\ln(\beta^2\lambda/\mu-\alpha^2)]/(\beta-\alpha) > \nu_{do}$

$\dot{D}(\nu)$  has a unique inflection point at  $\nu_{di} = [\ln(\beta^3\lambda/\mu-\alpha^3)]/(\beta-\alpha) > \nu_{dmax}$

$\dot{D}(\nu)$  is increasing for  $\nu < \nu_{dmax}$  and decreasing for  $\nu > \nu_{dmax}$

$\dot{D}(\nu)$  converges to zero as  $\nu$  increases to infinity.  $\square$

LEMMA 2: The negative of marginal money holding cost  $-\dot{A}(\nu|\rho)$  exhibit

$-\dot{A}(\nu|\rho)$  depends on  $\pi$

$-\dot{A}(\nu|\rho)$  is nonnegative

- $\dot{A}(v|\rho)$  has a unique root at  $v_{ko}=0$
- $\dot{A}(v|\rho)$  has a unique maximum at  $v_{kmax}=\rho/2$
- $\dot{A}(v|\rho)$  has a unique inflection point on each side of its maximum,  
 $v_{ki1}=\rho/(3+\sqrt{3}) < v_{kmax}$  and  $v_{ki2}=\rho/(3-\sqrt{3}) > v_{kmax}$
- $\dot{A}(v|\rho)$  is increasing for  $v < v_{kmax}$ , and decreasing for  $v > v_{kmax}$
- $\dot{A}(v|\rho)$  converges to zero for  $v \rightarrow \infty$ .  $\square$

LEMMA 3:

- (a) For any given inflation index  $\rho$ , there is a capacity threshold  $\kappa(\rho)$  such that for all  $(y/n) > \kappa(\rho)$  we have:  $-\dot{A}(v|\rho) > \dot{D}(v)$  for all  $v$ .
- (b) For any given payment frequency  $v$  and with increasing  $\rho$ ,  $-\dot{A}(v|\rho)$  decreases for  $v < \rho$ , and it increases for  $v > \rho$ .
- (c) For any given payment frequency  $v$ ,  $-\dot{A}(v|\rho)$  converges to zero both as  $\rho$  increases to infinity and as  $\rho$  decreases to zero.
- (d) The maximum  $-\dot{A}(v_{kmax}(\rho)|\rho)$  is a hyperbola of  $\rho$ ,  $-\dot{A}(v_{kmax}(\rho)|\rho) = (y/n)e^{-2} 4/\rho$ .  $\square$

Now we use Lemmas 1 to 3 to determine (a) the intersection points of  $\dot{D}$  and  $-\dot{A}$  and (b) how these intersection points vary with  $\rho$ . For simplicity we assume that capacity  $(y/n)$  is so large in relation to the disbursement and monitoring cost parameter  $\mu$  that for some inflation index  $\rho^+$  we have  $-\dot{A}(v|\rho^+) > \dot{D}(v)$  for all  $v$ . This is possible according to Lemma 3a. Hence, for some inflation index  $\rho^+$  overhead cost  $X$  are everywhere decreasing in  $v$ ,  $\dot{X} = \dot{D} + \dot{A} < 0$ .

Next we reduce the inflation index  $\rho$ . Then, according to Lemma 3b,  $-\dot{A}(v|\rho)$  increases in  $\rho$  for  $v < \rho$ , and it decreases for  $v > \rho$ . Since the second inflection point of  $-\dot{A}(v|\rho)$ ,  $v_{ki2}(\rho)$ , is smaller than  $\rho$ , the decreasing part of  $-\dot{A}(v|\rho)$  is concave,  $-[d^2 A/dv^2] > 0$ . We note that the support of this decreasing part of the graph,  $(\rho, \infty)$ , is itself moving to the left as  $\rho$  decreases. In accordance with Lemma 3c there is an inflation index  $\rho^* < \rho^+$  for which  $-\dot{A}(v|\rho^*)$  is tangential to  $\dot{D}(v)$  for some frequency  $\bar{v}(\rho^*) > v_{dmax}$  as it is depicted in Figure A2-1. As can also be seen from this figure, for any  $\rho < \rho^*$  there are always two intersection points  $v_1^*(\rho) < \bar{v}(\rho^*)$  and  $v_2^*(\rho) > \bar{v}(\rho^*)$ . With decreasing  $\rho$ ,  $v_1^*(\rho)$  decreases to  $v_{do} > 0$  and  $v_2^*(\rho)$  increases to infinity. As can be seen from Figure A2-1,  $v_1^*(\rho)$  describes local minima of payment cost  $X$  while  $v_2^*(\rho)$  describes local maxima. The resulting shape of the frequency curves  $v_1^*$  and  $v_2^*$  as functions of  $\rho$  for  $\rho < \rho^+$  is described in Figure 2.

FIGURE A2-1: Intersections of  $\dot{D}$  with  $-\dot{A}$  for  $\rho < \rho^+$

Now we increase the inflation index  $\rho$  above  $\rho^+$  which is depicted in Figure A2-2. Again there is a unique index  $\rho^{**}$  for which  $-\dot{A}$  is tangential to  $\dot{D}$  at some frequency  $\bar{\nu}(\rho^{**}) \in (\nu_{do}, \nu_{dmax})$ . As  $\rho$  increases further above  $\rho^{**}$  there are always two intersection points,  $\nu_1^{**}(\rho) < \bar{\nu}(\rho^{**})$  and  $\nu_2^{**}(\rho) > \bar{\nu}(\rho^{**})$ , where  $\nu_1^{**}(\rho)$  decreases to  $\nu_{do} > 0$  and  $\nu_2^{**}(\rho)$  increases to infinity. Again  $\nu_1^{**}(\rho)$  describes local minima and  $\nu_2^{**}(\rho)$  local maxima. For  $\rho > \rho^+$ , too, the frequency curves  $\nu_1^{**}$  and  $\nu_2^{**}$  are described in Figure 2 as a function of  $\rho$ .

FIGURE A2-2: Intersections of  $\dot{D}$  with  $-\dot{A}$  for  $\rho > \rho^+$

Finally we show that the frequency curves  $(\nu_1^*, \nu_2^*)$  are separated from the frequency curves  $(\nu_1^{**}, \nu_2^{**})$  by inflation rays  $\nu = (1/\gamma)\rho$  introduced in (5) as is shown in Figure 2.

LEMMA 4: There are unique deficit indices  $\gamma_Q$ ,  $\gamma_S$ ,  $\gamma_L$  and  $\gamma_Z$  with  $0 < \gamma_Q < \gamma_S < \gamma_L < \gamma_Z < \infty$  such that:

- for  $\gamma \in (0, \gamma_Q)$  the inflation ray has a unique intersection both with  $\nu_1^*$  and with  $\nu_2^*$ , but no intersection with  $\nu_1^{**}$  or  $\nu_2^{**}$ ,
- for  $\gamma \in (\gamma_Q, \gamma_S)$  the inflation ray has exactly two intersections with  $\nu_1^*$ , and none with  $\nu_2^*$ ,  $\nu_1^{**}$  or  $\nu_2^{**}$ ,
- for  $\gamma \in (\gamma_S, \gamma_L)$  the inflation ray does not intersect any payment frequency curve,
- for  $\gamma \in (\gamma_L, \gamma_Z)$  the inflation ray intersects  $\nu_2^{**}$  only,
- for  $\gamma \in (\gamma_Z, \infty)$  the inflation ray has a unique intersection with  $\nu_1^{**}$ .  $\square^{20}$

PROOF: It suffices to show that there exists a  $\gamma_x$  such that the corresponding inflation ray  $\nu = (1/\gamma_x)\rho$  does not intersect with any payment frequency curve. All other properties of Lemma 4 follow immediately from Figure 2. We show that  $\gamma_x := 2$  exhibits the required nonintersection property:

1. If  $P = (\rho_P, \nu_P)$  is on the  $\nu_1^*$ -curve then  $\rho_P \leq \rho^*$  and - in accordance with Figure A2-1 - the graph of  $-\dot{A}(\cdot | \rho_P)$  intersects  $\dot{D}(\cdot)$  in  $\nu_P$  from above. Hence, since  $-\dot{A}(\cdot | \rho_P)$  has its maximum at  $\nu = \rho_P/2$ , we have  $\rho_P/2 < \nu_P$  or  $\rho_P/\nu_P < 2$ .
2. If  $R = (\rho_R, \nu_R)$  is on the  $\nu_2^{**}$ -curve then  $\rho_R \geq \rho^{**}$  and - in accordance with Figure A2-2 -  $-\dot{A}(\cdot | \rho_R)$  intersects  $\dot{D}(\cdot)$  in  $\nu_R$  from below. Hence  $\rho_R/2 > \nu_R$  or  $\rho/\nu > 2$ .  $\square$

### APPENDIX 3: Stability properties under conditionally rational expectations

To analyze the stability properties under conditional rational expectations and bounded flexibility we first show that the economy will always remain within the inflation pipe. The proof is analogous to that of

Proposition 1.

PROPOSITION 4: For any starting point  $P_0=(v_0, \rho_0)$  and any sequence of feasible a priori expectations  $(\hat{P}_i)_{i \geq 1}$  at any date  $i \geq 1$  the economy  $P_i=(v_i, \rho_i)$  will be within the inflation pipe  $G_S:=\{(v, \rho): |\rho - \gamma v| \leq S\}$ .  $\square$

Next we describe movements within the inflation pipe. In the vicinity of the low inflation equilibrium the economy moves towards this equilibrium.

PROPOSITION 5: Let  $(v^+, \rho^+)$  be a low inflation equilibrium and for some  $i \geq 1$  let  $v_{i-1} < v^+ - |r-1|S$ ,  $(v_{i-1} > v^+ + |r-1|S$ , respectively), where  $r=b/(1-\gamma b) > 0$ . Then for any feasible a priori expectation  $\hat{P}_i \in \Psi_i$  the frequency increases (decreases),  $v_i (\geq) v_{i-1}$ .  $\square$

PROOF: For any date  $i$  we have  $v_i (\geq) v_{i-1}$  if the corresponding inflation possibility set  $\Psi_i$  is strictly below (above) the frequency line  $\phi$ . Thus it suffices to show that  $\Psi_i$  is strictly below (above)  $\phi$  if and only if  $v_{i-1} < v^+ - |r-1|S$ , (respectively if and only if  $v_{i-1} > v^+ + |r-1|S$ ). This equivalence follows in correspondence with Figure A3-1 from the relations  $v_{So} - v^+ = v^+ - v_{Su} = S[b/(1-\gamma b)]$  and  $v_{So} - v_{Sa} = v_{Sb} - v_{Su} = S$ , both in case of  $\bar{\Psi}$  being flatter than  $\phi$  (i.e.  $r > 1$ ) and in case of  $\bar{\Psi}$  being steeper than  $\phi$  (i.e.  $r < 1$ ).  $\square$

FIGURE A3-1: Inflation pipe movement and attractor  $R_S$ , case  $b > 1/(1+\gamma)$

FIGURE A3-2: Inflation pipe movement and attractor  $R_S$ , case  $b < 1/(1+\gamma)$

The following proposition formulates the stability result for low inflation equilibria.

PROPOSITION 6: Let  $(v^+, \rho^+)$  be a low inflation equilibrium and  $(v_i, \rho_i; \hat{v}_i, \hat{\rho}_i)_{i \geq 0}$  any adaptation process with conditional rational expectations and feasible a priori expectations starting in some point  $P_0=(v_0, \rho_0)$ . Then the economy will reach the attractor  $R_S$

$$(A3-1) \quad R_S := \{(v, \rho) \in G_S: |(v - v^+)| \leq rS\}, \quad r = [b/(1-\gamma b)] > 0,$$

after a finite number  $j$  of subperiods and it will remain there forever,  $j := \min\{k \geq 0: P_k \in R_S\} < \infty$  and  $P_i \in R_S$  for all  $i \geq j$ .

Furthermore, if the dynamic inflation line  $\Psi$  is steeper than the frequency line  $\phi$ , i.e.  $1/(1+\gamma) > b$ , then the adaptation process converges to the low inflation equilibrium  $(v^+, \rho^+)$ .  $\square$

PROOF: The finite arrival time property follows immediately from Propositions

4 and 5 as in Proposition 3 since the economy approaches the attractor in  $\nu$ -steps of length  $S$ . First we consider the case of  $b \geq 1/(1+\gamma)$ . Then in Figure A3-1  $R_S$  is given by the rhombus ABCD where  $\nu_{So} - \nu^+ = \nu^+ - \nu_{Su} = rS$  and  $\nu_{Sa} - \nu^+ = \nu^+ - \nu_{Sb} = S|r-1| = (r-1)S$ . We show that the economy remains within  $R_S$  for all dates  $i > j$ . For some date  $i$  let  $P_{i-1} \in R_S$ . It suffices to show that  $P_i \in R_S$ . We have either  $\nu_{i-1} > \nu_{Sa}$ ,  $\nu_{i-1} \in [\nu_{Sa}, \nu_{Sb}]$ , or  $\nu_{i-1} < \nu_{Sb}$ . In case of  $\nu_{i-1} > \nu_{Sa}$ ,  $\bar{\Psi}_i$  is strictly above  $\Phi$ . Thus, for any a priori expectation,  $\nu$  decreases and  $P_i$  is at the left endpoint of  $\bar{\Psi}_i$  which implies  $P_i \in R_S$ . In case of  $\nu_{i-1} \in [\nu_{Sa}, \nu_{Sb}]$ ,  $\bar{\Psi}_i \subset R_S$ . This implies immediately  $P_i \in R_S$ . In case of  $\nu_{i-1} < \nu_{Sb}$ ,  $\bar{\Psi}_i$  is strictly below  $\Phi$ . Thus  $\nu$  increases and  $P_i$  is at the right endpoint of  $\bar{\Psi}_i$  which proves  $P_i \in R_S$ .

Next we consider the case of  $b < 1/(1+\gamma)$  which is depicted in Figure A3-2. Again  $R_S$  is the rhombus ABCD. Let  $P_{i-1} \in R_S$  for some  $i$ . Then  $\bar{\Psi}_i$  intersects with  $\Phi$ . Now, for any feasible a priori expectation  $\hat{P}_i \in \bar{\Psi}_i$ ,  $P_i$  will be in the intersection point of  $\Phi$  and  $\bar{\Psi}_i$ , and thus  $|\nu_i - \nu^+| < |\nu_{i-1} - \nu^+|$ . Hence the process converges to  $(\nu^+, \rho^+)$ .  $\square$

For high inflation equilibria  $(\nu^+, \rho^+)$ , local stability depends in general on a priori expectations. For small disturbances and with a suitable sequence of a priori expectations the system may converge to equilibrium. However, once the deviation of frequencies is large enough the adaptation process is destabilizing for any sequence of feasible a priori expectations.

PROPOSITION 7: Let  $(\nu^+, \rho^+)$  be a high inflation equilibrium. If the initial disturbance is large enough,  $\nu_0 > \nu^+ + S[1+(b/(\gamma b-1))]$ ,  $(\nu_0 < \nu^+ - S[1+(b/(\gamma b-1))])$ , respectively) then for any sequence of feasible a priori expectations the adaptation process destabilizes such that  $\nu \rightarrow +\infty$ .  $\square$

PROOF: Let  $\nu_{Sc}$  and  $\nu_{Sd}$  be given in Figure A3-3 by the intersection points of the inflation ray  $\Gamma$  with those inflation possibility sets  $\bar{\Psi}$  and  $\bar{\Psi}'$  each of which has a common intersection with both  $\Phi$  and one of the edges of  $G_S$ . Then we have  $\nu_{Sc} = \nu^+ + S[1+(b/(\gamma b-1))]$  and  $\nu_{Sd} = \nu^+ - S[1+(b/(\gamma b-1))]$  which follows from the slope relations  $b = FE/DE$ ,  $(1/\gamma) = FC/AC$ ,  $1/(1+\gamma) = DB/AB$  and the identities  $\nu_{Su} - \nu_{Sd} = S$  and  $\nu_{Sc} - \nu^+ = \nu^+ - \nu_{Sd}$ . If  $\nu_{i-1} > \nu_{Sc}$  then  $\bar{\Psi}_i$  is strictly below  $\Phi$  and hence  $\nu$  increases for any a priori expectation. If  $\nu_{i-1} < \nu_{Sd}$  then  $\bar{\Psi}_i$  is strictly above  $\Phi$  and  $\nu$  decreases for any a priori expectation.  $\square$

FIGURE A3-3: Instability at the high inflation equilibrium

## NOTES

1 For too high budget deficits no equilibrium with constant inflation rates will exist.

2 Bruno and Fischer derive the stability properties of these equilibria under two alternative dynamic hypotheses. Under the first hypothesis, expectations about inflation adapt with finite speed while under the second hypothesis, expectations are rational but demand for real balances adapts with finite speed. - We note that Bruno and Fischer derive an alternative solution by introducing bond financing.

3 Other models in the literature have analogous problems. For a discussion compare Bernholz and Gersbach (1992). While Bruno and Fischer give an analysis in continuous time form, Bernholz and Gersbach discuss conditions under which the unintuitive comparative-static effects remain valid in discrete time form. These conditions refer to end-of-period and beginning-of-period formulations of the fiscal budget constraints.

4 There are other models of hyperinflation within the monetary approach with more intuitive comparative-static properties. But even in equilibrium these models abandon either rational expectations or market clearing. Evans and Yarrow (1981) consider equilibria based on adaptive expectations. Bernholz, Gärtner and Heri (1985) and Bruno (1989) use expectation formation schemes which depend on the stage of inflation. Cukierman (1988) assumes lags of inflationary expectations due to privacy of government information. Kiguel (1989) considers equilibria based on lagged money demand. Gersbach (1989) uses price stickiness.

5 At even higher deficits there is the possibility that inflation abides at high but constant inflation rates; here, too, inflation increases with the fiscal deficit.

6 This kind of stability is often referred to as corridor stability.

7 In Barro's model the rate of inflation is exogenous and payment frequencies increase with the square root of inflation. In my model, in contrast, inflation rates can have more than proportional effect on payment frequencies and both payment behavior and inflation are endogenous. It is this difference which turns out to explain how accelerating hyperinflation is triggered by increasing fiscal deficits.

8 In recent decades, however, the financial systems seems increasingly able to adapt to situations of high inflation such that near money funds are offered which are both indexed to inflation rates and liquid. See Dornbusch et al. (1990).

9 We note that the rate of money growth is equal to the inflation rate,  $M_{i+v}/M_i = \prod_{j=i}^{i+v-1} (M_{j+1}/M_j) = (y/y-g)^v = 1+\pi$ . According to (1) the real quantity of money is always equal to  $M_i/P_i = y\tau$ . *i.e.  $\frac{M}{P}$  decreases, if  $\tau$  becomes shorter!*

10 According to Appendix 1 we can choose parameters  $K_T$ ,  $K_0$ ,  $T$  with  $0 < K_T < K_0 < \infty$  and  $0 < T < \infty$  such that  $C(v) \geq K_0$  for  $v=0$  and  $C(v) \leq K_T$  for  $v \geq T$ . These conditions imply  $\lambda(1-e^{-\beta T}) \geq K_0 - K_T$  and thus  $\beta\lambda \geq (K_0 - K_T)/T$ . With suitable  $K_T$ ,  $K_0$ ,  $T$  we have  $\beta\lambda > \alpha\mu$ .

11 All minima could be conjectured to be global if we had applied other (but computational more difficult) disbursement and incentive cost functions, e.g. if  $B$  increased to infinity as  $v \rightarrow \infty$  and  $C$  increased to infinity as  $v \rightarrow 0$ .

12 For  $\gamma > \gamma_2$  we have  $d\rho/d\gamma > 0$  but for  $\gamma \in (\gamma_L, \gamma_2)$  we have  $d\rho/d\gamma < 0$ .

13 Here  $v_i$  is defined as a function of  $\rho_{i-1}$  and not of  $\rho_i$  because  $v_i$  determines date  $t_i$  of market  $i$  and there price level  $P_i$  and inflation index  $\rho_i$  are determined. As it is discussed in Section 4c, this assumption may contradict with rational expectations if the adaptation process has approached close enough to equilibrium.

14 A formal analysis yields that the system (8), (9) can be described by  $\begin{pmatrix} v(i) \\ \rho(i) \end{pmatrix} = \begin{pmatrix} 0 & b \\ -1 & (1+\gamma)b \end{pmatrix} \begin{pmatrix} v(i-1) \\ \rho(i-1) \end{pmatrix}$  the characteristic roots of which are  $b(1+\gamma)/2 \pm [b^2(1+\gamma)^2/4 - b]^{1/2}$ . Hence at the high inflation equilibrium the system will be unstable due to  $b\gamma > 1$ , i.e. due to  $v^{**}$  being steeper than  $\Gamma$ . At the low inflation equilibrium, it will be stable if  $b(1+\gamma) < 1$ , that is if  $\Phi$  is flatter than  $\Psi_i$ , and it will be unstable if  $b(1+\gamma) > 1$ .

15 In Figures 3 and 4 the inflation line  $\Psi_i(v_{i-1})$  intersects with the inflation ray  $\Gamma$  at  $v_{i-1}$ . Thus, for any previous state  $P_{i-1} = (v_{i-1}, \rho_{i-1})$  the new payment frequency  $v_i$  is given by the intersection of the frequency line  $\Phi$  and the vertical line through  $\rho_{i-1}$ , and the new state  $P_i = (v_i, \rho_i)$  is given by the intersection of the inflation line  $\Psi_i(v_{i-1})$  and the horizontal line through  $v_i$ .

16 Formally, for any date  $i$ , any previous  $v_{i-1}$  and any feasible a priori expectation  $\hat{P}_i = (\hat{v}_i, \hat{\rho}_i) \in \bar{\Psi}_i$  we define the payment frequency consistent with conditionally rational expectations by  $v_i := \sup\{v' \geq \hat{v}_i : \text{for all } v \in [\hat{v}_i, v'] \text{ we have } |v - v_{i-1}| \leq S \text{ and } v < a + b[(1+\gamma)v - v_{i-1}]\}$  if  $\hat{v}_i < a + b\hat{\rho}_i$ ,  $v_i := \inf\{v' \leq \hat{v}_i : \text{for all } v \in [v', \hat{v}_i] \text{ we have } |v - v_{i-1}| \leq S \text{ and } v > a + b[(1+\gamma)v - v_{i-1}]\}$  if  $\hat{v}_i > a + b\hat{\rho}_i$ , and  $v_i := \hat{v}_i$  if  $v = a + b\hat{\rho}_i$ , while  $\rho_i = (1+\gamma)v_i - v_{i-1}$ .

17 Without linearization the payment frequency  $v_i$  consistent with conditional rational expectations with respect to a priori expectations  $(\hat{v}_i, \hat{\rho}_i) \in \bar{\Psi}_i$  is given by  $v_i = \sup\{v' \geq \hat{v}_i : \text{for all } v \in [\hat{v}_i, v'] \text{ we have } |v - v_{i-1}| \leq S \text{ and } v < v^*(\rho) \text{ where } \rho = v[\gamma + \ln(v/v_{i-1})]\}$  if  $\hat{v}_i < v^*(\hat{\rho}_i)$ ,  $v_i = \inf\{v' \leq \hat{v}_i : \text{for all } v \in [v', \hat{v}_i] \text{ we have } |v - v_{i-1}| \leq S \text{ and } v > v^*(\rho) \text{ where } \rho = v[\gamma + \ln(v/v_{i-1})]\}$  if  $\hat{v}_i > v^*(\hat{\rho}_i)$ , and  $v_i = \hat{v}_i$  if  $\hat{v}_i = a + b\hat{\rho}_i$ .

18 In the Bortkiewicz-Debate (compare Bresciani-Turroni, 1937, pp 175) Bortkiewicz argued that money velocity is not (decisively) rising if the speed increases with which households spend their money on goods and services. The analysis of the present paper supports this view insofar as - in the extreme of immediate spending of household money - money velocity remains finite and it is then determined by firms payment behavior.

19 The stability result developed in this paper seems to be extendable to a large number of other linear dynamic systems provided an upper bound for the adaptation speed is introduced and provided without such an upper bound the characteristic roots of the original linear system have imaginary parts to a sufficient degree.

20 We note that for  $\rho=0$  the derivative of  $v_1^*$  with respect to  $\rho$  is given by  $(dv_1^*/d\rho)|_{\rho=0} = (y/n)/[\alpha v_{do}^2]$  where  $\alpha = (d\dot{D}/dv)(v_{do})$ . Hence  $b$ , the slope of the frequency line  $\Phi$ , is proportional to output per firm and period.

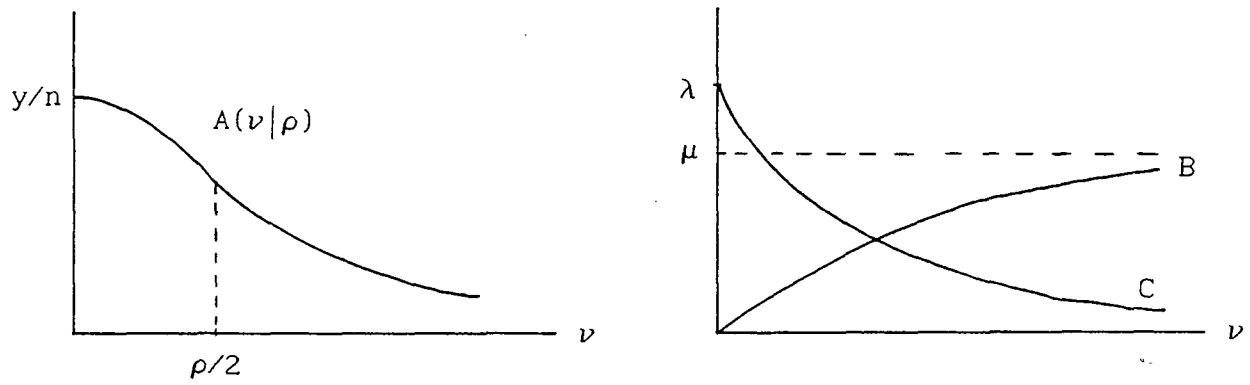


FIGURE 1: Money holding, disbursement and monitoring cost curves

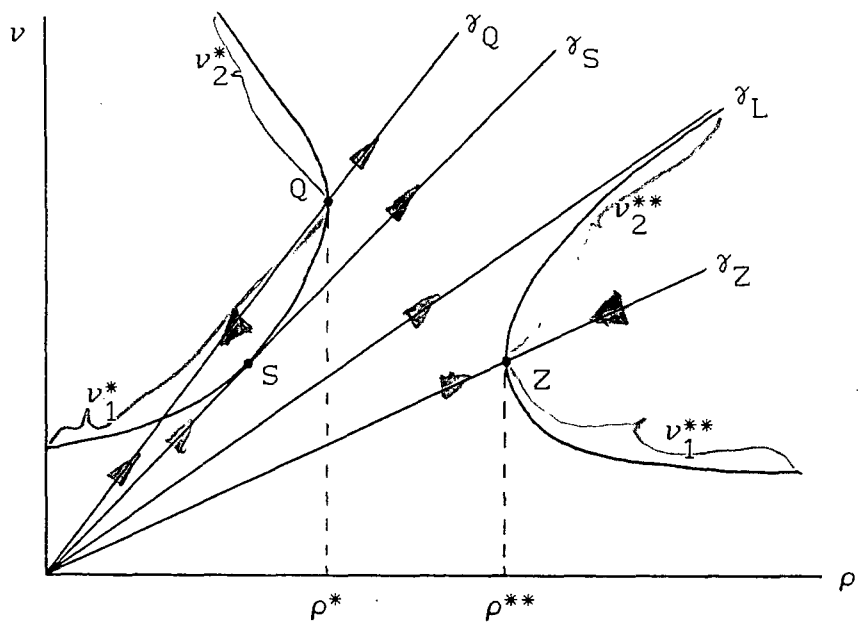


FIGURE 2: Frequency curves and inflation rays

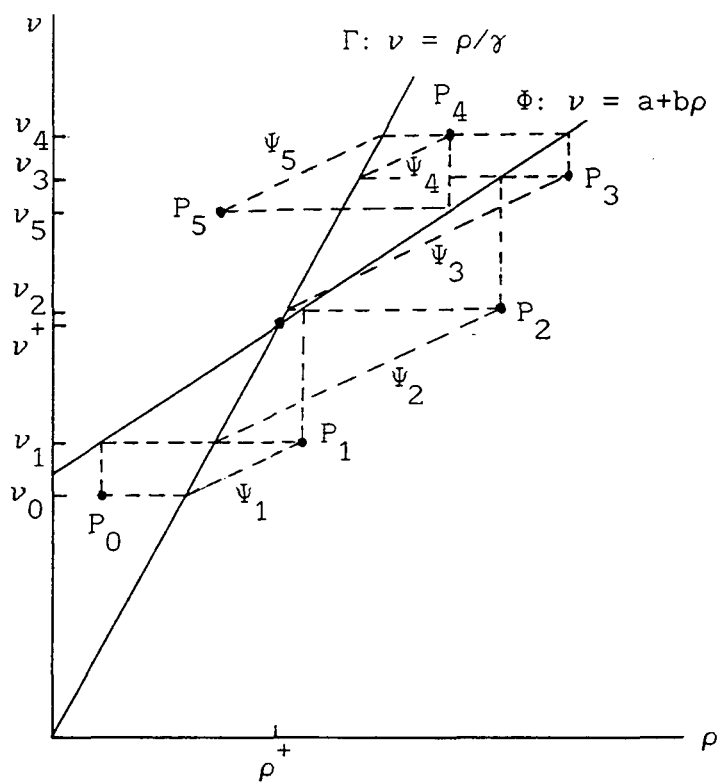


FIGURE 3: Early stage adaptation with unbounded flexibility.  
Case: Low inflation equilibrium,  $b < 1/\gamma$

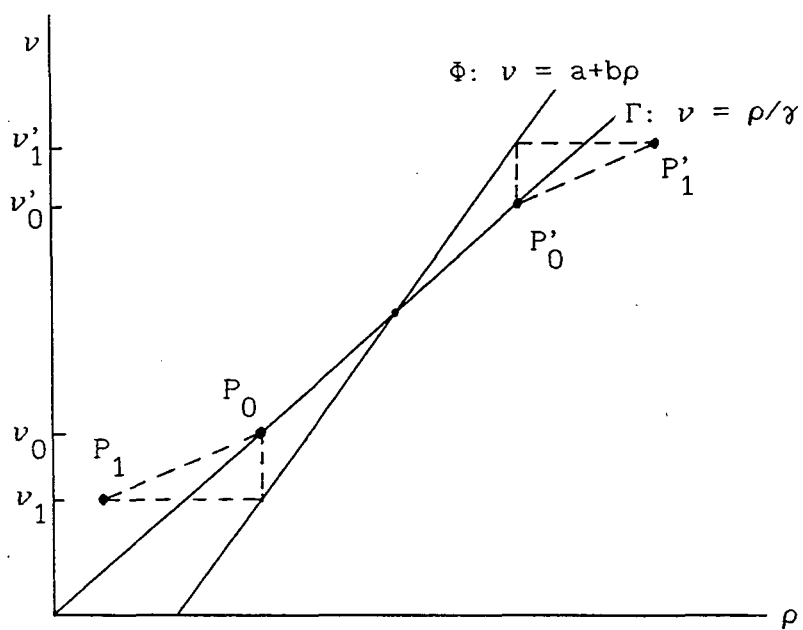


FIGURE 4: Early stage adaptation with unbounded flexibility.  
Case: High inflation equilibrium,  $b > 1/\gamma$

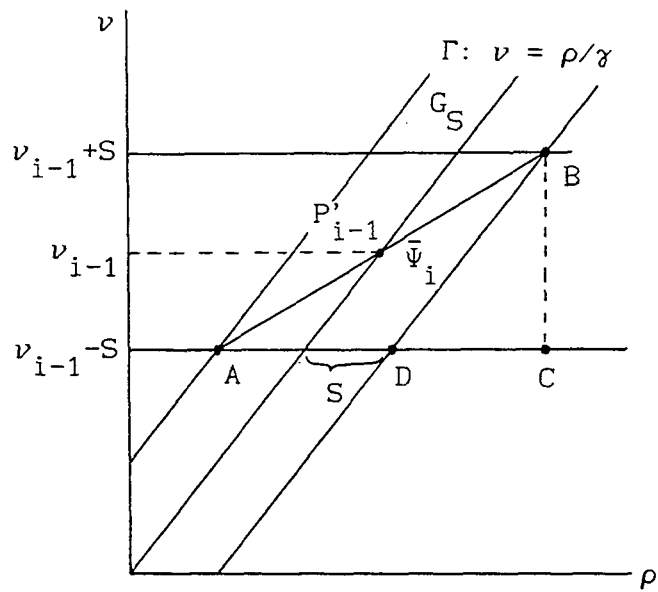


FIGURE 5: The inflation pipe  $G_S$

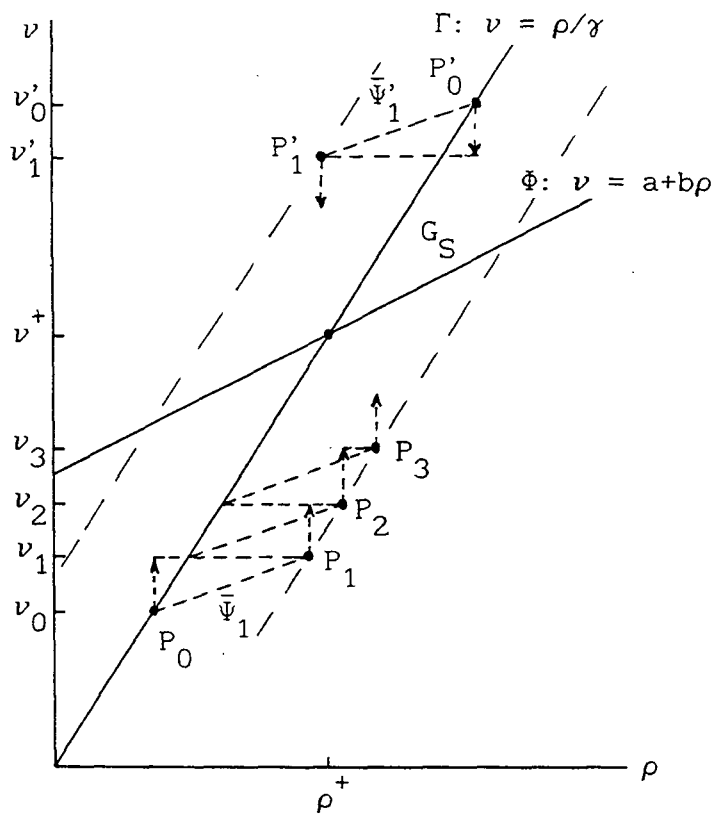


FIGURE 6: Movement in inflation pipe, low inflation equilibrium case

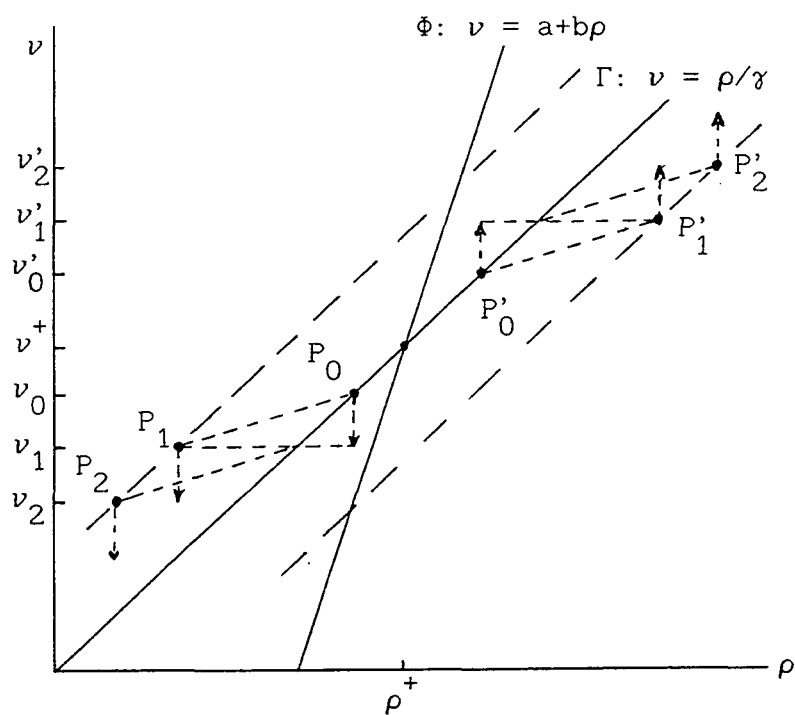


FIGURE 7: Movement in inflation pipe, high inflation equilibrium case

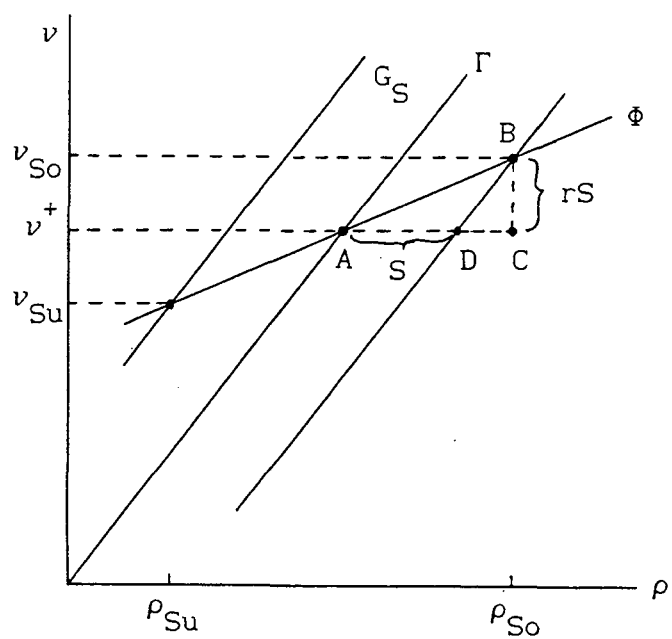


FIGURE 8: Attractor  $R_S$

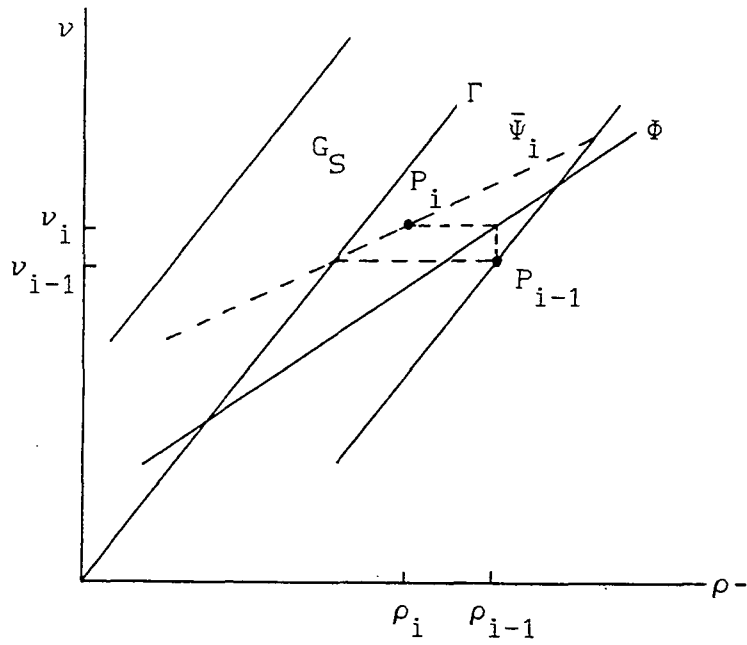


FIGURE 9: Stepwise adaptation without rational expectations

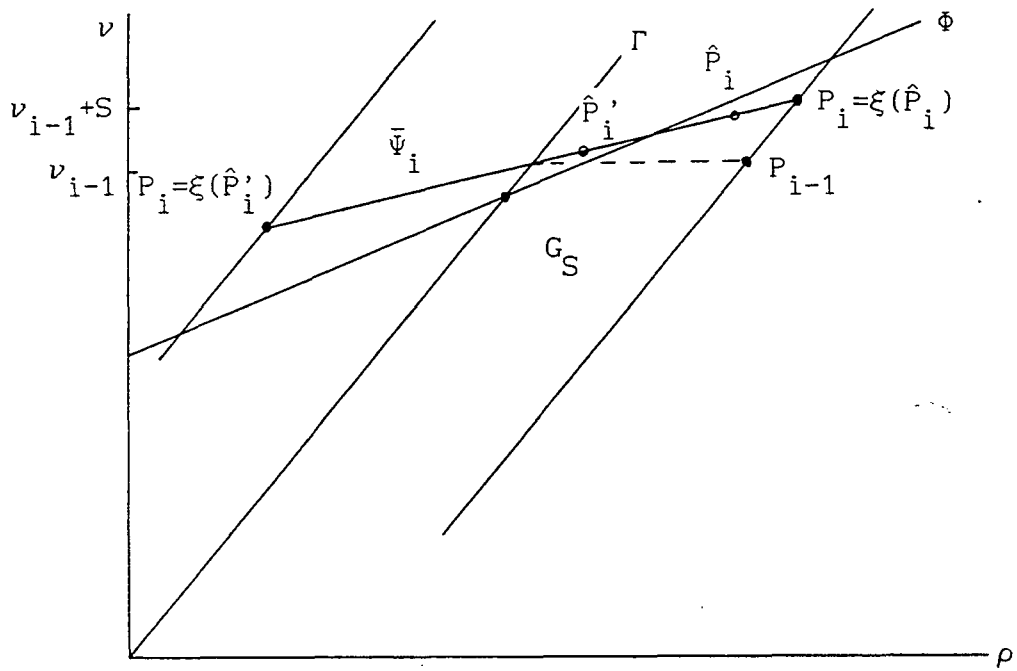


FIGURE 10: A priori expectations and conditional rational expectations equilibria, case:  $b > 1/(1+\gamma)$

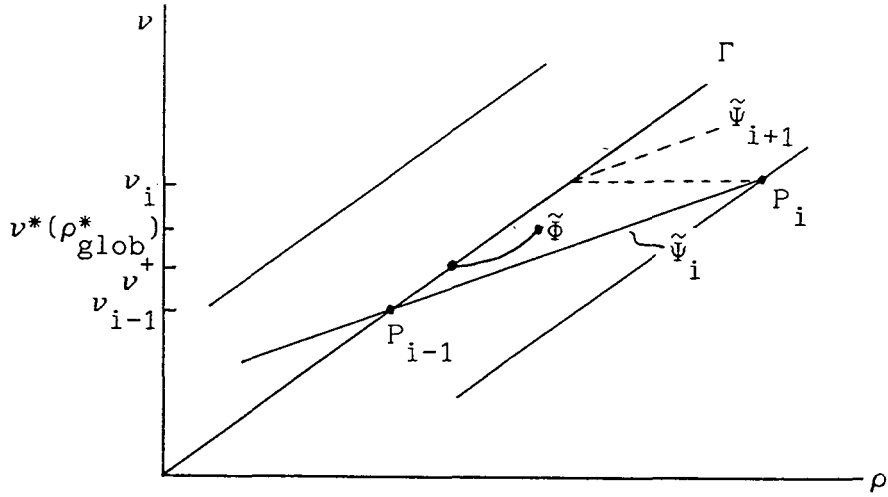


FIGURE 11: Instability of low inflation equilibria with high  $S$  and without linearization

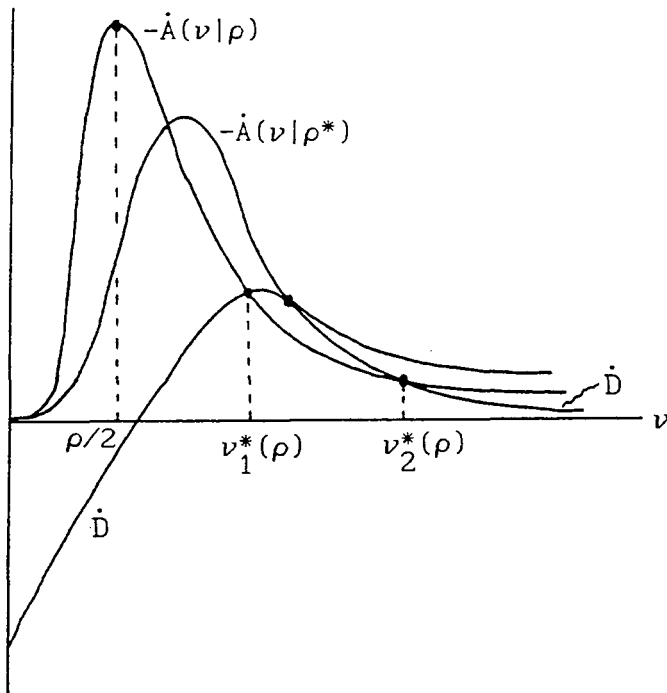


FIGURE A2-1: Intersections of  $\dot{D}$  with  $-\dot{A}$  for  $\rho < \rho^+$

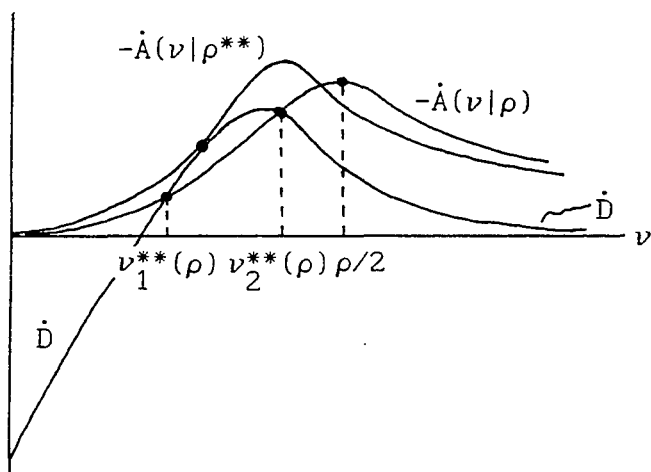


FIGURE A2-2: Intersections of  $\dot{D}$  with  $-\dot{A}$  for  $\rho > \rho^+$

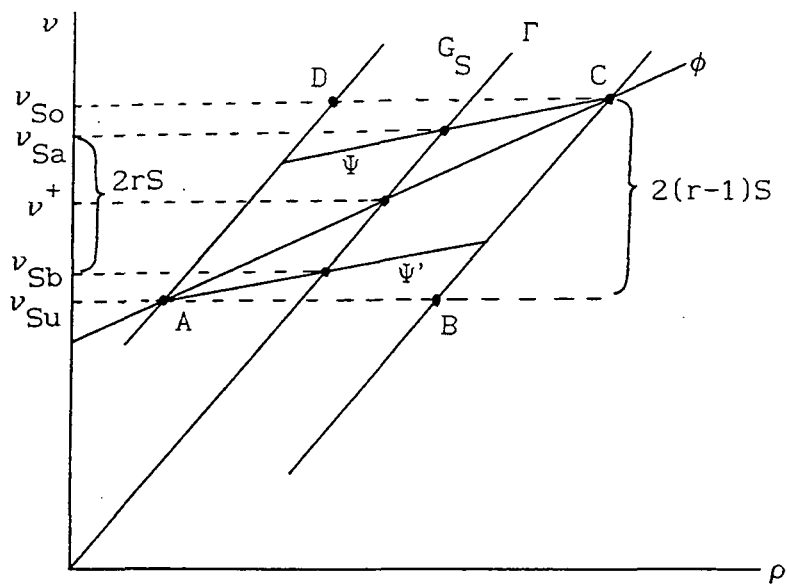


FIGURE A3-1: Inflation pipe movement and attractor  $R_S$ , case  $b > 1/(1+\gamma)$

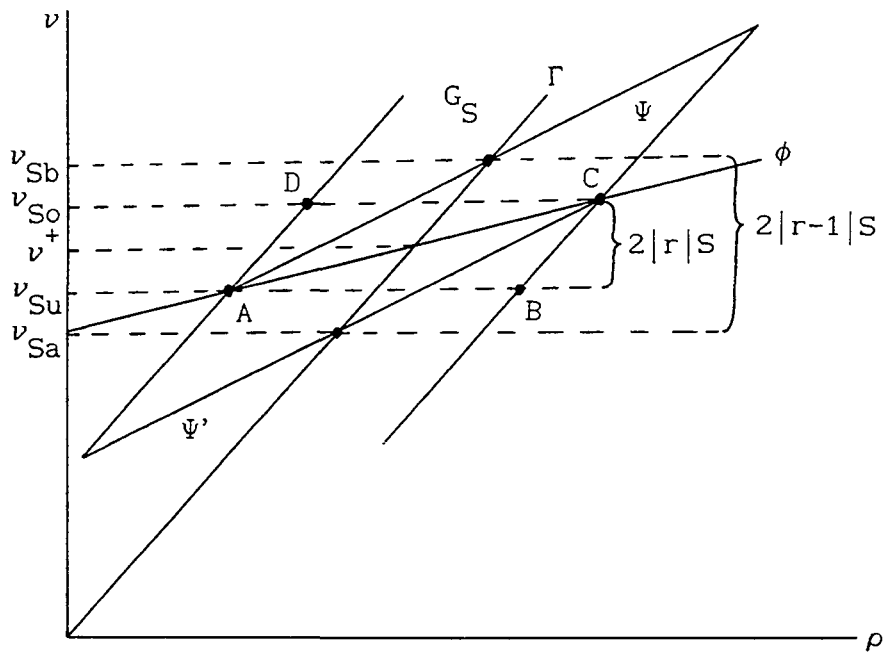


FIGURE A3-2: Inflation pipe movement and attractor  $R_S$ , case  $b < 1/(1+\gamma)$

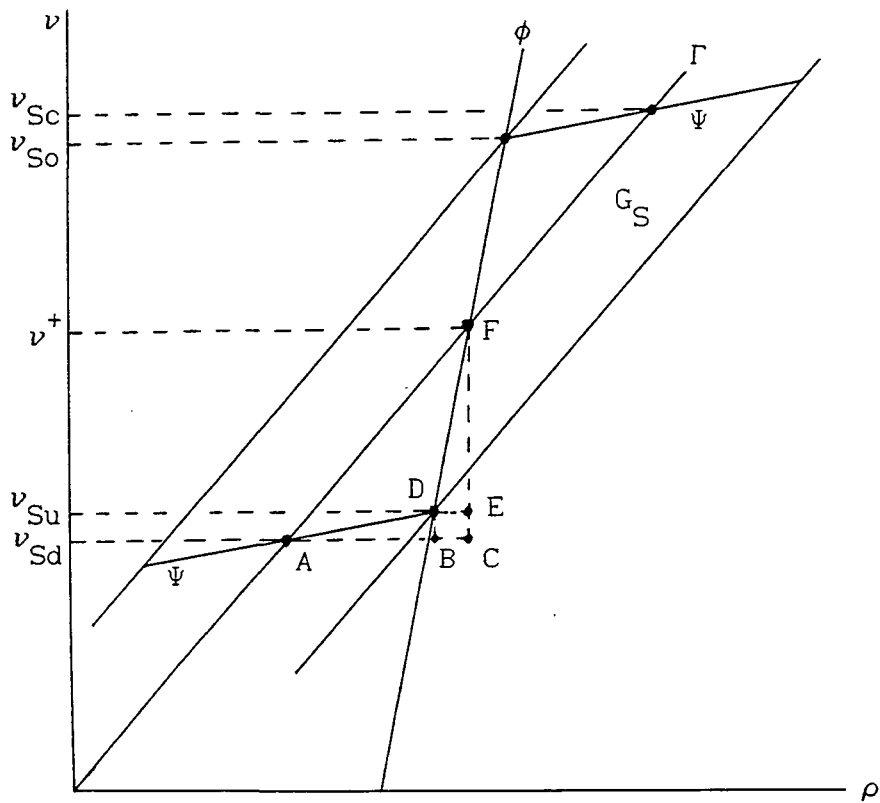


FIGURE A3-3: Instability at the high inflation equilibrium