

Vetschera, Rudolf

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A preference-preserving projection technique for MCDM

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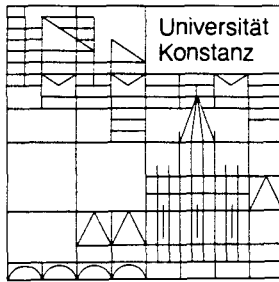
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**Fakultät für
Wirtschaftswissenschaften
und Statistik**

Rudolf Vetschera

**A Preference-Preserving
Projection Technique for MCDM**

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W 284 - 256 *RV*

Postfach 5560
D-7750 Konstanz

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Rudolf Vetschera

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A Preference-Preserving Projection Technique for MCDM *

Rudolf Vetschera

August 30, 1991

Abstract

The paper introduces a new method for the graphical representation of multi-criteria decision problems with discrete sets of alternatives. The approach proposed here extends projection techniques based on principal component analysis by incorporating information about preferences between alternatives in the graphical display. The advantages of principal component projection, which provides a clear view of data values, and a projection orthogonal to indifference hyperplanes, which provides preference information, are both made available to the user through a linear combination of projection matrices with interactively changed weights. The approach developed here can further be extended to more complex, nonlinear preference structures and strategies for that extension are also discussed in the paper.

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1 Introduction

One problem arising in practical application of multicriteria analysis is to convey the information available on decision alternatives to the user. The simultaneous display of several outcome dimensions easily leads to information overload and thus reduces decision quality.

It has been suggested in the literature that graphical representations of information will facilitate holistic interpretation of the data available. Although the use of graphics in general is sometimes over-valued and their effect has to be studied in more detail (DeSanctis, 1984), graphical interfaces for MCDM techniques have gained considerable popularity (e.g. Korhonen/Laakso, 1986).

Graphical methods for displaying multicriterial data can roughly be classified into two groups. The first consists of approaches that mainly provide a holistic, detailed image of all criteria for single or few alternatives. In many MCDM software packages, standard graphical displays like bar graphs are used to represent (current) solutions (e.g. in VIG: Korhonen, 1987). Other possible types of displays include Chernoff faces (Chernoff, 1973), Star Diagrams (Kasanen et al., 1989) and Korhonen's Harmonious Houses (Korhonen, 1991). Since these techniques provide a detailed representation of all criteria, their use is limited to a few decision alternatives.

In contrast, the second group is aimed at a representation of a larger set of alternatives. This means that information cannot be displayed in detail on all criteria, but criteria information has to be condensed in some way. This can be achieved by projecting the data points from high dimensional criteria space to some lower dimensional (usually 2- or 3-dimensional) projection space. Projection techniques were used to represent MCDM problems by Korhonen et al. (1980) and Lehert/de Wasch (1983). Specific methods based on projection techniques are the GAIA method (Mareschal/Brans, 1988) and BiPlot (Lewandowski/Granat, 1991).

Both GAIA and BiPlot are purely data driven techniques. They generate a projection matrix via principal component analysis directly from the data matrix or, as in the case of GAIA, intermediate results like a preference flow

matrix. While this leads to an optimal approximation of data in the projection space, important preference information might be lost in the projection. Consider, for example, the case in which preferences are given by a simple linear utility function. Unless the projection matrix used is orthogonal to the vector of scaling factors, points between which the user is indifferent according to the linear utility function can end up at any location in the projection space. The graphical representation of alternatives therefore gives no indication of preferences.

In this paper, we discuss extensions to these projection techniques which avoid the problem outlined above. The paper proceeds as follows: In section two, we give a short review of the standard principal component analysis approach. Section three extends this approach by incorporating weights of a linear utility function to create a projection matrix orthogonal to indifference structures. Section four illustrates the technique by a numerical example. In section five, we outline possible extensions to the case of nonlinear utility functions. Section six concludes the paper by identifying topics for further work.

2 A Review of Basic PCA

In this section, we give only a brief introduction to principal component analysis (PCA). The reader is referred to standard textbooks on multivariate statistics (e.g. Bamberg/Baur, 1984) for further reference.

The goal of PCA is to provide a compact representation of multi-dimensional data through a small number of so-called "factors". Let us consider a data matrix \mathbf{X} of N observations (alternatives) in K attributes (criteria). \mathbf{X} is assumed to be standardized so that the mean of each attribute (column) is zero and its variance one. We can represent the entire matrix \mathbf{X} by M general and K specific factors as

$$x_{nk} = \sum_{m=1}^M a_{km} f_{nm} + d_k u_{nk} \quad (1)$$

where x_{nk} is the value of variable k in the n th observation, the a_{km} are the *factor weights* for factor m and variable k , the f_{nm} are the *factor values* for alternative n in factor m and d_k^i and u_{nk} are the weight and value for the specific factor for attribute k . The goal of providing a compact representation is obtained by keeping M small. In matrix notation, we can rewrite equation (1) as

$$\mathbf{X} = \mathbf{FA}' + \mathbf{UD} \quad (2)$$

The goal of principal component analysis is to explain as much of the variance in each attribute of \mathbf{X} as possible through the factors. Ideally, we would like to obtain

$$\mathbf{X} = \mathbf{FA}' \quad (3)$$

Since \mathbf{X} is standardized,

$$\mathbf{R} = \frac{1}{N} \mathbf{X}'\mathbf{X} \quad (4)$$

is the correlation matrix of \mathbf{X} . It is also easy to show that

$$\begin{aligned} \mathbf{R} &= \frac{1}{N} \mathbf{X}'\mathbf{X} = \\ &= \frac{1}{N} [\mathbf{AF}'\mathbf{FA}' + \mathbf{D}'\mathbf{U}'\mathbf{FA}' + \mathbf{AF}'\mathbf{UD} + \mathbf{D}'\mathbf{U}'\mathbf{UD}] = \\ &= \mathbf{AA}' + \mathbf{DD} \end{aligned} \quad (5)$$

Ignoring the residual term \mathbf{DD} , we obtain the fundamental theorem of principal component analysis:

$$\mathbf{R} = \mathbf{AA}' \quad (6)$$

It can also be shown that the columns of \mathbf{A} consist of the eigenvectors of \mathbf{R} .

For the purpose of this paper, we are interested in obtaining the matrix \mathbf{F} of factor values, which are the projections of alternatives into lower dimensional space. Once we have determined \mathbf{A} , we obtain:

$$\begin{aligned} \mathbf{X} &= \mathbf{FA}' \\ \mathbf{XA} &= \mathbf{FA}'\mathbf{A} \\ \mathbf{XA}(\mathbf{A}'\mathbf{A})^{-1} &= \mathbf{F} \end{aligned} \tag{7}$$

We therefore obtain the projection of \mathbf{X} by postmultiplying \mathbf{X} by a projection matrix $\mathbf{P} = \mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}$.

The criteria themselves can also be represented in projection space by projecting the criteria unit vectors. This makes it possible to give a condensed representation of how well each alternative performs in each criterion. Figure 1 in section four provides an example of such a representation.

3 Preference-Preserving Projection

3.1 Introduction

In the standard PCA approach described above, the projection matrix \mathbf{P} is determined solely from the correlation matrix \mathbf{R} of the standardized problem data.

But when a two-dimensional graphical representation of the problem data is intended to support the user's decision process, basic concepts of the user's preference structure like preference or indifference between two alternatives should also have an easily observable, intuitive graphical representation. For example, indifference could be represented by a set of indifference curves connecting points between which the user is indifferent.

For simplicity, we will assume in this section that the user's preference system can be represented by a linear utility function of the form

$$U(\mathbf{x}_n) = \sum_{k=1}^K w_k x_{n,k} \quad (8)$$

where \mathbf{x}_n is a suitably scaled attribute vector describing decision alternative n , w_k is the decision weight (scaling factor) given to attribute k and $x_{n,k}$ is the k -th component of \mathbf{x}_n . Vector \mathbf{x}_n will also be interpreted as the n -th row of a decision matrix \mathbf{X} .

It is obvious that alternatives between which the user is indifferent, i.e. for which $U(\mathbf{x}_n)$ is equal to some constant c , lie on a hyperplane in K -dimensional space. However, for arbitrary projection matrices \mathbf{P} of dimension $K \times 2$, these points can be projected anywhere in the two-dimensional graphical representation.

The indifference hyperplanes of K -dimensional space will, however, be projected onto straight indifference lines in two-dimensional space if the projection is orthogonal to the hyperplanes. This is the case, for example, if vector \mathbf{w} is used as one column in \mathbf{P} .

However, incorporating the scaling factors into the projection matrix is not straightforward. The usual approach of PCA, as outlined above, is based on standardized data, while the factors refer to a specific scaling of the data. It is also necessary to take the first factor into account when determining the remaining column(s) of the projection matrix. The following two subsections will deal successively with these problems.

3.2 Non-Standardized Data

It can easily be seen that the indifference relation implied by (8) is preserved when a constant term is added to any column of the decision matrix \mathbf{X} . We can therefore still assume that \mathbf{X} is normalized to zero mean in all columns. But (8) is not invariant to the multiplication of columns by a constant term, so the column variances will be different from one. We denote the standard deviation of attribute k by σ_k and define the matrix S as:

$$\mathbf{S} = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_K \end{bmatrix} \quad (9)$$

We still assume that the factor values have a standard deviation of one, so we obtain the following general model:

$$\mathbf{X} = \mathbf{FA}'\mathbf{S} + \mathbf{UDS} \quad (10)$$

The correlation matrix is then given by

$$\mathbf{R} = \frac{1}{N}(\mathbf{XS}^{-1})'(\mathbf{XS}^{-1}) \quad (11)$$

and we still have

$$\mathbf{R} = \mathbf{AA}' + \mathbf{DD} \quad (12)$$

so the first factor is still determined by the largest eigenvector of \mathbf{R} . But the factor values and the projection matrix are now determined differently from above.

$$\begin{aligned} \mathbf{X} &= \mathbf{FA}'\mathbf{S} \\ \mathbf{XS}^{-1} &= \mathbf{FA}' \\ \mathbf{XS}^{-1}\mathbf{A} &= \mathbf{FA}'\mathbf{A} \\ \mathbf{F} &= \mathbf{XS}^{-1}\mathbf{A}(\mathbf{A}'\mathbf{A})^{-1} \\ \mathbf{P} &= \mathbf{S}^{-1}\mathbf{A}(\mathbf{A}'\mathbf{A})^{-1} \end{aligned} \quad (13)$$

The interpretation of equation (13) is obvious: instead of standardizing the data matrix to \mathbf{XS}^{-1} and then obtaining the projection matrix through post-multiplication by $\mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}$, the standardizing process can also be incorporated into the projection matrix \mathbf{P} .

3.3 Incorporating Preference Evaluations

From now on, we will only consider projections into two dimensional space, which are all we need for representing data on a computer screen. Our goal is to represent the preference relation implied by (8) via indifference lines.

Since we project into R^2 , the projection matrix is a $K \times 2$ matrix. We will indicate a projection matrix generating indifference lines in R^2 by \mathbf{Q} . A direct way to generate indifference lines is to use vector \mathbf{w} itself as one column of \mathbf{Q} . We therefore obtain the following structure for \mathbf{Q} :

$$\mathbf{Q} = \begin{bmatrix} w_1 & q_{12} \\ \vdots & \vdots \\ w_K & q_{K2} \end{bmatrix} \quad (14)$$

To determine the second column of \mathbf{Q} , we note that the factorial representation of \mathbf{X} as \mathbf{FA}' can be decomposed for the two-dimensional case as follows:

$$\begin{aligned} \mathbf{FA}' &= \begin{bmatrix} f_{11} & f_{12} \\ \vdots & \vdots \\ f_{N1} & f_{N2} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & \dots & a_{1K} \\ a_{21} & \dots & a_{2K} \end{bmatrix} = \\ &= \begin{bmatrix} f_{11} \\ \vdots \\ f_{N1} \end{bmatrix} \cdot [a_{11} \dots a_{1K}] + \begin{bmatrix} f_{12} \\ \vdots \\ f_{N2} \end{bmatrix} \cdot [a_{21} \dots a_{2K}] = \\ &= \mathbf{F}_{(1)}\mathbf{A}'_{(1)} + \mathbf{F}_{(2)}\mathbf{A}'_{(2)} \end{aligned} \quad (15)$$

Therefore, we can obtain the *residual data matrix* $\mathbf{X}_{(1)}$ containing the data not explained by the first column of \mathbf{Q} as:

$$\begin{aligned}
\mathbf{X} &= \mathbf{FA}'\mathbf{S} \\
\mathbf{XS}^{-1} &= \mathbf{F}_{(1)}\mathbf{A}'_{(1)} + \mathbf{F}_{(2)}\mathbf{A}'_{(2)} \\
\mathbf{XS}^{-1} - \mathbf{F}_{(1)}\mathbf{A}'_{(1)} &= \mathbf{F}_{(2)}\mathbf{A}'_{(2)} \\
\mathbf{X}_{(1)} &= \mathbf{F}_{(2)}\mathbf{A}'_{(2)}
\end{aligned} \tag{16}$$

From the residual matrix $\mathbf{X}_{(1)}$, the matrices $\mathbf{A}_{(2)}$, $\mathbf{F}_{(2)}$ and thus $\mathbf{Q}_{(2)}$ can be computed as outlined above.

In order to calculate $\mathbf{X}_{(1)}$, we need both the factor values $\mathbf{F}_{(1)}$ and the factor loads $\mathbf{A}_{(1)}$. The factor values are the projections of the data points and correspond to the utility scores of alternatives. We now have to solve a problem that is exactly the opposite of the usual problem of PCA: while one is usually mainly interested in factor loads and, after computing factor loads, might wish to compute factor values according to equation (7), we now want to determine factor loads from given factor values.

Assuming that just the first factor provides a sufficient approximation of the data, we can directly derive $\mathbf{A}_{(1)}$ from $\mathbf{F}_{(1)}$ as:

$$\begin{aligned}
\mathbf{X} &\approx \mathbf{F}_{(1)}\mathbf{A}'_{(1)}\mathbf{S} \\
\mathbf{XS}^{-1} &= \mathbf{F}_{(1)}\mathbf{A}'_{(1)} \\
\mathbf{F}_{(1)}'\mathbf{XS}^{-1} &= \mathbf{F}'_{(1)}\mathbf{F}_{(1)}\mathbf{A}'_{(1)} \\
\mathbf{A}'_{(1)} &= (\mathbf{F}'_{(1)}\mathbf{F}_{(1)})^{-1}\mathbf{F}_{(1)}'\mathbf{XS}^{-1}
\end{aligned} \tag{17}$$

Alternatively, we could try to find an "optimal" $\mathbf{A}_{(1)}$ which provides a best approximation to $\mathbf{X}_{(1)}$ in least squares sense by minimizing a residual matrix \mathbf{E} . Solving the regression problem

$$\mathbf{XS}^{-1} = \mathbf{F}_{(1)}\mathbf{A}'_{(1)} + \mathbf{E} \tag{18}$$

leads to normal equations of the form

$$\mathbf{A}'_{(1)} = (\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'\mathbf{X}\mathbf{S}^{-1} \quad (19)$$

and thus exactly to the same result as above.

Therefore, it is possible to use the utility scores of alternatives as factor values $\mathbf{F}_{(1)}$, obtain from these the factor loads $\mathbf{A}_{(1)}$ and finally the residual data matrix $\mathbf{X}_{(1)}$.

3.4 Combined Projection

Both representations, the one using \mathbf{P} based on the standard PCA approach and the one using matrix \mathbf{Q} , have certain advantages and disadvantages. The projection with \mathbf{Q} allows the user to visualize indifference and preference relations between alternatives along one axis. Furthermore, from

$$\mathbf{IQ} = \mathbf{I} \begin{bmatrix} w_1 & p_{12} \\ \vdots & \vdots \\ w_K & p_{K2} \end{bmatrix} \quad (20)$$

it can easily be seen that the projection of criteria vectors will represent the corresponding criteria weights on the same axis as the scores of alternatives.

On the other hand, providing utility values on one axis leaves only one dimension to represent all other aspects of the data. As can be seen from the figures in section four, the projection onto the two principal factors provides a more powerful representation of criteria values of alternatives than the preference preserving projection.

It is, however, possible to generate a unified representation by forming a linear combination of both projection matrices as:

$$\mathbf{T} = \lambda\mathbf{P} + (1 - \lambda)\mathbf{Q} \quad (21)$$

By changing the parameter λ in this linear combination, the user can move

freely between the two forms of projection. For example, the user may start with $\lambda = 0$ and progressively “untangle” the data values by increasing λ until a sufficiently clear picture of criteria values appears without completely destroying the preference information from the \mathbf{Q} -projection.

3.5 Indifference Regions

As soon as λ is increased above 0, the projection is no longer strictly orthogonal to the indifference hyperplanes. Each indifference hyperplane is therefore projected onto the entire R^2 . Since the data range in practical applications is finite, the projections of feasible indifferent data values onto R^2 will form a finite indifference region. In this subsection, we will develop a method for determining the extent of this region for a given alternative.

We assume that data values in attribute k fall within a lower bound \underline{x}_k and an upper bound \overline{x}_k . The boundaries of a region in R^2 containing all possible points of indifference to an alternative x_k can be determined by solving the following two parametric linear programs:

$$\begin{aligned} & \text{maximize } \mathbf{T}^{(1)}\mathbf{y} + \theta\mathbf{T}^{(2)}\mathbf{y} \\ & \text{s.t.} \\ & \mathbf{y}\mathbf{w} = \mathbf{x}_n\mathbf{w} \\ & \underline{x}_k \leq y_k \leq \overline{x}_k \end{aligned} \tag{22}$$

and

$$\begin{aligned} & \text{minimize } \mathbf{T}^{(1)}\mathbf{y} + \theta\mathbf{T}^{(2)}\mathbf{y} \\ & \text{s.t.} \\ & \mathbf{y}\mathbf{w} = \mathbf{x}_n\mathbf{w} \\ & \underline{x}_k \leq y_k \leq \overline{x}_k \end{aligned} \tag{23}$$

where $\mathbf{T}^{(1)}$ and $\mathbf{T}^{(2)}$ are the first and second column of \mathbf{T} . The first constraint of these linear programs considers only attribute vectors indifferent to \mathbf{x}_n , the remaining constraints take into account the possible data range. By

| Car | Price | Power | Tax | Speed | Consumption 1 | Consumption 2 |
|-------------|-------|-------|------|-------|---------------|---------------|
| 2 CV | 28.3 | 29.0 | 3.00 | 115. | 6.80 | 5.40 |
| Visa 11 | 44.3 | 50.0 | 5.00 | 140. | 6.30 | 4.80 |
| 205 GR | 44.8 | 50.0 | 4.00 | 142. | 5.80 | 4.30 |
| R4 | 29.8 | 34.0 | 4.00 | 110. | 6.30 | 5.60 |
| R5 GTL | 44.9 | 45.0 | 4.00 | 137. | 6.30 | 4.50 |
| Samba GL | 43.0 | 50.0 | 4.00 | 143. | 5.80 | 4.60 |
| Fiesta 1.1 | 33.3 | 53.0 | 4.00 | 145. | 8.80 | 6.00 |
| Corsa 1.25 | 38.0 | 55.0 | 5.00 | 152. | 8.70 | 5.10 |
| Polo 1100 | 43.4 | 50.0 | 4.00 | 146. | 9.20 | 6.10 |
| MG Metro | 47.5 | 73.0 | 6.00 | 163. | 9.00 | 5.50 |
| Fiat 126 | 21.6 | 24.0 | 3.00 | 105. | 7.30 | 6.10 |
| Fiat Panda | 34.3 | 45.0 | 4.00 | 140. | 7.10 | 4.80 |
| A112 Elite | 37.8 | 48.0 | 5.00 | 137. | 7.80 | 5.80 |
| Innocenti | 36.4 | 53.0 | 5.00 | 145. | 7.20 | 5.50 |
| Honda Civic | 44.6 | 60.0 | 5.00 | 145. | 8.00 | 5.20 |
| Mazda 323 | 44.9 | 55.0 | 5.00 | 160. | 8.40 | 5.60 |
| Colt | 46.3 | 70.0 | 6.00 | 165. | 7.60 | 5.90 |
| Tercel | 45.3 | 65.0 | 6.00 | 155. | 8.80 | 5.60 |
| Skoda 120 | 26.2 | 58.0 | 7.00 | 139. | 8.30 | 6.40 |
| Lada 2105 | 30.6 | 65.0 | 7.00 | 147. | 9.50 | 7.50 |

Table 1: Criteria values of alternatives

parametric variation of θ from $-\infty$ to $+\infty$, the first program generates the upper half and the second program the lower half of the boundary.

4 Numerical Example

Data for the following numerical example is taken from the literature (Mareschal/Brans, 1988) to allow for easy comparison with other, existing approaches. Table 1 contains data on 20 different cars used in this example.

We further assume that vector \mathbf{w} is given as $(0.25, 0.125, 0.125, 0.25, 0.125, 0.125)$.

Figure 1 shows the resulting projections for $\lambda = 0.9$. This projection is very close to the standard representation from PCA. The criteria vectors point into widely dispersed directions, so it is easy to identify the relative performance of each alternative in each criterion. On the other hand, the indifference region shown here for one alternative (alternative 9, the Polo 1100, represented in the display by a small rectangle) covers almost all other alternatives. This indicates that the display cannot be used to infer the ranking of alternatives from their position in the graph.

When λ is reduced to 0.1, the arrangement of alternatives on screen more closely reflects their ranking. The indifference regions become smaller and the user can, for example, determine that alternative 17 is definitely better than alternative 9, since the indifference regions shown in figure 2 do not overlap.

A very low level of λ finally corresponds to an almost completely orthogonal projection (figure 3). Here, the indifference regions are almost straight lines and the alternatives are arranged according to their ranking. This figure also clearly shows that the weights given to criteria 1 and 4 are twice the weights given to the other attributes. On the other hand, it is almost impossible to tell from this figure which alternatives perform well in which criteria.

5 Nonlinear Preference Structures

5.1 Introduction

In the above analysis, we considered the simple case of a linear preference structure. In this section we will examine possible extensions to the nonlinear case, specifically additive utility functions and then completely nonlinear structures.

Generally, we can specify two requirements which a graphical representation of preferences should fulfil:

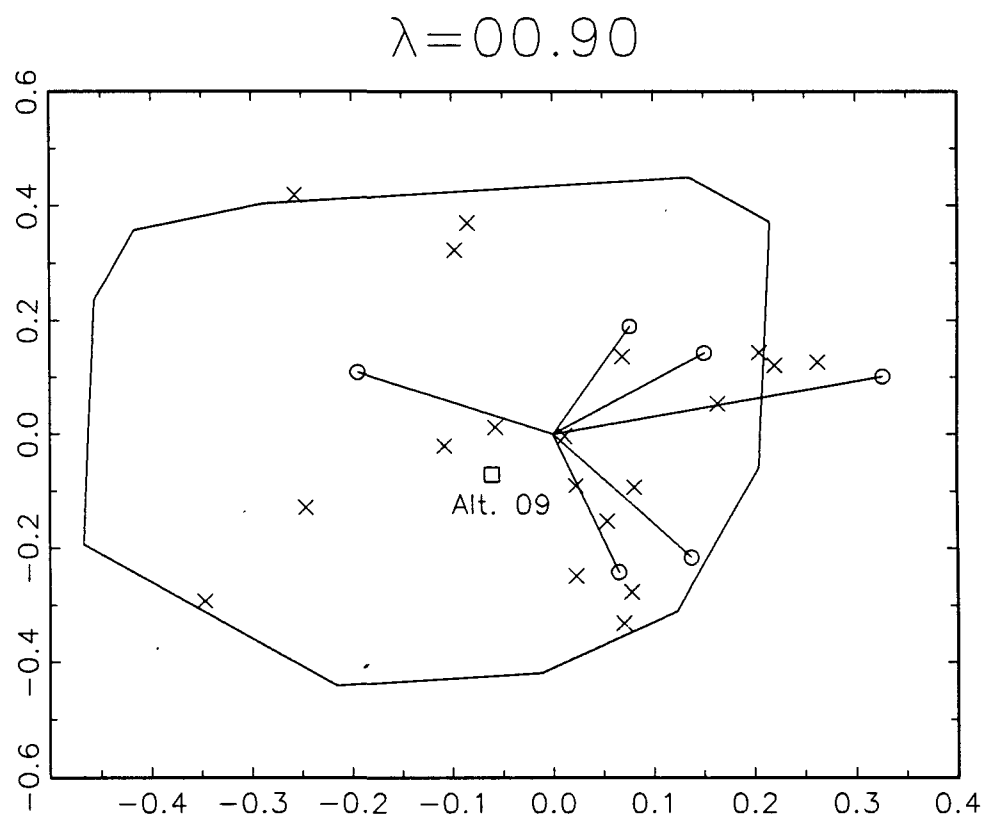


Figure 1: Projection close to standard PCA

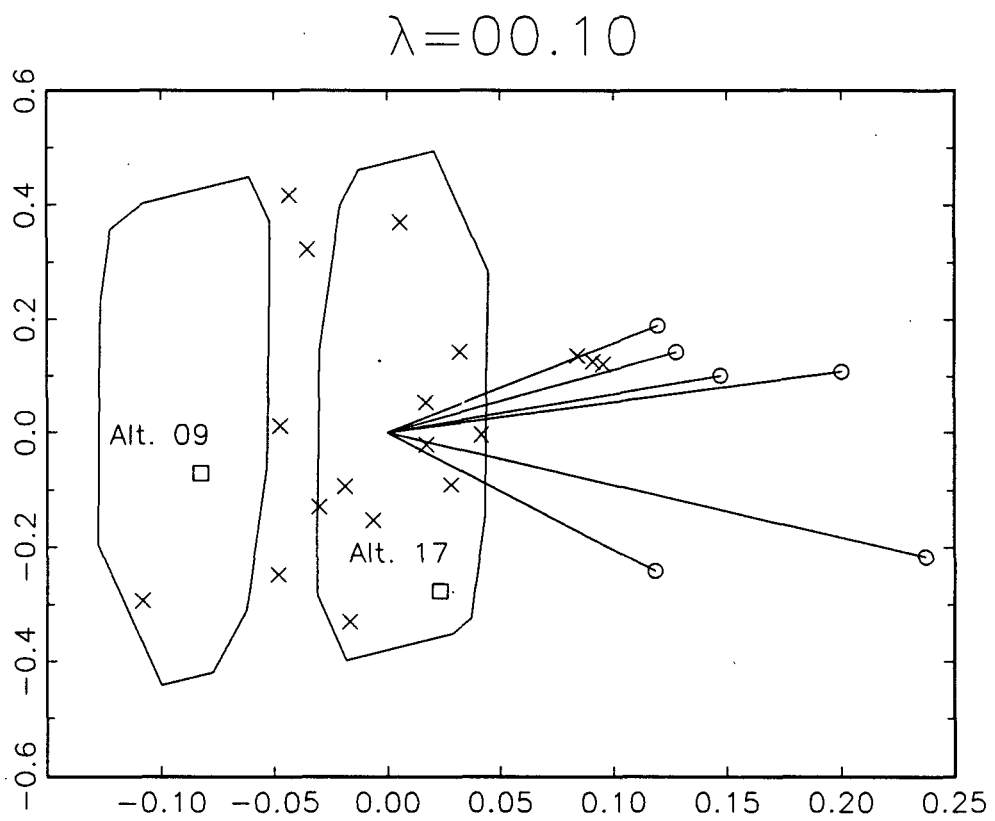


Figure 2: Combined projection showing distinct indifference regions

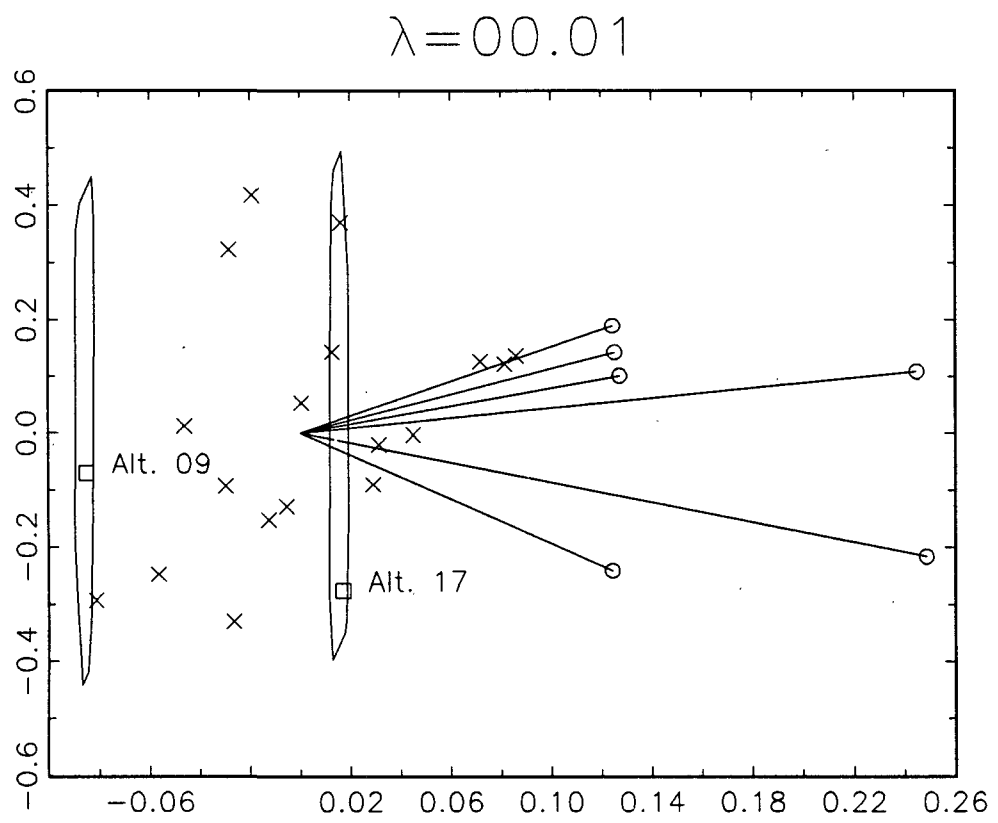


Figure 3: Orthogonal projection

- 1) Only alternatives between which the user is indifferent should be projected onto the same point in projection space.
- 2) All alternatives which are preferred to a given alternative should be projected onto points with higher values on the axis representing preference evaluation.

Evidently, the linear projection technique used so far fulfils those requirements for strictly linear utility functions. On the other hand, it is also possible to show that linear projections fulfil these requirements only for linear utility functions. Requirement 1) can be formulated (for linear projection matrices \mathbf{Q}) as follows:

Consider two criteria vectors \mathbf{x}_1 and \mathbf{x}_2 , which are projected onto the same point, therefore $\mathbf{x}_1\mathbf{Q} = \mathbf{x}_2\mathbf{Q}$. Requirement 1) states that for any two such vectors, $U(\mathbf{x}_1) = U(\mathbf{x}_2)$ should hold. The condition $\mathbf{x}_1\mathbf{Q} = \mathbf{x}_2\mathbf{Q}$ actually defines two linear equations between the criteria vectors, from which two constant trade-off rates between any three criteria can be derived. Since we can select any criteria, the utility function must exhibit constant trade-off rates between all criteria and therefore must be linear. Since it is not possible to strictly fulfil the requirements by linear projections for arbitrary utility functions (or more general multi-criteria evaluation techniques), we will now discuss several alternative strategies for overcoming these problems for different types of evaluations.

5.2 Additive Utility Functions

Under rather general conditions (Keeney/Raiffa, 1976), multi-attribute utility functions take the additive form

$$\sum w_k u_k(x_k) \tag{24}$$

where w_k is the criterion weight and $u_k()$ is the partial utility function of criterion k .

In this case linearity can formally be maintained when partial utility values $u_k(x_k)$ are used instead of the criteria values x_k themselves. But the question arises whether graphs generated by this formal substitution can be meaningfully interpreted and thus are a useful decision aid.

Graphs as those shown in section four serve three purposes: they provide information about:

- 1) the ranking of alternatives;
- 2) the importance of criteria;
- 3) the performance of alternatives in criteria.

Functions 1) and 2) are not affected at all by projecting partial utility values instead of actual criteria values. Function 3) is changed: a projection no longer indicates how well alternatives perform in different criteria, but rather how the performance is *valued* according to the partial utility function. But this information is also an important decision aid, so the third function is also maintained. Graphs based on partial utility values rather than criteria values are therefore meaningful decision aids and the methodology presented above can be applied to the case of additive utility functions.

5.3 Nonlinear preference structures

If arbitrary utility functions are considered, no such formal substitutions are possible. In this case, it is either possible to strictly adhere to the requirements specified in the beginning of this section and consequently use nonlinear projections or to maintain linear projections while deviating from the requirements.

5.3.1 Nonlinear projection

The concept of preference-preserving projection embedded in \mathbf{XQ} can directly be extended to nonlinear evaluations. Instead of calculating utility

scores of alternatives through postmultiplication by vector \mathbf{w} , they can be directly incorporated into the image matrix. We define a matrix \mathbf{V} as:

$$\mathbf{V} = \begin{bmatrix} U(\mathbf{x}_1) & v_{12} \\ \vdots & \vdots \\ U(\mathbf{x}_N) & v_{N2} \end{bmatrix} \quad (25)$$

where $U(\mathbf{x}_n)$ is the utility value of alternative \mathbf{x}_n and v_{n2} is obtained by projection of \mathbf{x}_n onto a single factor.

Instead of combining projection matrices through linear combination as in the preceding section, a linear combination of the projected images given by $\lambda\mathbf{XP} + (1-\lambda)\mathbf{V}$ could then be used to combine standard principal component analysis with the preference-preserving projection.

This technique will fulfil both requirements specified above. Since one coordinate of the projection is given by the utility score, only alternatives having identical utility scores can be projected onto the same point and the ordering requirement is also clearly fulfilled.

But it is no longer possible to provide a representation of criteria as in the case of linear projections. One could use the evaluation of hypothetical alternatives having the best score in the attribute to be represented and the lowest score in all other attributes to represent criteria. But such alternatives would receive an overall score of zero in many evaluation methods and thus be projected onto a straight vertical line through the origin. The main advantage of a graphical representation of criteria would thus be lost.

5.3.2 Approximation

An alternative to nonlinear projection with actual scores is a linear approximation of the nonlinear evaluation system. Given the evaluation of alternatives, a set of "pseudo-weights" could be determined which, when used in a linear utility function, would represent these evaluations as closely as possible. This approximation can, for example, be based on least square regression or on the minimization of absolute distances.

This approach has several advantages over the nonlinear projection approach outlined above: It still leads to a linear projection, in which it is possible to provide an adequate representation of criteria along with the alternatives. The “pseudo-weights” computed during the approximation would graphically be represented as shown in the previous section and thus provide a valuable insight into the importance with which criteria enter the evaluation process.

Using a linear approximation instead of actual evaluations from the decision technique employed, makes it also possible to represent results of decision techniques which generate only ordinal (and possibly even incomplete) evaluations, e.g. outranking methods (Roy/Vincke, 1984).

6 Conclusions and Further Research

In this paper, we have developed a method of simultaneous graphical representation of performance data on alternatives and their preference evaluations. This method allows for easy and direct evaluations of a large number of discrete alternatives in several criteria.

While application of this concept to simple linear utility functions is straightforward, its extension to nonlinear structures requires the development of linear approximation techniques, which is the focus of ongoing research efforts.

Another important topic is the integration of this concept into a general, graphic-oriented decision support environment. Such an environment would not only use the projection technique developed here for providing information about alternatives, but would also exploit graphic interaction techniques for data manipulation and evaluation tasks. The development of such an integrated software system, which is envisioned as a common “front-end” to several multi-attribute decision techniques, is the main task of a research project currently being undertaken.

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