Matsuyama, Kiminori

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Kiminori Matsuyama
Department of Economics
Northwestern University
IL 60208 Evanston
U.S.A.

**Institut für Höhere Studien (IHS), Wien**
**Institute for Advanced Studies, Vienna**

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# Abstract

This paper presents models of growth, which put the neoclassical and neo-Schumpetarian growth models in a unified framework. In doing so, it is argued that these two views of growth, one based on factor accumulation and the other based on innovation, are complementary in that they may capture different phases of a single growth experience. It is shown that, under an empirically plausible condition, the economy achieves sustainable growth through cycles, perpetually moving back and forth between two phases. One phase is characterized by higher output growth, higher investment, no innovation and a competitive market structure. The other phase is characterized by lower output growth, lower investment, high innovation, and a more monopolistic market structure. Both investment and innovation are essential in sustaining growth indefinitely, and yet the only one of them appears to play a dominant role in each phase.



## Keywords

Endogenous growth, endogenous fluctuations, global analysis of nonlinear dynamical systems

## JEL-Classifications

031, E22, E32

**Comments**

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1. Introduction.

The old growth literature, which goes back at least to von-Neumann, focused on factor accumulation as an engine of growth. One important contribution of the neoclassical growth literature is to point out that the process of growth based solely on factor accumulation must stop eventually, as it runs into diminishing returns. The recent literature on the neo-Schumpetarian growth stressed the innovation of new products, motivated by monopoly profits, as a way of avoiding diminishing returns, and of sustaining growth indefinitely. Many recent studies on growth accounting have attempted to access empirically the relative importance of these two alternative views of growth. What is implicit in these studies is the assumption that the relative contribution of the two sources of growth is stable over time.

This paper presents models of growth, which put the neoclassical and neo-Schumpetarian growth models in a unified framework. In doing so, it is argued that these two views of growth, one based on factor accumulation and the other based on innovation, are complementary in that they may capture different phases of a single growth experience. It is shown that, under an empirically plausible condition, the economy achieves sustainable growth through cycles, perpetually moving back and forth between two phases. One phase is characterized by high output growth, higher investment, no innovation and a competitive market structure. The other phase is characterized by lower output growth, lower investment, high innovation, and a more monopolistic market structure. In the long run, both investment and innovation grow at the same rate, but the economy alternates between the periods of high investment and the periods of high innovation. Both investment and innovation are essential in sustaining growth indefinitely, and yet the only one of them appears to play a dominant role in

each phase. The results in this paper should thus provide at least a caution when interpreting a country's growth experience. A country may have grown in the past solely based on factor accumulation, and yet, its growth may not come to an end, as the economy may enter a period of innovations, once it builds up sufficiently large resource base. Or a country may record a faster output growth during the period of less innovative activities, from which one should not conclude that the innovation contributes less to the growth.

The intuition behind the emergence of cycles is easy to grasp. The critical departure from the existing studies of neo-Schumpetarian growth made in this paper is that the innovators of new products enjoy a temporary monopoly power. This assumption plays a dual role in generating cycles. First, the degree of monopoly prevailing in the economy can change over time. Second, a potential innovator wants to enjoy its temporary monopoly power, when the degree of monopoly prevailing in the economy is higher. This is because the potential innovator needs to enter when the market for its product is large enough to recover the cost of innovation. The size of the market depends in part on how the products with which it competes with are priced. This leads to a synchronization of innovative activities. If the innovator chooses to introduce its product when others do, some of its competing products are monopolistically priced. On the other hand, if the innovator enters after others have innovated, the market for its product would be too small to recover the cost of innovation, because the competing products would become more competitively priced, as their innovators lose their monopoly power. As a result, the economy experiences the period of high innovation with a monopolistic market structure, followed by the period of no innovation with a competitive market structure. Once innovation stops, the output and investment growth go up, partially because the resources

are now redirected from innovative activities to manufacturing activities and partially because the competitive market structure allocates the resources more efficiently among the existing products. And, as a result of high investment growth, the economy will eventually build up enough resource base to enter another period of innovative activities.

The intuition may be easy to grasp, yet its formal demonstration presents a challenge for the theorist. The main difficulty is that only a global analysis of nonlinear dynamical systems can reveal the possibility that the economy switches back and forth between phases with markedly differing properties. Looking at the steady state, or even at a neighborhood of the steady state would not suffice. A rigorous and explicit analysis of nonlinear global dynamics is possible only when the dimensionality of a system is kept sufficiently low. The challenge is to make a carefully chosen set of simplifying assumptions so that a model is simple enough to be tractable, rich enough to generate cycles, and has sufficient structures that would enable us to gauge the empirical plausibility of the condition for the cycles. In this paper, the following assumptions are chosen in view of these requirements.[1]

First, it is assumed that there is only one type of the capital stock; that the capital stock is converted to differentiated intermediate products, which enter symmetrically into the production of the final good. Furthermore, following Romer (1987), a restriction is imposed between two parameters; the degree of substitution among differentiated products and the factor share. This restriction helps to ensure the existence of a balanced growth steady state. Second, the time is assumed to be discrete, where the period length is taken as

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[1]A wide range of alternative assumptions are discussed in Matsuyama (in progress), which presents an unified treatment of dynamic monopolistic competitions in macroeconomics.

the duration of the monopoly power enjoyed by the innovators of new products. In other words, monopoly lasts for only one period. This assumption, borrowed from Deneckere and Judd (1992), obviates the need to evaluate the market value of innovating firms, and hence helps to simplify the analysis drastically.[2] As shown in section 2, all these assumptions jointly make it possible to summarize the levels of output and innovation in each period effectively as a function of a single variable.

Another critical modelling choice is on the formulation of intertemporal consumption-saving decision. If the economy is populated by the infinitely-lived representative agent with homothetic preferences, the equilibrium dynamics is described by a discrete-time, two-dimensional system, whose global properties are difficult, if not impossible, to analyze. Instead, if we assume two-period overlapping generations of consumers with Cobb-Douglas preferences, the equilibrium dynamics can be described in a discrete-time, one-dimensional system, whose global properties can be investigated thoroughly. Adopting the overlapping generations framework, however, makes it difficult to interpret the period length, which is already taken to be the duration of the innovator's monopoly power. Furthermore, some readers might suspect that the overlapping generations structure may be responsible for the cycles. In view of the pluses and the minuses, both formulations are used in this paper. First, section 3 deals with the overlapping generations (OG) economy, and its global properties are examined in detail. Then, section 4 looks at the representative agent (RA) economy. Much of the results obtained in the OG economy is shown to carry over to the RA

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[2]If we add another period of monopoly, the dimensionality of the system would increase by two. If a finite duration of the monopoly power were introduced in a continuous time model, the dynamical system would become infinite-dimensional.

economy, in a slightly weaker form. Having demonstrated the robustness of the results, section 5 gauges the empirical plausibility of the condition for cycles.

Although mainly motivated by the question of growth, this paper can also be viewed as a contribution to the literature on endogenous fluctuations (see surveys by Boldrin and Woodford 1990 and Guesnerie and Woodford 1992). Among these studies, Deneckere and Judd (1992), Gale (1996) and Shleifer (1986) all constructed models of innovation cycles, based on the temporary nature of the monopoly enjoyed by the innovators. Like the Deneckere-Judd model and those in the neo-Schumpetarian growth literature, it is assumed that there is free entry to innovative activities. This means that an incentive to delay implementing innovations, a crucial element in the models of Gale and of Shleifer, plays no role in generating cycles in the present study.

Evans, Honkapohja, and Romer (1996) recently developed a model of growth cycles, which also builds upon the model of Romer (1987). Their study differs from the present study in two crucial respects. First, the cycles in their model are based on sunspots, expectational indeterminacy, and the multiplicity of equilibrium. In contrast, the results in this paper do not rely on multiple equilibria; the cycles appear here, as the unique steady state loses its stability. Second, in their model, the investment and the rate of innovation always move together, as the economy alternates between the states of high and low growth. On the other hand, the investment grows faster during the period of no innovation than during the period of innovation in the models developed below.

Jovanovic and Rob (1990) is closest in spirit to this paper. They developed a model of the economy, which could grow through two different forms of innovation: intensive search and extensive search. They identified the condition under which the economy alternates between the period of intensive

search and the period of extensive search. Their underlying model differs so much from the models in this paper, which makes it difficult to make any direct comparision.

## 2. The Structure of Production.

The time is discrete and extends from one to infinity: $t \in T = \{1,2,3,\ldots\}$. There is a single final good, which is produced competitively. The final good, taken as a numeraire, can either be consumed or invested. For the notational convenience, let $K_t$ denote the capital stock at the end of period t, i.e., the amount of the final good left unconsumed in period t, and carried over to period t+1. Note that this means that the amount of capital stock available for use in period t is denoted by $K_{t-1}$.

There are two primary factors of production; capital (K) and labor (L). Labor goes directly into the production of the final good, and the supply of labor is inelastic and equal to L in each period. Capital is first converted into a variety of differentiated intermediate products. These intermediates are aggregated into the composite by a symmetric CES, as in Dixit and Stiglitz. Labor and the composite of intermediates are combined with a Cobb-Douglas technology. More specifically, the technology of the final goods producer is expressed as

$$Y_t = A(L)^{\frac{1}{\sigma}} \left\{ \int_0^{N_t} [x_t(z)]^{1-\frac{1}{\sigma}} dz \right\}, \qquad (1)$$

where $x_t(z)$ is the amount of variety z employed in period t, $\sigma \in (1,\infty)$ is the direct partial elasticity of substitution between every pair of intermediate products, and $[0,N_t]$ represents the range of intermediates available in the marketplace in period t. Some features of this specification, borrowed from

Romer (1987), deserve comments. First, for a given availability of intermediate products, $N_t$, the technology of the final goods production satisfies the property of constant returns to scale, and hence it is consistent with the competitiveness of the final goods industry. Second, the final goods producer's demand for each intermediate product has a constant price elasticity equal to $\sigma$. Third, the labor share of the economy is equal to $1/\sigma$, and hence the total wage income of this economy is equal to $w_tL = Y_t/\sigma$.

One significant departure from Romer (1987) lies in the market structure of the intermediate inputs sector. At the beginning of period t, the economy inherits all the intermediate inputs of variety $z \in [0,N_{t-1}]$. These "old" intermediates are manufactured by converting *a* units of capital into one unit of an intermediate, and sold competitively. At the same time, the intermediate inputs of variety $z \in [N_{t-1},N_t]$ may be introduced and sold exclusively by their innovators in period t. These "new" intermediates require F units of capital to innovate. The process of manufacturing new intermediates, just as old ones, requires *a* units of capital per output.

Let $r_t$ denote the cost of capital. Then, the marginal cost of manufacturing intermediates in period t is equal to $ar_t$. The old products are produced competitively and hence sold at the marginal cost; $p_t(z) \equiv p_t^c = ar_t$ for $z \in [0,N_{t-1}]$. On the other hand, all the new products, if they exist, are sold at $p_t(z) \equiv p_t^m = a\sigma r_t/(\sigma-1)$, where $z \in [N_{t-1},N_t]$, because of the constant price elasticity, $\sigma$. Since all the intermediate products enter symmetrically in the production function of the final goods, we have $x_t(z) \equiv x_t^c$ for $z \in [0,N_{t-1}]$, and $x_t(z) \equiv x_t^m$ for $z \in [N_{t-1},N_t]$, and they satisfy

$$\frac{x_t^c}{x_t^m} = \left[\frac{p_t^c}{p_t^m}\right]^{-\sigma} = \left[1-\frac{1}{\sigma}\right]^{-\sigma} , \qquad (2)$$

and

$$\frac{p_t^c x_t^c}{p_t^m x_t^m} = \left[1-\frac{1}{\sigma}\right]^{1-\sigma} \equiv \theta > 1 .$$

The parameter, $\theta$, which plays an important role in the following analysis, depends positively on $\sigma$ and its value can range from 1 to e = 2.71828..., as one varies $\sigma$ from 1 to infinity.

The one-period monopoly enjoyed by the innovator gives an incentive for innovation, and there is no barrier to entry for innovative activities. The period t monopoly profit, net of the fixed cost, is $\pi_t = p_t^m x_t^m - r_t(ax_t^m + F)$; it is negative if and only if $x_t^m < (\sigma-1)F/a$. Thus, free entry ensures the following condition must hold in equilibrium:

$$x_t^m \le (\sigma-1)\frac{F}{a} , \quad N_t \ge N_{t-1} , \quad \left(x_t^m - (\sigma-1)\frac{F}{a}\right)\left(N_t - N_{t-1}\right) = 0 . \qquad (3)$$

This is to say that, when potential innovators do not expect the sale of a new product to reach the break-even point (i.e., $x_t^m < (\sigma-1)F/a$), there is no incentive for innovating new products (i.e., $N_t = N_{t-1}$). On the other hand, when innovation occurs and some new products are introduced (i.e., $N_t > N_{t-1}$), the innovator cannot earn any excess profit and must break even, due to the free entry.

The resource constraint on capital in period t is expressed as

$$K_{t-1} = N_{t-1} a x_t^c + \left(N_t - N_{t-1}\right)\left(a x_t^m + F\right) .$$

Note that the left hand side of this equation is $K_{t-1}$, which represents the amount of capital carried over from period t-1, and hence available for use in

period t. Using eqs. (2) and (3), the above constraint becomes

$$a x_t^c = a\left[1-\frac{1}{\sigma}\right]^{-\sigma} x_t^m = Min\left\{\frac{K_{t-1}}{N_{t-1}}, \theta\sigma F\right\}, \tag{4}$$

and

$$N_t = N_{t-1} + Max\left\{0, \frac{K_{t-1}}{\sigma F} - \theta N_{t-1}\right\}. \tag{5}$$

From eq. (1), the total output is equal to

$$Y_t = A(L)^{\frac{1}{\sigma}}\left[N_{t-1}\left(x_t^c\right)^{1-\frac{1}{\sigma}} + \left(N_t - N_{t-1}\right)\left(x_t^m\right)^{1-\frac{1}{\sigma}}\right]$$

which can further be rewritten to, by using eqs. (3), (4) and (5),

$$Y_t = \begin{cases} A\left[N_{t-1}L\right]^{\frac{1}{\sigma}}\left[\frac{K_{t-1}}{a}\right]^{1-\frac{1}{\sigma}} & if \quad \frac{K_{t-1}}{N_{t-1}} \le \theta\sigma F \\ A\left[\frac{K_{t-1}L}{\theta\sigma F}\right]^{\frac{1}{\sigma}}\left[\frac{K_{t-1}}{a}\right]^{1-\frac{1}{\sigma}} & if \quad \frac{K_{t-1}}{N_{t-1}} \ge \theta\sigma F \end{cases}$$

Eqs. (5) and (6) summarize what takes place on the production side of the economy in period t, when the economy inherits $K_{t-1}$ and $N_{t-1}$ at the beginning of the period. If $K_{t-1}/N_{t-1} \le \theta\sigma F$, the resource base of the economy, K, is too small relative to the number of the products, N, and there is no innovation. All the products are competitively produced, and the reduced form aggregate production function, given in eq. (6), has the standard neoclassical properties, including the law of diminishing returns in capital. If $K_{t-1}/N_{t-1} > \theta\sigma F$, on the other hand, the resource base of the economy is sufficiently large relative to the number of the products, and some new products are introduced. And the aggregate output is linear in capital, as in many endogenous growth models.

Before proceeding, let us minimize the notational burden in the ensuing analysis, by choosing the units of measurement in labor, in the quantity of intermediate inputs, and in the variety of intermediate inputs, so as to have

$$L = 1 , \quad a = 1 , \quad F = \frac{1}{\theta\sigma}$$

Furthermore, define

$$k_t \equiv \frac{K_t}{N_t}$$

With this normalization, the critical value of k, below which there is no innovation, is $k_c = 1$, and eqs. (5) and (6) are simplified to

$$\frac{N_t}{N_{t-1}} = Max\{ 1 , 1 + \theta ( k_{t-1} - 1 ) \} \tag{7}$$

and

$$\frac{Y_t}{K_{t-1}} = \frac{A}{\sigma} Max\left\{ (k_{t-1})^{-\frac{1}{\sigma}} , 1 \right\} \tag{8}$$

Needless to say, nothing of substance would be affected by such choices of the units.

In order to close the model, it is necessary to specify the mechanism of determining the capital accumulation, which comes in two different forms. One is based on the overlapping generations consumers (section 3), and the other is based on the infinitely-lived representative consumer (section 4).

## 3. The Overlapping Generations (OG) Economy.

The economy is populated by overlapping generations of the equal size, and each generation lives for two periods. Every period, a new generation of workers enters the economy and supplies L units of labor inelastically. The workers who enter the economy in period t earns the wage income, $w_tL$, some of which is consumed in the first period, $C_t^1$, and the rest is saved to finance their second period consumption, $C_{t+1}^2$. Their preferences are given by $U^t = (1-s)\log(C_t^1) +$

$s\log(C_{t+1}^2)$, which yields a simple saving function, $S_t = w_tL - C_t^1 = sw_tL = (s/\sigma)Y_t$. Since only this generation has an incentive to save in period t, the asset market equilibrium condition is $K_t = S_t = (s/\sigma)Y_t$. (Note that the only asset that the consumer holds is the capital stock. The ownership share of intermediate input producing firms is valueless, because their monopoly power lasts for only one period, in which they break even.)

Hence, from eqs. (7) and (8), the dynamics of the economy is uniquely determined by the following system of the difference equations in K and N:

$$K_t = \frac{sA}{\sigma} Max\left\{ \left( \frac{K_{t-1}}{N_{t-1}} \right)^{-\frac{1}{\sigma}} , 1 \right\} K_{t-1} \qquad \textbf{(9a)}$$

and

$$N_t = N_{t-1} + Max\{ 0 , \theta ( K_{t-1} - N_{t-1} ) \} \qquad \textbf{(9b)}$$

for an initial condition, $K_0$ and $N_0$.

Note that, in deriving eqs. (9a)-(9b), no reference was made concerning the physical depreciation rate of the capital stock, $\delta$. The dynamics is independent of $\delta$ in this model for two reasons. First, in the two-period OG economy, capital accumulation is determined solely by the gross savings by the young, the wage-earner. Second, due to the Cobb-Douglas preferences, the savings by the young is independent of the rate of returns. It is precisely these features of the model that would enable us to describe the equilibrium dynamics into an one-dimensional system, as will be seen below.

### 3.A. The Steady State Analysis.

Let us first look at the steady state of the economy, which is defined as an equilibrium path (for a particular set of initial conditions), in which $K_t/N_t$

= $k_t$ stays constant over time.

First, suppose that $k_{t-1} \leq 1$ in a steady state. Then, from eq. (9b), $N_t = N_{t-1}$ and hence $K_t = K_{t-1}$. In this steady state, there is no innovation, all goods are competitively supplied and the economy does not grow: it is a neoclassical stationary path. Eq. (9a) shows that, in a neoclassical stationary path, $k_{t-1} = (sA/\sigma)^{\sigma}$. The existence of such a stationary path thus requires $sA/\sigma \leq 1$.

Suppose now that $k_{t-1} > 1$ holds in a steady state. Then, from (9a) and (9b),

$$\frac{sA}{\sigma} = \frac{K_t}{K_{t-1}} = \frac{N_t}{N_{t-1}} = 1 + \theta(k_{t-1} - 1) > 1$$

Thus, in this steady state, where the resource base of the economy is sufficiently large relative to the number of existing products, there is always an incentive to innovate new products. The existence of such a balanced growth path requires that $sA/\sigma > 1$.

The above argument can be summarized as follows.

Preposition 1.

Let the growth potential of the OG economy be defined by $G \equiv sA/\sigma$. Then,

i) if $G \leq 1$, the economy is stationary in all the steady state paths. They are given by $\{K_t, N_t\} = \{k^*\Lambda, \Lambda\}$ for any $\Lambda > 0$, where $k^*$ is defined by

$$k^* \equiv G^{\sigma} = \left(\frac{sA}{\sigma}\right)^{\sigma} \leq 1$$

ii) if $G > 1$, the economy grows at the same rate in all the steady states paths. They are given by $\{K_t, N_t\} = \{k^{**}\Lambda G^t, \Lambda G^t\}$ for any $\Lambda > 0$, where $k^{**}$ is defined by

$$k^{**} \equiv 1 + \frac{G - 1}{\theta} > 1$$

The content of Proposition 1 is twofold. First, whether the economy grows in a steady state depends on $G = sA/\sigma$. If it is greater than one, the economy grows. If it is less than one, the economy stays stationary. For this reason, G is called the growth potential of the economy. Second, the steady state value of k is uniquely determined, although the dynamical system governing the evolution of the economy has two state variables, K and N, and a continuum of the steady state paths exist. When $G \leq 1$ (the case of stationary paths), it is equal to $k^*$. When $G > 1$ (the case of balanced growth paths), it is equal to $k^{**}$. The next subsection looks at the stability of the steady state by examining the asymptotic behavior of $k_t = K_t/N_t$, for an arbitrary initial value, $k_0 = K_0/N_0$.

3.B. <u>The Global Dynamical Analysis</u>.

It is straightforward to show that, from (9a)-(9b), the dynamics of $k_t$ is governed by the following one-dimensional map, $\Phi: R_+ \to R_+$,

$$k_t = \Phi(k_{t-1}) \equiv \begin{cases} G(k_{t-1})^{1-\frac{1}{\sigma}} & \text{if } k_{t-1} \leq 1 \\ G\dfrac{k_{t-1}}{1+\theta(k_{t-1}-1)} & \text{if } k_{t-1} \geq 1 \end{cases}$$

Recall that the critical value of k, below which there is no innovation, is given by $k_c = 1$. Viewed as a dynamical system of k, the equilibrium of the economy has a unique steady state. If $G \leq 1$, the unique steady state of the economy is a neoclassical stationary path, and equal to $k^* \equiv G^\sigma < k_c = 1$. If $G > 1$, the unique steady state is a balanced growth path, where the gross growth rate is G and $k^{**} \equiv 1 + (G-1)/\theta > k_c = 1$.

Let us define the <u>iteration</u> of the map as follows: $\Phi^1(k) \equiv \Phi(k)$ and $\Phi^t(k)$

$\equiv \Phi(\Phi^{t-1}(z))$. Then, $\{\Phi^t(k_0);\ t \in T\}$ represents the equilibrium trajectory for an initial condition, $k_0$. For any integer $n \geq 2$, <u>period-n cycles</u> of the map are defined as $\{\Phi^t(k);\ t \in T\}$, such that $\Phi^t(k) \neq k$ for all $1 \leq t < n$ and $\Phi^t(k) = k$. It is also useful to note that

$$\Phi^1(k_c) = G, \qquad \Phi^2(k_c) = \frac{G^2}{1+\theta(G-1)}, \qquad \Phi^3(k_c) = \left[\frac{G^2}{1+\theta(G-1)}\right]^{1-\frac{1}{\sigma}} G.$$

As always, the graphical technique is useful for analyzing a nonlinear dynamical system. Figures 1-3 all depict the dynamical system of $k_t$. The map is continuous and uni-modal. It is increasing in $(0,1)$ and decreasing in $(1,\infty)$. When $k_{t-1} \leq k_c = 1$, there is no innovation in period t, hence $N_t = N_{t-1}$. In this region, all the goods are supplied competitively and the economy grows solely by the accumulation of the capital stock, which is subject to the law of diminishing returns. The map is hence concave in this region, and the dynamic behavior mimics that of the neoclassical growth model. Let us call this region <u>the Solow regime</u>. On the other hand, when $k_{t-1} > k_c = 1$, new products are introduced. The economy grows partially due to the innovation, and partially due to the fact that capital accumulation is no longer subject to the diminishing returns. The dynamic behavior in this region thus resembles those of the neo-Schumpetarian model of endogenous growth. Let us call this region <u>the Romer regime</u>.

In Figure 1, $G < 1$ holds. This is the case where the steady state, $k_t = k^*$, is stationary. For any initial condition, the economy is trapped into the Solow regime, after at most one period, and then innovation stops. Once trapped, the growth of the economy solely depends on the accumulation of capital, and eventually peters out. The economy converges monotonically to the stationary state, $k^*$.

Figures 2 and 3 both depict the case where $G > 1$ holds, or equivalently

**Figure 1**

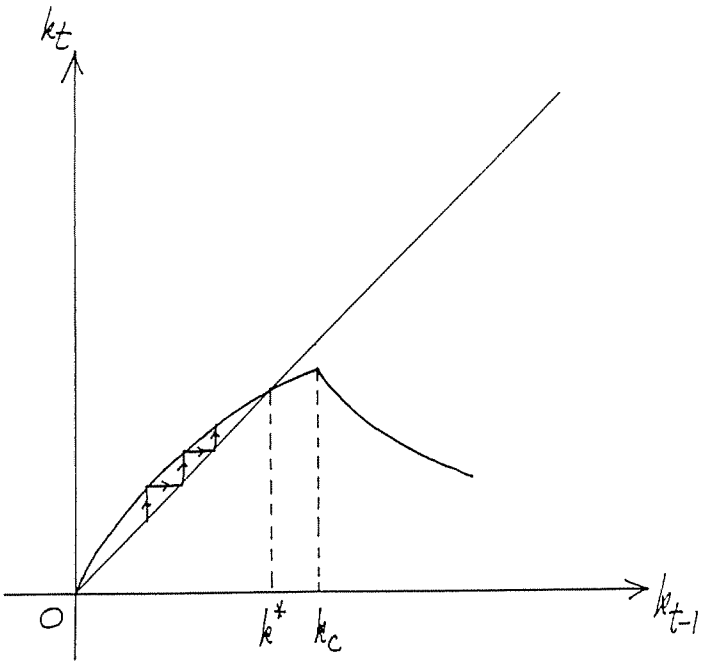



**Figure 2**

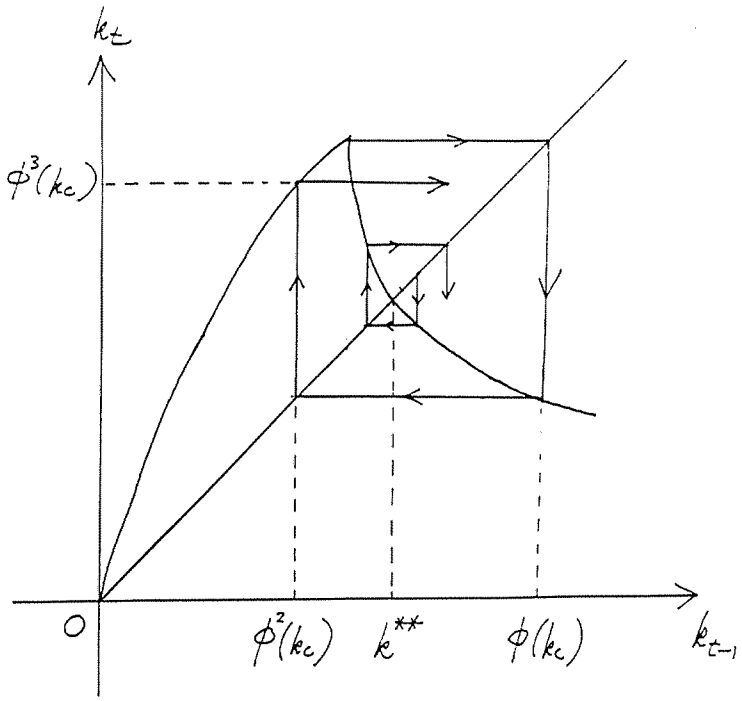
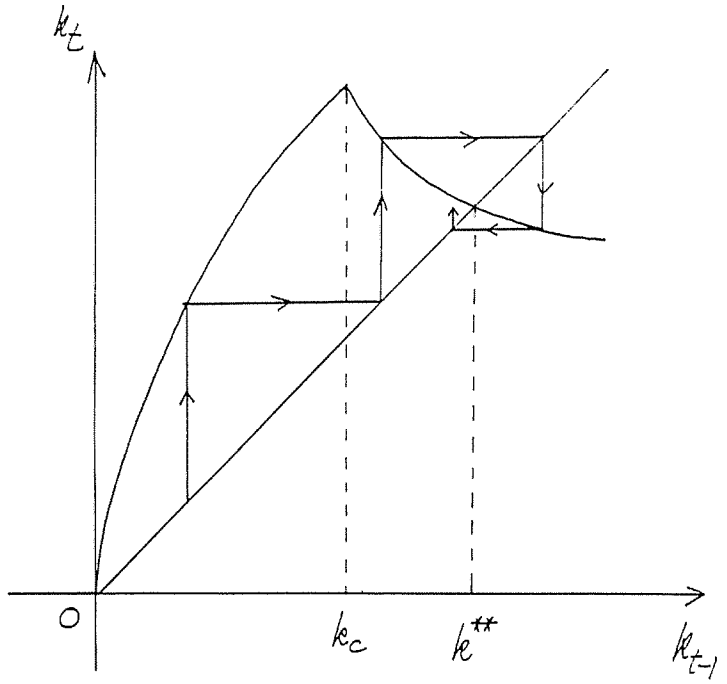



**Figure 3**



$\Phi(k_c) > k_c$. This is the case where the steady state, $k_t = k^{**}$, is a balanced growth path. Two figures differ in the local stability of the steady state, determined by the slope of the map at the steady state, $k^{**}$,

$$\frac{d\Phi}{dz}(k^{**}) = \frac{1-\theta}{G} < 0 .$$

Figure 2 shows the case of $1 < G < \theta - 1$, the case where the steady state is locally unstable. The interval, $[\Phi^2(k_c), \Phi(k_c)]$ represents the trapping region, i.e., the region which the economy enters eventually, and once entered, it will never leave. Some algebra can show that the condition, $G < \theta - 1$, is equivalent to $\Phi^2(k_c) < k_c$, which is to say that the trapping region covers both the Solow and Romer regimes. In the Solow regime, there is no innovation, and the economy grows solely by capital accumulation. If started with a small $k_0$, the economy may stay in the Solow regime for many periods, but it eventually accumulates enough capital to enter the Romer regime, and innovation begins. This way, the economy starts sustainable growth through cycles, by bouncing back and forth between the Romer and Solow regimes.

In Figure 3, $G > \theta - 1$ holds, and hence the slope of the map at $k = k^{**}$ is less than one, and $k_c < \Phi^2(k_c)$. As in the previous case, the economy may initially stay in the Solow regime for many periods, but it eventually enters the Romer regime. Then, the economy stays forever in the Romer regime and oscillates around and converges toward the steady state, or a balanced growth path. This is the case where the steady state is globally stable.

Thus, the graphical analysis suggests that the dynamics of $k_t$, given by the one-dimensional map, $k_t = \Phi(k_{t-1})$, has the three distinct asymptotic behaviors, depending on parameter values. The following proposition states it more

formally.[3]

Proposition 2.

Let the growth potential of the OG economy be defined by $G \equiv sA/\sigma$.

i) If $G < 1$, then, for any $k_0 \in R_+$, $\{k_t;\ t \in T\} \subset (0,k_c]$ and $\lim_{t\to\infty} k_t = k^*$. That is, the economy immediately settles down to the Solow regime and converges to a neoclassical stationary path.

ii) if $1 < G < \theta - 1$, there are period-2 cycles; $k_t$ fluctuates forever between the Solow and Romer regimes, for $k_0 \in R_+\backslash D$, where D is at most countable subset of $R_+$. That is, the economy almost surely moves back and forth between the Solow and Romer regimes.

iii) Suppose $G > \theta - 1$. Then, for any $k_0 \in R_+$, there exists a $t'$ such that $\{k_t;\ t \geq t'\} \subset [k_c,\ \Phi(k_c)]$ and $\lim_{t\to\infty} k_t = k^{**}$. That is, the economy eventually settles down to the Romer regime, and then oscillates around and eventually converges to a balanced growth path.

Proof.

i). The graphical analysis would suffice.

ii). First, in order to show the existence of period-2 cycles, it suffices to show that $H(k) \equiv \Phi^2(k) - k = 0$ has a solution other than $k = k^{**}$. Since $[\Phi^2(k_c),\Phi(k_c)]$ is the trapping region, $H(\Phi^2(k_c)) = \Phi^4(k_c) - \Phi^2(k_c) \geq 0$ and $H(k_c) = \Phi^2(k_c) - k_c < 0$, hence $H(k) = 0$ has a solution in $[\Phi^2(k),k_c]$. This proves the existence of period-2 cycles. Next, suppose that $\Phi^t(k_0)$ converges. Let the

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[3]The proposition ignores the two non-generic cases: $G = 1$, and $G = \theta-1$. The former case is similar to case i). In the latter case, there are a continuum of period-2 cycles, and for any initial condition, the economy converges to one of the period-2 cycles.

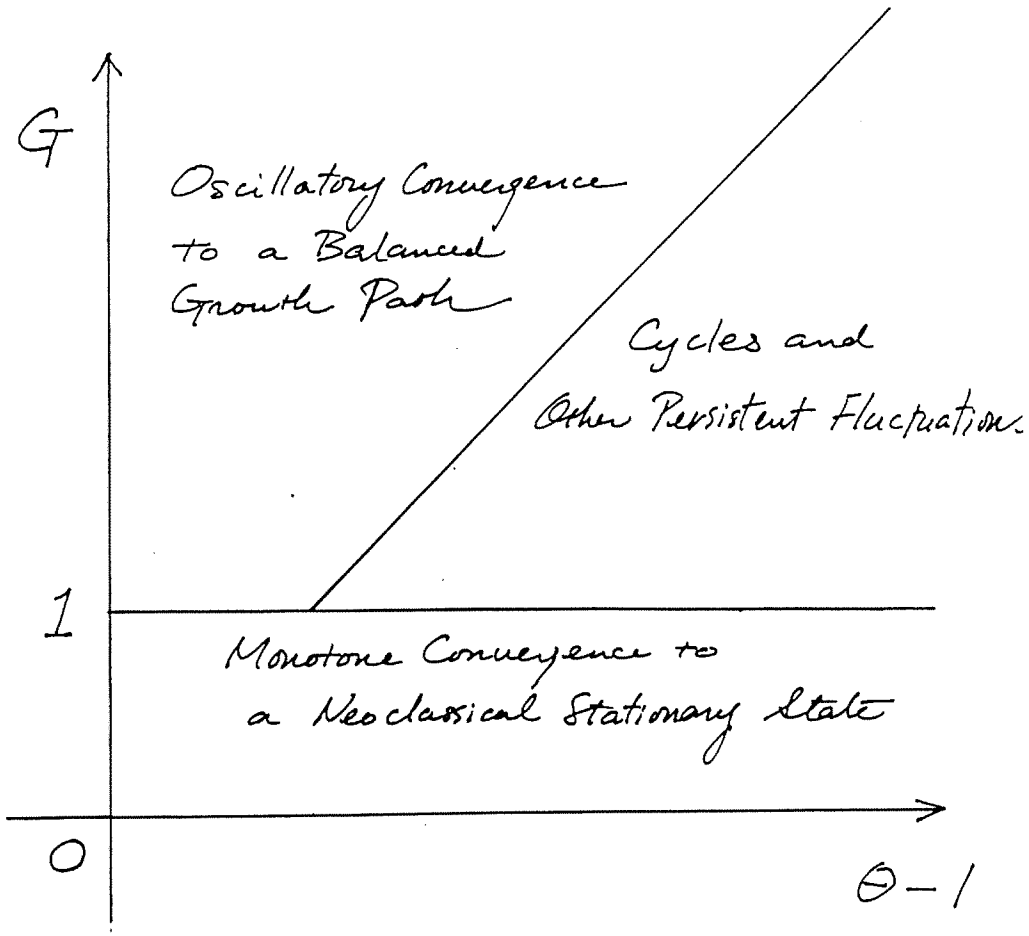
limit point be denoted by $k^{\infty}$. Then, from the continuity of $\Phi$, $k^{\infty} = \lim_{t\to\infty}\Phi^{t+1}(k_0) = \Phi(\lim_{t\to\infty}\Phi^{t}(k_0)) = \Phi(k^{\infty})$, therefore $k^{\infty} = k^{**}$. Let $D \equiv \{k_0 \in R_+ \mid \lim_{t\to\infty}\Phi^{t}(k_0) = k^{**}\}$. Since $k^{**}$ is locally unstable, $\Phi^{t}(k_0)$ cannot approach it asymptotically. Hence, $D = \{k_0 \in R_+ \mid \Phi^{t'}(k_0) = k^{**}$ for a finite $t'\}$, which is at most countable, since the uni-modality of $\Phi$ implies that, for any k, there are at most two x's that solve $k = \Phi(x)$. Hence, for almost all initial conditions, $k_t$ indefinitely fluctuates. The graphical analysis would suffice that the economy cannot be trapped in the Solow regime. To prove that the economy cannot be trapped in the Romer regime unless $k_t \neq k^{**}$, note that, as long as the economy stays in the Romer regime, the dynamics can be described by $1/k_{t+s} - 1/k^{**} = \{(1-\theta)/G\}^{s}[1/k_t - 1/k^{**}]$. Hence, $|(1-\theta)/G| > 1$ implies that, after a finite number of periods, $k_{t+s} < k_c$.

iii) First, for any $k_0 \in R_+$, there exists a $t'$, such that $k_{t'} \in [k_c, \Phi(k_c)]$. Then, $k_c < \Phi^2(k_c)$ implies that $k_{t'+1} \in [\Phi^2(k_c), \Phi(k_c)] \subset [k_c, \Phi(k_c)]$. Hence, for all $t \geq t'$, $k_t \in [k_c, \Phi(k_c)]$. Hence, from $|(1-\theta)/G| < 1$, $1/k_t - 1/k^{**} = \{(1-\theta)/G\}^{t-t'}[1/k_{t'} - 1/k^{**}] \to 0$, as $t \to \infty$. Q.E.D.

Remark. Some readers may wonder whether the equilibrium dynamics of this model is chaotic if $1 < G < \theta - 1$.[4] It is straightforward to show that the dynamical system is not chaotic in the sense of Li-York, by demonstrating the non-existence of period-3 cycles. For this, it suffices to show that $\Phi^3(k_c) > k_c$. This inequality takes the form

[4]See Grandmont (1986) for an accessible review of mathematics of chaos, and Boldrin and Woodford (1990) and Guesnerie and Woodford (1992) for overviews of chaotic dynamics in the economics literature.

Figure 4


G
Oscillatory Convergence
to a Balanced
Growth Path
Cycles and
Other Persistent Fluctuations
1
Monotone Convergence to
a Neoclassical Stationary State
0
θ − 1

$$h(G) \equiv G^{2+\frac{\sigma}{\sigma-1}} - 1 - \left[1-\frac{1}{\sigma}\right]^{1-\sigma}(G-1) > 0 ,$$

Since h is convex and h(1) = 0, h(G) > 0 for all G > 1 follows from

$$h'(1) \equiv 2 + \frac{\sigma}{\sigma-1} - \left[1-\frac{1}{\sigma}\right]^{1-\sigma} > 3 - e > 0$$

where e = 2.71828.... This does not rule out, however, the possibility that some trajectories are chaotic; that is to say, for some initial conditions, the economy indefinitely fluctuates, and yet may never converge to any cycles. To rule out such a possibility, one needs to show that all the cycles have the period length of a power of 2, a property that is difficult to demonstrate analytically. Another difficulty is that the map, $\Phi$, does not belong to any class of the functions studied in mathematics. For example, its Schwartzian derivative is <u>not</u> negative, which means, among other things, that the iteration of the critical point, $\Phi^t(k_c)$, may fail to converge to stable cycles, even if they exist.[5]

Proposition 2 is depicted in Figure 4. Although G is not a primitive parameter of the model, s or A can affect the property of the dynamics only through their effects on G, so that we directly look at the effect of a change in G. For any $\theta$, a sufficiently large value of G leads to an oscillatory convergence to a balanced growth path, and a sufficiently small value of G leads to a monotone convergence to a neoclassical stationary path. When $\theta > 2$, there is an intermediate range of G, for which the economy fluctuates for almost all initial conditions. Although the convergence to a balanced growth path is

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[5]The author would like to thank Clark Robinson for his lecture on these subtleties of the chaotic dynamics.

possible (with zero probability), this is an unstable situation and cannot be observed with occasional perturbations to the system.

To understand the mechanics of cycle-generating processes, let us consider the following thought experiment. Initially, the parameter satisfies $G > \theta - 1$, and the economy is in a balanced growth rate. Then, all of a sudden, there is a decline in G caused by, say, a decline in the saving rate, s. As G becomes smaller than $\theta - 1$, the balanced growth path loses its stability. This bifurcation generates stable cycles of period-2. According to the simulation exercises by the author, the period-2 cycles remain stable for a wide range of parameter values, but, as G is made even smaller, period-2 cycles eventually lose their stability and this bifurcation leads to stable cycles of period-4. A further decline in G causes another bifurcation to generate stable period-8 cycles, and then stable period-16 cycles. However, it seems that G must reach 1 and that the cycles should be replaced by a stable stationary state, before such a series of period-doubling bifurcations would lead to a chaos.

Figure 4 is also useful for thinking about the effect of policies designed to affect the growth potential of the economy. For example, suppose that $G < 1$ initially, and the government adopts some policies to increase G, say, a wage subsidy, financed by a lump-sum tax. If such a measure leads to $1 < G < \theta - 1$, then it generates sustainable growth, but also generate persistent fluctuations. And even when such a pro-growth policy is effective enough to make G greater than $\theta - 1$, the convergence to a balanced growth path is oscillatory, and the economy will experience fluctuations for long time.

The instability of the balanced growth path and the emergence of cyclical behavior are due to the complementarity in the <u>timing</u> of entry/innovation decisions. The timing matters in this model, because innovators could enjoy only

a temporary monopoly power. Innovations take place only when the market for a new product is sufficiently large that the innovator can reach the break-even level of the output. The size of the market partially depends on how the products with which it competes are priced. If the innovator enters when other firms also enter, some of the products are monopolistically priced. If it enters in the following period, then these products become competitively priced, as their innovators lose the monopoly power. This consideration gives an incentive for firms to enter when other firms also enter. This effect is stronger when different products are highly substitutable, i.e., when $\theta$ is high. At the same time, a growing resource base gives an offsetting force of spreading innovative and entry activities, whose effect is stronger when G is high. When the former effect dominates the latter, there is a bunching of the entry activity, and the economy moves back and forth between the Romer regime (the period of innovation) and the Solow regime (the period of no innovation).

3.C. Properties of the Period-2 Cycles.

To understand further the nature of cyclical behaviors, let us focus on the period-2 cycles, in which the economy alternates between the Solow regime, $k^L < 1$, and the Romer regime, $k^H > 1$, where $k^L$ and $k^H$ are determined jointly by

$$k^H = \Phi(k^L) = G(k^L)^{1-\frac{1}{\sigma}} \qquad \textbf{(10a)}$$

and

$$k^L = \Phi(k^H) = \frac{Gk^H}{1+\theta(k^H-1)} \qquad \textbf{(10b)}$$

Proposition 2 states that the period-2 cycles exist if and only $1 < G < \theta - 1$. And they are stable for a wide range of parameter values in this region.

Let us consider how the growth rate of the key variables, such as the innovation rate, N, the capital stock, K, and the output, Y, change over the cycles. If the economy is in the Solow regime in period t along the period-2 cycles, $k_{t-2} = k^H$, $k_{t-1} = k^L$, and $k_t = k^H$. From eq. (7),

$$\frac{N_t}{N_{t-1}} = 1 \quad ; \quad \frac{N_{t+1}}{N_t} = 1 + \theta\,(k^H - 1) \tag{11a}$$

Hence,

$$\frac{K_t}{K_{t-1}} = \frac{k^H}{k^L}\frac{N_t}{N_{t-1}} = \frac{k^H}{k^L} \quad ; \quad \frac{K_{t+1}}{K_t} = \frac{k^L}{k^H}\frac{N_{t+1}}{N_t} = \frac{k^L}{k^H}\{1 + \theta\,(k^H - 1)\} \tag{11b}$$

From eq. (8),

$$\begin{aligned} \frac{Y_t}{Y_{t-1}} &= \frac{A(k^L)^{-\frac{1}{\sigma}}K_{t-1}}{AK_{t-2}} = \frac{(k^L)^{1-\frac{1}{\sigma}}}{k^H}\{1 + \theta\,(k^H - 1)\} \quad ; \\ \frac{Y_{t+1}}{Y_t} &= \frac{AK_t}{A(k^L)^{-\frac{1}{\sigma}}K_{t-1}} = \frac{k^H}{(k^L)^{1-\frac{1}{\sigma}}} \end{aligned} \tag{11c}$$

Inserting eqs. (10a)-(10b) into eqs. (11a)-(11c) verifies that, in the Solow regime,

$$g_N = 1 \quad ; \quad g_K = g_Y = G(k^L)^{-\frac{1}{\sigma}} > G$$

and, in the Romer regime,

$$g_N = 1 + \theta(k^H - 1) \quad ; \quad g_K = g_Y = G$$

where $g_X$ denotes the (gross) growth rate of variable X. The average growth rate over the cycles are hence given by

$$g_N = g_K = g_Y = \{1 + \theta\,(k^H - 1)\}^{1/2} = G(k^L)^{-\frac{1}{2\sigma}} > G$$

Hence, we can conclude

Proposition 3.

Suppose that $1 < G < \theta - 1$. Along the period-2 cycles, both the output and the investment grow faster in the Solow regime (the period of no innovation) than in the Romer regime (the period of innovation). The average growth rate of the economy over the period-2 cycles exceeds G, the growth rate along the balanced growth path.

The first half of Propostion 3 states that, although the innovation of new goods is a crucial way of avoiding the diminishing returns and of sustaining growth indefinitely, the economy actually experiences a lower growth in the output and in the investment during the period of innovation. Only after the innovation stops, and when the market structure becomes competitive, the economy enjoys benefits of the innovation. The second half states that the cycles are indeed growth enhancing; they allow the economy to grow even faster than along the balanced growth path.

The assumption of the overlapping generations of consumers with Cobb-Douglas preferences enabled us to describe the equilibrium dynamics in a one-dimensional system, whose global properties can be examined in detail. But, it has some drawbacks. The period length can be interpreted in two different ways: the duration of the monopoly power enjoyed by the innovators, and the duration of the working life. Some readers might also suspect that the overlapping generations, rather than the temporary nature of monopoly, may be responsible for the emergence of cycles. To respond such a criticism, the next section turns to the model of the representative agent consumer, and demonstrates that the assumption of overlapping generations is not critical for the results obtained in this section.

4. The Representative Agent (RA) Economy.

The economy is populated by the infinitely lived representative consumer. The consumer supplies labor, L = 1, inelastically and choose the optimal consumption path that maximizes the discounted utility defined by

$$U = \sum_{t=1}^{\infty} \beta^t \ln(C_t) \qquad (\beta < 1)$$

subject to the initial asset holding, $K_0$, the flow budget constraint,

$$K_t = w_t L + r_t K_{t-1} + (1-\delta) K_{t-1} - C_t$$

and the intertemporal solvency condition, which rules out a Ponzi-scheme,

$$\lim_{T\to\infty} \frac{K_T}{\prod_{t=1}^{T} (1-\delta+r_t)} \geq 0$$

Recall that the only asset that the consumer holds across periods is the capital stock.

As is well-known, the optimal consumption path is characterized by the Euler equation

$$\frac{C_{t+1}}{C_t} = \beta(1-\delta+r_{t+1}) \tag{12}$$

and the binding intertemporal solvency condition

$$\lim_{T\to\infty} \frac{K_T}{\prod_{t=1}^{T} (1-\delta+r_t)} = \lim_{T\to\infty} \beta^T \frac{K_T}{C_T} = 0 \tag{13}$$

In equilibrium, $Y_t = w_t L + r_t K_{t-1}$ and $r_t K_{t-1} = (1-1/\sigma) Y_t$ hold. Hence, using the flow budget constraint, eqs. (7) and (8), eq. (12) can be written to

$$\beta\left[1-\delta+\left(1-\frac{1}{\sigma}\right)A\phi(k_t)\right] = \frac{(1-\delta+A\phi(k_t))k_t - k_{t+1}\psi(k_t)}{(1-\delta+A\phi(k_{t-1}))k_{t-1} - k_t\psi(k_{t-1})}\psi(k_{t-1}) \tag{14}$$

and eq. (13) becomes

$$\lim_{T\to\infty} \beta^T \frac{k_T \psi(k_{T-1})}{(1-\delta+A\phi(k_{T-1}))k_{T-1}-k_T\psi(k_{T-1})} = 0 \qquad (15)$$

where

$$\phi(k) \equiv Max\{k^{-\frac{1}{\sigma}}, 1\} \; ; \; \psi(k) \equiv Max\{1, 1+\theta(k-1)\} \; .$$

For a given initial condition, $k_0$, the equilibrium of the economy is a trajectory, $\{k_t;\ t \in T\}$, which satisfies eqs. (14) and (15). Note that any bounded trajectory satisfies eq. (15), so that only eq. (14) needs to be checked to see whether a bounded trajectory is an equilibrium.

The equilibrium dynamics of the RA economy differs in many ways from that of the OG economy. First, it depends on the physical depreciation rate of the capital stock, $\delta$. Second, the dynamics of k is now described by a two-dimensional system, as eq. (14) determines $(k_t, k_{t+1})$ as a function of $(k_{t-1}, k_t)$. Third, the equilibrium may not be unique. In spite of these differences, however, much of the results obtained for the OG economy carries over to the RA economy, in a slightly weaker form.

4.A. <u>The Steady State</u>.

Let us now look at the steady state of the economy, and find out the appropriate definition of the growth potential for the RA economy. Since any steady state satisfies eq. (15), one needs to check only eq. (14). First, suppose that a steady state is in the Solow regime, $k_t = k^* \leq 1$. Eq. (14) becomes

$$\beta\left[1-\delta+\left(1-\frac{1}{\sigma}\right)A(k^*)^{-\frac{1}{\sigma}}\right] = 1 \geq \beta\left[1-\delta+\left(1-\frac{1}{\sigma}\right)A\right]$$

Now, suppose that a steady state is in the Romer regime, $k_t = k^{**} > 1$. Then,

from (7) and (14), the economy is growing at the rate equal to

$$\frac{K_t}{K_{t-1}} = \frac{N_t}{N_{t-1}} = 1 + \theta(k^{**} - 1) = \beta\left[1 - \delta + \left(1 - \frac{1}{\sigma}\right)A\right] > 1$$

The above argument can be summarized as follows:

Proposition 4.

Let the growth potential of the RA economy be defined by

$$G \equiv \beta\left[1 - \delta + \left(1 - \frac{1}{\sigma}\right)A\right]$$

Then,

i) if $G \leq 1$, the economy is stationary in all the steady state paths. They are given by $\{K_t, N_t\} = \{k^*\Lambda, \Lambda\}$ for any $\Lambda > 0$, where $k^* \leq 1$ is defined by

$$\beta\left[1 - \delta + \left(1 - \frac{1}{\sigma}\right)A(k^*)^{-\frac{1}{\sigma}}\right] = 1$$

ii) if $G > 1$, the economy grows at the same rate in all the steady states paths. They are given by $\{K_t, N_t\} = \{k^{**}\Lambda G^t, \Lambda G^t\}$ for any $\Lambda > 0$, where $k^{**}$ is defined by

$$k^{**} \equiv 1 + \frac{G - 1}{\theta} > 1$$

Just as Proposition 1 for the case of the OG economy, the content of Proposition 4 is twofold. First, it shows the appropriate definition of the growth potential for the RA economy. Second, it states that a steady state value of k is uniquely determined.

4.B. The Local Stability of the Steady State.

Now, let us study the local stability of the unique steady state. That is, when the initial condition is sufficiently close to the steady state, is there an equilibrium path, along which the economy stays close to the steady state and converges to it?

Let us first suppose $G < 1$, or $k^* < 1$, so that the stationary state is in the interior of the Solow regime. Then, in a neighborhood of the stationary state, $\psi(k) = 1$ and hence, eq. (14) becomes

$$k_{t+1} = (1-\delta+A\phi(k_t))k_t + \beta\left[1-\delta+\left(1-\frac{1}{\sigma}\right)A\phi(k_t)\right]\{k_t-(1-\delta+A\phi(k_{t-1}))k_{t-1}\}$$

Linearizing around $k^*$ yields the following second-order difference equation in $Dk_t \equiv k_t - k^*$:

$$Dk_{t+1} - \left\{\frac{\beta}{\sigma-1}\left[\frac{1}{\beta}-(1-\delta)\right]^2+\frac{1}{\beta}+1\right\}Dk_t + \frac{1}{\beta}Dk_{t-1} = 0$$

The two characteristic roots of this equation are both positive: one of them is greater than one, and the other smaller than one. In other words, $k^*$ has the one-dimensional locally stable manifold. If the initial condition, $k_0$ is in a neighborhood, there exists a trajectory, $\{k_t;\ t \in T\}$, that stay in the neighborhood and converges to $k^*$ monotonically.

Now, suppose $G > 1$. In a neighborhood of $k^{**}$, $\phi(k) = 1$ and $\psi(k) = 1 + \theta(k-1)$, so that eq. (14) becomes

$$k_{t+1} = \left[(1-\delta+A+G)k_t - \frac{(1-\delta+A)Gk_{t-1}}{1+\theta(k_{t-1}-1)}\right]\frac{1}{1+\theta(k_t-1)}$$

Linearization around $k^{**}$ yields the second-order difference equation in $Dk_t \equiv k_t - k^{**}$:

$$Dk_{t+1} - \frac{(1-\delta+A)+(1-\theta)}{G}Dk_t + \frac{(1-\delta+A)(1-\theta)}{G^2}Dk_{t-1} = 0$$

which has one positive and one negative characteristic roots,

$$\frac{1-\delta+A}{G} > 1 \; ; \; \frac{1-\theta}{G} < 0$$

If $1 < G < \theta - 1$, the absolute values of the two roots are both greater than one, hence there exists no locally stable manifold around $k^{**}$. (i.e., in other words, $k^{**}$ is a source). Hence, the equilibrium dynamics, which starts in a neighborhood, will not stay in the neighborhood. If $\theta - 1 < G$, then the negative root has the absolute value smaller than one, while the positive root is greater than one. In other words, $k^{**}$ has a one-dimensional locally stable manifold, and if the initial condition, $k_0$ is in a neighborhood, there exists a trajectory, $\{k_t;\ t \in T\}$, that stays in the neighborhood and converges to $k^{**}$ oscillatorily.

By virtue of the Local Manifold Theorem (see, e.g., Guckenheimer and Holmes (1983, p.16)), one can translate the above findings into the following form.

Proposition 5.

Let the growth potential of the RA economy be defined by

$$G \equiv \beta\left[1-\delta+\left(1-\frac{1}{\sigma}\right)A\right]$$

i) If $G < 1$, the neoclassical stationary state, $k^*$, is locally stable in that there exists a neighborhood of $k^*$, U, such that, if $k_0 \in U$, there exists an equilibrium, whose entire trajectory stays in U, and along which the economy converges monotonically to $k^*$; $\lim_{t\to\infty} k_t = k^*$.

ii) If $1 < G < \theta - 1$, the balanced growth path, $k^{**}$, is locally unstable in that there exists a neighborhood of $k^{**}$, U, such that, if $k_0 \in U$, there exists some t such that $k_t \notin U$, along any equilibrium path. That is, when the economy starts close to the balanced growth path, it will move away

from it.

iii) If $\theta - 1 < G$, the balanced growth path, $k^{**}$, is locally stable in that there exists a neighborhood of $k^{**}$, U, such that if $k_0 \in U$, there exists an equilibrium path, whose entire trajectory stays in U, and along which the economy converges oscillatorily to $k^{**}$; $\lim_{t\to\infty} k_t = k^{**}$.

Proposition 5 states that, once the growth potential, G, is appropriately redefined, the condition for the local stability of the steady state in the RA economy takes the same form as in the OG economy, given in Proposition 2. Unlike Proposition 2, however, Proposition 5 deals only with the local dynamics, so that it does not tell us whether the neoclassical stationary state is globally stable when $G < 1$, nor whether the balanced growth is globally stable when $\theta - 1 < G$. For the case of $1 < G < \theta - 1$, however, the following statement about the global dynamics can be made as a direct corollary of Proposition 5.

Corollary. If $1 < G < \theta - 1$, the economy fluctuates forever, for $k_0 \in R_+ \backslash D$, where D is at most countable subset of $R_+$.

Proof. The two-dimensional dynamical system, defined by eq. (14), is continuous, and its unique steady state, $(k^{**}, k^{**})$, is a source. Hence, if the economy converges along the equilibrium path, $(k_0, k_1)$ must be mapped into the steady state, $(k^{**}, k^{**})$, in a finite time, from which the result follows. Q.E.D.

4.C. Period-2 Cycles.

Characterizing the equilibrium dynamics for an arbitrary initial condition for the RA economy is beyond the scope of this paper. Nevertheless, one can

obtain some ideas about the global equilibrium dynamics, by studying the period-2 cycles, whose existence is demonstrated below.

Suppose that the economy alternates between $k^L < 1$ and $k^H > 1$. Setting $k_{t-1} = k^L$, $k_t = k^H$, and $k_{t+1} = k^L$ in eq. (14) yields

$$G = \frac{(1-\delta+A)k^H - k^L\psi(k^H)}{(1-\delta+A\phi(k^L))k^L - k^H} \quad . \tag{16a}$$

Likewise, setting $k_{t-1} = k^H$, $k_t = k^L$, and $k_{t+1} = k^H$ in eq. (14) yields

$$\beta\left[1-\delta+\left(1-\frac{1}{\sigma}\right)A\phi(k^L)\right] = \frac{(1-\delta+A\phi(k^L))k^L - k^H}{(1-\delta+A)k^H - k^L\psi(k^H)}\psi(k^H) \quad . \tag{16b}$$

Since eq. (15) holds automatically along such a path, the period-2 cycles exist if eqs. (16a)-(16b) have a solution, $k^L < 1$ and $k^H > 1$.

The following proposition states that some parts of Proposition 3 can be extended to the RA economy.

Proposition 6.

Let the growth potential of the RA economy be defined by

$$G \equiv \beta\left[1-\delta+\left(1-\frac{1}{\sigma}\right)A\right]$$

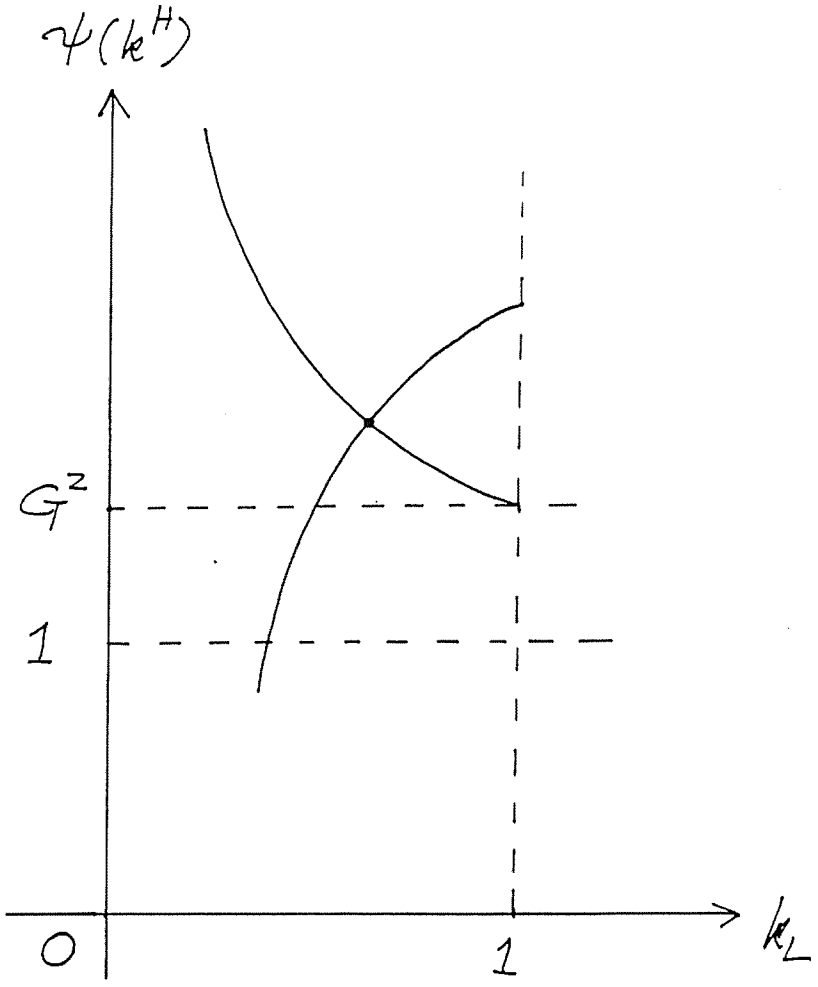
then, the period-2 cycles, characterized by eqs. (16a)-(16b), exist if and only if $1 < G < \theta - 1$. Furthermore, the average growth rate of the economy over the period-2 cycles exceeds G, the growth rate along the balanced growth path.

Proof. Since

$$\phi(k^L) = (k^L)^{-\frac{1}{\sigma}} \quad ; \quad \psi(k^H) = 1 + \theta(k^H - 1)$$

eqs. (16a)-(16b) are equivalent to

**Figure 5**



$$\psi(k^H) = \frac{\theta G\left\{(1-\delta)k^L + A(k^L)^{1-\frac{1}{\sigma}}\right\} + (1-\theta)(1-\delta+A+G)}{1-\delta+A+G-\theta k^L} \qquad (17a)$$

and

$$\psi(k^H) = G\beta\left[1-\delta+\left(1-\frac{1}{\sigma}\right)A(k^L)^{-\frac{1}{\sigma}}\right] . \qquad (17b)$$

As depicted in Figure 5, the RHS of eqs. (17a) is increasing in $k^L$, while the RHS of eq. (17b) is decreasing in $k^L$ and goes to infinity as $k^L \to 0$. Hence, the two curves, intersects in the relevant range, if and only if the RHS of eq. (17b), when evaluated at $k^L = 1$, is greater than one and less than the RHS of eq. (17a), evaluated at $k^L = 1$. In other words, the period-2 cycles exist if and only if

$$1 < G^2 < \frac{\theta G(1-\delta+A)+(1-\theta)(1-\delta+A+G)}{1-\delta+A+G-\theta} = G^2 + \frac{(1-\delta+A+G)(G-1)(\theta-1-G)}{1-\delta+A+G-\theta}$$

which is equivalent to $1 < G < \theta - 1$.

Since the growth rate in N is equal to one in the Solow regime and it is equal to $\psi(k^H)$ in the Romer regime, the average growth rate of the economy over the cycles satisfies

$$g_N = g_K = g_Y = (\psi(k^H))^{\frac{1}{2}} > G$$

as depicted in Figure 5. Q.E.D.

Remark: Proposition 4 through 6 can be extended for general intertemporally homothetic preferences of the form:

$$U = \sum_{t=1}^{\infty} \beta^t \frac{(C_t)^{1-1/\gamma} - 1}{1-1/\gamma}$$

if the growth potential is redefined as

$$G \equiv \left\{\beta\left[1-\delta+\left(1-\frac{1}{\sigma}\right)A\right]\right\}^{\gamma}$$

as long as the equilibrium exist. Allowing for such general preferences, however, causes a technical difficulty of ensuring the existence of equilibrium. This is because, when $\gamma \neq 1$, a bounded sequence, $\{k_t;\ t \in T\}$, does not necessarily satisfy the intertemporal solvency condition. To ensure the existence, one need to impose an additional restriction, which takes the form that A cannot be too large for a given $\gamma$, or that $\gamma$ cannot be too large for a given A. Assuming $\gamma = 1$, or equivalently Cobb-Douglas preferences, would suffice for this purpose. It also has an advantage of simplifying the exposition drastically.

To see whether the rest of Proposition 3 can be extended for the RA economy, note that eqs. (11a)-(11c) are also applicable to the RA economy. Hence, one need to demonstrate the following inequalities

$$\left[\frac{k^H}{k^L}\right]^2 > 1+\theta\left(k^H-1\right) > \left[\frac{k^H}{\left(k^L\right)^{1-\frac{1}{\sigma}}}\right]^2$$

for $k^L$ and $k^H$ that solve eqs. (16a)-(16b), or equivalently, eqs. (17a)-(17b), in order to show that both the output and the investment grow faster in the Solow regime than in the Romer regime along the period-2 cycles. For each set of parameter values we examined, the above inequalities turn out to hold. Hence, it appears that the rest of Proposition 3 carries over to the RA economy, as well. However, we are able to <u>prove</u> it only for a special case, in spite of our best efforts.

<u>Proposition 7</u>.

Let $\delta = 1$. Then, along the period-2 cycles, which exist if and only if $1 < G < \theta - 1$, both the output and the investment grow faster in the Solow regime (the period of no innovation) than in the Romer regime (the period of innovation).

Proof. When $\delta = 1$, the growth potential is equal to

$$G \equiv \beta\left(1-\frac{1}{\sigma}\right)A$$

and one can verify that a pair, $k^L < 1$ and $k^H > 1$, defined by

$$k^H = \Phi(k^L) = G(k^L)^{1-\frac{1}{\sigma}}$$

and

$$k^L = \Phi(k^H) = \frac{Gk^H}{1 + \theta(k^H-1)}$$

solves eqs. (16a)-(16b). Inserting these expressions into eqs. (11a)-(11c) verifies that, in the Solow regime,

$$g_N = 1 \quad ; \quad g_K = g_Y = G(k^L)^{-\frac{1}{\sigma}} > G$$

and, in the Romer regime,

$$g_N = 1 + \theta(k^H-1) \quad ; \quad g_K = g_Y = G$$

The result follows from $k^L < 1$ and $k^H > 1$. Q.E.D.[6]

In summary, this section has demonstrated that much of the results obtained in the OG economy indeed carries over to the RA economy, once the parameter G is

[6]The proof makes use of the fact that the two equations, which characterizes the period-2 cycles in the RA economy for $\delta = 1$, have exactly the same form with eqs. (10a)-(10b), which characterizes the period-2 cycles in the OG economy. It is not clear, however, if there is any significance to this equivalence.

appropriately redefined. In particular, Figure 4 is applied to the RA economy, without any modification.

## 5. Empirical Plausibility of Cycles.

Only a global analysis can demonstrate the possibility that the economy experiences a persistent growth by moving back and forth between the Solow regime and the Romer regime. And a rigorous and explicit analysis of the global equilibrium dynamics is possible only when the dimensionality of a dynamical system is kept sufficiently low. Much of the restrictions imposed in the models were chosen due to the need to keep the models tractable, rather than due to their realism. Hence, these restrictions may well be rejected by the data. Nor should one expect the two models developed above to be capable of generating the equilibrium behavior that closely mimics the time series observed in the real world, due the low dimensionality of the systems. Nevertheless, one could still gauge the empirical plausibility of the condition for cycles, $1 < G < \theta - 1$.

To answer this question, let $G = (1+g)^{\tau}$, where $g$ and $\tau$ represent the annual rate of the growth potential, and the period length in years, respectively. Then, the condition can be rewritten as

$$0 < g < (\theta-1)^{\frac{1}{\tau}} - 1 .$$

Table 1 shows the upper bound of this range, for some values of $\theta$ and $\tau$. To consider a reasonable range for $\theta$, recall that $\sigma$ plays two distinctive roles in this model. First, $1-1/\sigma$ is the capital share in GNP. It is now the standard practice in the growth literature to interpret capital to include not only physical capital, but also human capital, because the capital stock in growth models represent a broad category of factors, which can be accumulated to enhance

TABLE 1:

(theta-1)^(1/tau)-1

| | 2.40 | 2.45 | 2.50 | 2.55 | 2.60 | 2.65 | 2.70 |
|---|---|---|---|---|---|---|---|
| 3 | 11.87% | 13.19% | 14.47% | 15.73% | 16.96% | 18.17% | 19.35% |
| 6 | 5.77% | 6.39% | 6.99% | 7.58% | 8.15% | 8.70% | 9.25% |
| 9 | 3.81% | 4.21% | 4.61% | 4.99% | 5.36% | 5.72% | 6.07% |
| 12 | 2.84% | 3.14% | 3.44% | 3.72% | 3.99% | 4.26% | 4.52% |
| 15 | 2.27% | 2.51% | 2.74% | 2.96% | 3.18% | 3.39% | 3.60% |
| 18 | 1.89% | 2.09% | 2.28% | 2.46% | 2.65% | 2.82% | 2.99% |

our income in future. The standard estimates for the share of human capital is approximately one-half. To this, if we add one-third for physical capital, a plausible value for $\sigma$ would be around six, or $\theta \fallingdotseq 2.49$. Second, $\sigma$ represents the price elasticity, and the monopoly margin enjoyed by the innovator of a new product is given by $1/(\sigma-1)$. Conservatively, this suggests that $\sigma$ should be in the range from five to twenty, or that $2.44 < \theta < 2.65$. In Table 1, $\theta$ ranges from 2.40 to 2.70. It is much harder to come up with a tight range for $\tau$, the period length, which represents the duration of monopoly power enjoyed by the innovator of a new product. It all depends on what sort of products and what sort of industries one may have in mind. The agnosticism leads us to use a wide range of $\tau$ in Table 1, from 3 years to 18 years. Table 1 hence suggests that, for a reasonable annual rate of the growth potential, it is quite plausible that the condition for cycles would be satisfied.

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**Institut für Höhere Studien**
**Institute for Advanced Studies**
Stumpergasse 56
A-1060 Vienna
Austria

Phone: +43-1-599 91-145
Fax: +43-1-599 91-163