

Usher, Dan

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# Duty, Self-interest, and the Chance of Casting a Pivotal Vote

Dan Usher  
Queen's University

Department of Economics  
Queen's University  
94 University Avenue  
Kingston, Ontario, Canada  
K7L 3N6

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## Duty, Self-interest, and the Chance of Casting a Pivotal Vote

Dan Usher  
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**Abstract:** One votes from self-interest or from a sense of duty. Voting from self-interest requires there to be some chance, however small, that one's vote swings the outcome of an election from one political party to another. This paper is a discussion of three models of what that chance might be: the *common sense* model inferring the probability of a tied vote today from the distribution of outcomes in past elections, *person-to-person randomization* where each voter looks upon the rest of the electorate as analogous to drawings from an urn with given proportions of red and blue balls, and *nation-wide randomization* where voters are lined up according to their preferences for one party or the other but where chance shifts the entire schedule of preferences up or down. Emphasis is on the third model about which the paper may have something new to say. Nation-wide randomization may be helpful in connecting private benefits from a win for one's preferred party with a duty to vote, and in comparing the pros and cons of compulsory voting.

JEL Classification: D72

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## Duty, Self-interest, and the Chance of Casting a Pivotal Vote

“In the nineteenth century, economics was often called the “dismal science” largely because the equilibria predicted from price theory were not palatable to those who called it dismal. In what seems to me a deeper sense, however, politics is *the* dismal science because there are no fundamental equilibria to predict.”

William H Riker, 1980, page 443.

A vote is said to be pivotal if it swings the outcome of an election from one party to another. The probability of casting a pivotal vote is important for two reasons: i) When the act of voting is at all costly, the probability may be too small to provide a purely self-interested person with a sufficient incentive to vote rather than to abstain, so that a sense of duty must be invoked to explain why people vote at all and ii) Electoral may be designed or evaluated according to the distribution of “political power” among eligible voters or among legislators, where political power is interpreted as the probability of casting a pivotal vote.<sup>1</sup> This paper is about the first reason. The main contribution of this paper is a model of uncertainty about the outcomes of an election allowing for permanent preferences among eligible voters for one party over another, identifying benefits to others that voting from a sense of duty might convey and comparing benefits of voting, rather than abstaining, to oneself and to others. A secondary contribution is the discussion of voting pacts that make voting advantageous to their participants.

As formulated by Riker and Ordeshook (1968), one’s decision to vote or abstain depends upon whether or not

$$\pi B + D > C \quad (1)$$

where B is one’s personal benefit if one’s preferred party wins the election

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<sup>1</sup> Only if one’s vote might be pivotal is there any chance of influencing the outcome of an election; if one’s vote is not pivotal, it is as though one did not vote at all, Shapley and Shubik (1954, page 787) define the power of a voter as “the chance he has of being critical to the success of the winning coalition” where all arrangements of voters are assumed to be equally likely and where a voter in any particular arrangement is deemed to be pivotal when his appearance in the sequence of like-minded voters generates a majority for the party he favours. The Shapley-Shubik index of voting power can be thought of as a generalization of the probability of casting a pivotal vote in which, for example, a person who is part of a large majority is not deemed powerless despite the fact that he is never pivotal as defined here. For a review of the literature on voting power see, Felsenthal and Machover (1998).

D is the value one places upon voting as a duty to the rest of the community.<sup>2</sup>  
 C is one's cost of voting  
 and  $\pi$  is the probability of casting a pivotal vote.

An implicit premise in the formulation of equation (1) is that a tie between two competing parties is automatically broken by the flip of a coin. Then, for a person's vote to be pivotal, there must be one of two combinations of events. In the first, there is an even number of other voters divided equally between competing political parties, so that, if this person abstains, the election would be decided by the flip of a coin to break the tie. In the second, there is an odd number of other voters with a one-vote advantage for the party this person opposes, so that, if this person abstains, the party he opposes would win but, if this person votes, there is a tie that must be broken by a flip of a coin. In neither case can this person's probability of being pivotal exceed fifty percent. Since the probabilities that other voters are exactly tied or that, but for one's vote, the party one favours would lose by exactly one vote, a person's probability of being pivotal is typically very, very small. It turns out that  $\pi$  equals half the probability that other voters are tied in the event that there is an even number of other voters.<sup>3</sup>

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<sup>2</sup>That voting may not be personally advantageous was recognized by Downs (1957) and by Tullock (1967). Tullock suggested people's willingness to vote rather than to abstain might be explained by the possibility that voting is not costly at all. Voting for a political party might be like supporting a local football team or voting for the American Idol or congregating in Times Square on New Year's Eve. Incorporating this possibility converts equation (1) to

$$\pi B + D + N > C \quad (1)$$

where N is the value one attaches to participating in voting as a public event. For some purposes, N might be incorporated into D, but duty to participate and enjoyment of participation are not the same. One is about the welfare of the community, while the other is about one's own pleasure.

<sup>3</sup> Consider a person choosing whether to vote or to abstain in circumstances where it is certain that N other people will vote but uncertain what the outcome of their vote will be. The probability,  $\pi$ , of this person's vote becoming pivotal depends on how the N other voters are expected to divide up between the two political parties. Call the parties left and right, and suppose for convenience that this person favours the left party. Then,

$\pi$  = the probability of this person's vote being pivotal, turning a win by the right party into a win by the left party  
 = {(the probability that N is even)  
 x (the probability of a tie among the N sure voters)  
 x (the probability that the right party wins the toss of a coin to break the tie created when the dithering person abstains)}  
 + {(the probability that N is odd)}

To *homo economicus*, to the entirely self-interested citizen as postulated in most economic analysis, there can be no sense of duty, and voting can only be advantageous if  $\pi B > C$ , regardless of D. What has come to be called “the paradox of not voting” is the difficulty in reconciling the fact that a great many people do vote with the presumption that the probability of casting a pivotal vote is much too small to justify voting except as an altruistic gesture for the benefit of one’s social class or the community at large.

A simple example shows what is at stake. The competing political parties are called right and left. A person who happens to favour the left party is considering whether to vote or abstain, where the cost of voting is \$20 and where it is worth \$10,000 to this person for the left party to win the election. The cost of voting is the dollar value of the time and trouble required to go to the ballot box and to cast one’s vote. Compensation of \$10,000 would be required to keep this person equally well off in the event of a win for the right party as this person would be if the left party won instead. If a sense of duty played no role in this person’s decision whether to vote or to abstain and if strict self-interest were the only consideration, it would be disadvantageous for this person to vote unless the chance of his vote switching the outcome of the election - turning a tie into a win for the left party, or a loss by just one vote into a tie - exceeds 1 in 500 [The required  $\pi = C/B = 20 \div 10,000 = 1/500$ .] For most elections, this probability seems much too high. Suppose that this person is choosing to vote or abstain in a constituency where a million other people are expected to vote and where the left party is confidently expected to obtain somewhere between 45% and 55% of the votes, with equal chances of outcomes anywhere between these limits. If so, the chance of a tie between left and right parties becomes 1 in 100,000, and the probability of this person’s vote being pivotal becomes 1 in 200,000.

In this example, the chance,  $\pi$ , of the person’s vote altering the outcome of the election is no more than 1 in 200,000, but the minimal chance required to make voting advantageous to the strictly self-interested voter is 1 in 500. This person’s chance of swinging the election is only a quarter of a percent of the chance required to make voting personally advantageous. Of course,

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x (the probability that the right party wins by one vote among the N sure voters)  
 x (the probability that the left party wins the toss of a coin to break the tie created when the dithering person votes)}

Since the first and third expressions in both triplets are equal to  $\frac{1}{2}$ , and since the values of both middle expressions are the virtually the same, it follows that

$\pi =$  one half the probability that other voters are tied when the number of other voters is even.

As there is no sharp distinction between the N sure voters and the 1 dithering person, all voters are equally likely to be pivotal. Everybody’s chance of being pivotal depends upon how many other people are expected to vote and upon the probability of a tie. The size of that chance is the subject of the three models to follow.

these numbers are pulled out of thin air, but the gap between the estimated chance and the required chance is so large that it seems unlikely for any reasonable choice of numbers to induce more than a very small minority of eligible voters with enormous stakes in the outcome of the election to vote rather than to abstain. That is the paradox of not voting.

Two questions arise immediately: What is the probability of casting a pivotal vote? What exactly is the sense of duty that might induce people to vote even when it is not in their immediate self-interest to do so? There is a vast literature on both of these questions. This paper is an attempt to add a few strokes to the picture.

### **Three Models of How a Person's Vote Might Turn Out to be Pivotal**

For there to be a probability of casting a pivotal vote, there must be uncertainty in the voter's mind about the outcome of the election.<sup>4</sup> Such uncertainty can only be a reflection of uncertainty about some aspect of society. Thus, any attempt to determine, analytically or empirically, whether  $\pi$  is large enough to be accounted for voting by self-interest alone - whether  $\pi > C/B$  - must be based upon a model of how some aspect of society is unpredictable. Three such models are to be discussed and compared: the "common sense" model, "person-by-person randomization" and "nation-wide" randomization. This paper is focused upon nation-wide randomization not because it is unambiguously more realistic or more useful than the other two, but because it is the model about which the paper may have something new to say. Nation-wide randomization has special implications about the duty to vote where self-interest alone is an insufficient incentive.

**a) The Common Sense Model:** The numerical example in the preceding section - with a million voters and a range of 100,000 equally likely outcomes - is an application of the common sense model, relying on the assumptions that observed pluralities will never be exceeded and that all outcomes within the observed range are equally likely. On these assumptions, the probability of a tie is 1 in 100,000, implying that the probability of one's vote being pivotal is 1 in 200,000, and, with a cost of voting of \$20, the strictly self-interested citizen would prefer to abstain unless his valuation of a win for his preferred party exceeds four million dollars. Not many citizens' valuations would be as high as that. It is hard to cook plausible numbers for which even a significant minority of eligible voters would choose to vote rather than to abstain if pure self-interest were the only consideration.

The great advantage of the common sense model is in supplying a rough and ready way to estimate the chance that a vote turns out to be pivotal. There are several disadvantages. The assumption that the distribution of votes is uniform between well-specified limits is convenient but completely arbitrary. It is hard to tell what the appropriate limits, assumed in the example to

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<sup>4</sup>Strictly speaking, one would be sure that one's vote is pivotal if one knew for sure that the rest of the electorate is tied, but nobody can ever be expected to know that.

be 45% and 55%, might be. A more plausible distribution of outcomes might be inferred from information about the history of outcomes in past elections, a distribution with more weight in the middle and less at the tails. There is also some question as to whether outcomes in past elections are really predictive of the outcome in an election today. Regardless of the history of elections, the probability of a voter being pivotal may be relatively high in some elections and relatively low in others, high when an election is expected to be close, and low when one party is almost certain to win. Most importantly, the probability of a voter being pivotal is observed rather than derived from assumptions about voters' behaviour. The model has nothing to say about why  $\pi$  is what it is inferred to be. The model is most useful as a check on other models to be discussed below, raising suspicion whenever outcomes are too far from what the common sense model would suggest.<sup>5</sup>

### **b) Person-by-Person Randomization**

Person-by-person randomization is based upon an analogy between voting and sampling.<sup>6</sup> Think of all voters as lined up at the ballot box, and voting one by one. Just before a person casts his vote, the angel of chance assigns that person a preference, for the left party with a probability of  $p$  and for the right party with a probability  $(1 - p)$ . In an electorate of  $N$  voters, the number,  $n$ , of votes for the left party is analogous to the number of blue balls in  $N$  drawings from an urn containing blue and red balls in proportions  $p$  and  $(1 - p)$ . The distribution of  $n$  is binomial with mean  $pN$  and standard deviation  $[p(1 - p)N]^{1/2}$ . Suppose for convenience that  $N$  is an even number.

The probability,  $T$ , of a tie among the  $N$  voters is precisely the probability that the number of blue balls,  $n$ , is equal to  $N/2$ . It is the probability of picking exactly  $N/2$  blue balls in  $N$  draws from this urn, where

$$T = \alpha\beta \tag{2}$$

when  $\alpha$  = the probability that the first  $N/2$  balls are blue and the remaining  $N/2$  balls are red

$$= (p)^{N/2}(1 - p)^{N/2} \tag{3}$$

and

$$\begin{aligned} \beta &= \text{the number of ways to place } N/2 \text{ balls in } N \text{ slots} \\ &= N!/[N/2!N/2!] \end{aligned} \tag{4}$$

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<sup>5</sup>The common sense model is employed in Edlin, Gelman and Kaplan(2008) and is discussed at greater length in Gelman, Katz and Bafumi (2004)

<sup>6</sup>The first such models I know of are those of Beck(1975) and of Good and Mayer (1975).



The probability of a tie becomes

$$T = \{N! / [(N/2)!(N/2)!]\} (p)^{N/2} (1 - p)^{N/2} \quad (5)$$

which, when simplified by means of Stirling's formula <sup>7</sup>, reduces to

$$T = \{2/\Pi N\}^{1/2} [(2p)^{N/2} (2(1-p))^{N/2}] \quad (6)$$

where  $\Pi$  is the ratio of the circumference to the diameter of a circle.

Now imagine of a society with  $(N + 1)$  people, where the first  $N$  are assumed to vote in accordance with preferences as assigned by the angel of chance and the  $(N + 1)^{st}$  person is deciding whether to vote or abstain. The probability,  $\pi$ , that the vote of the  $(N + 1)^{st}$  person is pivotal is half the probability,  $T$ , of a tie among the other  $N$  people.

Suppose once again that there are a million voters, i.e.  $N = 1,000,000$ . As the binomial distribution is bell shaped, the probability of any particular  $n$  depends on its distance from the mean, highest when  $n$  is the mean, and steadily lower the farther from the mean  $n$  happens to be. Thus, the probability that  $n = N/2$  is greatest when  $N/2$  is itself the mean, that is, when  $p = 1/2$ . With  $p = 1/2$ , it follows from equation (6) that the probability of a tie between political parties - the probability  $n = N/2 = 500,000$  - is just under one tenth of 1%. With a cost of voting of \$20, the benefit from a win for one's preferred party sufficient to make voting advantageous must be at least \$32,000, which is high but not impossible.

The probability of a tie diminishes sharply when  $p$  deviates from  $1/2$ , causing the mean of the distribution of votes for the left party to deviate from  $N/2$ . Even a deviation of as little as 1% reduces the probability of a tie dramatically. Suppose the probability of a randomly-chosen person voting for the left party rises from .5 to .51, raising the mean,  $pN$ , of the distribution of  $n$  from 500,000 to 510,000, a difference of 10,000 votes. The effect of this difference upon the

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<sup>7</sup>Stirling's formula is

$$N! = (2 \Pi N)^{1/2} (N/e)^N$$

from which it follows that

$$\begin{aligned} \alpha &= N! / [(N/2)!(N/2)!] = \{(2 \Pi N)^{1/2} (N/e)^N\} / \{(2 \Pi N/2)^{1/2} (N/2e)^{N/2}\} \{(2 \Pi N/2)^{1/2} (N/2e)^{N/2}\} \\ &= [ \{(2 \Pi N)^{1/2}\} / \{(2 \Pi N/2)^{1/2}\} \{(2 \Pi N/2)^{1/2}\} ] [ \{(N/e)^N\} / \{(N/2e)^{N/2}\} \{(N/2e)^{N/2}\} ] \\ &= [(2/\Pi N)^{1/2}] [2^N] \end{aligned}$$

from which equation (6) follows immediately.

probability of a tied vote depends upon the standard deviation of  $n$ ,  $[p(1 - p)N]^{1/2}$ , which is equal in this example to 500. Thus, the distance between the expected number of votes for the left party (550,000) and the number of votes for which the election is tied (500,000) amounts to 20 standard deviations  $[10,000/500 = 20]$ . The probability of a variable being 20 standard deviations from its mean is infinitely minute. Beck (1975, table 1) computes that probability to be about  $10^{-90}$ . The entire national income of the world would not be sufficient to compensate for such a small chance of a pivotal vote. As there is no particular reason for supposing  $p$  to be exactly  $1/2$ , a sense of duty would surely be required to explain why people vote at all.

Some features of person-to-person randomization should be noted. Hidden within person-to-person randomization is a special assumption about sampling. Imagine a population of exactly one million voters, with no abstentions and where exactly half a million people vote left and the other half vote right. Clearly, on these assumptions, the outcome is a tie, and any additional voter must be pivotal despite the fact that the probability of a randomly-chosen person voting for the left party remains equal to  $1/2$ . With  $p = 1/2$ , the probability of a tied election is 100% in this case, as opposed to only a tenth of 1% in Beck's calculation. The discrepancy arises from the distinction between sampling with and without replacement. A ball picked randomly from an urn may or may not be returned to the urn before the next ball is drawn. Beck's formula is for sampling with replacement. Voting may be more like sampling without replacement. Enormous in this example, the difference between estimated probabilities of a tie would be less pronounced when the proportion of population that votes is significantly less than 1.

As discussed so far, person-to-person randomization is of *interests*. Everybody knows what is best for himself, but some people would be made better off with a win by the left party while other people would be made better off with a win by the right. Alternatively, randomization may be of *opinions*. One of two parties may be better for everybody, but people may differ in their judgments about which party that is. Everybody gains the same fixed amount  $B$  if the better party wins, but a proportion,  $p$ , of eligible voters thinks the left party is better and a proportion  $(1 - p)$  thinks the right party is better. If people are correct on average - in the sense that  $p > 1/2$  if and only if the left party is preferable - and as long as the value of  $p$  is not common knowledge, the probability of the better party winning the election is an increasing function of the number of people who vote rather than abstain. The optimal proportion of voters in the population becomes that for which the cost of voting is just equal the difference between the expected gain from switching the outcome of the election advantageously and the expected loss from switching the election disadvantageously. It is also possible that some people know what is best, others do not, and everybody knows who is who. In such situations, public decision-making is best left to the experts rather than to the ballot box, or voting may be for people whose task it would be to identify experts.

There is an apparent discrepancy among the assumptions. The  $(N + 1)^{\text{th}}$  person has a genuine option to vote or to abstain. The other  $N$  people are given no such choice. They must all vote left or right, as the case may be, in accordance with preferences assigned by the angel of chance. Alternatively, if it does not pay to vote when the number of people actually voting is large, each eligible voter might be inclined to adopt a mixed strategy, voting or abstaining at

random. In a mixed strategy, each voter's chance of becoming pivotal increases together with the number of abstentions. As long as eligible voters act from self-interest alone so that  $D = 0$ , the mixed strategy would contain a smaller and smaller chance of voting until the probability of a tie rises to the point where  $\pi B > C$ . in equation (1) above when  $D$  is absent.

There is also something unsatisfactory about supposing voters' preferences to be assigned in the first instances over political parties themselves rather than over policies that political parties may come to adopt. The angel of chance might more appropriately be seen as conveying tastes randomly on the understanding that people's votes would be determined accordingly. The implicit assumption that policies of political parties are fixed and inflexible is at variance with the median voter theorem according to which pressure to win elections drives all political parties to set platforms in accordance with the preferences of the median voter.

Nor does the model contain any explanation of why the probability,  $p$ , of voting for the left party is what it is assumed to be. Presumably  $p$  would be a reflection of economic and political circumstances in the economy, but the model contains no list of relevant circumstances or mechanism connecting these to political behaviour.

Uncomfortable with Beck's (1975) assumption that every person's probability,  $p$ , of voting for the left party is fixed and immutable, Chamberlain and Rothschild (1981) assume instead that  $p$  itself is chosen from a probability distribution of possible values between 0 and 1. There are two angels of chance, the first choosing  $p$  for the entire population, and the second choosing each person's vote for the given value of  $p$ . Chamberlain and Rothschild show that, if the distribution of  $p$  is uniform (and in some other cases as well), the probability of a tie is proportional to  $1/N$ . The probability of a tie is still very small, but not as small as in Beck's computation when  $p$  differs from  $\frac{1}{2}$  by as much as 1%.

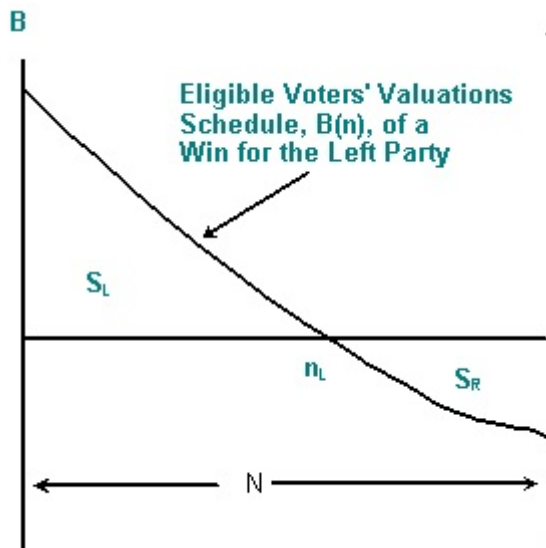
Some of the difficulties in person-to-person randomization are circumvented in Ledyard (1984) by redirecting voters' preferences from political parties to the platforms that political parties might adopt and by shifting randomization from preferences for one party over the other to the cost of voting. A political equilibrium is constructed by introducing a variable  $z$  which plays a double role in voters' utility functions and in platforms of political parties. People are assumed to differ in two dimensions, in their cost of voting,  $C$ , and in their preferred values of  $z$ , as though each person's  $C$  is chosen randomly from one urn, and each person's  $z$  is chosen randomly from another. Other things being equal, a person's utility is highest when society's choice of  $z$  conforms to his preferred value, and diminishes steadily as  $z$  varies above or below the most preferred level. Each of two political parties chooses a value of  $z$ , and each person, if he votes at all, would vote for the party providing the value of  $z$  that is closest to his preferred value. The outcome is that a) both parties choose the same value of  $z$ , and b) all eligible voters abstain (so that the election is decided by the flip of a coin), but c) the parties' common choice of  $z$  (which may be the first preference of the median voter) is such that any party deviating from the common value of  $z$  would be sure to lose the election because some voters would be roused to vote against it..

Missing from both person-to-person randomization and the common sense approach is a way to measure self-interest,  $B$ , and duty,  $D$ , on the same scale. To whom is there a sense of duty? What exactly is the benefit conferred by one person's decision to vote rather than to abstain upon others to whom one's duty is due? There is no hint in person-to-person randomization or in the common sense approach of the content or magnitude of  $D$ , no basis for inferring whether  $D$  is large or small or whether a sense of duty is likely to make up for the deficiency in self-interest alone. Nation-wide randomization fills the gap, but only at the cost of some rather strong assumptions.

### Nation-wide Randomization

The logic of nation-wide randomization is that elections may be won or lost due to unpredictable swings in the climate of opinion. Everybody occupies a place on a left-right scale according to his valuation, positive or negative, of a win for the left party, where the absolute value each person's valuation can be represented by  $B$  in equation (1). Uncertainty in the outcome of elections arises from unpredictable events in the economy, the world at large or public administration, seen as actions of the angel of chance that raise or lower everybody's value of  $B$  simultaneously.

**Figure 1: The Distribution of Voters' Valuations of a Win for the Left Party**



With person-to-person randomization, everybody would be exactly alike but for the angel of chance. By contrast, with nation-wide randomization, people are intrinsically different in their political preferences and can be ranked from top to bottom according to their valuations,  $B(n)$ , of a win for the left party. For example,  $B(n_\alpha) = 1,000$  means that person  $n_\alpha$  places a value of \$1,000 on a win for the *left* party rather than the right, while  $B(n_\beta) = -1,000$  means that person  $n_\beta$  places

a value of \$1,000 on a win for the *right* party rather than the left. All eligible voters may be lined up from left to right according to the intensity of their preferences for the left party;  $B(n)$  is highest at  $n = 0$ ,  $B(n)$  is lowest at  $n = N$ , and  $B'(n) < 0$ .

A schedule of eligible voters' valuations of a win for the left party is illustrated in figure 1 with voters' benefits,  $B(n)$ , on the vertical axes and with the  $N$  eligible voters lined up appropriately on the horizontal axis. The number of people,  $n_L$ , who prefer the left party to the right party, for whom  $B(n) > 0$ , is indicated by the crossing of the schedule  $B(n)$  with the horizontal axis. If voting were costless, the left party would win whenever  $n_L > N/2$  and the right party would win whenever  $n_L < N/2$ .

The areas designated as  $S_L$  and  $S_R$  respectively are the total of the valuations by all left-supporters of a win for the left party rather than the right and the total of the valuations by all right-supporters of a win for the right party rather than the left. Specifically,

$$S_L = \text{sum of all } B(n) \text{ for which } B(n) > 0 \quad (7a)$$

and 
$$S_R = \text{sum of all } B(n) \text{ for which } B(n) < 0 \quad (7b)$$

As the figure is drawn,  $n_L > N/2$  and  $S_L > S_R$  signifying that the left party wins the election as long as everybody votes and that a win for the left party is best for the nation as a whole. A different postulated shape of the voters' valuation curve - with the same value of  $n_L$  but flatter to the left of  $n_L$  but steeper to the right - could create a discrepancy between number of votes and aggregate benefits. A majority of the population might vote for the left party despite the fact that people would be better off on average if the right party won instead <sup>8</sup>.

Thus far, as voting is portrayed in figure 1, there can be no pivotal voter because everybody knows which party would win the election and because voting is costless. For there to be a probability of a person's vote swinging the outcome of an election, these constraints must be relaxed. They will be relaxed in turn.

Uncertainty is introduced through nation-wide randomization in which the angel of chance chooses a schedule,  $B(n)$ , of voter valuations out of a set of possible schedules, all with

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<sup>8</sup> Drawn for convenience as a continuous function in figure 1, the voter's valuation schedule must really be a step function with  $B$  defined only for integral values of  $n$ . The curve in the figure should really be a collection of dots. The  $n^{\text{th}}$  person's value of a win for the left party would be a dot placed at a distance  $n$  from the left-hand vertical axis and at a distance  $B(n)$  above or below the horizontal axis. When  $N$  is even, there is a tie between left and right parties if and only if  $B(N/2) > 0$  but  $B((N/2) + 1) < 0$ . Otherwise a party must win by at least two votes. When  $N$  is odd, the right party wins by one vote (so that an extra vote would create a tie) if and only if  $B((N + 1)/2) < 0$  but  $B((N + 1)/2 - 1) > 0$ . With  $n$  looked upon as continuous, there is a tie when  $B(N/2) = 0$ , or, equivalently, when  $n_L = N/2$ .

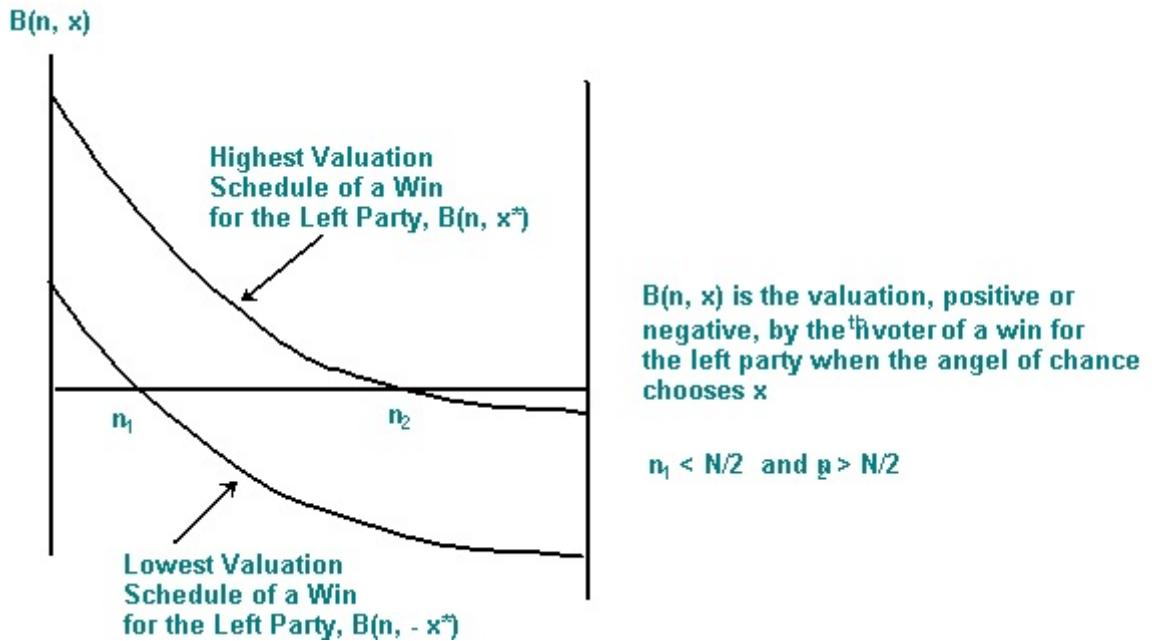
the general shape of that in figure 1, but differing in their locations, up or down to reflect a general bias toward one party or the other.

Imagine a “basic” schedule  $B(n)$  such as is illustrated in figure 1 that is shifted up or down in accordance with a random variable  $x$  selected by the angel of chance and converting the valuation schedule of a win for the left party from  $B(n)$  to  $B(n, x)$  where

$$B(n, x) = B(n) + x \tag{8}$$

and where  $x$  may be positive or negative. A positive  $x$  pushes the schedule up, raising all valuations of a win for the left party and lowering all valuations of a win for the right party accordingly. A negative  $x$  pushes the schedule down, raising valuations of a win for the right party and lowering all valuations of a win for the left party accordingly. It is difficult to say *a priori* what the distribution of the random variable,  $x$ , might be, but, in the interest of simplicity, it is assumed here to be uniform, equally likely to take on any value from  $-x^*$  to  $x^*$  where  $x^*$  is some fixed limit assigned to the angel of chance.

**Figure 2: Highest and Lowest Voters’ Valuation Schedules**



On these assumptions, the voters’ valuation schedule is equally likely to lie anywhere between the highest schedule,  $B(n, x^*)$ , generating, for each person  $n$ , the largest possible valuation of a win for the left party (or the smallest possible valuation of a win for the right party), and the lowest schedule,  $B(n, -x^*)$ , generating, for each person  $n$ , the largest possible valuation of a win for the right party. Highest and lowest schedules are illustrated in figure 2.

As long as voting is costless, there is no harm in supposing that everybody votes. If the angel of chance picks the highest valuation schedule, the left party wins with  $n_2$  out of  $N$  votes (because, by construction,  $n_2 > N/2$ ). If the angel of chance picks the lowest valuation schedule, the right party wins with  $N - n_1$  out of  $N$  votes (because, by construction,  $N - n_1 > N/2$ ).

Intervention by the angel of chance creates two distinct electoral probabilities, the probability  $\pi$  of any person's vote being pivotal and the probability  $P$  of the left party winning the election. With reference to figure 2 and on the assumption that all numbers of left supporters between  $n_1$  and  $n_2$  are equally likely, the probability of a person's vote becoming pivotal depends upon the *width* of the band between  $n_1$  and  $n_2$  and the probability of a win for the left party depends upon the *location* of the band, whether it is mainly to the right or mainly to the left of the center point,  $N/2$ .

Since all schedules within the band between  $n_2$  and  $n_1$  are equally likely, the chance of a tie must be  $1/[n_2 - n_1]$ , and a voter's chance of being pivotal must be

$$\pi = 1/2 [n_2 - n_1] \quad (9)$$

Since, the left party wins whenever the voters' valuation schedule intersects the horizontal axis beyond  $N/2$  and the right party otherwise, the probability,  $P$ , of a win for the left party becomes the ratio of two distances, the distance between  $N/2$  and  $n_2$  and the distance between  $n_1$  to  $n_2$ . The probability,  $P$ , of the left party winning the election becomes

$$P = [n_2 - N/2]/[n_2 - n_1] \quad (10)$$

which, as the figure is drawn is somewhat less than  $1/2$ .

The introduction of a cost of voting induces everybody for whom  $B$  is neither very large nor very small to abstain. Assume that the cost of voting,  $C$ , is the same for everybody, and that the angel of chance is equally likely to select any  $x$  within the range from  $-x^*$  to  $+x^*$ . As all duty to vote is so far abstracted away, it is immediately evident from equation (1) above that a person votes if and only if one's gain from a win for one's preferred party exceeds the *expected* cost,  $C/\pi$ , of swinging the election. The probabilities,  $\pi$  and  $P$ , of one's vote being pivotal and of a win for the left party but equations (9) and (10) must be modified to allow for abstention of eligible voters for whom the benefit of a win for one's preferred party falls short of its expected cost.

With  $\pi$  correctly anticipated, a person votes for the left party if and only if

$$B > C/\pi \quad (11a)$$

would vote for the right party if and only if

$$-B > C/\pi \quad (11b)$$

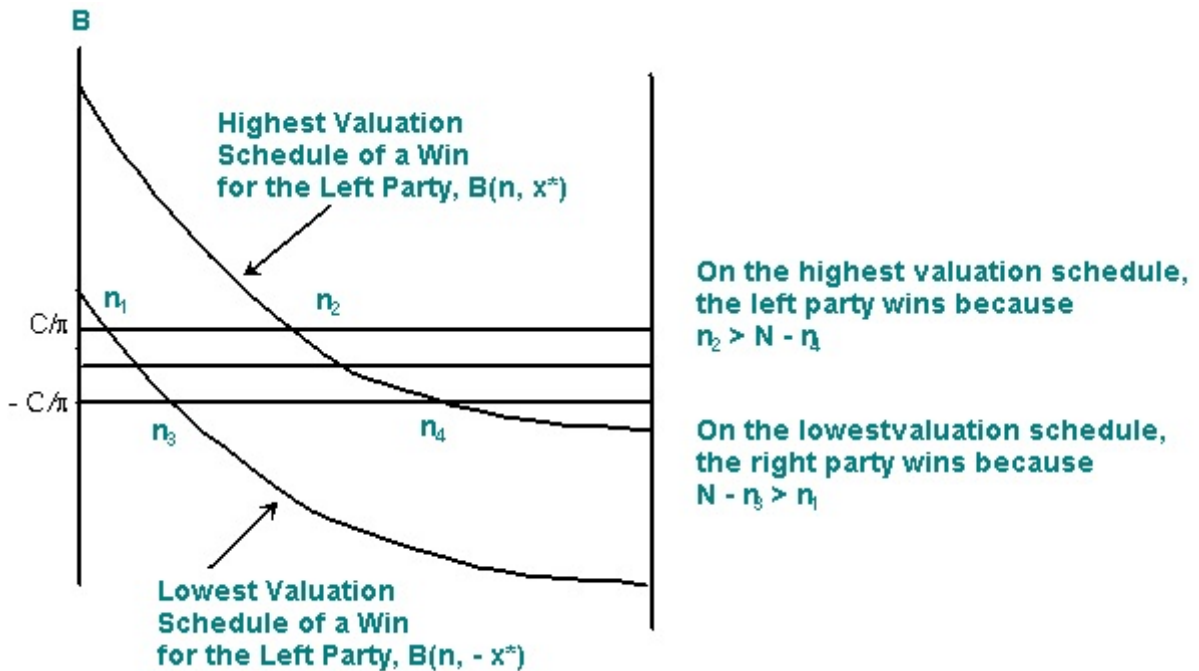
and would abstain if and only if

$$-C/\pi < B < C/\pi \quad (11c)$$

One's chance of casting a pivotal vote is in some respects like a market price of an ordinary good or service. Like market prices,  $\pi$  is at once a signpost for each and every voter and a characteristic of the community of voters as a whole. A person votes or abstains in accordance with the electorate's value of  $\pi$ , but the electorate's value of  $\pi$  depends on what voters choose to do. Each voter sees himself as having free choice, but sees the community of voters as determinate.

Numbers of voters are illustrated in figure 3, a reproduction of figure 2 with the addition of two horizontal lines at distances  $C/\pi$  above and below the horizontal axis. One votes or abstains in accordance with equation (11), for the left party of one's value of  $B$  is above the higher line, one votes for the right party if one's value of  $B$  is below the lower line and one abstains in between.

**Figure 3: Uncertainty about Others' Preferences, the Cost of Voting and the Probability of Casting a Pivotal Vote**



For any given  $\pi$  and for any given valuation schedule, the number of votes for each party and the number of abstentions can be inferred from equation (11). If the highest valuation schedule (with  $x = x^*$ ) is chosen, then  $n_2$  people vote for the left party,  $N - n_4$  people vote for the right party,  $n_4 - n_2$  people abstain and the left party wins the election. If the lowest valuation schedule (with  $x = -$



$x^*$ ) is chosen, then  $n_1$  people vote for the left party,  $N - n_3$  people vote for the right party,  $n_3 - n_1$  people abstain and the right party wins the election. A situation could arise where one party is so much preferred to the other that it wins the election regardless of which valuation schedule is chosen. Were that so, no voter could ever be pivotal and the outcome of the election would depend upon the mobilization of each party's supporters as will be discussed below. Assume for the present that that is not so. Assume the left party wins when the highest valuation schedule is chosen, and the right party wins when the lowest valuation schedule is chosen. The assumption is that

$$n_2 > N - n_4 \quad (12a)$$

and  $n_1 < N - n_3 \quad (12b)$

yielding a unique value of the probability of casting a pivotal vote. Steps in the computation of  $\pi$  are set out in table 1

**Table 1: Numbers of Votes and Abstentions**  
(On the assumption that each party has some chance of winning the election)

	highest valuation schedule	lowest valuation schedule
abstentions	$n_4 - n_2$	$n_3 - n_1$
votes for the left party	$n_2$	$n_1$
votes for the right party	$N - n_4$	$N - n_3$
total votes	$N - n_4 + n_2$	$N - n_3 + n_1$
plurality (for the left party in the first column and for the right party in the second)	$n_2 + n_4 - N > 0$	$N - n_3 - n_1 > 0$

Depending on the angel of chance, the outcome of the election varies from a win for the left party by at most  $n_2 + n_4 - N$  votes to a win for the right party by at most  $N - n_1 - n_3$  votes. The total number of votes between the largest possible plurality for the left party and the largest possible plurality for the right party is

$$[n_2 + n_4 - N] + [N - n_1 - n_3] = [n_2 + n_4 - n_1 - n_3]$$

As long as the plurality for one party or the other has equal chances of lying anywhere within that range, the probability of a tie must be  $1/[n_2 + n_4 - n_1 - n_3]$  and the probability that one's vote is pivotal becomes

$$\pi = 1/2[n_2 + n_4 - n_1 - n_3] \quad (13)$$

and the probability, P, that the left party wins the election must be the ratio of the maximal win for the left party to the full range between the maximal win for the left and the maximal win for the right, i.e.

$$P = [n_2 + n_4 - N] / [n_2 + n_4 - n_1 - n_3] \tag{14}$$

Equation (13) completes the equilibrium in figure 3. The numbers in the table -  $n_1$ ,  $n_2$ ,  $n_3$  and  $n_4$  - are dependent upon the value of  $\pi$ , but  $\pi$  itself is dependent on these numbers. Note the family resemblance between the what was called “common sense model” and “nation-wide randomization” in the determination of the probability of casting a pivotal vote. The common assumption in these models is that if one party might win by up to Q votes and the other party might win by up to R votes and if all outcomes in between are equally likely, the probability of a person’s vote being pivotal must be  $1/2(Q + R)$ . Nation-wide randomization can be thought of as a rationalization of the outcome in the common sense model.

### A Numerical Example

Consider a society with a population, N, of 10,000 eligible voters where a person’s (positive or negative) valuation, B, of a win for the left party is

$$B(n, x) = 105,000 - 20n + x \tag{15}$$

where n is an ordering of people from less to more right wing and where the angel of chance picks x from a minimum of -15,000 to a maximum of 15,000. In the event that  $x = 0$ , people’s valuation of a win for the left party varies from a maximum of 105,000 when  $n = 0$  to a minimum of -95,000 when  $n = 10,000$ , meaning that the most right wing person values a win for the right party at 95,000. The outcome when voting is costless is shown in table 2.

**Table 2: Electoral Outcomes Dependant on the Angel of Chance when Voting is Costless**

	highest valuation schedule $x = 15,000$	lowest valuation schedule $x = -15,000$
votes for the left party	6000	4500
votes for the right party	4000	5500
plurality for the winning party	2,000 (for the left party)	1,000 (for the right party)
probability of a pivotal vote	$\pi = 1/\{2[\text{sum of both pluralities}]\} = 1/6,000$	
probability of win for left party (P)	$= [\text{maximal plurality for left party}]/[\text{sum for both pluralities}] = 2/3$	

Equation (15) becomes a linear representation of the valuation schedules in figure 2. The highest schedule is that for which  $x = 15,000$  and the lowest schedule is that for which  $x = -15,000$ . On the highest schedule, the number of votes for the left party is  $n_1$  and the number of votes for the right party is  $10,000 - n_1$  because, by assumption, everybody for whom  $n < n_1$  votes left and everybody for whom  $n_1 > n$  votes right. On the lowest schedule, the number of votes for the left party is  $n_2$  and the number of votes for the right party is  $10,000 - n_2$ . Values of  $n_1$  and  $n_2$  can be read off equation (15) with the appropriate values of  $x$ . Specifically, when voting is costless,

$$B(n_2, 15,000) = 105,000 - 20n_2 + 15,000 = 0 \quad (16a)$$

and 
$$B(n_1, -15,000) = 105,000 - 20n_1 - 15,000 = 0 \quad (16b)$$

from which it follows immediately that  $n_1 = 4,500$  and  $n_2 = 6,000$ . Depending upon the angle of choice, the outcome for the left party may with equal probabilities lie anywhere between a loss by 1,000 votes [5,500 - 4,500] to a win by 2,000 votes [6,000 - 4,000], a range of 3,000 votes. The probability of a tie becomes 1 in 3,000, and, since the breaking of a tie by the flip of a coin can be to the advantage of either party, the probability of a person's vote being pivotal must be 1 in 6,000 in accordance with equation (9) above. These outcomes are presented in table 2 below.

Of no consequence when voting is costless, the probability,  $\pi$ , of one's vote being pivotal becomes important when voting is costly. The decision whether to vote or abstain becomes a trade off between the benefit of a win for one's preferred party and the expected cost of procuring it, where expected cost depends upon  $\pi$  as well as  $C$ . When the upper and lower valuation schedules in figure 3 are straight lines in accordance with equation (15), the critical values,  $n_1$ ,  $n_2$ ,  $n_3$  and  $n_4$  in figure 3 are all functions of  $\pi$ , and all five variables become determinate because  $\pi$  is a function of  $n_1$ ,  $n_2$ ,  $n_3$  and  $n_4$  in accordance with equation (13).

Since a portion of eligible voters abstain, it becomes necessary to derive each of  $n_1$ ,  $n_2$ ,  $n_3$  and  $n_4$  in a separate equation, equating a variant of  $B(n,x)$  in equation (15) to the voter's expected cost,  $C/\pi$ , of turning the election from a loss to a win for the party he favours. With the linear valuation curves in equation (15), this becomes unconscionably simple, for the probability of a tied election turns out to be independent of  $C$ , so that the value of  $\pi$  in table 2 carries over to the more complex case where voting is costly.<sup>9</sup>

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<sup>9</sup>Deduction of the same number of votes from each of two competing political parties need have no effect upon their probabilities of winning the election or the probability that a person's vote is pivotal. As long as the valuation schedules are linear, an increase in  $C$  leaves the distribution of pluralities unchanged, inducing equal numbers of abstentions by people who would otherwise vote left and by people who would otherwise vote right, so that, regardless of what value of  $x$  the angle of chance has selected, the winner of the election is unchanged when the expected cost of voting is increased, and a voter's chance of being pivotal is unchanged as well.

For the assumed value, 10, of  $C$  and the estimated value,  $1/6,000$ , of  $\pi$  in table 2, the value of  $C/\pi$  becomes 60,000 and the numbers of votes for the left party in accordance with the highest and lowest valuation schedules become

$$B(n_1, 15,000) = 105,000 - 20n_2 + 15,000 = C/\pi = 60,000 \quad (17a)$$

and  $B(n_2, -15,000) = 105,000 - 20n_1 - 15,000 = C/\pi = 60,000 \quad (17b)$

For characterizing votes for the right party, it is convenient to construct two new variables,  $m$  for the number of people who vote for the right party and  $D$ , as a function of  $m$ , for the value placed by a right supporter on a win for the right party. Specifically, corresponding to  $n_3$  and  $n_4$  in figure 3 are  $m_3$  and  $m_4$ , numbers of votes for the right party corresponding respectively to the lowest and the highest valuation schedules when there is a cost of voting,  $C$ , and a probability of being pivotal,  $\pi$ . It follows immediately that  $m_3 = 10,000 - n_3$ , that  $m_4 = 10,000 - n_4$ , that  $D(m, x) = -B(10,000 - n, x)$  and that

$$\begin{aligned} D(m_4, 15,000) &= -105,000 + 20n_4 - 15,000 \\ &= 95,000 - 20m_4 - 15,000 = C/\pi = 60,000 \end{aligned} \quad (17c)$$

and  $D(m_3, -15,000) = 95,000 - 20m_3 + 15,000 = C/\pi = 60,000 \quad (17d)$

From the four components of equation (17) it follows that

$$n_1 = 1,500, n_2 = 3,000, m_3 = 2,500 \text{ and } m_4 = 1,000$$

On the higher valuation curve, 4,000 people ( $n_2 + m_4$ ) vote, 6,000 people abstain and the left party wins by 2,000 votes, while on the lower valuation curve, 4,000 people ( $n_1 + m_3$ ) vote, 6,000 people abstain and the right party wins by 1,000 votes. Since the range of outcomes of the election varies from a win by 2,000 by the left party to a win by 1,000 votes by the right party, the probability of a tie must be 1 in 3,000, and the probability,  $\pi$ , that a person's vote is pivotal (when everybody in whose interest it is to vote does so) becomes 1 in 6,000, exactly as computed in table 2 for the case where voting is costless. The numbers are brought together in table 3, and the valuation curves themselves are shown as figure 4.

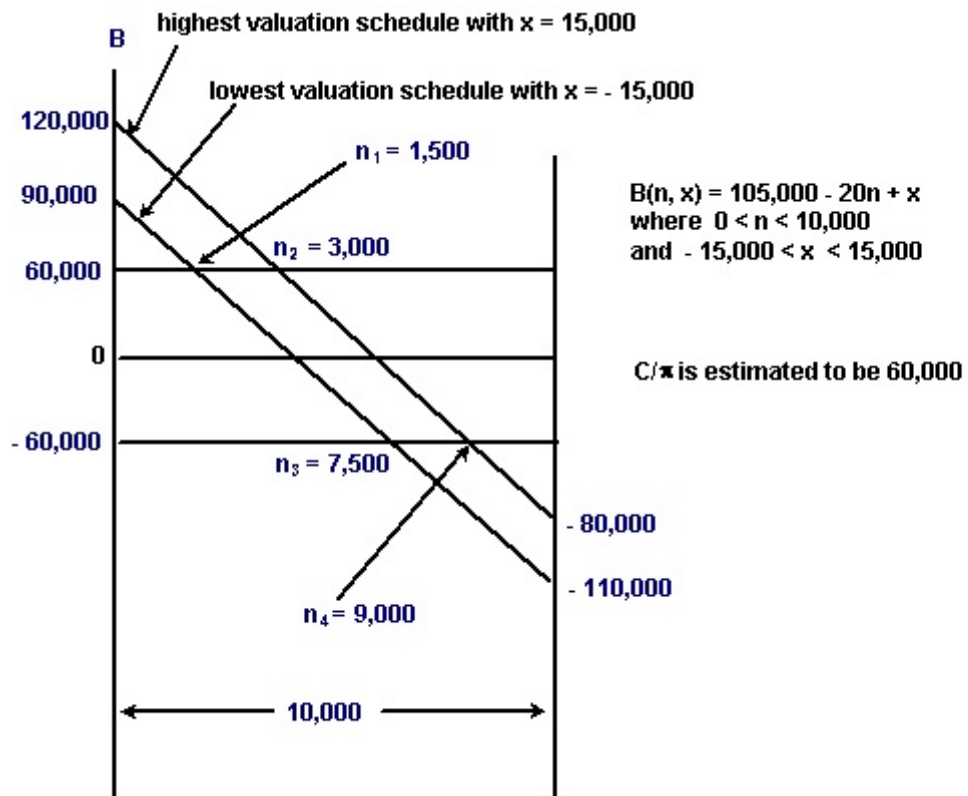
The information in table 3 and figure 4 can be looked upon from two points of view. It is, on the one hand, a picture of how the collectivity behaves, showing everybody's probability of being pivotal, the range of possible pluralities for each party, the number of abstentions, and the valuations of a win for one's preferred party required to induce a person to vote. It is, on the other hand, a guide for deciding whether to vote or to abstain, supplying the critical value of  $\pi$  in equation (1). These points of view are consistent as long as nobody can infer from his own value of  $B$  what valuation schedule the angel of chance has selected. Nobody must know his own ranking,  $n$ , on the scale from most left to most right. The required ignorance might be secured by the imposition of two randomization processes, one setting the value of  $x$  for the electorate as a whole and the other assigning places on the chosen valuation schedule. Or the angel of chance

may be seen as engineering that part of the behaviour of the rest of the community which no particular voter can infer from his own circumstances.

**Table 3: Electoral Outcomes Dependant on the Angel of Chance when Voting is Costly**

	highest valuation schedule $x = 15,000$	lowest valuation schedule $x = - 15,000$
votes for the left party	3000	1500
votes for the right party	1000	2500
abstentions	6000	6000
plurality for the winning party	2,000 (for the left party)	1,000 (for the right party)
probability of a pivotal vote ( $\pi$ ) = $1/\{2[\text{sum of both pluralities}]\} = 1/6,000$		

**Figure 4: Votes and Abstentions when Voting is Costly**



Artificial as it is, this example suggests an important principle about who votes and who abstains. In so far as voting is from self-interest rather than from a sense of duty, it is the extremists who vote and the moderates who abstain. One votes if and only if the value of a win for the party one favours exceeds 60,000. Everybody else abstains. Symmetries in the example prevent this consideration from influencing the outcome of the election, but, as will be discussed below, there is no guarantee of such symmetry in other, more realistic situations.

### Qualifications and Exceptions

The numerical example serves to establish that there *may* be an equilibrium – with some people eligible voters voting for the left party, others voting for the right party and still others choosing to abstain - in circumstances where each voter knows his own preference but has only a rough idea about the distribution of preferences in the rest of the population, and, above all, where those who vote do so out of pure self interest-rather than from a sense of duty. The parameters of the example were chosen to generate such an outcome. The example does not establish that there *must* be an equilibrium or that the outcome is necessarily a reflection of the will of the electorate.

a) A Sure Winner: The locations of maximal and minimal valuation schedules in figures 2, 3 and 4 were chosen to ensure that both parties have some chance of winning the election. The maximal schedule delivers a win the left party, the minimal valuation schedule delivers a win for the right party and some schedule in between delivers a tie. That need not always be so. Regardless of what value of  $x$  is chosen by the angel of chance, significantly higher schedules than those in equation (15) above would deliver a sure win to the left party, and significantly lower schedules would deliver a sure win to the right.

Suppose, for example, the constant term (105,000) in equation (15) were reduced to 75,000, converting the schedule to

$$B(n, x) = 75,000 - 20n + x \tag{18}$$

while everything else remains the same. The right party would then be sure to win the election regardless of the value of  $x$  selected by the angel of chance. The most favourable outcome for the left is when  $x = 15,000$ . Even so, there would be only 1,500 out of the 10,000 eligible voters for whom  $B \geq 60,000$ , as against 2,500 eligible voters for whom  $B \leq 60,000$ . The best the left party can hope for is a three-to-five loss. The right party's proportion of the votes can only increase when the angel of chance picks a lower value of  $x$ .

Recall the derivation of  $\pi$  in equation (13). The formula required that the set of options open to the angel of chance be such that some lead to a win by the left party and others lead to a win by the right. Figure 4 demonstrates how this may occur, but not that it must always do so. If not, the formula for  $\pi$  becomes inapplicable. Despite the randomness introduced by the angel of

choice, the situation may remain essentially as illustrated in figure 1 where  $n_L$  is substantially larger than  $N/2$  so that nobody's vote can be pivotal unless a substantial but unknown number of other people abstain.

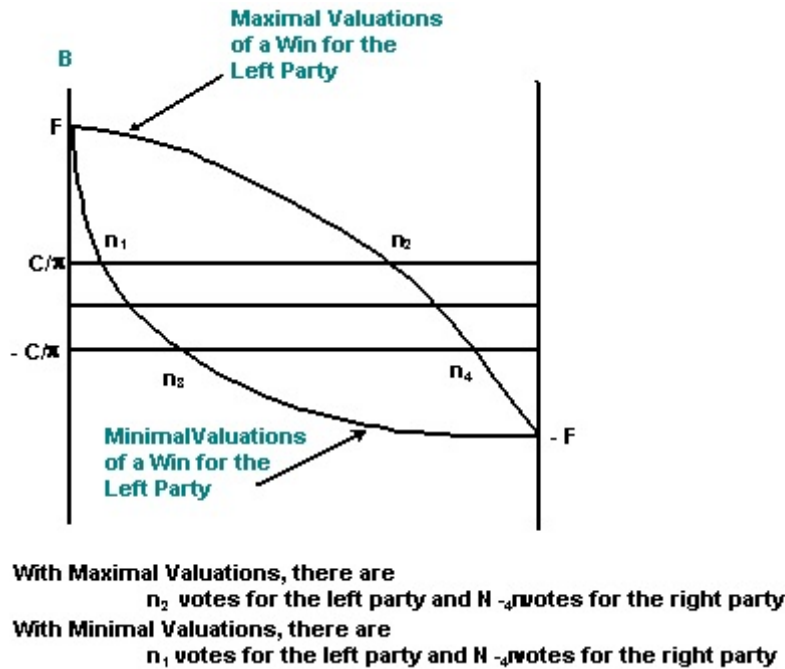
Though it is probably rare for a political party to have absolutely no chance of winning in a two-party race, the possibility cannot be ruled out altogether. Duty aside, there are reasons why people might vote regardless. Political parties are ongoing institutions. A good showing for a political party in this election might pave the way for a win in the next. A close race may say something about the preferences of the population, causing the party in power to take more account of the policies of the losing party than if the win had been by an overwhelming majority. The British Labour party hung on for generations before finally attaining office. In Canada, the Cooperative Commonwealth Federation is said to have provoked the ruling Liberal party to adopt many social programs.

b) Evidence about the Angel of Chance: A major premise of nation-wide randomization is that nobody can infer from his own preferences what the choice of the angel of chance might be. Nobody should be able to deduce the value of  $x$  from his own value of  $B$ . There are circumstances where that cannot be so. With reference to figure 4, consider a person for whom  $B = 100,000$ . That person must know that the angel of chance cannot have chosen the lowest valuation curve because the highest  $B$  consistent with that schedule is only 90,000. To be consistent with  $B = 100,000$ , the chosen valuation curve must be among top half of all valuation curves, curves corresponding to values of  $x$  between 0 and 15,000. Not knowing his ranking among all eligible voters, that person cannot know which among the top half of all values of  $x$  the angel of chance has chosen, but the person does know that  $x$  is chosen from that limited set. More importantly, knowing that  $x > 0$ , this voter must also know that the left party is sure to win the election, meaning that his own vote cannot be pivotal and that it cannot be in his interest to vote. Despite the high value placed upon a win for the left party, this person prefers to abstain.

There are two ways around this problem: The first is to suppose that a voter who sees his own value of  $B$  to be unexpectedly high (or low as the case may be) presumes not that  $x$  is high, but that the constant term (110,000) in equation (15) has increased, leaving the variability of possible outcomes unchanged. An increase in the constant term increases the left party's chance of winning the election without affecting the probability of a tie. As can be read off figure 4, the value of  $P$  in equation (14) would be increased, but the value of  $\pi$  in equation (13) would not.

The other way to preserve all voters' ignorance about the value of  $x$  is to change the postulated shapes of the voters' valuations curves. The parallel straight lines in equation (15) could be replaced by curved lines meeting at common points on the vertical axes as shown in figure 5. Then, as long as no eligible can be sure of his ranking on the valuation curve, no feasible value of a person's  $B$  conveys information about which  $x$  the angel of chance has chosen.

**Figure 5: Valuation Schedules with Common Maximal and Minimal Values**



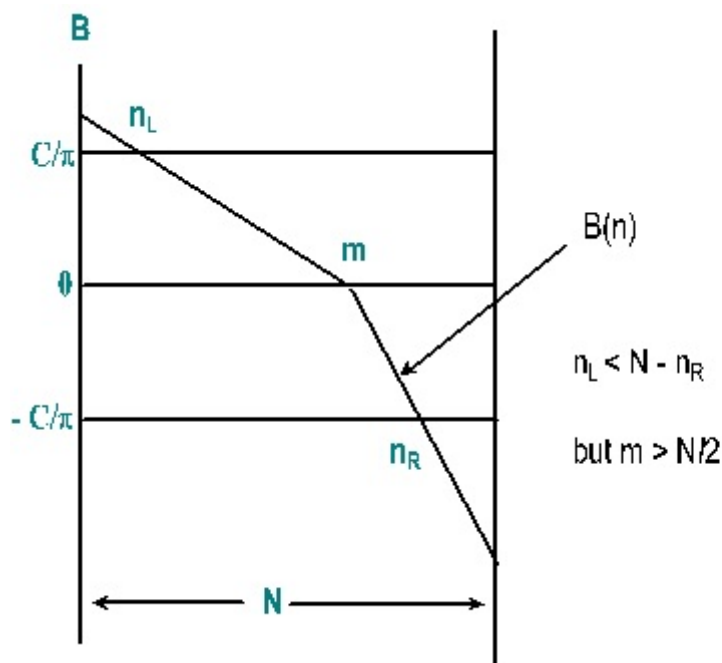
c) Bias in Favour of Small Groups with Strong Preferences: A situation can easily arise where a majority of the population prefers one party, but where the other party wins the election because its supporters place higher values on a win for their preferred party and are therefore less likely to abstain. This possibility is illustrated in figure 6 where the basic valuation schedule,  $B(n)$ , is kinked at the horizontal axis. In interpreting the figure, suppose that a juggling of the basic valuation schedule by the angel of chance supplies both parties with some chance of winning the election, creating some probability,  $\pi$ , of a person's vote becoming pivotal. As the figure is drawn, over half the population prefers the left party, but more people vote for the right party than for the left. Over half the population prefers the left party because  $B(n) > 0$  for all  $n < m$  where  $m > N/2$ . More people vote for the right party than for the left because  $(N - n_R) > n_L$  where  $(N - n_R)$  and  $n_L$  are the numbers of people for whom the value of a win for their preferred parties exceeds the expected cost of swinging the election. The electoral triumph of a small group with strong preferences is exemplified by the supporters of the right party when the basic valuation schedule,  $B(n)$ , is kinked at the horizontal axis as shown in Figure 6.

A majority of the population prefers the left party to the right, but a “decisive minority” with a sufficiently strong preference for the right party enables the right party to win the election.<sup>10</sup>

<sup>10</sup>“...for large electorates, no matter how great the proportion of the electorate that prefers a particular outcome, that outcome is likely to lose the election if the opposing outcome is preferred by an expected majority of voters with a sufficiently large incentive to vote”



**Figure 6: The More Popular Party Loses the Election**



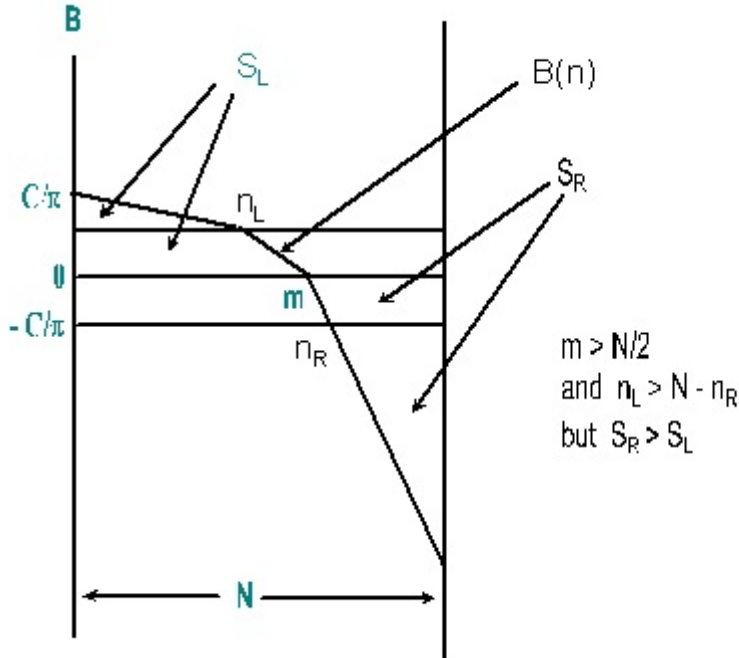
That cannot be the whole of the story for there can be no incentive for anybody to vote if one particular party is sure to win. Here, once again, this conundrum is circumvented by the angel of chance. The ultimate valuation schedule is not  $B(n)$  as shown in figure 6, but  $B(n, x)$  in equation (15) where the variability of  $x$  is presumably large enough for the left party to retain some chance, however small, of winning the election. The postulated symmetry of  $x$ , with equal chances of being positive or negative for any given absolute value, ensures that the right party is the more likely to win despite the small chance that it may not.

d) Social Welfare: Parallel to the possible discrepancy between numbers of supporters of each party and number of votes is a second possible discrepancy between number of voters and social welfare. The party with the most supporters need not convey the greatest social welfare. This is illustrated in figure 7 where the voters' valuation schedule is such that more people favour the left party than the right despite the fact that the right party conveys the greater social welfare;  $m > N/2$  despite the fact that  $S_R > S_L$ . A relatively indifferent majority of the population may prefer that shopping on Sunday be allowed, while a passionate minority prefers that shopping on Sunday be forbidden. The dollar value of the potential loss to the minority may well exceed the dollar value of the potential gain to the majority.

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On decisive minorities" Campbell (1999, 1203).

**Figure 7: The Party Supplying the Larger Aggregate Benefit Loses the Election**



Implicitly, surplus has so far been graduated in dollars to ensure that  $B$  and  $C/\pi$  are commensurate. Nevertheless, if one party tends to favour the rich while the other tends to favour the poor and if the disputed public policy is to narrow the distribution of income, then it may be of some interest to measure surplus in utils as well. For any given voters' valuations curve, the surpluses,  $S_L$  and  $S_R$ , measured in dollars and as illustrated in figure 6 above are

$$S_L = \int_0^m B(n) dn \quad S_R = \int_m^N -B(n) dn \quad (19)$$

Transformed into utils, the surpluses become  $W_L$  and  $W_R$  where

$$W_L = \int_0^m u(B(n)) dn \quad W_R = \int_m^N u(-B(n)) dn \quad (20)$$

and where  $u$  is a transformation of  $B$  such that, in a choice between two risky prospects, a person always chooses the prospect with the larger expected value of  $u$  rather than  $B$ .

e) The Cost of Voting: It is assumed here that the cost of voting is the same for everybody. The extreme opposite assumption is that cost varies from one person to the next but there is no correlation between the cost of voting and anything else, in which case, for example, half of society's cost of voting could be saved through a random abstention by half of all eligible voters. If the coin comes up heads, you do not vote; if the coin comes up tails, you do exactly what you

would otherwise have done. As mentioned above, Ledyard (1984) derives conditions where the outcome is the first preference of the median voter when the cost of voting varies from 0 to some maximal amount and where people's costs of voting are uncorrelated with their preferences for one party over another.

f) The Political Environment: To focus upon paradox of not voting, much of the circumstances of democratic politics has been assumed away:

i) Party platforms are fixed. There is no discussion or explanation of how political parties go about choosing their platforms or of why the voters' valuations schedule is what it is assumed to be. No account is taken of political parties' incentives to adopt platforms appealing to the largest possible share of the electorate. Between the lines of this paper one may detect that a right party is favoured by the rich and a left party is favoured by the poor, but left and right may have many dimensions, each contributing to the schedule  $B(n)$ .<sup>11</sup>

ii) There are only two parties, one of which will form the government. A paradox of not voting could be discussed in the framework of proportional representation, but that lead is not followed here.

iii) Nothing is said about the motives, interests or competence of politicians. In particular, no distinction is drawn between support of a political party because of its platform and support because its leaders, once in office, can be expected to serve nation-wide common interests effectively,

iv) Voters are never mistaken in their valuations,  $B$ , of their benefits or costs from the election of one party rather than the other. There is no campaign financing or electioneering because there is nothing about which the voter can be persuaded.

v) Eligible voters choose to vote or to abstain individually and self-interestedly, with no sense of duty and no coordination among them. Duty will be discussed presently.

g) Conflicting Interests vs Conflicting Opinions about Common Interests

The distinction between interests and opinions, discussed already in connection with

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<sup>11</sup>Parties need brand names, but it is often hard to see why the range of products in each brand is what it happens to be, why particular bundles of policies get lumped together as left or as right. Some policies might be identified as pro-rich or pro-poor, but not all policies can be classified that way. Why, for example, would the party favouring private provision of health care be opposed to gay marriage. Why might one party favour the killing of people before they are born, while the other favours the killing of people afterwards.

person-by-person randomization, has its counterpart in nation-wide randomization. When valuations reflect *interests* as has been assumed so far,  $B(n_\alpha) > B(n_\beta)$  means that person  $n_\alpha$  has more to gain than person  $n_\beta$  from a win by the left party. If  $B(n_\alpha) = 10$  and  $B(n_\beta) = -7$ , then everybody expects a win for the left party to make person  $n_\alpha$  better off by 10 and to make person  $n_\beta$  worse off by 7 than if the right party had won instead. By contrast, when valuations reflect *opinions* about common interests, everybody expects a win for the left party to convey the same benefit, or cost as the case may be, to everybody, but people may disagree about what that common benefit or cost would be. If  $B(n_\alpha) = 10$  and  $B(n_\beta) = -7$ , then person  $n_\alpha$  would expect a win for the left party to make everybody better off by 10 and person  $n_\beta$  would expect a win for the left party to make everybody worse off by 7.

When valuations represent opinions rather than interests and if each person is absolutely sure of his judgment, voting becomes an enormous act of charity to one's fellow citizens not observant or clever enough to appreciate what the true consequence of the election might be.<sup>12</sup>

If people differ in their opinions about common benefits or costs of a win for the left party, it would be odd, though not impossible, for everybody to hold their opinions with certainty. One may be certain about benefits to oneself. It is difficult to be certain about benefits or costs to others, especially when one knows that others, with the same interests as oneself, disagree. People might more appropriately be thought to have probability distributions of outcomes with different means or variances. There may also be a tendency to believe that what is best for oneself is best for society as well.<sup>13</sup>

Surplus becomes ambiguous. With  $B(n)$  interpreted as person  $n$ 's opinion about a common benefit, the surpluses,  $S_L$  and  $S_R$ , would be meaningless unless it is supposed that each person is equally likely to be correct, converting  $S_L$  and  $S_R$  into expected benefits as seen by different groups of people, those for whom  $B > 0$  and those for whom  $B < 0$ .

With one exception below, this paper adheres to the assumption that  $B(n)$  refers to person  $n$ 's assessment of his own benefit from a win for the left party rather than to person  $n$ 's opinion about the benefit to everybody.

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<sup>12</sup>See Edlin, Gelman, and Kaplan, (2008). If I am one of 10 million an eligible voters, if I am absolutely sure that a win for the right party conveys a benefit of \$1,000 to each of my fellow citizens, and, if, by chance, just enough of my fellow citizens mistakenly vote for the left party to create a tie or a win by one vote for the left party, then my vote conveys a benefit of \$10 billion to my fellow citizens. If I am at all altruistic, the required probability of my vote being pivotal may be very, very small.

<sup>13</sup>Upton Sinclair is alleged to have said that, "It is difficult to get a man to understand something when his salary depends on his not understanding it."

## Voting Pacts

Among like-minded people, voting is the private provision of a public good. Nobody pays tax voluntarily because everybody's valuation of the *extra* services of the army or the police made possible by his tax alone is almost certainly less than the tax he pays. Everybody favours compulsory taxation because everybody's valuation of the *entire* services of the army and the police is almost certainly greater than the tax he pays. Similarly, within a group of like-minded eligible voters, all pro-left or all pro-right, everybody might agree to vote on condition that everybody else within the group is required to do so too. Such an agreement may be called a "voting pact".<sup>14</sup>

In the numerical example, each person's cost of voting is \$10, each person's probability of casting a pivotal vote  $1/6,000$  and, consequently, each person's expected cost of swinging the election is \$60,000. A person abstains unless his benefit from a win by his preferred party exceeds \$60,000. The required benefit can be cut in half by a pact between any two like-minded voters who would not otherwise vote at all. Consider two eligible voters - both supporters of the left party or both supporters of the right - each of whom values a win for their preferred party at, say, \$45,000. Acting separately and self-interestedly, they would both abstain. Acting together, each voting on condition that the other does so too, they double the chance of being pivotal, from  $1/6,000$  to  $1/3,000$ , cutting each of their expected costs of swinging the election from \$60,000 to \$30,000 at which it becomes individually advantageous for both of them to vote rather than to abstain. Acting together, 100 like-minded eligible voters who would be unwilling to vote in the absence of any such arrangement, raise the probability of swinging the election from  $1/6,000$  to  $1/60$  at which voting becomes individually advantageous as long as the benefit of a win for one's preferred party exceeds not \$60,000, but only \$600.

Voting pacts differ from ordinary public goods in two main respects: Their benefits accrue not to society as a whole, but to a part of society, people favouring one of two political parties, at the expense of the rest, with no assurance that pacts are beneficial on balance to society as a whole. More importantly, there is no legally-binding mechanism to ensure that all beneficiaries of voting pacts contribute their share of the cost. Agreements to share the cost of ordinary public goods are enforced by the state. Agreements to vote can only be enforced imperfectly and incompletely, through social pressure exerted by friends, neighbours, churches,

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<sup>14</sup>Voting pacts are first cousins, if not closer, to the voting behaviour of groups as modeled by Uhlander (1989) and Morton (1991). Their emphasis is upon leaders and followers, upon the interaction between candidates seeking office and citizens choosing whether and for whom to vote, but they must postulate some material or just psychic private advantage in voting to overcome the free rider problem. Uhlander discusses rewards to by successful candidates to groups small enough to make each person's vote advantageous. There is also a family resemblance between voting behaviour of groups and the probabilistic voting theorem demonstrating an equilibrium when candidates distribute goodies to groups so as to maximize their probability of being elected. See Mueller (2003, chapter 12).

labour unions and so on. Voting pacts rely on nothing more than the exhortation to “get out and vote” among people who recognize one another to be on the same side of the fence. That may be, but is not necessarily, sufficient depending on the cost of voting and the degree of social cohesion.

Imperfect though they may be, voting pacts can have a profound effect upon the outcome of an election. Consider the extreme example summarized in table 4 of a bifurcated society with 100 rich people each placing a value of \$300 on a win for the right party, and with 900 poor people each placing \$100 on a win for the left party. Suppose, once again, that each person’s cost of voting is \$10. The angel of chance may juggle the numbers a bit, but not enough to make any voter pivotal except possibly within the context of a voting pact. Surpluses are  $S_L = 9,000$  and  $S_R = 3,000$ . If everybody votes, the left party would win nine-to-one.

**Table 4: A Two-class Society**  
[The cost of Voting,  $C$ , is \$20]

	number of eligible voters favouring	benefit per person of a win for the favoured party	total surplus
left	900 [n = 1 to 900]	$B(n) = \$100$	$S_L = \$90,000$
right	100	$B(n) = \$300$	$S_R = \$30,000$

In this society, 101 votes by left supporters is enough to ensure a win for the left party, and one extra vote, raising the total to 102, is sufficient to ensure that no voter, left or right, can be pivotal. When more than 101 votes for the left party, there is no chance of the right party winning the election, and supporters of the right party might as well abstain to save themselves the cost of voting in circumstances where their votes can have no effect.

All left-supporters are locked into a complex prisoners’ dilemma where each person is made better off by abstaining than by voting, but only on condition that enough others vote as well. Voters gain \$90 but abstainers gain \$100 as long as enough left-leaning people vote to ensure that the left party wins the election. How then are the voters selected? In a sense, the voters are suckers, bearing a burden that could very well be borne by somebody else instead. Each voter may well think to himself that, if he abstains, some other left-leaning person will be induced to vote in his place for fear of the emergence of a voting pact on the right. Perhaps supporters of one party or the other can devise a punishment for abstention. Exclusion from common activities or mere disapproval may be sufficient. Perhaps, if and when potential pact members can identify one another, they can devise a grim-trigger strategy of dissolving the entire pact if any member abstains. Pacts may be easier to enforce among all like-minded beneficiaries than among a proportion of them.

Despite the left’s overwhelming majority, it remains possible for cooperation among left-

leaning people to break down altogether and for the right party to win instead. Failure is especially likely if voting is expensive. Suppose the cost of voting were \$110 rather than \$10. Then it would be in no left-leaning person to vote without an agreement among a number of left-leaning people to share the cost. The right party could easily win despite the fact that there is a net gain of \$60,000 to society as a whole from a win for the left. There may be no equilibrium comparable to that illustrated in figure 4, and the outcome of the election may turn on who can cooperate with whom.

Suppose participation in a voting pact is costly. Abstracting from any sense of duty, a voting pact is advantageous to its members if and only if

$$\Pi B > C + K \quad (21)$$

where  $\Pi$  is the probability that the pact is pivotal and  $K$  is the per person cost of participation. The value of  $\Pi$  would be 1 if the pact were sure to generate a win for the party it supporters, and the value of  $\Pi$  would be less than 1 if there were just a chance of doing so. It is not unreasonable to suppose that the cost to each participant rises together with the size,  $s$ , of the pact and falls together with the proportion,  $p$ , of the beneficiaries of the pact. Thus,

$$K = K(s, p) \quad (22)$$

where  $K_s > 0$  because it is more expensive per member to hold together a large pact than a small one, and  $K_p < 0$  because it is more expensive per member to maintain a pact with a small proportion of the beneficiaries than with a large one.

With reference to the example in table 4, it is at least possible that the right party wins the election because there is some pact of size,  $s$ , such that  $\Pi = 1$  and

$$300 > 10 + K(s, 100) \quad (23)$$

but there is no pact of size,  $s$ , such that  $\Pi = 1$  and

$$100 > 10 + K(s, 900) \quad (24)$$

Despite the left's overwhelming majority, it is possible for the right party to win the election because a sufficient pact is feasible on the right side alone. .

## Duty

There are two questions here: To whom might one owe a duty to vote? and What precisely are the benefits to others that voting conveys?

A duty to vote may be to one's social class or to society as a whole. In the context of this

paper, one's own social class is everybody inclined to vote for the same political party as oneself, everybody for whom  $B$  is positive if one's own  $B$  is positive and everybody for whom  $B$  is negative if one's own  $B$  is negative. Duty to one's social class is nothing more than an obligation to vote for the party one favours, to vote for the left party if  $B > 0$ , to vote for the right party if  $B < 0$ , and not to abstain.

Duty to one's social class is straightforward. Duty to society as a whole is more complex. Consider a person for whom  $B > 0$  living in a society where the right party is best for society as a whole, meaning that  $S_R > S_L$  or, better still, that  $W_R > W_L$  where  $W_R$  and  $W_L$  are defined in equation (20) as social gains in utility as distinct from income. Though this person is made better off by a win for the left party, he knows that society as a whole is made better off by a win for the right. Duty to one's class warrants a vote for the left party, but duty to society as a whole would seem to warrant a vote for the right.

What about abstention? Of the three options open to this person - vote left, abstain, or vote right - what is their ordering from most to least dutiful to society as a whole? This question must be answerable if  $D$  in equation (1) is to have any meaning at all. One's first thought is that abstention is more dutiful than a vote for the left party - that the ordering of actions on the scale from more to less dutiful is <"vote right", "abstain", "vote left">, rather than <"vote right", "vote left", "abstain"> - because the chance of the right party winning is greater if one abstains than if one votes left. Better, it would seem, to abstain than to vote for the party that is not best for society as a whole.

Abstention may be the right course of action in a race between two political parties where knowledge rather than interest is the foundation of voters' support. I cannot do better than to abstain if I believe that one of the two competing parties is best for both of us and that you are the better judge of which party that is. Then, abstention becomes analogous to the commissioning experts to decide which drugs to allow onto the market. On the other hand, if I firmly believe that I know which party is best for everybody and that you do not, then, as discussed in Edlin et.al. (2008), voting becomes an act of charity on my part, enhancing your welfare even if you do not have the sense to do so yourself.

Otherwise, when support for political parties rests upon voters' interests rather than on voters' opinions about common interests, there are two strong reasons not to abstain. The first is that wide-spread abstention creates conditions where outcomes of elections depend more on voting pacts than on the underlying preference of the electorate. This can be especially unfortunate when parties differ in their ability to persuade or bribe their supporters to vote. Majority rule voting is intended to reflect the will of the majority not just of the party faithful, but of society as a whole.

The other reason pertains to possibilities implicitly assumed away in the description of voting so far. Elections have been described as contests between two, and only two, political parties with given platforms. Were that so, to abstain when one really favours the left party is, implicitly, to support the right party, and abstention becomes more dutiful than voting for the



party one prefers. That need not be so. Especially under proportional representation, there may be third, fourth or fifth political parties narrowly focused upon religion, locality, race or some other divisive criterion, parties that become more successful when moderates abstain. The effect of abstention may then be to strengthen the extremists. More important than whether the left party or the right party wins the election is that one or the other of the main parties does so without concessions to a selection of small, narrowly-focused groups. To be sure, new parties emerge from time to time, and not all such parties are pernicious, but democracy would seem to be safer when the influence of such parties depends more upon their support among the entire population than upon their ability to muster the votes. A duty to vote rather than to abstain is to keep the Nazis away.

Abstention can also be harmful by shifting the first preference of the median voter. Consider a radical change in assumptions. The language of left and right can be taken seriously by reducing all politics to the choice of a single variable  $x$ , where each eligible voter has a preferred value of  $x$ , and where each of two competing parties chooses some value of  $x$  as its platform. These are the assumptions where the “median voter theorem” applies. Both parties choose the preferred  $x$  of the median voter because the party failing to do so must necessarily lose the election. A host of well-known qualifications and exceptions do not concern us here.

Expositions of the median voter theorem typically ignore abstentions, but, strictly-speaking, the theorem applies to the subset of the population that actually votes rather than to the population as a whole. The equilibrium  $x$  varies up or down depending on whether abstentions are more prevalent among voters with high preferred  $x$  than among voters with low preferred  $x$ . This is illustrated in the appendix for the redistribution of income. Ignoring altruism, the poor want more redistribution and the rich want less. The  $x$  in the appendix is the rate of a negative income tax. It is shown how the equilibrium rate depends upon propensities to vote.

### **Compulsory Voting**

If and in so far as voting is a public good not just to like-minded voters, but to society as a whole, might it not be desirable to make voting compulsory? Perhaps people should be obliged to vote, just as they are obliged through taxation to pay a share of the cost of ordinary public goods. The case for compulsory voting depends very much on the strength of the duty to vote. If everybody votes from a sense of duty, because voting is fun or as a way of identifying oneself with the community of which one is a part, then compulsory voting is at best superfluous. At the other extreme where nobody votes unless it is individually advantageous to do so, compulsory voting could be as necessary as taxation to finance public goods. In between, the case for compulsory voting depends on several considerations some of which have so far been ignored in this paper.

1) A Duty to be Informed: Duty to vote is more than an obligation to show up at the ballot box. It is also an obligation to vote well, to investigate the platforms of the competing parties and to choose carefully among the available options. Compulsory voting could be harmful if people who vote voluntarily now take the trouble to be well-informed, while people who abstain now,

if obliged to vote, would cast their ballots carelessly, allowing themselves to excessively influenced by attack ads and prejudicial propaganda. These, sharply contrasting categories are extremes within which most citizens are to be found.

Also, when voters differ in knowledge rather than in interests, it may sometimes be in the common interest for certain voters to abstain because they know themselves to be less well-informed than their fellow citizens and less able to judge what is best for society as a whole. I may want to abstain because I do not know the candidates and would rather leave the decision to others who do. Compulsory voting might stop such behaviour (though it would always be possible to turn in a blank ballot).

2) Systematic vs. Random Abstention: Abstentions would not matter much and there would be no strong case for compulsory voting if the decision to vote or abstain were taken at random, if each person's decision were like the drawing of different coloured balls from an urn. Vote if the ball turns out to be yellow. Abstain if the ball turns out to be green. There would still be some chance of random abstentions affecting the outcome of the election, but that chance would be very small unless the proportion of abstentions is very large. There might be a socially-optimal rate of abstentions at which the marginal gain from reducing the total cost voting just equals the marginal expected loss from the chance of influencing the outcome of the election.<sup>15</sup>

The harm from abstentions could be much greater if abstention were systematic as illustrated in figures 3 where, for any given value of  $\pi$ , everybody for whom  $B > C/\pi$  votes and everybody for whom  $B < C/\pi$  abstains. Residual voters might then have very different interests from the interests of the population as a whole. In particular, people with high  $B$  are likely to be relatively wealthy, while people with low  $B$  are likely to be relatively poor. A situation could easily arise where, as shown in figures 6 and 7, a majority of the population favours one party, a win for that party creates the larger social welfare, but that party loses regardless because the other party appeals to the minority of the population that actually votes. Redistribution of income is a transfer from the few rich to the many poor. Systematic abstentions could easily swing the outcome of an election from the majority party favouring extensive redistribution to the minority party opposing it, where majority and minority in this context refer not just to people who vote, but to the population as a whole.<sup>16</sup>

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<sup>15</sup>On optimal abstention with costly voting, see Borgers (1984). If everybody has the same probability of being correct and as long as that probability exceeds  $\frac{1}{2}$ , then, the more people who vote, the greater is the chance that the right decision will be made. On this "Condorcet Jury Theorem", see Young (1995).

<sup>16</sup>Compulsory voting is advocated by Lijphart (1997). Lijphart claims that turnout is declining in most countries, that the propensity to vote rather than to abstain increases with education and wealth, and that there is a corresponding bias in public policy against the uneducated and the poor.

3) Political Parties' Choice of Platforms: The emphasis in this article has so far been almost entirely upon voters' responses to political parties with given platforms rather than political parties' responses to the preferences of voters. The question of how platforms of political parties are chosen has been largely ignored. For some purposes, this is not unreasonable, just as it is not unreasonable for some purposes to focus upon either the demand side or the supply side of the market. But just as both sides of the market must be considered in the formation of economic policy, so too must behaviour of political parties be recognized in the formation of political policy such as compulsory voting.<sup>17</sup>

Political parties may be inclined to choose platforms appealing to the people who actually vote rather than to the population as a whole. Thus, if the population of voters favours less redistribution than does the population as a whole, both political parties can be expected to include less redistribution in their platforms. Redistribution may be curtailed not so much because people vote for the party with less redistribution in its platform, but because redistribution may be abandoned by both parties. If so, then compulsory voting may restore the redistribution of income onto the political agenda. The effect of bias in abstentions upon the median voter's optimal redistribution is illustrated in the appendix.

4) Frustration of Voting Pacts: Compulsory voting would automatically block small, passionate and well-organized groups of citizen from magnifying their influence on political outcomes by casting a larger proportion of votes than their proportion in the population as a whole. The importance of this consideration depends very much on the total rate of abstention. It is irrelevant if almost everybody votes regardless of whether or not voting is compulsory. With approximately half the population abstaining in many elections and with the increasing polarization of many societies, this may well be the most important consideration.

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<sup>17</sup>Strategic choice of platforms is postulated in Ledyard (1984) where, as discussed above, all politics is about the choice of one number,  $z$ , which may be thought of as the rate of a negative income tax. Every voter has a preferred value of  $z$ , and both political parties choose a value of  $z$  as its platform to maximize support among the electorate. In equilibrium, both parties choose the same  $z$  which turns out to be the first preference of the median voter. Alternatively, in the "citizen-candidate" model, political parties are replaced by individual politicians seeking office, while politicians have policy preferences that they cannot disguise and that they must act upon if they are elected. The outcome is, once again, the first preference of the median voter because only the median voter himself can successfully run for office. On this model, see Osborne and Slivinski (1999) and Besley and Coate (1999). A third such model relies upon "probabilistic voting". Platforms in this model are allocations among voters of the national income, voters are combined into interest groups and each group's proportion of votes for each party is a concave function of the party's offer of income to that group. For a simple description, see Mueller (2004).

## Conclusion

In contrast to person-to-person randomization and the common sense model of how votes may turn out to be pivotal, nation-wide randomization allows people to be permanently left-leaning or permanently right-leaning without removing the uncertainty in the outcome of an election. Nation-wide randomization places self-interest and duty on a common scale, shows how the preference of a majority of eligible voters may fail to be reflected in the outcome of an election, leads naturally to the distinction between duty to other like-minded voters and duty to society as a whole and helps to clarify arguments for and against compulsory voting.

But the “paradox of not voting” is not resolved. Nothing in this paper yields a political analogy to the general equilibrium in the economy where a Pareto optimum emerges from universal self-interested behaviour. Duty rather than self-interest must be invoked to explain why rates of abstention are not very much higher than they are observed to be and why outcomes of elections are not intolerably distant from the preference of the majority of the electorate.

## References

- Beck, N., "A Note on the Probability of a Tied Election", *Public Choice*, 1975, 75-79.
- Besley, T and Coate, S. "An Economic Model of Representative Democracy", *Quarterly Journal of Economics*, 65-95.
- Borgers, T. "Costly Voting", *American Economic Review*, 2004, 57-66.
- Campbell, C.M. "Large Elections and Decisive Minorities", *Journal of Political Economy*, 1999, 1199-1217.
- Chamberlain, G. and Rothschild, M., "A Note on the Probability of Casting a Decisive Vote". *Journal of Economic Theory*, 1981, 152-162.
- Downs, A., *An Economic Theory of Democracy*, Harper and Row, 1957.
- Edlin, A. S., Gelman, A. and Kaplan, N., "Vote for Charity's Sake", *Economists' Voice*, [www.bepress.com/ev](http://www.bepress.com/ev) October, 2008.
- Edlin, A. S., Gelman, A. and Kaplan, N., "Voting as Rational Choice: Why and How People Vote to Improve the Well-being of Others", *Rationality and Society*, 2007, 293-314
- Gelman, A., Katz, J.N. and Bafumi, J., "Standard Voting Power Indexes Do Not Work: An Empirical Analysis", *British Journal of Political Science*, 2004, 657-674.
- Ledyard, J.O., "The Pure Theory of Large Two-candidate Elections", *Public Choice*, 1984, 7-41.
- Lijphart, A., "Unequal Participation: Democracy's Unresolved Dilemma", *The American Political Science Review*, 1997, 1-14.
- Mueller, D.C., *Public Choice III*, Cambridge University Press, 2003.
- Morton, R., "Groups in Rational Turnout Models", *American Journal of Political Science*, 1991, 758-76.
- Osborn, M, and Slivinski, A., "A Model of Political Competition with Citizen-Candidates", *Quarterly Journal of Economics*, 1996, 65-96.
- Riker, W., "Implications from the Disequilibrium of Majority Rule for the Study of Institutions", *The American Political Science Review*, 1980, 432-446.
- Riker, W. and Ordeshook, P.C., "A Theory of the Calculus of Voting", *The American Political Science Review*, 1968, 25-42.

Shapley L. S. and Shubik, M., "A Method for Evaluating the Distribution of Power in a Committee System", *The American Political Science Review*, 787-93.

Uhlener, C.J., "Rational Turnout: the Neglected Role of Groups", *American Journal of Political Science*, 1989, 390-4226

Woodhouse, A.S.P., *Puritanism and Liberty*, 1947.

Young, P., "Condorcet's Theory of Voting", *American Political Science Review*. 1995, 51-64.

## Appendix: Why There is Less Redistribution of Income when the Propensity to Vote is Higher among the Rich than among the Poor.

It is commonly believed that abstentions influence political outcomes, in particular, that a relatively high rate of abstention among the poor tends to produce the amount of redistribution that the government supplies. Presented in this appendix is an example of precisely that. The ingredients of the example are a propensity to vote that increases steadily with income, a skewed distribution of pre-tax income, redistribution by a negative income tax, tax evasion and a tax rate determined by majority rule voting where the first preference of the median voter prevails. The chosen tax rate is lower, and the corresponding redistribution is diminished. Society's chosen tax rate is lower, and redistribution is diminished, when the propensity to vote increases with income than when everybody's propensity to vote is the same.

These are the assumptions:

a) People's pre-tax incomes,  $Y$ , vary steadily from 0 to 1. There is no harm in thinking of units as millions of dollars. Specifically, the density of population,  $p(Y)$ , is inversely proportional to income.

$$p(Y) = 1 - Y \quad (A1)$$

so that the average income is  $\frac{1}{2}$ .

b) The higher a person's income, the more likely is that person to vote rather than to abstain. Specifically, people's propensity to vote is directly proportional to income.

$$v(Y) = Y \quad (A2)$$

A person with no income does not vote at all, a person with the highest income, 1, always votes and people's propensity to vote varies steadily in between. This is a very strong assumption chosen the simplify calculation and to produce a strong result.

c) Redistribution of income is arranged through a negative income tax. Everybody is taxed at a fixed rate,  $t$ , and the entire revenue from the tax is redistributed in equal amounts to everybody. All other uses of public funds are ignored. The preferred tax rate,  $t(Y)$ , of a person with income  $Y$  is the rate at which that person's disposable income,  $I$ , is maximized, where disposable income is pre-tax income, less tax paid, less expenditure to reduce the amount of income declared, plus the equalized transfer of all public revenue. It will be shown that  $t(Y)$  falls as income rises. The exact form of the function,  $t(Y)$ , will be derived.

d) People evade tax to the extent that it is in their interest to do so. Only a portion of one's income is hidden from the tax collector because the cost of hiding income increases steadily with the amount of income one hides. Specifically the cost of hiding the first 1% of one's income is 1% of the amount hidden, the cost of hiding the second 1% is two percent of the amount

hidden, and so on until the cost of concealing the final 1% is 100% of the amount hidden. Tax evasion is one of several ways of reducing one's tax bill. Others are to work less and to engage in more do-it-yourself activities, but these are ignored here. The important consideration is that each person hides income up to the point where cost of hiding an extra dollar of income the cost of the tax that would otherwise be paid. This assumption ensures that nobody, not even the very poorest person, wants a tax rate as high as 100%.

e) The tax rate is chosen by majority rule voting where the first preference of the median voter prevails.

f) In voting about the tax rate and in choosing how much of one's income to declare, everybody acts in his own interest exclusively.

On these assumptions, a distinction can be drawn between two median incomes, the median,  $M$ , in the entire population and the median,  $R$ , among people who choose to vote rather than to abstain. The median  $M$  is the income of the person in the middle when everybody, regardless of whether or not he votes, is lined up in accordance with pre-tax income  $Y$ . The median  $R$  is the income of the person in the middle when only those people who vote are lined up in accordance with pre-tax income.

It will be shown that, on these assumptions,

- the income of the median *person*,  $M$ , is less than the income of the median *voter*,  $R$ . Specifically, the pre-tax income of the median person is .293, while the pre-tax income of the median voter is .5.
- the preferred tax rate of the median person is 29.3%, raising the disposable income,  $I$ , of the median person by about 10% from .239 to .324. [That the preferred tax rate of the median person is the same as his pre-tax income is an accidental consequence of the assumptions, a quirk with no economic significance.]
- The preferred tax rate of the median voter is 0%.

In this example, abstentions decreasing with income block redistribution altogether. The effect of abstentions would be in the same direction but less dramatic if the correlation between income and the propensity to vote were less pronounced.

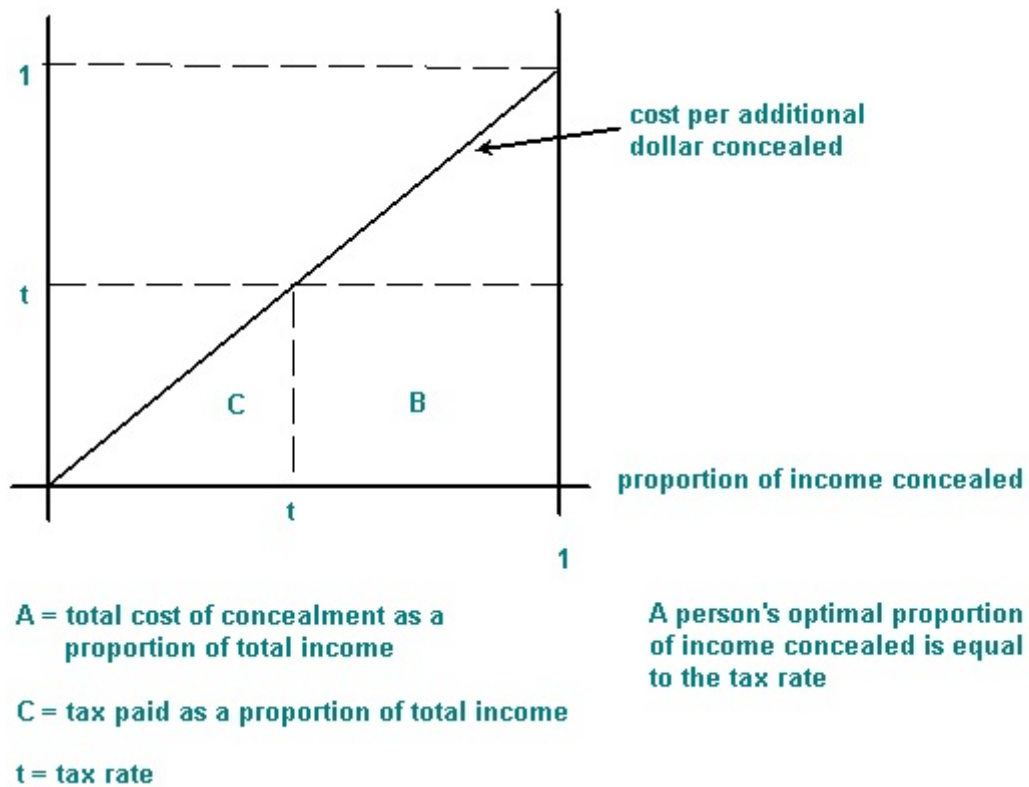
The median voter's preferred tax rate,  $t(M)$ , is computed in three stages. First the gap between disposable income,  $I$ , and declared income,  $Y$ , is computed as a function of the tax rate. Then a person's preferred tax rate is computed as a function of income, i.e. the function  $t(Y)$  is derived. Finally, the income,  $M$ , of the median voter is computed together with the median voter's preferred tax rate.

In accordance with assumption (d), the taxpayer's cost of tax evasion is shown in figure



A. The premise behind assumption (d) is that there are inexpensive ways to conceal small amounts of income, but, the more income is concealed, the more expensive does extra concealment become. Both axes are proportions between 0 and 1, the horizontal axis of income concealed, the vertical axis of the cost of concealment per dollar concealed. The height of the diagonal line is the marginal cost of concealment as a function of the proportion concealed. In choosing the proportion of income to conceal, the taxpayer equates the marginal cost of concealment to the tax that would otherwise be paid. A person conceals income up to the point where it would be more costly to conceal an extra dollar of income than to pay the tax on that amount. It then follows that a tax rate of  $t$  induces people to conceal a fraction  $t$  of their incomes, so that declared income becomes a fraction  $(1 - t)$  of true income and tax paid as a proportion of true income becomes  $t(1 - t)$ , shown as the area B on the figure. The full cost of concealment per dollar of income is indicated by the area C which is equal  $t^2 / 2$ . Denote the average income by  $Y^A$ . It follows from assumption (a) that  $Y^A$  equals  $1/2$ . Total tax revenue per person is  $t(1 - t)Y^A$ .

**Figure A: Taxpayers' Cost and Benefit of Concealment of income from the Tax Collector**



It then follows that the disposable income,  $I$ , of a person with pre-tax income,  $Y$ , becomes

$$I = \text{pre-tax income} - \text{tax paid} - \text{cost of concealment} - \text{transfer (equal to average tax revenue)}$$

$$= Y - t(1 - t)Y - (t^2 / 2)Y - t(1 - t)Y^A \quad (A3)$$

If empowered to choose the tax rate all by himself, a person with income Y would choose a rate, called t(Y), to maximize disposable income, I. The rate would be set so that  $\delta I / \delta t = 0$ . Specifically,

$$\delta I / \delta t = -Y + 2tY - tY + (1 - 2t)Y^A = 0 \quad (A4)$$

implying that  $t(Y) = (Y^A - Y) / (2 Y^A - Y) \quad (A5)$

$$= (1 - 2Y) / (2 - 2Y) \quad \text{Because } Y^A = 1/2$$

This is the exact specification of the function t(Y) in assumption ©) above. It follows at once that the preferred tax rate of a person for whom Y = 0 is 50%, and that the preferred tax rate declines steadily with income until it falls to 0 at Y<sup>A</sup>. Anybody with more than the average income would prefer the tax rate to be negative, but that is not relevant here.

Now consider the median incomes, M and R, for all the entire population and for voters only. The median income, M, for the entire population is the solution of the equation

$$\int_0^M p(Y) dY = \int_M^1 p(Y) dY \quad (A6)$$

or equivalently

$$\int_0^M (1 - Y) dY = \int_M^1 (1 - Y) dY \quad (A7)$$

Since the integral of (1 - Y) is (Y - Y<sup>2</sup> / 2), equation (7) implies that

$$[M - M^2 / 2] = 1/2 - [M - M^2 / 2] \quad (A8)$$

or  $2 M^2 - 4M + 1 = 0$

so that  $M = 1 - 2^{-1/2} = .293$

and the median person's preferred tax rate in accordance with equation (A5) is

$$t(M) = (Y^A - M) / (2 Y^A - M) = (1/2 - .293) / (1 - .293) = .293$$

If everybody voted, causing redistribution to reflect the preference of the median voter, the tax rate would be .293 and the per capita transfer from the government .1036, raising the income of the poorest person from nothing to .1036 equivalent to 20.72% of average income. From equation (A3), it follows that redistribution raises the disposable income of the median person from .293 to .323, but lowers the disposable income of the richest person from 1 to .856.

By contrast, the income,  $R$ , of the median income among all voters is identified by the equation

$$\int_0^R p(Y)v(Y)dY = \int_R^1 p(Y)v(Y)dY \quad (A9)$$

where  $p(Y)v(Y)$  is the density of voters with income  $Y$ . With  $p(Y)$  and  $v(Y)$  in assumptions (a) and (b), equation (A9) becomes

$$\int_0^R (1-Y)YdY = \int_R^1 (1-Y)YdY \quad (A10)$$

Since the integral of  $(1 - Y)Y$  is  $(Y^2 / 2 - Y^3/3)$ , equation (A10) reduces to

$$[R^2/2 - R^3/3] = (1/2 - 1/3) - [R^2/2 - R^3/3]$$

or 
$$4 R^3 - 6 R^2 + 1 = 0$$

implying that  $R = 1/2$  which just happens to equal the pre-tax, pre-transfer income in the population as a whole. The median income among all voters just happens to equal the average income among all people, whether or not they vote. But, as is immediately evident from equation (A5), the person with the average income prefers a tax rate of 0% with no redistribution at all.

On the assumptions in this example, redistribution of income is blocked altogether by the greater propensity to vote rather than to abstain among the rich than among the poor. Milder assumptions would yield a less dramatic result, but the general direction of this result would remain. The higher the correlation between income and the propensity to vote, the less redistribution there would be.