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A terminological note on cyclotomic polynomials and Blaschke matrices

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Abstract

In a recent paper, Mertens and Ravn (2010) study the effects of anticipated fiscal policy shocks in a structural vector autoregressive model. The authors maintain that (i) the lag polynomial $\Theta^{(q-1)}(L) = \omega^{q-1} + \omega^{q-2}L + \ldots + \omega L^{q-2} + L^{q-1}$ is a cyclotomic polynomial and (ii) the matrix $B(L)$ which transforms a non-fundamental MA representation into a fundamental one is a Blaschke matrix. Though, the results in Mertens and Ravn (2010) are correct, we find that the terms 'cyclotomic' and 'Blaschke matrix' are misused.

JEL classification: C32, E32
Keywords: Nonfundamentalness, Cyclotomic polynomial, Blaschke matrix

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1 Introduction

In a recent paper published in *The Economic Journal*, Mertens and Ravn (2010) study the effects of anticipated fiscal policy shocks in a structural vector autoregressive model. Due to the anticipation, the moving average (MA) representation of this model is nonfundamental, i.e. some of the roots of the MA determinant lie inside the unit circle. The associated lag polynomial of the anticipated shock is given by

$$\Theta^{(q-1)}(L) = \omega^{q-1} + \omega^{q-2}L + \ldots + \omega L^{q-2} + L^{q-1}, \tag{1}$$

where $q \geq 1$ and $\omega \in (0,1)$ are the length of anticipation and the anticipation rate, respectively. Based on Lippi and Reichlin (1993, 1994), the authors use the transformation matrix

$$B(L) = K\, M_1(L) M_2(L) \ldots M_{q-1}(L) \tag{2}$$

to flip the roots of the MA determinant and transform the nonfundamental MA representation into a fundamental one. Mertens and Ravn (2010) maintain that $\Theta^{(q-1)}(L)$ is a *cyclotomic polynomial* and $B(L)$ is a *Blaschke matrix*. However, we find that $\Theta^{(q-1)}(L)$ is neither cyclotomic nor $B(L)$ of Blaschke-type.

2 Cyclotomic Polynomials

We begin by stating the definition of a cyclotomic polynomial as it can be found in many textbooks.[1]

Definition 1 *Let n be a positive integer. Then the nth cyclotomic polynomial, denoted as $\Phi_n(x)$, is defined as*

$$\Phi_n(x) = \prod_{k=1}^{\varphi(n)} (x - \zeta_k), \tag{3}$$

where $\zeta_1, \ldots, \zeta_{\varphi(n)}$ are the $\varphi(n)$ primitive nth roots of unity and $\varphi(n)$ denotes Euler's totient function.

Remark 1 *A primitive nth root satisfies $\zeta_k^n = 1$ with $ord(\zeta_k) = n$, where $ord(\cdot)$ denotes the order of ζ_k, i.e. the smallest positive integer n that satisfies $\zeta_k^n = 1$.*

Remark 2 *Euler's totient function $\varphi(n)$ can be written as*

$$\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d} \tag{4}$$

[1] See e.g. Lang (2002). Readers who are interested in the historical background of cyclotomic polynomials are referred to Sury (1999).

with $\mu(d)$ as Möbius function

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1 \\ (-1)^m & \text{if } d \text{ is squarefree and } m \text{ is the number of prime divisors of } d \\ 0 & \text{else} \end{cases} \quad (5)$$

Clearly, the polynomial $\Theta^{(q-1)}(L)$ does not satisfy definition 1, but it can be related to cyclotomic polynomials in the following way: Let $x = L/\omega$, then the polynomial equation $\Theta^{(q-1)}(L) = 0$ is equivalent to

$$\tilde{\Theta}^{(q-1)}(x) = \sum_{j=0}^{q-1} x^j = \frac{x^q - 1}{x - 1} = 0. \quad (6)$$

Obviously, the $q-1$ roots of $\tilde{\Theta}^{(q-1)}(x) = 0$ are the qth roots of unity, satisfying $x^q - 1 = 0$. In Euler form, these roots can be explicitly expressed as

$$x_j^{(q-1)} = \exp\left(\frac{2\pi i}{q} j\right) \qquad (j = 1, \ldots, q-1), \quad (7)$$

where $i = \sqrt{-1}$ is the imaginary unit. The roots of $\Theta^{(q-1)}(L) = 0$ are then implicitly given by $\omega_j = \omega x_j^{(q-1)}$ $(j = 1, \ldots, q-1)$.

From definition 1 and equation (6) follows that the polynomial $\tilde{\Theta}^{(q-1)}(x)$ can be written as the *product* of cyclotomic polynomials. Since $x^q - 1 = \prod_{d|q} = \Phi_d(x)$, we can write

$$\tilde{\Theta}^{(q-1)}(x) = \prod_{d|q,\, d\geq 2} \Phi_d(x). \quad (8)$$

Note that (8) reduces to $\tilde{\Theta}^{(q-1)}(x) = \Phi_q(x)$ if q is *prime*.

If q is *not* prime, then $\tilde{\Theta}^{(q-1)}(x)$ is not cyclotomic. For example, if $q = 4$, then

$$\tilde{\Theta}^{(3)}(x) = 1 + x + x^2 + x^4 = \frac{x^4 - 1}{x - 1} = \Phi_2(x)\Phi_4(x) \quad (9)$$

with the cyclotomic polynomials $\Phi_2(x)$ and $\Phi_4(x)$ given by

$$\Phi_2(x) = \prod_{d|2,\, d\geq 1} (x^{2/d} - 1)^{\mu(d)} = (x^2 - 1)(x - 1)^{-1} = x + 1 \quad (10)$$

$$\Phi_4(x) = \prod_{d|4,\, d\geq 1} (x^{4/d} - 1)^{\mu(d)} = (x^4 - 1)(x^2 - 1)^{-1} = x^2 + 1 \quad (11)$$

3 Blaschke matrices

As in Lippi and Reichlin (1993, 1994), the Blaschke matrix is defined as follows:

Definition 2 *$A(L)$ is a Blaschke matrix if it satisfies*

1. *$A(L)$ has no poles of modulus smaller or equal to unity.*
2. *If $A^*(L)$ denotes the transpose of the complex conjugate of $A(L)$, then $A(L)^{-1} = A^*(L^{-1})$.*

In Mertens and Ravn (2010) K is a constant orthogonal matrix and $M_i(L)$ is given by

$$M_i(L) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1-\omega_i L}{L-\bar{\omega}_i} \end{pmatrix}, \quad (i = 1, \ldots, q-1), \tag{12}$$

where ω_i are the roots of $\Theta^{(q-1)}(L) = 0$ and $\bar{\omega}_i$ denotes the complex conjugate of ω_i. The matrix $B(L)$ has the property to flip the stable roots of the MA representation. However, it is not a Blaschke matrix. Since all roots of $\Theta(L)$ are stable, i.e. $|\bar{\omega}_i| < 1$, the pole of $|M_i| = 0$ (i.e. $\bar{\omega}_i$) lies inside the unit circle. Hence, $B(L)$ does *not* satisfy condition 1 of definition 2 und therefore is *not* a Blaschke matrix.

A Blaschke matrix would be of the form

$$\tilde{B}(L) = \tilde{K}\, \tilde{M}_1(L) \ldots \tilde{M}_{q-1}(L), \tag{13}$$

where, in line with Lippi and Reichlin (1994),

$$\tilde{M}_i(L) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{L-\omega_i}{1-\bar{\omega}_i L} \end{pmatrix}, \quad (i = 1, \ldots, q-1).^2 \tag{14}$$

However, the Blaschke matrix $\tilde{B}(L)$ flips the *unstable* roots of the MA determinant and therefore transforms a fundamental MA representation into a nonfundamental one. Contrarily, the transformation matrix $B(L)$ used by Mertens and Ravn (2010) works in the opposite direction. It flips the *stable* roots of the MA determinant and therefore transforms a nonfundamental MA representation back to a fundamental one.

4 Conclusion

The purpose of this short terminological note is to clarify the terms cyclotomic polynomial and Blaschke matrix and relate it to Mertens and Ravn (2010) and to the topic of news shocks in general.

[2]Note that the term $\frac{L-\omega_i}{1-\bar{\omega}_i L}$ is a Blaschke factor which was first introduced by Blaschke (1915).

References

Blaschke, W. (1915) "Eine Erweiterung des Satzes von Vitali über Folgen analytischer Funktionen" *Leipziger Berichte* 67, 194-200.

Lang, S. (2002) *Algebra*, Springer-Verlag: Berlin.

Lippi, M. and Reichlin, L. (1993) "The dynamic effects of aggregate demand and supply disturbances: comment" *American Economic Review* 83(3), 644-652.

Lippi, M. and Reichlin, L. (1994) "VAR analysis, nonfundamental representations, Blaschke matrices" *Journal of Econometrics* 73(1), 307-325.

Mertens, K. and Ravn, M.O. (2010) "Measuring the impact of fiscal policy in the face of anticipation: A structural VAR approach" *The Economic Journal* 120, 393-413.

Sury, B. (1999) "Cyclotomy and cyclotomic polynomials" *Resonance* **4**(12), 41-53.