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Stability of Linear Stochastic Difference Equations in Controlled Random Environments

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Abstract

We consider the stochastic sequence $\{Y_t\}_{t \in \mathbb{N}}$ defined recursively by the linear relation $Y_{t+1} = A_t Y_t + B_t$ in a random environment. The environment is described by the stochastic process $\{(A_t, B_t)\}_{t \in \mathbb{N}}$ and is under the simultaneous control of several agents playing a discounted stochastic game. We formulate sufficient conditions on the game which ensure the existence of Nash equilibrium in Markov strategies which has the additional property that, in equilibrium, the process $\{Y_t\}_{t \in \mathbb{N}}$ converges in distribution to a unique stationary sequence.

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1 Introduction

This paper considers the stochastic sequence $\{Y_t\}_{t \in \mathbb{N}}$ defined recursively by the linear relation

$$Y_{t+1} = A_t Y_t + B_t \quad (t \in \mathbb{N}) \quad (1)$$

in the random environment $\{(A_t, B_t)\}_{t \in \mathbb{N}}$. The dynamics of the environment is under the simultaneous control of several agents who play a discounted stochastic game. We formulate sufficient conditions on the game which guarantee the existence of Nash equilibria in Markov strategies which have the additional property that, in equilibrium, the solution to (1) converges in distribution as $t \rightarrow \infty$.

Stochastic sequences of the form (1) have been extensively investigated under a mean contraction condition and under the assumption that the driving sequence $\{(A_t, B_t)\}_{t \in \mathbb{N}}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is stationary under the *exogenous* measure \mathbb{P} . For example, Vervaat (1979) considers the case where the environment consists of i.i.d. random variables. Brandt (1986) assumes that the driving sequence is stationary and ergodic under the law \mathbb{P} ; see also Borovkov (1998). Horst (2001) studies the case where the environment is asymptotically stationary in the sense that the process $\{(A_t, B_t)\}_{t \in \mathbb{N}}$ is stationary under some law \mathbb{Q} on (Ω, \mathcal{F}) which coincides with \mathbb{P} on the tail-field generated by $\{(A_t, B_t)\}_{t \in \mathbb{N}}$.

In view of many applications in economics it seems natural to extend the analysis of stochastic difference equations to situations in which the environment is under the simultaneous control of several agents. In such a situation the measure \mathbb{P} is no longer exogenous, but is derived through a game-theoretic solution concept. For example, the process $\{Y_t\}_{t \in \mathbb{N}}$ could be sequence of temporary equilibrium prices of a risky asset generated by the microeconomic interaction of investors who are active on a financial market. In such a model, the sequence

$\{(A_t, B_t)\}_{t \in \mathbb{N}}$ may be driven by the changes in the behavioral characteristics of the agents. If the investors are ‘large’, i.e., if their behavior influences the dynamics of the random environment, then it seems natural to assume that the agents anticipate their impact on the formation of stock prices and, therefore, interact in a strategic manner. In such a context, the process $\{Y_t\}_{t \in \mathbb{N}}$ may be viewed as the state sequence associated to a stochastic game, and so the probabilistic structure of the random environment specified by the measure \mathbb{P} is no longer exogenous. Instead, it is defined through an equilibrium strategy implemented by the individual investors. In order to analyze the dynamics of the price process in equilibrium, it is now desirable to have sufficient conditions which guarantee the existence of Nash equilibria which ensure that the sequence $\{Y_t\}_{t \in \mathbb{N}}$ settles down in the long run.

In this paper we consider discounted stochastic games with weakly interacting players in which the conditional distribution of the random variable (A_t, B_t) only depends on the average action taken by the players. We formulate conditions on the game which guarantee the existence of a Nash equilibrium in Markov strategies τ such that under the induced measure \mathbb{P}^τ on (Ω, \mathcal{F}) the solution to (1) converges as $t \rightarrow \infty$. In a first step we show that stochastic games in which the state sequence follows a linear dynamics and in which the interaction between different agents is sufficiently weak have stationary equilibria in Markov strategies that depend in a Lipschitz continuous manner on the current state. This part of the paper is inspired by the work of Curtat (1996) and uses a perturbation of a *Moderate Social Influence* condition introduced in Horst and Scheinkman (2002). Under a suitable mean contraction condition on the random environment we then prove that the sequence $\{(A_t, B_t)\}_{t \in \mathbb{N}}$ defined on $(\Omega, \mathcal{F}, \mathbb{P}^\tau)$ has a nice tail structure in the sense of Horst (2001). This allows us to show that the shifted sequence $\{Y_{t+T}\}_{t \in \mathbb{N}}$ converges in law to a uniquely

determined stationary process as $T \rightarrow \infty$.

The remainder of this paper is organized as follows. In Section 2 we formulate our main results. Section 3 proves the existence of a Lipschitz continuous equilibrium in Markov strategies. The convergence result for the solution to (1) is proved in Section 4.

2 Assumptions and the main results

Let $\psi := \{(A_t, B_t)\}_{t \in \mathbb{N}}$ be a sequence of \mathbb{R}^2 -valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{Y_t\}_{t \in \mathbb{N}}$ be the sequence in (1) driven by the “input” ψ . In this section we specify a probabilistic framework which allows us to analyze the asymptotic behavior of the solution to the linear stochastic difference equation (1) in a situation in which the evolution of the random environment is controlled by several strategically interacting agents.

The long run behavior of the sequence $\{Y_t\}_{t \in \mathbb{N}}$ has been intensively investigated under a mean contraction condition and under the assumption that ψ is stationary under some *exogenous* measure \mathbb{P} ; see, e.g., Brandt (1986) or Vervaat (1979). Horst (2001) assumes that the environment is asymptotically stationary and that it has a nice tail structure in the sense of the following definition.

Definition 2.1 (*Horst (2001)*) *Let $\hat{\mathcal{F}}_t := \sigma(\{(A_s, B_s)\}_{s \geq t})$ and let*

$$\mathcal{T}_\psi := \bigcap_{t \in \mathbb{N}} \hat{\mathcal{F}}_t, \tag{2}$$

be the tail- σ -algebra generated by ψ . A driving sequence ψ is called nice with respect to a probability measure \mathbb{Q} on (Ω, \mathcal{F}) if the following properties are satisfied:

(i) ψ is stationary and ergodic under \mathbb{Q} and satisfies

$$\mathbb{E}_{\mathbb{Q}} \ln |A_0| < 0 \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}} (\ln |B_0|)^+ < \infty \quad (3)$$

where $\mathbb{E}_{\mathbb{Q}}$ denotes the expectation with respect to the measure \mathbb{Q} .

(ii) The asymptotic behavior of ψ is the same under \mathbb{P} and \mathbb{Q} , i.e.,

$$\mathbb{P} = \mathbb{Q} \quad \text{on} \quad \mathcal{T}_{\psi}. \quad (4)$$

Remark 2.2 We denote by $\|\cdot\|_{\mathcal{E}}$ the total variation of a signed measure on a measurable space (E, \mathcal{E}) . Since

$$\lim_{t \rightarrow \infty} \|\mathbb{P} - \mathbb{Q}\|_{\hat{\mathcal{F}}_t} = \|\mathbb{P} - \mathbb{Q}\|_{\mathcal{T}_{\psi}}, \quad (5)$$

a driving sequence ψ satisfies (4) if and only if it becomes stationary in the long run. This is equivalent to the existence of a sequence $\{c_t\}_{t \in \mathbb{N}}$ satisfying $\lim_{t \rightarrow \infty} c_t = 0$ and

$$\sup_{l \geq t} \|\mathbb{P} - \mathbb{Q}\|_{\hat{\mathcal{F}}_{t,l}} \leq c_t, \quad (6)$$

where $\hat{\mathcal{F}}_{t,l} := \sigma(\{(A_s, B_s)\}_{t \leq s \leq l})$. Here, both (5) and (6) follow from the continuity of the total variation distance along increasing and decreasing σ -algebras.

In the sequel it will be convenient to denote by $\text{Law}(Y, \mathbb{P})$ the law of a random variable Y on $(\Omega, \mathcal{F}, \mathbb{P})$ and to write \xrightarrow{w} for weak convergence of probability measures.

Let us turn to the solution $\{Y_t\}_{t \in \mathbb{N}}$ of (1). For any initial value $Y_0 = y \in \mathbb{R}$, we have the explicit representation

$$Y_t = y_t(y, \psi) := \sum_{j=0}^{t-1} \left(\prod_{i=t-j}^{t-1} A_i \right) B_{t-j-1} + \left(\prod_{i=0}^{t-1} A_i \right) y \quad (t \in \mathbb{N}). \quad (7)$$

In the stationary setting $\mathbb{P} = \mathbb{Q}$ analyzed by Brandt (1986), we may as well assume that the driving sequence is defined for all $t \in \mathbb{Z}$, due to Kolmogorov's extension theorem. Under the *mean contraction condition* (3), there exists a unique stationary solution of (1) under \mathbb{Q} driven by ψ . That is, there is a unique stationary process $\{Y_t^*\}_{t \in \mathbb{Z}}$ which satisfies the recursive relation (1) for all $t \in \mathbb{Z}$. The random variable Y_t^* is \mathbb{Q} -a.s. finite, takes the form

$$Y_t^* = \sum_{j=0}^{\infty} \left(\prod_{i=t-j}^{t-1} A_i \right) B_{t-j-1} \quad (t \in \mathbb{Z}), \quad (8)$$

and, for any initial value $y \in \mathbb{R}$, the solution $\{y_t(y, \psi)\}_{t \in \mathbb{N}}$ to (1) converges almost surely to the stationary solution in the sense that

$$\lim_{t \rightarrow \infty} |y_t(y, \psi) - Y_t^*| = 0 \quad \mathbb{Q}\text{-a.s.}$$

In the non-stationary case $\mathbb{P} \neq \mathbb{Q}$ studied in Horst (2001) the shifted sequence $\{y_{t+T}(y, \psi)\}_{t \in \mathbb{N}}$ driven by a nice input ψ converges in distribution to the unique stationary solution $\{Y_t^*\}_{t \in \mathbb{N}}$ to (1) under \mathbb{Q} :

$$\text{Law}(\{y_{t+T}(y, \psi)\}_{t \in \mathbb{N}}, \mathbb{P}) \xrightarrow{w} \text{Law}(\{Y_t^*\}, \mathbb{Q}) \quad (T \rightarrow \infty). \quad (9)$$

So far, the asymptotics of the sequence $\{Y_t\}_{t \in \mathbb{N}}$ have only been analyzed in situations where the probabilistic structure of the random environment ψ is described by an *exogenous* measure. Our aim is study to dynamics of the solution to the linear stochastic difference equation (1) in a situation in which the evolution of ψ is controlled by strategically interacting agents who play a stochastic game.

2.1 The stochastic game

The infinite-horizon discounted stochastic games $\Sigma = (I, X, (U^i), \beta, Q, y)$ which we consider in this paper are defined in terms of the following objects:

- $I = \{1, 2, \dots, M\}$ is a finite set of *players*.
- $X \subset \mathbb{R}$ is a common compact and convex *action space* for the players.
- $U^i : \mathbb{R} \times \prod_{i \in I} X \rightarrow \mathbb{R}$ is the *utility function* for player $i \in I$.
- $\beta \in (0, 1)$ is a common discount factor.
- Q is a stochastic kernel from X to \mathbb{R}^2 .
- $y \in \mathbb{R}$ is the starting point of the state sequence $\{Y_t\}_{t \in \mathbb{N}}$.

A typical action of player $i \in I$ is denoted $x^i \in X$. The actions taken his competitors are denoted $x^{-i} \in X^{-i} := \{x^{-i} = (x^j)_{j \in I \setminus \{i\}}\}$, and $\overline{X} := \{\underline{x} = (x^i)_{i \in I} : x^i \in X\}$ is the compact set of all *action profiles*. To each action profile $\underline{x} \in \overline{X}$, we associate the average action $x := \frac{1}{M} \sum_{i \in I} x^i$.

At each time $t \in \mathbb{N}$, the players observe the current position Y_t of the state sequence $\{Y_t\}_{t \in \mathbb{N}}$. They take their actions $x_t^i = \tau^i(Y_t)$ independently of each other according to a *stationary Markov strategy* $\tau^i : \mathbb{R} \rightarrow X$ and the selected action profile $\underline{x}_t = (x_t^i)_{i \in I} = (\tau^i(Y_t))_{i \in I}$ along with the present state Y_t yields the instantaneous payoff $U^i(Y_t, \underline{x}_t) = U^i(Y_t, x_t^i, x_t^{-i})$ to the agent $i \in I$. We assume that the law of motion only depends on the average action taken by the individual players. More precisely,

$$Y_{t+1} = A_t Y_t + B_t \quad \text{with} \quad (A_t, B_t) \sim Q(c x_t; \cdot) \quad \text{for some} \quad c > 0 \quad \text{where} \quad x_t := \frac{1}{M} \sum_{i=1}^M x_t^i. \quad (10)$$

In this sense we assume that the indirect interaction between different agents, i.e., the interaction via the state sequence $\{Y_t\}_{t \in \mathbb{N}}$, is *global*. Thus, in games with many players or in

games with a small $c > 0$, the impact of an individual agent on the dynamics of the random environment $\{(A_t, B_t)\}_{t \in \mathbb{N}}$ for the evolution of the state sequence $\{Y_t\}_{t \in \mathbb{N}}$ is weak.

Remark 2.3 *For our subsequent analysis it will be essential that the impact of an individual player on the law of the random environment is sufficiently weak. Assuming that the conditional distribution of the random variable (A_t, B_t) depends on the current action profile \underline{x}_t only through the quantity $cx_t = \frac{c}{M} \sum_{i \in I} x_t^i$ simplifies the formulation of an appropriate weak interaction condition. Of course, weak dependence assumptions can also be formulated differently.*

A stationary Markov strategy $\tau = (\tau^i)_{i \in I}$ along with an initial distribution μ for the starting point of the state sequence and together with the law of motion Q induces a probability measure \mathbb{P}_μ^τ on (Ω, \mathcal{F}) in the canonical way. Under the measure \mathbb{P}_μ^τ the state sequence $\{Y_t\}_{t \in \mathbb{N}}$ is a Markov chain on the state space \mathbb{R} and the expected discounted reward to player $i \in I$ is given by

$$J^i(y, \tau) = J^i(y, \tau^i, \tau^{-i}) := \mathbb{E}_\mu^\tau \left[\sum_{t=0}^{\infty} \beta^t U^i(Y_t, \underline{x}_t) \right]. \quad (11)$$

Here the expectation is taken with respect to the measure \mathbb{P}_μ^τ and $\tau^{-i} = (\tau^j)_{j \neq i}$. In what follows we shall write \mathbb{P}_y^τ for $\mathbb{P}_{\delta_y}^\tau$.

Definition 2.4 *A stationary Markov strategy profile τ is a Nash equilibrium for Σ if no player can increase his payoff by unilateral deviation from τ , i.e., if*

$$J^i(y, \tau) \geq J^i(y, \sigma^i, \tau^{-i}) \quad (12)$$

for all Markov strategies $\sigma^i : \mathbb{R} \rightarrow X$ and each $i \in I$.

Our objective is to formulate conditions which guarantee the existence of a Nash equilibrium in Markov strategies τ such that the Markov chain $(\{Y_t\}, \mathbb{P}_y^\tau)$ converges in law to a unique limiting distribution. To this end, we need to assume strong concavity of an agents' utility function with respect to his own action and we have to place a quantitative bound on the dependence of the instantaneous utility for player $i \in I$ on the actions taken by his competitors.

Assumption 2.5 (i) *Uniformly in $i \in I$, the utility functions $U^i : \mathbb{R} \times \overline{X} \rightarrow \mathbb{R}$ are bounded, Lipschitz continuous and twice continuously differentiable.*

(ii) *There exists constants $L^{i,i}(y) > 0$ and $L > 0$ such that*

$$\frac{\partial^2}{\partial (x^i)^2} U^i(y, x^i, x^{-i}) \leq -L^{i,i}(y) \leq -L < 0. \quad (13)$$

In particular, the function $U^i(y, \cdot, x^{-i})$ is strongly concave on X .

(iii) *The law $Q(x; \cdot)$ has a density $q(x; \cdot)$ with respect to some measure ν on \mathbb{R}^2 . The maps $x \mapsto q(x; \eta)$ are uniformly bounded, uniformly Lipschitz continuous and twice continuously differentiable on an open set containing X . Moreover, $|\frac{\partial}{\partial x} q(x, \eta)| \leq \varrho_1(\eta)$ and $|\frac{\partial^2}{\partial x^2} q(x, \eta)| \leq \varrho_2(\eta)$ for some functions $\varrho_1, \varrho_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ with are integrable with respect to μ .*

The Lipschitz continuity condition on the conditional densities $q(x; \cdot)$ translates into a norm-continuity assumption on the transition probabilities $Q(x; \cdot)$:

$$\|Q(x_n; \cdot) - Q(x; \cdot)\|_{\mathcal{B}} \xrightarrow{n \rightarrow \infty} 0 \quad \text{if} \quad x_n \xrightarrow{n \rightarrow \infty} x,$$

where \mathcal{B} denotes the Borel- σ -field on \mathbb{R}^2 . Such a norm-continuity condition has also been imposed by, e.g., Nowak (1985) and Duffie, Geanakoplos, MasColell, and McLennan (1994).

Let us consider an example where our assumptions on the densities $q(x; \cdot)$ can indeed be verified.

Example 2.6 *Let $f : X \rightarrow \mathbb{R}^2$ be a two times continuously differentiable function and let $\varphi_m(\cdot)$ be the density with respect to Lebesgue measure of the two-dimensional standard normal distribution with mean $m = (m_1, m_2)$. It is easy to show that $q(x; \eta) := \varphi_{f(x)}(\eta)$ satisfies Assumption 2.5 (ii).*

In order to establish the existence of Lipschitz continuous Nash equilibria in the stochastic game under consideration, we have to control the strength of interactions between different players. To this end, we introduce the constants

$$L^{i,j}(y) = \sup_{\bar{x} \in \bar{X}} \left| \frac{\partial^2}{\partial x_i \partial x_j} U^i(y, \bar{x}) \right| \quad (i \neq j) \quad \text{and} \quad L^i = \sup_{y \in \mathbb{R}, \bar{x} \in \bar{X}} \left| \frac{\partial^2}{\partial x_i \partial y} U^i(y, \bar{x}) \right|. \quad (14)$$

The quantity $L^{i,j}(y)$ may be viewed as a measure for the dependence of agent i 's instantaneous utility of the choice of player j , given the current state y . By analogy, L^i measures the dependence of his one-period utility of the current position on the state sequence. We also put

$$\hat{L} := \sup_{x \in X} \frac{c^2}{M^2} \left\| \frac{\partial^2}{\partial x^2} q(x, \cdot) \right\|_{L^1} \quad (15)$$

where $\|\cdot\|_{L^1}$ denotes the L^1 -norm with respect to the measure ν . In order to guarantee the existence of Lipschitz continuous equilibria in the discounted stochastic game Σ we need to assume that the interaction between different agents is not too strong. Since the players interact both through their instantaneous utility functions and through their individual impacts on the evolution of the state sequence, we have to control both the dependence of an action of player j on the instantaneous utility of the agent i and the dependence of the law of

motion on the actions taken by an individual player. We formulate this conditions in term of the following perturbation of the *Moderate Social Influence* in Horst and Scheinkman (2002).

Assumption 2.7 *There exists $\gamma < 1$ such that the following holds for all $i \in I$:*

$$\sum_{j \neq i} L^{i,j}(y) + \frac{\beta}{1-\beta} \hat{L} M \|U^i\|_{\infty} \leq \gamma L^{i,i}(y). \quad (16)$$

For games with many players, i.e., for $M \rightarrow \infty$, and for games in which the impact of an individual agent on the dynamics of the state sequence is very weak, i.e., for $c \rightarrow 0$, our condition (16) reduces to the *Moderate Social Influence* assumption $\sum_{j \neq i} L^{i,j}(y) < \gamma L^{i,i}(y)$ introduced in Horst and Scheinkman (2002).

Let us consider a simple case study where our Assumption 2.7 can easily be verified.

Example 2.8 *Consider a law of motion Q which is of the form*

$$Q\left(\frac{c \sum_{i \in I} x_i}{M}; \cdot\right) = \frac{c \sum_{i \in I} x_i}{M} Q_1(\cdot) + \left(1 - \frac{c \sum_{i \in I} x_i}{M}\right) Q_2(\cdot). \quad (17)$$

If $Q_i(\cdot)$ has a bounded density $q_i(\cdot)$ with respect to some law ν on \mathbb{R}^2 , then our Moderate Social Influence condition translates into an assumption on the marginal rates of substitution.

Indeed, (16) holds if

$$\sum_{j \neq i} \left| \frac{\partial^2}{\partial x_i \partial x_j} U^i(y, \underline{x}) \right| \leq \gamma \left| \frac{\partial^2}{\partial^2 x_i} U^i(y, \underline{x}) \right| \quad (18)$$

for all $i \in I$ and for some $\gamma < 1$. In addition, (13) requires $\inf_y \left| \frac{\partial^2}{\partial^2 x_i} U^i(y, \underline{x}) \right| > 0$.

Remark 2.9 *Observe that (18) is necessary for our Moderate Social Influence condition.*

On the other hand, if (18) holds, then we can always choose a small enough $c > 0$ such that Assumption 2.7 holds.

We are now going to state a result which guarantees the existence of Lipschitz continuous equilibria in the stochastic game Σ . It will be proved in Section 3 below.

Theorem 2.10 *Assume that the discounted stochastic game Σ satisfies Assumptions 2.5 and 2.7 and*

$$\sup_{x \in X} \int |A| \left| \frac{\partial^2}{\partial x^2} q(x, A, B) \right| \nu(dA, dB) < \infty. \quad (19)$$

Then there exists $C^ > 0$ such that, for all $c < C^*$ the game Σ has a stationary equilibrium in Markov strategies τ which is Lipschitz continuous. That is, there exists $L^* < \infty$ such that*

$$|\tau^i(y_1) - \tau^i(y_2)| \leq L^* |y_1 - y_2| \quad (i \in I).$$

Remark 2.11 *The proof of Theorem 2.10 will show that whenever the law of motion takes the linear form (17), the stochastic game has a Lipschitz continuous equilibrium for all $c > 0$.*

2.2 Convergence of the state sequence

Let us now return to the solution to the stochastic difference equation (1). For given a Nash equilibrium in Markov strategies τ , we denote by \mathbb{P}_y^τ the law on (Ω, \mathcal{F}) induced by τ and the starting point y on (Ω, \mathcal{F}) . In order to guarantee asymptotic stability of the process $\{Y_t\}_{t \in \mathbb{N}}$ under \mathbb{P}_y^τ , we need to assume that the following mean contraction condition for the dynamics defined by (1) is satisfied.

Assumption 2.12 *(i) There exists a constant $r < 1$ such that*

$$\int |A| Q(x; dA, dB) \leq r \quad \text{and} \quad \sup_x \int |B| Q(x; dA, dB) < \infty. \quad (20)$$

(ii) *There is a measure $\tilde{\nu}$ on \mathbb{R}^2 and a constant $c > 0$ such that*

$$Q(x; \cdot) \geq c\tilde{\nu}(\cdot) \quad \text{and} \quad \int |A|\tilde{\nu}(dA, dB) \leq r.$$

We are now ready to formulate the main result of this paper. It will be proved in Section 4 below.

Theorem 2.13 *Let τ be a Lipschitz continuous equilibrium in Markov strategies for the stochastic game Σ . If Assumption 2.12 is satisfied, then the following holds:*

(i) *There is a probability measure \mathbb{Q}^τ on (Ω, \mathcal{F}) such that the driving sequence ψ defined on $(\Omega, \mathcal{F}, \mathbb{P}_y^\tau)$ is nice with respect to \mathbb{Q}^τ .*

(ii) *Under \mathbb{Q}^τ there exists a unique stationary solution $\{Y_t^*\}_{t \in \mathbb{N}}$ to (1) and*

$$\text{Law}(\{Y_{t+T}\}_{t \in \mathbb{N}}, \mathbb{P}_y^\tau) \rightarrow \text{Law}(\{Y_t^*\}_{t \in \mathbb{N}}, \mathbb{Q}^\tau) \quad \text{for all } y \in \mathbb{R} \text{ as } T \rightarrow \infty.$$

We close with section with two case studies where the assumptions on Q in Theorem 2.13 can indeed be verified.

Example 2.14 (i) *Let us return to the law of motion Q introduced in Example 2.8 below.*

In this case, (19) is trivially satisfied and Assumption 2.12 holds if $Q_1(\cdot)$ and $Q_2(\cdot)$ satisfy (20) and if, for instance each Q_i is of the form $Q_i(\cdot) = \lambda \tilde{Q}_i(\cdot) + (1 - \lambda)\tilde{\nu}(\cdot)$ for a suitable measure ν on \mathbb{R}^2 .

(ii) *Consider now the densities $q(x; \eta) = \lambda \varphi_{f(x)}(\eta) + (1 - \lambda)\tilde{\nu}(\cdot)$ where $\varphi_{f(x)}(\eta)$ is introduced in Example 2.6. We assume that $f(x) = (f_1(x), x)$ for some $f_1 : X \rightarrow \mathbb{R}$. Since f is two times continuously differentiable and because $X \subset \mathbb{R}$ is compact, it is easy to show*

that (19) holds. Moreover, there exists $r_1 > 0$, such that the mean-contraction condition (20) holds whenever $f_1(x) \leq r_1$.

3 Existence of Lipschitz continuous Nash equilibria

This section is devoted to the proof of Theorem 2.10. Since the state space of our stochastic game Σ is not compact, we cannot prove the theorem directly. Instead, we shall first analyze the ‘truncated’ games

$$\Sigma^N = (I, X, (U^i), \beta, Q, y, [-N, N]) \quad (N \in \mathbb{N})$$

with the compact state space $[-N, N]$ in which the conditional dynamics of the new state Y_{t+1} , given Y_t and the action profile \bar{x}_t is of the form

$$Y_{t+1} = v^N(Y_t, A_t, B_t) := \max\{\min\{A_t Y_t + B_t, N\}, -N\} \quad \text{and} \quad (A_t, B_t) \sim Q\left(\frac{c}{M} \sum_{i \in I} x_t^i, \cdot\right).$$

Using our *Moderate Social Influence* assumption we show that the games Σ^N have Lipschitz continuous Nash equilibria in Markov strategies τ^N whose Lipschitz constants do not depend on $N \in \mathbb{N}$. A Lipschitz continuous equilibrium for Σ will then be determined as a suitable accumulation point of the sequence $\{\tau^N\}_{N \in \mathbb{N}}$.

3.1 Lipschitz continuous equilibria in truncated games

The aim of this section is to establish the existence of Lipschitz continuous Nash equilibria in the truncated games Σ^N .

Proposition 3.1 *Under the assumptions of Theorem 2.10 there exists $C^* > 0$ such that for all $c \leq C^*$ the following holds:*

(i) For all $N \in \mathbb{N}$, the truncated game Σ^N has a Lipschitz continuous Nash equilibrium τ^N in Markov strategies.

(ii) The Lipschitz constant of τ^N ($N \in \mathbb{N}$) does not depend on $N \in \mathbb{N}$.

The proof of Proposition 3.1 requires some preparation. We put $u := \max_i \|U^i\|_\infty$ and denote by $\mathcal{L}_{L,u}([-N, N], \mathbb{R}^M)$ the class of all Lipschitz continuous functions $f : [-N, N] \rightarrow \mathbb{R}^M$ with Lipschitz constant L that satisfy $\|f\|_\infty \leq u$. To each such average continuation function $f = (f_i)_{i \in I}$ we associate the reduced one-shot game $\Sigma_f^N := (I, X, (U_f^{i,N}), \beta, y, [-N, N])$ with payoff functions

$$U_f^{i,N}(y, \underline{x}) = (1 - \beta)U^i(y, \underline{x}) + \beta \int f_i \circ v^N(y, A, B)q(x, A, B)\nu(dA, dB) \quad (21)$$

viewed as a function from $[-N, N] \times \overline{X}$ to \mathbb{R} .

Remark 3.2 Observe that $U_f^{i,N}(y, \underline{x})$ is the payoff to player $i \in I$ in the discounted stochastic game Σ^N , if the game terminates after the first round, if the players receive rewards according to the payoff functions f_i in the second period, and if first period payoffs are discounted at the rate $1 - \beta$.

The following Lemma shows that the conditional best reply $\tau_f^{i,N}(y, x^{-i})$ of player $i \in I$ in the game Σ_f^N is uniquely determined and that Σ_f^N has a unique equilibrium $\tau_f^N(y)$. The map $\tau_f^N : [-N, N] \rightarrow \overline{X}$ is Lipschitz continuous. The Lipschitz constant can be specified in terms of the Lipschitz constant of f , the discount factor, the bounds for the utility functions and the quantities $L^{i,j}(y)$, L^i and \hat{L} introduced in (14) and (15), respectively. In particular, it can be chosen *independently* of the specific average continuation function f and independently of N . This turns out to be the key to the proof of Proposition 3.1.

Lemma 3.3 For $N \in \mathbb{N}$, let Σ_f^N be the reduced game with average continuation function $f \in \mathcal{L}_{L,u}([-N, N], \mathbb{R}^M)$. Under the assumptions of Theorem 2.10 the following holds:

- (i) The conditional best reply $\tau_f^{i,N}(y, x^{-i})$ of player $i \in I$ is uniquely determined and depends in a Lipschitz continuous manner on the actions of his competitors. More precisely,

$$|\tau_f^{i,N}(y, x_1^{-i}) - \tau_f^{i,N}(y, x_2^{-i})| \leq \frac{(1 - \beta)L^{i,j}(y) + \beta u \hat{L}}{(1 - \beta)L^{i,i}(y) - \beta u \hat{L}} |x_1^j - x_2^j| \quad (22)$$

if $x_1^k = x_2^k$ for all $k \neq j$. Moreover,

$$|\tau_f^{i,N}(y_1, x^{-i}) - \tau_f^{i,N}(y_2, x^{-i})| \leq \tilde{L} |y_1 - y_2| \quad (23)$$

for all $y_1, y_2 \in [-N, N]$ and each $x^{-i} \in \overline{X}^{-i}$. Here

$$\tilde{L} := \sup_{y,i} \frac{(1 - \beta)L^i + \beta L(f)q}{(1 - \beta)L^{i,i}(y) - \beta u \hat{L}},$$

$L(f)$ denotes the Lipschitz constant of f and the quantity q is defined in (19).

- (ii) The reduced game Σ_f^N has a unique equilibrium $\tau_f^N(y) = \{\tau_f^{i,N}(y)\}_{i \in I} \in \overline{X}$.
- (iii) The map $y \mapsto \tau_f^{i,N}(y)$ is Lipschitz continuous and the Lipschitz constant can be chosen independently of the average continuation function f and independently of N .
- (iv) The map $f \mapsto \tau_f^{i,N}(\cdot)$ is continuous.

PROOF:

- (i) Let us fix an average continuation function $f \in \mathcal{L}_{L,u}([-N, N], \mathbb{R}^M)$, an action profile $x^{-i} \in \overline{X}^{-i}$ and a state $y \in [-N, N]$. By Assumption 2.5, the map $U_f^{i,N}(y, \cdot, x^{-i})$ is two times continuously differentiable on an open set containing X and our *Moderate Social*

Influence condition yields

$$\frac{\partial^2}{\partial (x^i)^2} U_f^{i,N}(y, x^i, x^{-i}) \leq -(1 - \beta)L^{i,i}(y) + \beta u \hat{L} < 0.$$

Thus, an agent's utility function is strongly concave with respect to his own action, and so his conditional best reply given the choices of his competitors is uniquely determined. To establish the quantitative bound (22) on the dependence of player i 's best reply on the behavior of all the other agents, we fix a player $j \neq i$ and action profiles x_1^{-i} and x_2^{-i} which differ only at the j -th coordinate. Under the assumptions of Theorem 2.10 we have

$$\left| \frac{\partial}{\partial x^i} U_f^{i,N}(y, x^i, x_1^{-i}) - \frac{\partial}{\partial x^i} U_f^{i,N}(y, x^i, x_2^{-i}) \right| \leq \left\{ (1 - \beta)L^{i,j}(y) + \beta u \hat{L} \right\} |x_1^j - x_2^j|. \quad (24)$$

Thus, (22) follows from Theorem A.1. In view of (24) and because

$$\left| \frac{\partial}{\partial x^i} U_f^{i,N}(y_1, x^i, x^{-i}) - \frac{\partial}{\partial x^i} U_f^{i,N}(y_2, x^i, x^{-i}) \right| \leq \left\{ (1 - \beta)L^i + \beta L(f)q \right\} |x_1^j - x_2^j| \quad (25)$$

our estimate (23) also follows from Theorem A.1.

- (ii) The existence of an equilibrium in pure strategies for the static game Σ_f^N follows from strict concavity of the utility functions $U_f^{i,N}$ with respect to the player's own actions along with compactness of the action spaces using standard fixed points arguments. In order to prove uniqueness, we proceed as in the proof of Proposition 4.21 in Horst and Scheinkman (2002). In view of (13), our *Moderate Social Influence* condition yields

$$\bar{L} := \sup_{i,y} \sum_{j \neq i} \frac{(1 - \beta)L^{i,j}(y) + \beta u \hat{L}}{(1 - \beta)L^{i,i}(y) - \beta u \hat{L}} < 1.$$

Thus, given action profiles x_1^{-i} and x_2^{-i} of player i 's competitors, (22) shows that

$$|\tau_f^{i,N}(y, x_1^{-i}) - \tau_f^{i,N}(y, x_2^{-i})| \leq \bar{L} \max_j |x_1^j - x_2^j|.$$

For $x_1 \neq x_2$, we therefore obtain

$$\max_i |\tau_f^{i,N}(y, x_1^{-i}) - \tau_f^{i,N}(y, x_2^{-i})| < \max_i |x_1^i - x_2^i|.$$

Thus, the map $x \mapsto (\tau_f^{i,N}(y, x^{-i}))_{i \in I}$ has at most one fixed point. This proves uniqueness of equilibria in Σ_f^N .

(iii) Let $\tau_f^N(y)$ be an equilibrium. Then $\tau_f^{i,N}(y) = \tau_f^{i,N}(y, \{\tau_f^{j,N}(y)\}_{j \neq i})$, and so

$$\begin{aligned} |\tau_f^{i,N}(y_1) - \tau_f^{i,N}(y_2)| &\leq |\tau_f^{i,N}(y_1, \{\tau_f^{j,N}(y_1)\}_{j \neq i}) - \tau_f^{i,N}(y_1, \{\tau_f^{j,N}(y_2)\}_{j \neq i})| \\ &\quad + |\tau_f^{i,N}(y_1, \{\tau_f^{j,N}(y_2)\}_{j \neq i}) - \tau_f^{i,N}(y_2, \{\tau_f^{j,N}(y_2)\}_{j \neq i})| \\ &\leq \bar{L} \sum_{j \neq i} |\tau_f^{j,N}(y_1) - \tau_f^{j,N}(y_2)| + \tilde{L} |y_1 - y_2|. \end{aligned}$$

This yields

$$|\tau_f^{i,N}(y_1) - \tau_f^{i,N}(y_2)| \leq \frac{\tilde{L}}{(1 - \bar{L})} |y_1 - y_2|. \quad (26)$$

Hence the equilibrium mapping $\tau_f^N : [-N, N] \rightarrow \underline{\bar{X}}$ is Lipschitz continuous with a constant that does not depend on $N \in \mathbb{N}$ and which depends on the average continuation function f only through its Lipschitz constant $L(f)$.

(iv) We fix $y \in [-N, N]$ and $x^{-i} \in \underline{\bar{X}}^{-i}$ and apply Theorem A.1 to the map

$$(x^i, f) \mapsto U_f^{i,N}(y, x^i, x^{-i}).$$

Due to Assumption 2.5 (ii) there exists a constant $q^* < \infty$ such that

$$\left| \frac{\partial}{\partial x^i} U_{f_1}^{i,N}(y, x^i, x^{-i}) - \frac{\partial}{\partial x^i} U_{f_2}^{i,N}(y, x^i, x^{-i}) \right| \leq q^* \|f_1 - f_2\|_\infty,$$

for all $f, g \in \mathcal{L}_{L,u}([-N, N], \mathbb{R}^M)$, and so Theorem A.1 shows that there is $L_0 < \infty$ such that

$$|\tau_{f_1}^{i,N}(y, x^{-i}) - \tau_{f_2}^{i,N}(y, x^{-i})| \leq L_0 \|f_1 - f_2\|_\infty.$$

Thus, similar arguments as in the proof of (iii) yield the assertion.

□

Our *Moderate Social Influence* conditions appears to be rather strong. However, for the proof of Theorem 2.10 it will be essential that the maps $\tau_f^N : [-N, N] \rightarrow \overline{X}$ are Lipschitz continuous with a constant that depends on the average continuation function f only through its Lipschitz constant. For that, we need uniqueness of equilibria in the reduced one-shot games Σ_f^N . We guarantee this by assuming that the utility functions $U_f^{i,N}$ inherit enough concavity in the player's own actions from the original reward functions U^i .

We are now ready to prove Proposition 3.1.

PROOF OF PROPOSITION 3.1: Let $B_u([-N, N], \mathbb{R}^M)$ be the class of all measurable functions $f : [-N, N] \rightarrow \mathbb{R}^M$ with $\|f\|_\infty \leq u$. We introduce an operator T from $\mathcal{L}_{L,u}([-N, N], \mathbb{R}^M)$ to $B_u([-N, N], \mathbb{R}^M)$ by

$$(Tf)_i(y) = (1 - \beta)U^i(y, \tau_f^N(y)) + \beta \int f_i \circ v^N(y, A, B) q \left(\frac{c}{M} \sum_{i \in I} \tau_f^{i,N}(y), A, B \right) \nu(dA, dB). \quad (27)$$

A standard argument in discounted dynamic programming shows that for any fixed point F^N of T , the action profile $\tau_{F^N}^N(y)$ is an equilibrium in the non-zero sum stochastic game Σ^N . The equilibrium payoff to player $i \in I$ is given by $\frac{F^N(y)}{1-\beta}$, and the map $\tau_{F^N}^N : [-N, N] \rightarrow \overline{X}$ is Lipschitz continuous, due to Lemma 3.3. In order to establish our assertion it is therefore enough find $L < \infty$ and $C^* > 0$ such that the operator T has a fixed point in $\mathcal{L}_{L,u}([-N, N], \mathbb{R}^M)$ for all $c < C^*$. For this, we proceed in three steps:

- (i) For all $f \in \mathcal{L}_{L,u}([-N, N], \mathbb{R}^M)$, the mapping $y \mapsto (Tf)(y)$ is Lipschitz continuous:

Indeed, Lipschitz continuity of the utility functions and the densities along with Assumption 2.12 (i) yields a constant $c_1 < \infty$ such that

$$|Tf_i(y_1) - Tf_i(y_2)| \leq c_1 (|y_1 - y_2| + \|\tau_f^N(y_1) - \tau_f^N(y_2)\|_\infty). \quad (28)$$

Thus, Lipschitz continuity of the map $\tau_f^N(\cdot)$ yields Lipschitz continuity of $Tf(\cdot)$.

- (ii) The operator T is continuous: For any sequence $\{f^n\}_{n \in \mathbb{N}}$, $f^n \in \mathcal{L}_{L,u}([-N, N], \mathbb{R}^M)$, that converges to f in the topology of uniform convergence, Lemma 3.3 (v) yields $\lim_{n \rightarrow \infty} \|\tau_{f^n}^N - \tau_f^N\|_\infty = 0$. Thus, Lipschitz continuity of the reward functions and the densities gives us constants c_2 and c_3 such that

$$|Tf_i^n(y) - Tf_i(y)| \leq c_2 \|\tau_{f^n}^N - \tau_f^N\|_\infty + c_3 \{ \|f_i^n - f_i\|_\infty + \|\tau_{f^n}^N - \tau_f^N\|_\infty \},$$

and so $\lim_{n \rightarrow \infty} \|Tf^n - Tf\|_\infty = 0$.

- (iii) Lemma 3.3 and (26) show that there are constants d_1, d_2 and d_3 such that, for all $N \in \mathbb{N}$ and each $f \in \mathcal{L}_{L,u}([-N, N], \mathbb{R}^M)$, the equilibrium mappings $\tau_f^{i,N} : [-N, N] \rightarrow \overline{X}$ are Lipschitz continuous and that the Lipschitz constant $L^*(L)$ takes the form

$$L^* = \frac{d_1 + c \frac{d_2}{M} L}{d_3}.$$

In particular, (28) yields

$$|Tf_i(y_1) - Tf_i(y_2)| \leq c_1 \left[1 + \frac{d_1 + c \frac{d_2}{M} L}{d_3} \right] |y_1 - y_2|.$$

If we choose $C^* \leq \frac{d_3 M}{c_1 d_2}$, then for all sufficiently large $L \in \mathbb{N}$ and for each $c < C^*$, the operator T maps the set $\mathcal{L}_{L,u}([-N, N], \mathbb{R}^M)$ continuously into itself. Since $\mathcal{L}_{L,u}([-N, N], \mathbb{R}^M)$ is a compact set with respect to the topology of uniform convergence, T has a fixed point by Kakutani's theorem.

This finishes the proof □

Remark 3.4 *In case the law of motion Q takes the linear form (17), we have $d_2 = 0$, and so the proof of Proposition 3.1 shows that such games Σ^N have Lipschitz continuous equilibria for all $c > 0$.*

3.2 Lipschitz continuous equilibria in discounted stochastic games

We are now prepared to prove the existence of Lipschitz continuous Nash equilibria in Markov strategies for discounted stochastic games with affine state sequences.

PROOF OF THEOREM 2.10: For $N \in \mathbb{N}$, let τ^N and F^N be a Lipschitz continuous Nash equilibrium and the associated payoff function for the truncated game Σ^N . In view of Proposition 3.1 we may assume that these maps are Lipschitz continuous with a common Lipschitz constant, and so the sequences $\{\tau^N\}_{N \in \mathbb{N}}$ and $\{F^N\}_{N \in \mathbb{N}}$ are equicontinuous. Thus, by the theorem of Ascoli and Arzela, there exists a subsequence (N_k) and Lipschitz continuous functions $\tau : \mathbb{R} \rightarrow \overline{X}$ and $F : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\tau_{F^{N_k}}^{N_k}(y) \rightarrow \tau(y) \quad \text{and} \quad F^{N_k}(y) \rightarrow F(y) \quad \text{uniformly on compact sets as } k \rightarrow \infty.$$

Let us now fix $y \in \mathbb{R}$. For any compact set $K \subset \mathbb{R}^2$ we have

$$\lim_{n \rightarrow \infty} \sup_{(A,B) \in K} |F^N \circ v^N(y, A, B) - F(Ay + B)| = 0.$$

We also have $\lim_{N \rightarrow \infty} \|Q(\frac{c}{M} \sum_{i \in I} \tau^{i,N}(y); \cdot) - Q(\frac{c}{M} \sum_{i \in I} \tau^i(y); \cdot)\|_{\mathcal{B}} = 0$. Since the sequence $\{F^N \circ v^N\}_{N \in \mathbb{N}}$ is uniformly bounded, we obtain

$$F_i(y) = (1 - \beta)U^i(y, \tau(y)) + \beta \int F_i(Ay + B) q\left(\frac{c}{M} \sum_{i \in I} \tau^i(y), A, B\right) \nu(dA, dB).$$

It is easily seen that $\tau : \mathbb{R} \rightarrow \overline{\mathbf{X}}$ is a stationary Nash equilibrium in Markov strategies for the game Σ . By construction, τ is Lipschitz continuous. \square

4 Asymptotic stability of the equilibrium process

Throughout this section, we denote by $\tau^N : [-N, N] \rightarrow \overline{\mathbf{X}}$ and $\tau : \mathbb{R} \rightarrow \overline{\mathbf{X}}$ Lipschitz continuous equilibria in the stochastic games Σ^N and Σ , respectively. We may assume that

$$\tau^N(y) \rightarrow \tau(y) \quad \text{uniformly on compact sets as } N \rightarrow \infty. \quad (29)$$

For any initial value $y \in \mathbb{R}$ of the solution to (1), the strategies τ^N and τ induce the respective measures $\mathbb{P}_y^{\tau^N}$ and \mathbb{P}_y^τ on (Ω, \mathcal{F}) . To simplify notation, it is understood that under $\mathbb{P}_y^{\tau^N}$ the process $\{Y_t\}_{t \in \mathbb{N}}$ is the state sequence on $[-N, N]$ in the truncated stochastic game Σ^N . This sequence may also be viewed as the Markov chain on the compact state space $[-N, N]$ associated to a suitable random system with complete connections; see, e.g., Barnsley, Demko, Elton, and Geronimo (1988) or Norman (1972). In view of Theorem 4.2 in Norman (1972) weak convergence of the process $\{Y_t\}_{t \in \mathbb{N}}$ to a unique stationary measure follows from Assumption 2.12.

Proposition 4.1 *Under the assumption of Theorem 2.10 there exists unique probability measure μ^N such that*

$$\mu_t^{N,y} := \text{Law}(Y_t, \mathbb{P}_y^{\tau^N}) \xrightarrow{w} \mu^N \quad \text{for all } y \in \mathbb{R} \text{ as } t \rightarrow \infty.$$

Under $\mathbb{Q}^{\tau^N}(\cdot) := \int \mathbb{P}_y^{\tau^N}(\cdot) \mu^N(dy)$ the process $\{Y_t\}_{t \in \mathbb{N}}$ is stationary and ergodic.

Now our aim is to show that under \mathbb{P}_y^τ the sequence $\{Y_t\}_{t \in \mathbb{N}}$ converges in distribution to the unique weak limit of the sequence $\{\mu^N\}_{N \in \mathbb{N}}$. To this end, we modify arguments which

appear in Föllmer, Horst, and Kirman (2002). In a first step we prove weak convergence of the one dimensional distributions of the process $\{Y_t\}_{t \in \mathbb{N}}$ under $\mathbb{P}_y^{\tau, N}$ to the one dimensional distributions of the sequence $\{Y_t\}_{t \in \mathbb{N}}$ under \mathbb{P}_y^τ as $N \rightarrow \infty$

Lemma 4.2 *For all $t \in \mathbb{N}$ we have*

$$\mu_t^{N,y} \xrightarrow{w} \text{Law}(Y_t, \mathbb{P}_y^\tau) =: \mu_t^y \quad \text{as } N \rightarrow \infty. \quad (30)$$

PROOF: We prove (30) by induction over $t \in \mathbb{N}$. For $t = 1$, the assertion is trivial. Let us therefore fix a Lipschitz continuous function with compact support $f : \mathbb{R} \rightarrow \mathbb{R}$ and assume that

$$\mu_t^{N,y}(f) := \int f(y) \mu_t^{N,y}(dy) \rightarrow \mu_t(f) \quad \text{as } N \rightarrow \infty.$$

In view of the induction hypothesis, it is enough to show that

$$\int f \circ v^N(Ay + B) Q \left(\frac{c}{M} \sum_{i \in I} \tau^{i,N}(y); dA, dB \right) \rightarrow \int f(Ay + B) Q \left(\frac{c}{M} \sum_{i \in I} \tau^i(y); dA, dB \right)$$

uniformly on compact sets as $N \rightarrow \infty$. Since f has compact support, $f \circ v^N = f$ for all sufficiently large $N \in \mathbb{N}$, and so it suffices to prove that

$$\left| \int f(Ay + B) \left[Q \left(\frac{c}{M} \sum_{i \in I} \tau^{i,N}(y); dA, dB \right) - Q \left(\frac{c}{M} \sum_{i \in I} \tau^i(y); dA, dB \right) \right] \right| \rightarrow 0$$

uniformly on compact sets as $N \rightarrow \infty$. This, however, follows from Lipschitz continuity of the stochastic kernel Q and from (29) because

$$\left\| Q \left(\frac{c}{M} \sum_{i \in I} \tau^{i,N}(y); \cdot \right) - Q \left(\frac{c}{M} \sum_{i \in I} \tau^i(y); \cdot \right) \right\|_{\mathcal{B}} \leq C \max_{i \in I} |\tau^{i,N}(y) - \tau^i(y)|$$

for some $C < \infty$. This shows vague convergence of the sequence $\{\mu_{t+1}^{N,y}\}_{N \in \mathbb{N}}$ to the probability measure μ_{t+1}^y and, therefore, weak convergence. \square

We now prove convergence in distribution of the sequence $\{Y_t\}_{t \in \mathbb{N}}$ to the unique weak limit of the sequence $\{\mu^N\}_{N \in \mathbb{N}}$. This turns out to be the key to the proof of Theorem 2.13.

Proposition 4.3 *There exists a unique probability measure μ on \mathbb{R} such that*

$$\text{Law}(Y_t, \mathbb{P}_y^\tau) \xrightarrow{w} \mu \quad \text{for all } y \in \mathbb{R} \quad \text{as } t \rightarrow \infty.$$

PROOF: In order to prove our assertion, we proceed in two steps.

- (i) Let us first show that the family of probability measures $\{\mu_t^{N,y}, \mu_t^y\}_{N,t \in \mathbb{N}}$ is tight. For this, it suffices to prove that

$$\lim_{c \rightarrow \infty} \sup \left\{ \mathbb{P}_y^{\tau^N} [|Y_t| \geq c], \mathbb{P}_y^\tau [|Y_t| \geq c] : N, t \in \mathbb{N} \right\} = 0. \quad (31)$$

Due to Tchebychev's inequality, (31) follows from

$$\sup \left\{ \mathbb{E}_y^{\tau^N} [|Y_t|], \mathbb{E}_y^\tau [|Y_t|] : N, t \in \mathbb{N} \right\} < \infty. \quad (32)$$

In order to prove (32), we put

$$a := \sup_x \int |A| Q(x; dA, B) < 1 \quad \text{and} \quad b := \sup_x \int |B| Q(x; dA, dB) < \infty$$

and consider the deterministic sequence $\{\overline{Y}_t^y\}_{t \in \mathbb{N}}$ defined by the recursive relation

$$\overline{Y}_{t+1}^y = a \overline{Y}_t^y + b \quad (t \in \mathbb{N}, Y_0 = y).$$

Since $a < 1$, we have $\lim_{t \rightarrow \infty} \overline{Y}_t^y = \frac{b}{1-a} < \infty$. In particular, $\sup_t Y_t^y < \infty$, and so it suffices to show that

$$\mathbb{E}_y^{\tau^N} |Y_t| \leq \overline{Y}_t^y \quad \text{and that} \quad \mathbb{E}_y^\tau |Y_t^y| \leq \overline{Y}_t^y \quad \text{for all } y \geq 0. \quad (33)$$

Clearly, (33) holds for $t = 0$. We proceed by induction and assume that (33) holds for all $t \leq T$. For the induction step, note that

$$\begin{aligned}
\mathbb{E}_y^T |Y_{T+1}| &\leq \mathbb{E}_y^T [|A_T Y_T|] + \mathbb{E}_y^T [|B_T|] \\
&\leq \mathbb{E}_y^T [|Y_T| \mathbb{E}_y^T [|A_T| | |Y_T|]] + b \\
&\leq a \mathbb{E}_y^T [|Y_T|] + b \\
&\leq a \overline{Y}_T + b \\
&= \overline{Y}_{T+1}.
\end{aligned}$$

Similar arguments show that $\mathbb{E}_y^{T_N} |Y_t| \leq \overline{Y}_t^y$ for all $t \in \mathbb{N}$.

- (ii) Since the sequence $\{\mu_t^y\}_{t \in \mathbb{N}}$ is tight and because the Markov chain $(\{Y_t\}_{t \in \mathbb{N}}, \mathbb{P}_y^T)$ has the Feller property, a standard argument shows that it has a stationary measure μ^* . In order to prove weak convergence of $\{\mu_t^y\}_{t \in \mathbb{N}}$ to μ^* observe first that, by Prohorov's theorem and (i), there exists a sub-net $(N_k, t_k)_{k \in \mathbb{N}}$ and an accumulation point μ^y such that $\mu_{t_k}^{N_k, y} \xrightarrow{w} \mu^y$ as $k \rightarrow \infty$. Thus, Proposition 4.1 along with Lemma 4.2 shows that

$$\lim_{t_k \rightarrow \infty} \lim_{N_k \rightarrow \infty} \mu_{t_k}^{N_k, y}(f) = \mu^y(f) = \lim_{N_k \rightarrow \infty} \lim_{t_k \rightarrow \infty} \mu_{t_k}^{N_k, y}(f) = \lim_{N_k \rightarrow \infty} \mu^{N_k}(f),$$

for any bounded continuous function f on \mathbb{R} . Hence the accumulation point μ^y does not depend on the initial value, i.e., $\mu^y = \mu$. Thus, we may as well assume that the starting point is distributed according the stationary measure μ^* . In this case $\mu_t = \mu^*$, and so

$$\mu(f) = \int \left(\lim_{t_k \rightarrow \infty} \lim_{N_k \rightarrow \infty} \mu_{t_k}^{N_k, y}(f) \right) \mu^*(dy) = \lim_{t_k \rightarrow \infty} \int \mu_{t_k}^y(f) \mu^*(dy) = \mu^*(f)$$

by dominated convergence. Thus,

$$\mu^*(f) = \lim_{(N,t) \rightarrow \infty} \mu_t^{N,y}(f) = \lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \mu_t^{N,y}(f) = \lim_{t \rightarrow \infty} \mu_t^y(f).$$

This finishes the proof. \square

Proposition 4.1 gives us convergence in law of the process $\{Y_t\}_{t \in \mathbb{N}}$ under \mathbb{P}_y^τ to a unique limiting distribution μ^* . In order to establish our global convergence result stated in Theorem 2.10 it is now enough to show that the driving sequence $\{(A_t, B_t)\}_{t \in \mathbb{N}}$ defined on $(\Omega, \mathcal{F}, \mathbb{P}_y^\tau)$ is nice with respect to the measure

$$\mathbb{Q}^\tau(\cdot) := \int \mathbb{P}_y^\tau(\cdot) \mu^*(dy). \quad (34)$$

Proposition 4.4 *Assume that the assumptions of Theorem 2.10 are satisfied. Let τ be a Lipschitz continuous equilibrium in Markov strategies for the stochastic game Σ and denote by μ^* be unique limiting distribution of the Markov chain $\{Y_t\}_{t \in \mathbb{N}}$ under \mathbb{P}_y^τ . Then the driving sequence $\{(A_t, B_t)\}_{t \in \mathbb{N}}$ is nice with respect to the measure \mathbb{Q}^τ defined by (34).*

PROOF: Since the Markov chain $\{Y_t\}_{t \in \mathbb{N}}$ defined on $(\Omega, \mathcal{F}, \mathbb{P}_y^\tau)$ converges in law to the unique limiting distribution μ^* , the sequence $\{Y_t\}_{t \in \mathbb{N}}$ is stationary and ergodic under \mathbb{Q}^τ . Thus, it remains to show that the asymptotic distribution of the random environment $\{(A_t, B_t)\}_{t \in \mathbb{N}}$ is the same under \mathbb{P}_y^τ and under \mathbb{Q}^τ .

To this end, we denote by $C < \infty$ the Lipschitz constant associated to the stochastic kernel Q from X to \mathbb{R}^2 :

$$\left\| Q \left(\frac{c}{M} \sum_{i \in I} \tau^i(y_1); \cdot \right) - Q \left(\frac{c}{M} \sum_{i \in I} \tau^i(y_2); \cdot \right) \right\|_{\mathcal{B}} \leq C |y_1 - y_2|. \quad (35)$$

Let $\psi_t := \{(A_1, B_1), \dots, (A_t, B_t)\}$ and let $B_t \in \mathcal{B}_t$, the Borel- σ -field on \mathbb{R}^{2t} . Our aim is to show that there is a constant $L^* < \infty$ such that

$$|\mathbb{P}_{y_1}^\tau[\psi_t \in B_t] - \mathbb{P}_{y_2}^\tau[\psi_t \in B_t]| \leq L^*|y_1 - y_2| \quad \text{for all } y_1, y_2 \in \mathbb{R}. \quad (36)$$

For this, we proceed by induction. For $t = 1$, (36) follows from (35). Let us therefore assume that there is α_t such that

$$|\mathbb{P}_{y_1}^\tau[\psi_t \in B_t] - \mathbb{P}_{y_2}^\tau[\psi_t \in B_t]| \leq \alpha_t|y_1 - y_2| \quad \text{for all } y_1, y_2 \in \mathbb{R}$$

For the induction step observe first that it suffices to prove (36) for sets B_t of the form $B_t = B_t^1 \times \dots \times B_t^t$ with $B_t^l \in \mathcal{B}_1$. The induction hypothesis along with our mean contraction condition yields

$$\begin{aligned} & |\mathbb{P}_{y_1}^\tau[\psi_{t+1} \in B_{t+1}] - \mathbb{P}_{y_2}^\tau[\psi_{t+1} \in B_{t+1}]| \\ &= \left| \mathbf{1}_{B_{t+1}^1}(\{A, B\}) \mathbb{P}_{Ay_1+B}^\tau[\psi_t \in B_t] Q\left(\frac{c}{M} \sum_{i \in I} \tau^i(y_1); dA, dB\right) \right. \\ & \quad \left. - \mathbf{1}_{B_{t+1}^1}(\{A, B\}) \mathbb{P}_{Ay_2+B}^\tau[\psi_t \in B_t] Q\left(\frac{c}{M} \sum_{i \in I} \tau^i(y_2); dA, dB\right) \right| \\ &\leq \int |\mathbb{P}_{Ay_1+B}^\tau[\psi_t \in B_t] - \mathbb{P}_{Ay_2+B}^\tau[\psi_t \in B_t]| Q\left(\frac{c}{M} \sum_{i \in I} \tau^i(y_1); dA, dB\right) \\ & \quad + \sup_{B_1} |\mathbb{P}_{y_1}^\tau[\psi_1 \in B_1] - \mathbb{P}_{y_2}^\tau[\psi_1 \in B_1]| \\ &\leq C|y_1 - y_2| + \alpha_t \int |Ay_1 - Ay_2| Q\left(\frac{c}{M} \sum_{i \in I} \tau^i(y_1); dA, dB\right) \\ &\leq (C + r\alpha_t)|y_1 - y_2| \end{aligned}$$

where $r < 1$ is defined in (20). Thus, $\alpha_{t+1} \leq C + r\alpha_t$, and this yields (36) with $L^* := \frac{C}{1-r}$.

Since the maps $y \mapsto \mathbb{P}_y^\tau[\psi_t \in B_t]$ are Lipschitz continuous, because the Lipschitz constants do not depend on $t \in \mathbb{N}$ nor on $B_t \in \mathcal{B}_t$, Proposition 4.1 gives us

$$\|\mathbb{P}_y^\tau - \mathbb{Q}^\tau\|_{\hat{F}_{t,l}} \leq L^* d(\mu_t^y, \mu^\tau) \quad \text{for all } l \in \mathbb{N}.$$

Here $d(\mu_t^y, \mu^*)$ denotes the distance of the measures μ_t^y and μ^* in the Vasserstein metric. In view of Remark 2.2, this proves that ψ is nice with respect to \mathbb{Q}^τ . \square

We are now ready to prove Theorem 2.13.

PROOF OF THEOREM 2.13: Due to Theorem 2.10 there is a Lipschitz continuous equilibrium in Markov strategies τ for the stochastic game Σ . It follows from Proposition 4.4 that the random environment ψ defined on $(\Omega, \mathcal{F}, \mathbb{P}_y^\tau)$ is nice with respect to the measure \mathbb{Q}^τ on (Ω, \mathcal{F}) , and so the assertion follows from Theorem 2.4 in Horst (2001). \square

A Parameter dependent optimization problems

The proof of Lemma 3.3 uses a result about Lipschitz continuous dependence of solutions to parameter dependent optimization problem. It is a special case of Theorem 3.1 in Montruccio (1987):

Theorem A.1 *Let $X \subset \mathbb{R}$ be closed and convex and let $(Y, \|\cdot\|)$ be a normed space. Let $F : X \times Y \rightarrow \mathbb{R}$ be a finite function which satisfies the following conditions:¹*

(i) *For all $y \in Y$, the map $x \mapsto F(x, y)$ is concave and two times continuously differentiable*

on X . Moreover, there exists $\alpha > 0$ such that $\frac{\partial^2}{\partial x^2} F(x, y) \leq -\alpha$.

(ii) *The derivative $\frac{\partial}{\partial x} F(x, y)$ of F at (x, y) satisfies the Lipschitz continuity condition*

$$\left| \frac{\partial}{\partial x} F(x, y_1) - \frac{\partial}{\partial x} F(x, y_2) \right| \leq L \|y_1 - y_2\|$$

for all $y_1, y_2 \in Y$ and all $x \in X$.

¹Montruccio (1987) formulates this theorem under the additional assumption of Y being a closed and convex subset of a Hilbert space. His proof, however, shows that this assumptions is redundant.

Under the above assumptions there exists a unique map $\theta : Y \rightarrow X$ that satisfies $\sup_{x \in X} = F(x, y) = F(\theta(y), y)$. Moreover, θ is Lipschitz continuous and

$$|\theta(y_1) - \theta(y_2)| \leq \frac{L}{\alpha} \|y_1 - y_2\| \quad \text{for all } y_1, y_2 \in Y.$$

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