Riedle, Markus

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Lyapunov Exponents for Linear Delay Equations in arbitrary Phase Spaces

Markus Riedle*
Institute of Mathematics,
Humboldt–University of Berlin,
E-mail: riedle@mathematik.hu-berlin.de

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Abstract

A linear differential equation with infinite delay is considered in the generalized form as an integral equation. As usually, the function space $\mathcal{B}$ of the admissible initial conditions is only described axiomatically. Merely using this abstract description the long time behavior of the solutions is determined by calculating the Lyapunov exponents. The calculation is based on a representation of the solution in the second dual space of $\mathcal{B}$. The representation requires a modified version of the usual weak*-integral.

Key words: Lyapunov exponents, differential equations with infinite delay, weak*-integral, abstract phase space, variation of constants formula, stochastic delay differential equations

1 Introduction

For a function $x : (-\infty, a] \to \mathbb{C}^d$ with some $a \in \mathbb{R}$ and $d \in \mathbb{N}$ define the history of x by the function $x_t := \{x(t+u), u \leqslant 0\}$ for $t \leqslant a$. Consider the equation

$$\begin{aligned} x(t) &= \varphi(0) + \int_0^t L(x_s)ds + h(t), \quad t \geqslant 0, \\ x(u) &= \varphi(u), \quad u \leqslant 0, \end{aligned} \tag{1.1}$$

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where the initial condition φ is an element of the so-called phase space $\mathcal{B}$. The space $\mathcal{B}$ is a linear subspace of the set of $\mathbb{C}^d$-valued functions on the negative real line equipped with a semi-norm $\|\cdot\|_{\mathcal{B}}$. The operator L is linear and bounded on $\mathcal{B}$. The function $h : [0,\infty) \to \mathbb{C}^d$ is assumed to be continuous with $h(0) = 0$. A solution of (1.1) is a function $x : \mathbb{R} \to \mathbb{C}^d$ which is continuous on $[0,\infty)$ and satisfies equation (1.1). Since every history x_t of a solution x of (1.1) contains for $t \geqslant 0$ the initial data φ, there is no suitable function space which involves every history x_t independently of the initial data. Therefore, the phase space $\mathcal{B}$ is usually only described axiomatically in the theory of differential equations with infinite delay [2]. Moreover, this approach has the advantage that the phase space $\mathcal{B}$ and hence the semi-norm may be much better adapted to the application under consideration. In particular, the long time behavior of the history x_t in an appropriate semi-norm of $\mathcal{B}$ is relevant in applications. We determine the long time behavior by calculating the Lyapunov exponents of equation (1.1) only using the axiomatic description of $\mathcal{B}$.
In most of the literature treating linear differential equations with delay the differentiated form of equation (1.1) is considered:

$$\dot{x}(t) = L(x_t) + f(t) \quad \text{for almost every } t \geqslant 0, \quad x_0 = \varphi, \tag{1.2}$$

with a locally integrable function f. Obviously, equation (1.1) includes this one. Moreover, the integral equation (1.1) contains the stochastic case where the function h is replaced by a random, possibly non-differentiable, trajectory of a continuous stochastic process.
If there is a fixed $\rho \geqslant 0$ such that the operator L depends only on the bounded history $\{\varphi(u); u \in [-\rho, 0]\}$ for every function $\varphi \in \mathcal{B}$ we say that equations (1.1) and (1.2) are of finite delay of length ρ. Then, every history $\{x(t+u), u \in [-\rho, 0]\}$ of a solution x is a continuous function for $t \geqslant \rho$. Therefore, a canonical choice of the phase space $\mathcal{B}$ is $C([-\rho, 0], \mathbb{C}^d)$, the space of continuous $\mathbb{C}^d$-valued functions on $[-\rho, 0]$. These equations are considered in [3] but in the differentiated form (1.2).
In the case of a finite delay the Lyapunov exponents of the solution of (1.1) are studied in [6] on the space of measurable bounded functions on $[-\rho, 0]$. There, the equation is perturbed more generally by a locally integrable function h. The approach is based on the construction of a certain bilinear form, for which the specification of the phase space is necessary. For equations with infinite delay on a generalized form of the Delfour-Mitter space such a bilinear form is given in [7] and the same methods as in [6] apply to determine the Lyapunov exponents. But, if the phase space is only described axiomatically, such a bilinear form cannot be derived.
In this paper the major tool to determine the Lyapunov exponents in the general case of an axiomatically described phase space is a representation of the history of the solution of equation (1.1) in the second dual space

of $\mathcal{B}$. Such a representation is known for solutions of the differentiated equation (1.2) and is given in terms of the weak*-integral in the second dual space [5]. Similar to the usual theory of weak*-integrals we develop in section 3 a modified weak*- integral which is adapted to the problem for the more general equation (1.1). Using this weak*-integral we can derive the representation of the history of the solution in the second dual space. A well known result in the theory of functional differential equations allows to decompose the space $\mathcal{B}$ into the direct sum of two subspaces in a proper way. In section 4 we project the representation of the history in the second dual space onto these subspaces and estimate then their long time behavior. Applying these results, we can state the Lyapunov exponents in section 5. For equation (1.2) both, the representation in the second dual space and the decomposition, are well known. However, to the best of our knowledge, they have not been applied to calculate the Lyapunov exponents for the solution of the differentiated equation (1.2) and not even for the equation (1.1).
To begin with, we introduce in section 2 the axiomatic approach describing the phase space $\mathcal{B}$ and summarize some necessary prerequisites of the theory of differential equation with infinite delay. This section and the notation used are based on the monograph [5] and the paper [8].

2 Linear Autonomous Systems

Consider the linear autonomous differential equations with infinite delay:

$$\dot{x}(t) = L(x_t) \quad \text{for a.e. } t \geqslant 0, \quad x_0 = \varphi \in \mathcal{B}\,. \tag{2.3}$$

We say, that a solution is a function $x = x(\cdot, \varphi)$ on $\mathbb{R}$ which is locally absolutely continuous on $[0, \infty)$ and satisfies the equation (2.3) with $x_0 = \varphi$. One of the desirable properties which shall characterize a phase space $\mathcal{B}$ is that the history of each possible "candidate" for a solution, that is a continuous prolongation of an element of the phase space, is again included in the space $\mathcal{B}$. Such a "candidate" is called admissible:

Definition 2.1. *A function $x : (-\infty, a) \to \mathbb{C}^d$ for some $a > 0$ is said to be admissible for the space $\mathcal{B}$ if $x_0 \in \mathcal{B}$ and x is continuous on $[0, a)$.*

In the theory of differential equation with infinite delay some conditions on the phase space $\mathcal{B}$ are required. The Euclidean metric in $\mathbb{C}^d$ is denoted by $|\cdot|$.

Condition (A). *For all admissible functions $x : (-\infty, a) \to \mathbb{C}^d$ for $\mathcal{B}$ and with some $a > 0$ the following conditions hold:*

1) a) $x_t \in \mathcal{B}$ *for every* $t \in [0, a)$;

b) *there exists* $H > 0$, *such that* $|x(t)| \leqslant H \|x_t\|_{\mathcal{B}}$ *for every* $t \in [0, a)$;

c) *there exists* $N : [0, \infty) \to [0, \infty)$, *continuous, independent of* x, *there exists* $M : [0, \infty) \to [0, \infty)$, *locally bounded, independent of* x, *such that:*

$$\|x_t\|_{\mathcal{B}} \leqslant N(t) \sup_{0 \leqslant u \leqslant t} |x(u)| + M(t) \|x_0\|_{\mathcal{B}} \quad \text{for every } t \in [0, a).$$

2) $t \mapsto x_t$ *is a* $\mathcal{B}$*-valued continuous function for every* $t \in [0, a)$.

It is known that the homogeneous equation (2.3) has a unique solution if $\mathcal{B}$ satisfies Condition (A) and the operator L is bounded on $\mathcal{B}$. Moreover, the Condition (A) already implies that the phase space contains $C_c(\mathbb{R}_-, \mathbb{C}^d)$, the set of continuous functions with compact support. For, every function φ which is continuous on $[0, \infty)$ and vanishes on $(-\infty, 0]$ is an admissible function.
The Condition (A) gives an inner characterization of a phase space. For linear systems semi-group theory offers a powerful technique, but it requires some Banach space structures.

Condition (B). *The phase space* $\mathcal{B}$ *is complete.*

Condition (C). *For every Cauchy sequence* $\{\varphi_n\} \subseteq \mathcal{B}$ *with respect to the semi-norm* $\|\cdot\|_{\mathcal{B}}$, *which converges uniformly on every compact subset of* $\mathbb{R}_-$ *to a function* φ, *the function* φ *is in* $\mathcal{B}$ *and* $\|\varphi_n - \varphi\|_{\mathcal{B}}$ *tends to zero for* $n \to \infty$.

For $\varphi \in \mathcal{B}$ the symbol $\hat{\varphi}$ denotes the equivalence class $\{\psi : \|\psi - \varphi\|_{\mathcal{B}} = 0\}$. Hence, the quotient space $\hat{\mathcal{B}} := \mathcal{B} / \|\cdot\|_{\mathcal{B}}$ is a Banach space with the norm $\|\hat{\varphi}\|_{\hat{\mathcal{B}}} = \|\varphi\|_{\mathcal{B}}$ if the Condition (B) is fulfilled. For a bounded linear operator T on $\mathcal{B}$ let $\hat{T}$ be the induced operator $\hat{T}\,\hat{\varphi} = T\varphi$ for some $\varphi \in \hat{\varphi}$.
In the sequel suppose that the phase space $\mathcal{B}$ satisfies condition (A) and that L is a linear bounded operator from $\mathcal{B}$ into $\mathbb{C}^d$. Define the solution operators for $t \geqslant 0$

$$T(t) : \mathcal{B} \to \mathcal{B}, \quad T(t)\varphi = x_t,$$

where $x = x(\cdot, \varphi)$ is the solution of equation (2.3).
Since the space $\mathcal{B}$ includes $C_c(\mathbb{R}_-, \mathbb{C}^d)$ the Riesz representation theorem implies that a Borel prolongation $\widetilde{L}$ exists uniquely on the space of bounded measurable functions mapping $\mathbb{R}_-$ into $\mathbb{C}^d$ with compact support such that $\widetilde{L}\varphi = L\varphi$ for every function $\varphi \in C_c(\mathbb{R}_-, \mathbb{C}^d)$. We define the $\mathbb{C}^{d \times d}$-valued function

$$\mu(u) := \begin{cases} 0, & u = 0, \\ -\widetilde{L}(\mathbb{1}_{[u,0]}\,\mathrm{I}), & u < 0, \end{cases} \tag{2.4}$$

where $\mathrm{I} \in \mathbb{C}^{d\times d}$ is the identity matrix. According to Theorem 3.4.2 in [5] the function μ is a measurable function which is locally of bounded variation and is normalized, that is continuous from the left and satisfies $\mu(0) = 0$. Moreover, Theorem 3.4.2 in [5] states

$$L\varphi = \int d\mu(u)\varphi(u) \quad \text{for all } \varphi \in C_c(\mathbb{R}_-, \mathbb{C}^d), \tag{2.5}$$

$$\mathrm{Var}\,(\mu, [a_1, a_2]) \leqslant c\,\|L\|_{op}\, N(a_2 - a_1)M(-a_2) \quad \text{for } a_1 < a_2 \leqslant 0, \tag{2.6}$$

where $\mathrm{Var}\,(f, J)$ denotes the total variation of a function f on the interval J and c is a constant.
The representation of the solution of equation (1.1), given in the next section, strongly depends on the differential resolvent. This is a function $r : \mathbb{R}_+ \to \mathbb{C}^{d\times d}$ solving the matrix equation

$$\dot{r}(t) = \int_{[-t,0]} d\mu(u)r(t+u) \quad \text{for almost every } t \geqslant 0, \quad r(0) = \mathrm{I}\,. \tag{2.7}$$

There exists a unique locally absolutely continuous function r which is a solution of equation (2.7). Let $\mathbb{C}^{d*}$ be the space of d-dimensional row-vectors and consider the adjoint equation of (2.3)

$$y(s) + \int_s^0 y(u)\mu(s-u)du = b(s), \quad s \leqslant 0, \tag{2.8}$$

with the forcing function $b : \mathbb{R}_- \to \mathbb{C}^{d*}$ being locally of bounded variation. According to Theorem 4.1.4 in [5] equation (2.8) has a unique solution $y = y(\cdot, b)$, which is locally of bounded variation:

$$\mathrm{Var}\,(y, [s, 0]) \leqslant \mathrm{Var}\,(b, [s, 0]) + \sup_{s\leqslant u\leqslant 0} |b(u)| \left(e^{-cs\|L\|_{op}\|N\|_{C[0,-s]}} - 1\right) \tag{2.9}$$

with a constant $c > 0$. If the forcing function b is normalized then so is the solution $y(\cdot, b)$. Furthermore, by Corollary 4.1.7 in [5] the solution of (2.8) is given by

$$y(s) = b(0)r(-s) - \int_{[s,0]} db(u)r(u-s), \quad s \leqslant 0. \tag{2.10}$$

Let $\mathcal{B}^*$ and $\hat{\mathcal{B}}^*$ be the dual spaces of $\mathcal{B}$ and $\hat{\mathcal{B}}$, respectively, which are Banach spaces. We denote by $\langle\psi, \varphi\rangle$ the duality pairing of $\psi \in \mathcal{B}^*$ and $\varphi \in \mathcal{B}$. $\mathcal{B}^*$ is isometrically isomorph to $\hat{\mathcal{B}}^*$ by the mapping $\varphi^* \leftrightarrow \hat{\varphi}^*$ for $\varphi^* \in \mathcal{B}^*$, where $\hat{\varphi}^* \in \hat{\mathcal{B}}^*$ is defined by $\langle\hat{\varphi}^*, \hat{\varphi}\rangle = \langle\varphi^*, \varphi\rangle$ for every $\varphi \in \mathcal{B}$. In the same way one can identify the adjoint operator $\hat{T}^*$ of $\hat{T}$ with the adjoint operator T^* of T for a bounded linear operator T on $\mathcal{B}$.

By Theorem 3.4.2 in [5], for every $\psi \in \mathcal{B}^*$ one can find a unique $\widetilde{\psi} : \mathbb{R}_- \to \mathbb{C}^{d*}$ which is locally of bounded variation and normalized such that

$$\langle \psi, \varphi \rangle = \int d\widetilde{\psi}(u)\varphi(u) \quad \text{for every } \varphi \in C_c(\mathbb{R}_-, \mathbb{C}^d), \tag{2.11}$$

$$\text{and } \operatorname{Var}\left(\widetilde{\psi}, [-t, 0]\right) \leqslant c' N(t) \, \|\psi\|_{\mathcal{B}^*} \quad \text{for } t \geqslant 0, \tag{2.12}$$

with a constant c'. Now we can formulate the connection between the adjoint operators $\{T^*(t)\}_{t\geqslant 0}$ of the solution operators $\{T(t)\}_{t\geqslant 0}$ and the solution y of the adjoint equation (2.8). This formula is the basis for all of the following considerations.
For all $\psi \in \mathcal{B}^*$ the following equation holds

$$\widetilde{[T^*(t)\psi]}(0-) = y(-t, \widetilde{\psi}) \quad \text{for every } t > 0. \tag{2.13}$$

A proof is given in Theorem 4.2.2 in [5].

3 Representation in the second dual space

Denote by $BV([a, b], \mathbb{C}^{k\times l})$, $k, l \in \mathbb{N}$, the space of functions of bounded variation mapping the interval $[a, b]$ into $\mathbb{C}^{k\times l}$. Equipped with the norm $\|f\|_{BV[a,b]} := |f(a)| + \operatorname{Var}(f, [a, b])$, it is a Banach space. Recall, that for functions $f \in BV([a, b], \mathbb{C}^{k\times l})$ and $h \in C([a, b], \mathbb{C}^l)$ the integral on the left hand side of

$$\int_{[a,b]} f(s)dh(s) = -\int_{[a,b]} [df(s)]h(s) + f(b)h(b) - f(a)h(a) \tag{3.14}$$

is well defined by the right hand side, where the integral is understood as a Riemann-Stieltjes integral. Since h is continuous, the integral on the right hand side coincides with the Lebesgue integral. Define for $t \geqslant 0$ the operator

$$K(t) : C([0, t], \mathbb{C}^d) \to \mathcal{B},$$
$$(K(t)h)(u) := \begin{cases} \int_0^{t+u} r(t - s + u)dh(s), & u \in [-t, 0], \\ 0, & u < -t. \end{cases}$$

Since $K(t)h$ is a continuous function with compact support for $t \geqslant 0$ and $h \in C([0, t], \mathbb{C}^d)$, it is an element of the phase space $\mathcal{B}$. Similar to Theorem 4.2.1 in [5], one establishes that for all $t \geqslant 0$ the operator $K(t)$ is linear and bounded on $C([0, t], \mathbb{C}^d)$. Furthermore, for every $\varphi \in \mathcal{B}$ and $h \in C(\mathbb{R}_+, \mathbb{C}^d)$ with $h(0) = 0$ there exists uniquely a solution $x = x(\cdot, \varphi, h)$ of equation (1.1) which is given in the phase space $\mathcal{B}$ by

$$x_t = T(t)\varphi + K(t)h. \tag{3.15}$$

Evaluating x_t in zero yields the solution in the state space $\mathbb{C}^d$ at time t. By using the representation (2.5) of L on $C_c(\mathbb{R}_-, \mathbb{C}^d)$ the proof is similar to the one of Theorem 1 in [6].
Later on, we shall use the relation (2.13) of the adjoint operators T^* and the solution y of the adjoint equation (2.8) to estimate the history of the solution of equation (1.1). With a glance at the representation (3.15) of the history we need a relation of the operator $K(t)$ and the adjoint operator $T^*(t)$, which is provided by the next lemma.

Lemma 3.1. *For every $\psi \in \mathcal{B}^*$ and $h \in C([0,t], \mathbb{C}^d)$ the following relation holds for every $t \geqslant 0$*

$$\langle \psi, K(t)h \rangle = -\int_0^t [T^*(t-s)\psi]\tilde{}(0-)dh(s).$$

Proof. The integral is well defined because $[T^*(t-s)\psi]\tilde{}(0-) = y(s-t, \widetilde{\psi})$ holds by (2.13) and the solution $y(\cdot, \widetilde{\psi})$ of equation (2.8) is locally of bounded variation. Note that $K(t)h$ is an element of $C_c(\mathbb{R}_-, \mathbb{C}^d)$. First, we assume that h is a continuously differentiable function on $[0,t]$. Thus, one obtains for every $\psi \in \mathcal{B}^*$ by (2.11), (2.10) and (2.13)

$$\begin{aligned}
\langle \psi, K(t)h \rangle &= \int_{[-t,0]} d\widetilde{\psi}(u)[K(t)h](u) \\
&= \int_{[-t,0]} d\widetilde{\psi}(u) \left(\int_0^{t+u} r(t-s+u)dh(s) \right) \\
&= \int_{[-t,0]} d\widetilde{\psi}(u) \left(\int_0^{t+u} r(t-s+u)\dot{h}(s)ds \right) \\
&= \int_0^t \left(\int_{[s-t,0]} d\widetilde{\psi}(u) r(t-s+u) \right) \dot{h}(s)ds \\
&= -\int_0^t y(s-t, \widetilde{\psi})\dot{h}(s)ds \\
&= -\int_0^t [T^*(t-s)\psi]\tilde{}(0-)dh(s).
\end{aligned}$$

Since both sides of the equation depend continuously on h with respect to the supremum norm and the continuously differentiable functions are dense in $C([0,t], \mathbb{C}^d)$ the assertion follows. □

In the sequel we define a weak*-integral in the dual space of a Banach space, which corresponds to the integral (3.14) in a similar way as the usual weak*-integral to the Lebesgue integral. Let X be an arbitrary Banach space with the norm $\|\cdot\|_X$ and denote by $\langle x^*, x \rangle$ the dual pairing of $x^* \in X^*$ and $x \in X$.

Definition 3.2. *A function $f : [a,b] \to X^*$ is called (Riemann-Stieltjes) integrable on $[a,b]$ with respect to continuous functions in the weak* topology of X^* if*

1) *the function $t \mapsto \langle f(t), x\rangle$ is of bounded variation on $[a,b]$ for each $x \in X$;*

2) *the linear operator*

$$F : X \to BV([a,b],\mathbb{C}), \quad F(x)(s) := \langle f(s), x\rangle, \quad s \in [a,b],$$

is continuous.

Note that the first assumption already implies that the function $\langle f(\cdot), x\rangle$ is measurable.

Lemma 3.3. *Let $f : [a,b] \to X^*$ be a weak*-integrable function with respect to continuous functions and $h \in C([a,b],\mathbb{C})$. Then there exists a unique element $x^* \in X^*$ such that*

$$\langle x^*, x\rangle = \int_a^b \langle f(s), x\rangle dh(s) \quad \textit{for all } x \in X.$$

Proof. Define the operator $F : X \to BV([a,b],\mathbb{C})$ as in the Definition 3.2. Since the operator F is assumed to be continuous, one obtains

$$\begin{aligned}\left|\int_a^b \langle f(s), x\rangle dh(s)\right| &\leqslant 2\,\|h\|_{C[a,b]}\,\|F(x)\|_{BV[a,b]} \\ &\leqslant 2\,\|h\|_{C[a,b]}\,\|F\|_{op}\,\|x\|_X\,.\end{aligned}$$

Hence, the linear functional $x \mapsto \int_a^b \langle f(s), x\rangle dh(s)$ is bounded and is thus an element of $\mathcal{B}^*$. □

The Lemma 3.3 allows to define a weak*-integral with respect to a continuous function.

Definition 3.4. *Let $f : [a,b] \to X^*$ be a weak*-integrable function with respect to continuous functions and $h \in C([a,b],\mathbb{C})$. We define the weak*-integral of f with respect to h by the functional*

$$*\!\int_a^b f(s)dh(s) \in X^* : \quad \langle *\!\int_a^b f(s)dh(s), x\rangle := \int_a^b \langle f(s), x\rangle dh(s)$$

for all $x \in X$.

For $f = (f_1, \cdots, f_d)$ with $f_i : [a,b] \to X^*$ weak*-integrable with respect to continuous functions and $h \in C([a,b], \mathbb{C}^d)$ the weak*-integral $*\int f dh$ is defined componentwise.
In the classical definition of a weak*-integrable function the assumption 1) in Definition 3.2 is replaced by demanding that the function $\langle f(\cdot), x\rangle$ is measurable and integrable with respect to the Lebesgue measure. These properties are already sufficient to show that the operator F in the second assumption is closed when $BV([a,b], \mathbb{C})$ is replaced by the space of Lebesgue integrable functions, see Theorem 3.7.1 in [4]. Hence, the operator F is continuous in this case. However, if the weak*-integral shall correspond to the Riemann-Stieltjes integral (3.14) we have to require the continuity of the operator F in the Definition 3.2.
Now, let T be a bounded linear operator on X with values in X, then T^*f is weak*-integrable with respect to continuous functions for a weak*-integrable function $f : [a,b] \to X^*$ and it holds

$$T^* *\int_a^b f(s)dh(s) = *\int_a^b T^* f(s)dh(s). \tag{3.16}$$

This can be shown analogously to the same property of the usual weak*-integral, see e.g. Lemma 3.14 in the Appendix II of [1].
Define for $i = 1, \cdots, d$ the functionals

$$\gamma^i : \mathcal{B}^* \to \mathbb{C}, \quad \gamma^i(\psi) := -\widetilde{\psi}^i(0-), \tag{3.17}$$

where $\widetilde{\psi}^i$ denotes the i-th component of $\widetilde{\psi}$. In [5] it is shown that γ^i is an element of the second dual space $\mathcal{B}^{**}$ of the phase space $\mathcal{B}$ for $i = 1, \cdots, d$. Denote by γ the vector $\gamma(\psi) := (\gamma^1(\psi), \cdots, \gamma^d(\psi)) = -\widetilde{\psi}(0-)$ and we set $T\gamma := (T\gamma^1, \cdots, T\gamma^d)$ for an operator T on $\mathcal{B}^{**}$.
In the sequel we identify the space $\hat{\mathcal{B}}$ with the subspace of the second dual space $\mathcal{B}^{**}$ in the usual manner. The dual pairing of $\psi \in \mathcal{B}^*$ and $\varphi^{**} \in \mathcal{B}^{**}$ is denoted by $\langle \psi, \varphi^{**}\rangle$.
We obtain from Lemma 3.1 by use of the weak*-integral a representation of the history of the solution of equation (1.1) in the second dual space B^{**}. A major part of calculating the Lyapunov exponents will be based on this representation.

Theorem 3.5. *Let $x = x(\cdot, \varphi, h)$ be the solution of (1.1). Then $T^{**}(t-\cdot)\gamma^i : [0,t] \to \mathcal{B}^{**}$ is weak*-integrable with respect to continuous functions for all $i = 1, \cdots, d$ and*

$$\hat{x}_t = \hat{T}(t)\,\hat{\varphi} + *\int_0^t T^{**}(t-s)\gamma dh(s) \quad \textit{for } t \geqslant 0. \tag{3.18}$$

Proof. Denote by y^i the i-th component of the solution of the adjoint equation (2.8). For each $\psi \in \mathcal{B}^*$ we obtain by (3.17) and (2.13)

$$\langle \psi, T^{**}(t-s)\gamma^i \rangle = \langle T^*(t-s)\psi, \gamma^i \rangle = -y^i(s-t, \widetilde{\psi}).$$

Consequently, the function $s \mapsto \langle \psi, T^{**}(t-s)\gamma^i \rangle$ is of bounded variation on $[0,t]$. The equations (2.9) and (2.12) yield

$$\begin{aligned} \left\| \langle \psi, T^{**}(t-\cdot)\gamma^i \rangle \right\|_{BV[0,t]} &\leqslant \left| y(-t, \widetilde{\psi}) \right| + \mathrm{Var}\left(y(\cdot, \widetilde{\psi}),\ [-t, 0] \right) \\ &\leqslant c(t)\mathrm{Var}\left(\widetilde{\psi}, [-t, 0] \right) \\ &\leqslant c' c(t) N(t) \left\| \psi \right\|_{\mathcal{B}^*}, \end{aligned}$$

with constants c' and $c(t)$. Hence, the function $s \mapsto T^{**}(t-s)\gamma^i$ is weak*-integrable with respect to continuous functions for $i = 1, \cdots, d$. Moreover, Lemma 3.1 implies

$$\begin{aligned} \langle \psi, *\int_0^t T^{**}(t-s)\gamma dh(s) \rangle &= \int_0^t \langle T^*(t-s)\psi, \gamma \rangle dh(s) \\ &= -\int_0^t [T^*(t-s)\psi]\tilde{}(0-) dh(s) \\ &= \langle \psi, \hat{K}(t)h \rangle. \end{aligned}$$

By means of (3.15) the assertion follows. □

What is the advantage of the representation (3.18) in the second dual space $\mathcal{B}^{**}$ over the representation (3.15) in $\mathcal{B}$? In the latter the integral $K(t)h$ is an element of $\mathcal{B}$ but in general the integrand r is not. Consequently, one cannot estimate $\|K(t)h\|_{\mathcal{B}}$ by estimating the integrand r. Although in the representation (3.18) the integrand of the weak*-integral is in the second dual space $\mathcal{B}^{**}$ one can benefit from the close association of the solution operators T and T^{**} via the adjoint operator T^* and the adjoint equation (2.8).

4 Estimates on the subspaces

First, we report on a result concerning a decomposition of $\mathcal{B}$ as it is stated in Chapter 5 of [5]. Assume in the sequel that the phase space $\mathcal{B}$ satisfies the Conditions (A), (B) and (C) and that the operator L is bounded on $\mathcal{B}$. From Condition (A), the semi-groups $\{T(t)\}_{t\geqslant 0}$ and $\{\hat{T}(t)\}_{t\geqslant 0}$ are strongly continuous semi-groups of bounded operators on $\mathcal{B}$ and $\hat{\mathcal{B}}$, respectively. Denote the generator of $\{\hat{T}(t)\}_{t\geqslant 0}$ by $\hat{A}$. For the trivial equation $L = 0$,

the solution semi-group is given by

$$S(t) : \mathcal{B} \to \mathcal{B}, \quad (S(t)\varphi)(u) = \begin{cases} \varphi(0), & u \in [-t, 0], \\ \varphi(t+u), & u \leqslant -t, \end{cases} \tag{4.19}$$

for $t \geqslant 0$. Define a parameter β, depending only on the phase space $\mathcal{B}$, by the following relation

$$\beta := \lim_{t\to\infty} \frac{\log \alpha(\hat{S}(t))}{t},$$

where α is the Kuratowski measure of non-compactness of bounded operators, see e.g. Definition 4.20 in the Appendix II of [1]. For a fixed $\lambda \in \mathbb{C}$ define the function $(e(\lambda)b)(u) := e^{\lambda u}b$ for $u \leqslant 0$ and an arbitrary $b \in \mathbb{C}^d$. If $\mathrm{Re}\lambda > \beta$ then the functions $e(\lambda)b$ are elements of $\mathcal{B}$ for every $b \in \mathbb{C}^d$. The point spectrum of the generator $\hat{A}$ is given by

$$\sigma_P(\hat{A}) = \{\lambda \in \mathbb{C} : \exists\, b \in \mathbb{C}^d \setminus \{0\} :\ e(\lambda)b \in \mathcal{B} \text{ and } \lambda b - L(e(\lambda)b) = 0\}.$$

Let $\Lambda = \{\lambda_1, \cdots, \lambda_p\}$ be a finite set of eigenvalues of $\hat{A}$ such that $\mathrm{Re}\lambda_j > \beta$ for $j = 1, \cdots, p$. For every λ_j there exists a basis $\mathbf{\Phi}_j = (\Phi_1, \cdots, \Phi_{m_j})$ of the generalized eigenspace $M(\lambda_j) := \ker((\lambda_j \,\mathrm{Id} - \hat{A})^{k_j})$, $k_j \in \mathbb{N}$. For such a basis there are matrices $B_j \in \mathbb{C}^{m_j \times m_j}$ with the single eigenvalue λ_j and $\hat{A}\,\mathbf{\Phi}_j = \mathbf{\Phi}_j\, B_j$. Define $\mathbf{\Phi}_\Lambda = (\mathbf{\Phi}_1, \cdots, \mathbf{\Phi}_p)$, $B_\Lambda = \mathrm{diag}(B_1, \cdots, B_p)$ and $m = m_1 + \cdots + m_p$. For every vector $a \in \mathbb{C}^m$, $\hat{T}(t)\,\mathbf{\Phi}_\Lambda\, a$ with initial value $\mathbf{\Phi}_\Lambda\, a$ at $t = 0$ may be defined on $\mathbb{R}$ by the relation

$$\hat{T}(t)\,\mathbf{\Phi}_\Lambda\, a = \mathbf{\Phi}_\Lambda\, e^{B_\Lambda t}a, \quad \mathbf{\Phi}_\Lambda(u) = \mathbf{\Phi}_\Lambda(0)e^{B_\Lambda u},\ u \leqslant 0. \tag{4.20}$$

Furthermore, there exists a $\hat{T}(t)$-invariant subspace $\hat{Q}_\Lambda$ of $\hat{\mathcal{B}}$ such that

$$\hat{\mathcal{B}} = \hat{P}_\Lambda \oplus \hat{Q}_\Lambda, \tag{4.21}$$

where $\hat{P}_\Lambda = \{\hat{\varphi} \in \hat{\mathcal{B}} : \hat{\varphi} = \mathbf{\Phi}_\Lambda\, a \text{ for some } a \in \mathbb{C}^m\}$. In the special case of $\Lambda = \Lambda(s) := \{\lambda \in \sigma_P(\hat{A}) : \mathrm{Re}\lambda \geqslant s\}$ for some $s > \beta$ the set $\Lambda(s)$ is finite and one obtains for every $\varepsilon > 0$ the estimate

$$\left\| \hat{T}(t)\,\hat{\varphi} \right\|_{\hat{\mathcal{B}}} \leqslant k(\varepsilon)e^{(s'+\varepsilon)t}\, \|\hat{\varphi}\|_{\hat{\mathcal{B}}} \quad \text{for } t \geqslant 0,\ \hat{\varphi} \in \hat{Q}_\Lambda, \tag{4.22}$$

where $s' := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma_P(\hat{A}) \setminus \Lambda(s)\} \vee \beta$ and $k(\varepsilon)$ is a constant depending on ε. Let $\pi_P : \hat{\mathcal{B}} \to \hat{\mathcal{B}}$ be the projection on $\hat{P}_\Lambda$ along the direct sum (4.21): $\pi_P(\hat{\varphi}) = \hat{\varphi}^P$ if $\hat{\varphi} = \hat{\varphi}^P + \hat{\varphi}^Q$, $\hat{\varphi}^P \in \hat{P}_\Lambda$, $\hat{\varphi}^Q \in \hat{Q}_\Lambda$ and $\pi_Q = \mathrm{Id} - \pi_P$. Since the operator $\hat{T}(t)$ is invariant on the spaces $\hat{P}_\Lambda$ and $\hat{Q}_\Lambda$ the projections π_P and π_Q and their adjoints commute with the operator $\hat{T}(t)$ and its adjoints $T^*(t)$ and $T^{**}(t)$. Furthermore, $\pi_P^*\,\mathcal{B}^* = (\hat{Q}_\Lambda)^\perp$, where $(\hat{Q}_\Lambda)^\perp$

denotes the annihilator of $\hat{Q}_\Lambda$, and there exists a basis $\mathbf{\Psi}_\Lambda = (\Psi_1, \cdots, \Psi_m)^T$ for $\pi_P^* \mathcal{B}^* \subseteq \mathcal{B}^*$ such that $\langle \Psi_i, \Phi_j \rangle = \delta_{ij}$ for $i, j = 1, \cdots, m$. One obtains

$$\pi_P \hat{\varphi} = \sum_{i=1}^{m} \langle \Psi_i, \hat{\varphi} \rangle \Phi_i = \mathbf{\Phi}_\Lambda \langle \mathbf{\Psi}_\Lambda, \hat{\varphi} \rangle \quad \text{for all } \hat{\varphi} \in \hat{\mathcal{B}}, \tag{4.23}$$

$$\pi_P^{**} \varphi^{**} = \sum_{i=1}^{m} \langle \Psi_i, \varphi^{**} \rangle \Phi_i = \mathbf{\Phi}_\Lambda \langle \mathbf{\Psi}_\Lambda, \varphi^{**} \rangle \quad \text{for all } \varphi^{**} \in \mathcal{B}^{**}, \tag{4.24}$$

using the notations $\langle \mathbf{\Psi}_\Lambda, \hat{\varphi} \rangle := (\langle \Psi_1, \hat{\varphi} \rangle, \cdots, \langle \Psi_m, \hat{\varphi} \rangle)^T$ and similarly for $\langle \mathbf{\Psi}_\Lambda, \varphi^{**} \rangle$.
For such a decomposition of the phase space $\hat{\mathcal{B}}$ into two subspaces with respect to a given subset of eigenvalues of the generator $\hat{A}$ we project the representation (3.18) onto these subspaces. We denote by $\mathbb{C}_\beta$ the half plane $\{z \in \mathbb{C} : \mathrm{Re} z > \beta\}$.

Theorem 4.1. *Let $\hat{\mathcal{B}} = \hat{P}_\Lambda \oplus \hat{Q}_\Lambda$ be a decomposition with respect to $\Lambda \subseteq \sigma_P(\hat{A}) \cap \mathbb{C}_\beta$. Then the solution $x = x(\cdot, \varphi, h)$ of (1.1) can be represented by*

$$\pi_P \hat{x}_t = \hat{T}(t)(\pi_P \hat{\varphi}) + *\int_0^t T^{**}(t-s)(\pi_P^{**}\gamma) dh(s), \tag{4.25}$$

$$\pi_Q \hat{x}_t = \hat{T}(t)(\pi_Q \hat{\varphi}) + *\int_0^t T^{**}(t-s)(\pi_Q^{**}\gamma) dh(s). \tag{4.26}$$

Proof. It follows easily from Theorem 3.5 □

In the sequel we estimate these projections of the history onto the respective subspaces. By combining the estimates we obtain the Lyapunov exponents in the last section. For the estimate of the projection onto the finite dimensional subspace $\hat{P}_\Lambda$ we simplify the representation (4.25). The equations (4.23), (4.24) and (4.20) yield

$$\begin{aligned} \hat{T}(t)(\pi_P \hat{\varphi}) &= \hat{T}(t)(\mathbf{\Phi}_\Lambda \langle \mathbf{\Psi}_\Lambda, \hat{\varphi} \rangle) = \mathbf{\Phi}_\Lambda \, e^{B_\Lambda t} \langle \mathbf{\Psi}_\Lambda, \hat{\varphi} \rangle, \\ T^{**}(t-s)(\pi_P^{**}\gamma) &= T(t-s)(\mathbf{\Phi}_\Lambda \langle \mathbf{\Psi}_\Lambda, \gamma \rangle) = -\mathbf{\Phi}_\Lambda \, e^{B_\Lambda (t-s)} \widetilde{\mathbf{\Psi}}_\Lambda(0-). \end{aligned}$$

Therefore, we obtain the representation

$$\begin{aligned} \pi_P \hat{x}_t &= \mathbf{\Phi}_\Lambda \left(e^{B_\Lambda t} \langle \mathbf{\Psi}_\Lambda, \hat{\varphi} \rangle - \int_0^t e^{B_\Lambda (t-s)} \widetilde{\mathbf{\Psi}}_\Lambda(0-) dh(s) \right) \\ &=: \mathbf{\Phi}_\Lambda \, U_\Lambda(t). \end{aligned} \tag{4.27}$$

The function U_Λ defined in (4.27) is called the coordinate process as it represents the coordinate of the projection $\pi_p \hat{x}_t$ with respect to the basis

$\mathbf{\Phi}_\Lambda$ for a fixed $t \geqslant 0$. Note that the coordinate process U_Λ is the solution of the ordinary integral equation in $\mathbb{C}^m$:

$$u(t) = \langle \mathbf{\Psi}_\Lambda, \hat{\varphi} \rangle + \int_0^t B_\Lambda u(s) ds - \widetilde{\mathbf{\Psi}}_\Lambda(-0) h(s), \quad t \geqslant 0.$$

By the representation (4.27) we can establish the estimate for the projection $\|\pi_P \hat{x}_t\|_{\hat{\mathcal{B}}}$ in the next theorem in a similar way as in [6]. Let $\hat{\mathcal{B}} = \hat{P}_\Lambda \oplus \hat{Q}_\Lambda$ be a decomposition with respect to $\Lambda \subseteq \{\lambda \in \sigma_P(\hat{A}) : \mathrm{Re}\lambda \geqslant v\}$ with $v > \beta$. Assume $h(t) = \mathcal{O}(e^{\kappa t})$ for $t \to \infty$ with some $\kappa < v$. Then for $t \geqslant 0$ define

$$Y_\Lambda(t) := \int_t^\infty e^{B_\Lambda(t-s)} \widetilde{\mathbf{\Psi}}_\Lambda(0-) dh(s).$$

The indefinite integral is well defined in the sense of (3.14) since h is Lebesgue integrable on $\mathbb{R}_+$ with respect to $e^{B_\Lambda(t-\cdot)} \widetilde{\mathbf{\Psi}}_\Lambda(0-)$.

Theorem 4.2. *Let $\hat{\mathcal{B}} = \hat{P}_{\Lambda_i} \oplus \hat{Q}_{\Lambda_i}$ be the decompositions with respect to $\Lambda_i := \{\lambda \in \sigma_P(\hat{A}) : Re\lambda = v_i\}$ with $v_1 > \cdots > v_p > \beta$ for $i = 1, \cdots, p$. Denote the basis of $(\hat{Q}_{\Lambda_i})^\perp$ by $\mathbf{\Psi}_i$. Assume $h(t) = \mathcal{O}(e^{\kappa t})$ for $t \to \infty$ with some $\kappa < v_p$. Then*

1) *if $\langle \mathbf{\Psi}_i, \hat{\varphi} \rangle = Y_{\Lambda_i}(0)$ for $i = 1, \cdots, l-1$, $l \in \{1, \cdots, p\}$ and $\langle \mathbf{\Psi}_l, \hat{\varphi} \rangle \neq Y_{\Lambda_l}(0)$ it follows*

$$\lim_{t \to \infty} \frac{1}{t} \log \|\pi_P \hat{x}_t\|_{\hat{\mathcal{B}}} = v_l;$$

2) *if $\langle \mathbf{\Psi}_i, \hat{\varphi} \rangle = Y_{\Lambda_i}(0)$ for $i = 1, \cdots, p$ it follows*

$$\limsup_{t \to \infty} \frac{1}{t} \log \|\pi_P \hat{x}_t\|_{\hat{\mathcal{B}}} \leqslant \kappa.$$

Here π_P denotes the projection onto $\hat{P}_\Lambda$ if the phase space is decomposed according to $\hat{\mathcal{B}} = \hat{P}_\Lambda \oplus \hat{Q}_\Lambda$ with respect to $\Lambda := \Lambda_1 \cup \cdots \cup \Lambda_p$.

Proof. Similar to Theorem 8 in [6]. □

For calculating the Lyapunov exponents of the projection $\pi_Q \hat{x}_t$ onto the other subspace $\hat{Q}_\Lambda$ we use the representation (4.26). Determining the asymptotic behavior of the weak*-integral in the representation (4.26) results in estimating the variation of the solution of the adjoint equation (2.8). By doing this, we benefit from the restriction of the integrand T^{**} on $\pi_Q^{**} \mathcal{B}^{**}$ in (4.26). Therefore, we establish in the next lemma an upper bound of the variation of the solution $y(\cdot, b)$ of the adjoint equation, if the forcing function b lies in a subspace. This is a sharper estimate than it is provided by (2.9) for the general case of arbitrary forcing functions.

Lemma 4.3. *Let $\hat{\mathcal{B}} = \hat{P}_\Lambda \oplus \hat{Q}_\Lambda$ be a decomposition of $\hat{\mathcal{B}}$ with respect to $\Lambda = \{\lambda \in \sigma_P(\hat{A}) : Re\lambda \geqslant s\}$ for some $s > \beta$ and π_Q be the projection onto $\hat{Q}_\Lambda$. Then for every $\varepsilon > 0$ there exist some constants $k_0 = k_0(\varepsilon)$ and $k_1 = k_1(\varepsilon)$, such that the solution y of the equation (2.8) satisfies the following estimate for all $\psi \in \mathcal{B}^*$ and $t \geqslant 0$:*

$$Var\left(y(\cdot, [\pi_Q^*\psi]\tilde{}), [-t, 0]\right) \leqslant \left(k_0 N(t) + k_1 \int_0^t N(t-u) e^{(s'+\varepsilon)u} du\right) \|\psi\|_{\mathcal{B}^*},$$

where $s' := \sup\{Re\lambda : \lambda \in \sigma_P(\hat{A}) \setminus \Lambda(s)\} \vee \beta$.

Proof. We denote by c a generic constant only depending on ε. The estimate (4.22) yields for every $\varphi \in \mathcal{B}$ and $t \geqslant 0$

$$\left\|(\pi_Q\, \hat{T}(t))\, \hat{\varphi}\right\|_{\hat{\mathcal{B}}} \leqslant c \|\pi_Q\, \hat{\varphi}\|_{\hat{\mathcal{B}}}\, e^{(s'+\varepsilon)t} \leqslant c \|\pi_Q\|_{\hat{\mathcal{B}}\to\hat{\mathcal{B}}} \|\hat{\varphi}\|_{\hat{\mathcal{B}}}\, e^{(s'+\varepsilon)t}. \tag{4.28}$$

The equations (2.13) and (2.12) imply for the solution y of the adjoint equation (2.8) for every $\psi \in \mathcal{B}^*$ and $t > 0$ and $\delta > 0$

$$\begin{aligned}
\left|y(-t, [\pi_Q^*\psi]\tilde{})\right| &= \left|[T^*(t)(\pi_Q^*\psi)]\tilde{}(0-)\right| \\
&\leqslant \mathrm{Var}\left([T^*(t)(\pi_Q^*\psi)]\tilde{}(\cdot), [-\delta, 0]\right) \\
&\leqslant cN(\delta) \left\|T^*(t)(\pi_Q^*\psi)\right\|_{\mathcal{B}^*} \\
&\leqslant cN(\delta) \|\psi\|_{\mathcal{B}^*} \left\|T^*(t)\pi_Q^*\right\|_{\mathcal{B}^*\to\mathcal{B}^*} \\
&= cN(\delta) \|\psi\|_{\mathcal{B}^*} \left\|\pi_Q\, \hat{T}(t)\right\|_{\hat{\mathcal{B}}\to\hat{\mathcal{B}}} \\
&\leqslant cN(\delta) \|\psi\|_{\mathcal{B}^*}\, e^{(s'+\varepsilon)t}.
\end{aligned}$$

Since $\delta > 0$ is arbitrary this yields

$$\left|y(-t, [\pi_Q^*\psi]\tilde{})\right| \leqslant cN(0) \|\psi\|_{\mathcal{B}^*}\, e^{(s'+\varepsilon)t}. \tag{4.29}$$

Set $\mu(u) = 0$ for $u > 0$ and define with $y = y(\cdot, [\pi_Q^*\psi]\tilde{})$:

$$\begin{aligned}
F(s) :&= \int_s^0 y(u)\mu(s-u)du, \quad s \leqslant 0, \\
&= \int_{-t}^0 y(u)\mu(s-u)du \quad \text{for } s \in [-t, 0].
\end{aligned}$$

Due to the estimates (2.6) and (4.29) it follows

$$\begin{aligned}
\mathrm{Var}\,(F, [-t, 0]) &\leqslant \int_{-t}^0 |y(u)|\, \mathrm{Var}\,(\mu, [-t-u, 0])\, du \\
&\leqslant c \int_{-t}^0 N(0) \|\psi\|_{\mathcal{B}^*}\, e^{-(s'+\varepsilon)u} \|L\|_{op}\, N(t+u) M(0) du \\
&= c \|\psi\|_{\mathcal{B}^*} \int_0^t N(t-u) e^{(s'+\varepsilon)u} du.
\end{aligned} \tag{4.30}$$

From (2.12), for every $\psi \in \mathcal{B}^*$ and $t \geqslant 0$ we get

$$\operatorname{Var}\left([\pi_Q^*\psi]^{\sim}, [-t,0]\right) \leqslant cN(t)\left\|\pi_Q^*\psi\right\|_{\mathcal{B}^*} \leqslant cN(t)\left\|\pi_Q\right\|_{\mathcal{B}\to\mathcal{B}}\left\|\psi\right\|_{\mathcal{B}^*}. \quad (4.31)$$

The equations (4.30) and (4.31) establish the required estimate. □

Due to the upper bound of the variation of the solution of the adjoint equation (2.8) in Lemma 4.3 we can now provide an estimate of the projection $\pi_Q\hat{x}_t$ of the solution x of equation (1.1).

Theorem 4.4. *Let $\hat{\mathcal{B}} = \hat{P}_\Lambda \oplus \hat{Q}_\Lambda$ be a decomposition with respect to $\Lambda = \{\lambda \in \sigma_P(\hat{A}) : Re\lambda \geqslant s\}$ for some $s > \beta$. Then for all $\varepsilon > 0$ and $\varphi \in \mathcal{B}$ there exists a constant $k = k(\varepsilon, \hat{\varphi})$, such that the projection onto $\hat{Q}_\Lambda$ of the history of the solution $x(\cdot) = x(\cdot, \varphi, h)$ of equation (1.1) satisfies the following estimate:*

$$\left\|\pi_Q\hat{x}_t\right\|_{\hat{\mathcal{B}}} \leqslant k\left(e^{(s'+\varepsilon)t} + \left\|h\right\|_{C[0,t]}\left(N(t) + \int_0^t N(t-u)e^{(s'+\varepsilon)u}du\right)\right).$$

Proof. The Theorem 4.1 implies

$$\left\|\pi_Q\hat{x}_t\right\|_{\hat{\mathcal{B}}} \leqslant \left\|\hat{T}(t)(\pi_Q\hat{\varphi})\right\|_{\hat{\mathcal{B}}} + \left\|*\int_0^t T^{**}(t-s)(\pi_Q^{**}\gamma)dh(s)\right\|_{\hat{\mathcal{B}}}. \quad (4.32)$$

Due to Lemma 4.3 the second term in (4.32) can be estimated by

$$\begin{aligned}
&\left\|*\int_0^t T^{**}(t-s)(\pi_Q^{**}\gamma)dh(s)\right\|_{\hat{\mathcal{B}}} \\
&\quad= \sup_{\|\psi\|_{\mathcal{B}^*}\leqslant 1}\left|\langle\psi, *\int_0^t T^{**}(t-s)(\pi_Q^{**}\gamma)dh(s)\rangle\right| \\
&\quad= \sup_{\|\psi\|_{\mathcal{B}^*}\leqslant 1}\left|\int_0^t \langle\psi, T^{**}(t-s)(\pi_Q^{**}\gamma)\rangle dh(s)\right| \\
&\quad= \sup_{\|\psi\|_{\mathcal{B}^*}\leqslant 1}\left|\int_0^t \langle T^*(t-s)(\pi_Q^*\psi), \gamma\rangle dh(s)\right| \\
&\quad= \sup_{\|\psi\|_{\mathcal{B}^*}\leqslant 1}\left|-\int_0^t [T^*(t-s)(\pi_Q^*\psi)]^{\sim}(0-)dh(s)\right| \\
&\quad= \sup_{\|\psi\|_{\mathcal{B}^*}\leqslant 1}\left|-\int_0^t y(s-t, [\pi_Q^*\psi]^{\sim})dh(s)\right| \\
&\quad\leqslant 2\left\|h\right\|_{C[0,t]}\sup_{\|\psi\|_{\mathcal{B}^*}\leqslant 1}\left\{\operatorname{Var}\left(y(\cdot, [\pi_Q^*\psi]^{\sim}), [-t,0]\right)\right\} \\
&\quad\leqslant 2c\left\|h\right\|_{C[0,t]}\left(N(t) + \int_0^t N(t-u)e^{(s'+\varepsilon)u}du\right),
\end{aligned}$$

with a constant $c = c(\varepsilon) > 0$. Applying (4.28) for estimating the first term in (4.32) finishes the proof. □

5 The exponents and some examples

On applying the decomposition of the phase space $\hat{\mathcal{B}}$ with respect to

$$\begin{aligned} &\Lambda(s) = \Lambda_1 \cup \cdots \cup \Lambda_p, \quad p \in \mathbb{N}, \\ &\Lambda_i = \{\lambda \in \sigma_P(\hat{A}) : \mathrm{Re}\lambda = v_i\} \quad \text{for some } v_1 > \cdots > v_p \geqslant s, \end{aligned} \tag{5.33}$$

we determine the Lyapunov exponents of the solution of equation (1.1) subject to the initial condition $\varphi \in \mathcal{B}$. As in Theorem 4.2, the vector $\mathbf{\Psi}_i$ denotes the basis for $(\hat{Q}_{\Lambda_i})^{\perp}$ with respect to the decomposition $\hat{\mathcal{B}} = \hat{P}_{\Lambda_i} \oplus \hat{Q}_{\Lambda_i}$.

Theorem 5.1. *Let $\Lambda = \Lambda(s)$ for some $s > \beta \vee 0$ be given as in (5.33). Assume for every $\varepsilon > 0$ and for $t \to \infty$:*

$$\|h\|_{C[0,t]} = \mathcal{O}(e^{(\kappa+\varepsilon)t}) \quad \textit{for some } 0 \leqslant \kappa < s, \tag{5.34}$$

$$N(t) + \int_0^t N(t-u)e^{(s'+\varepsilon)u}du = \mathcal{O}(e^{(\theta-\kappa+\varepsilon)t}) \quad \textit{for some } 0 \leqslant \theta < s. \tag{5.35}$$

Then the following holds for the solution $x(\cdot, \varphi, h)$ of equation (1.1)

1) *if $\langle \mathbf{\Psi}_i, \hat{\varphi} \rangle = Y_{\Lambda_i}(0)$ for $i = 1, \cdots, l-1$, $l \in \{1, \cdots, p\}$, and $\langle \mathbf{\Psi}_l, \hat{\varphi} \rangle \neq Y_{\Lambda_l}(0)$ it follows:*

$$\lim_{t\to\infty} \frac{\log \|\hat{x}_t\|_{\hat{\mathcal{B}}}}{t} = v_l.$$

2) *if $\langle \mathbf{\Psi}_\Lambda, \hat{\varphi} \rangle = Y_\Lambda(0)$ it follows:*

$$\limsup_{t\to\infty} \frac{\log \|\hat{x}_t\|_{\hat{\mathcal{B}}}}{t} \leqslant \kappa \vee s' \vee \theta.$$

Proof. Independent of the assumptions on $\hat{\varphi}$, Theorem 4.4 implies

$$\limsup_{t\to\infty} \frac{1}{t} \log \|\pi_Q \hat{x}_t\|_{\hat{\mathcal{B}}} \leqslant s' \vee \theta. \tag{5.36}$$

If $\hat{\varphi}$ satisfies the assumptions under 1), Theorem 4.2 yields

$$\lim_{t\to\infty} \frac{1}{t} \log \|\pi_P \hat{x}_t\|_{\hat{\mathcal{B}}} = v_l.$$

Therefore, one obtains

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \log \|\hat{x}_t\|_{\hat{\mathcal{B}}} &= \lim_{t\to\infty} \frac{1}{t} \log \|\pi_P \hat{x}_t + \pi_Q \hat{x}_t\|_{\hat{\mathcal{B}}} \\ &= \lim_{t\to\infty} \frac{1}{t} \log \left(\|\pi_P \hat{x}_t\|_{\hat{\mathcal{B}}} \, |1 + o(1)| \right) \\ &= v_l. \end{aligned}$$

If the assumptions under 2) hold we have

$$\limsup_{t\to\infty} \frac{1}{t} \log \|\pi_P \hat{x}_t\|_{\hat{\mathcal{B}}} \leqslant \kappa$$

according to Theorem 4.2 which finishes the proof in combination with (5.36). □

Not only the Lyapunov exponents in the arbitrary norms are useful but also the results on the way to establish Theorem 5.1. For, the combination of the Theorems 4.2 and 4.4 allows to decompose the history of the solution in a bounded and a dominating – the growing – part, respectively, in the semi-norm under consideration. We give an example for this, which also displays the connection of the functions h and N in Theorem 5.1. For simplicity, we assume that the operator L of the equation (1.1) is given by

$$L\varphi = \int_{(-\infty,0]} \varphi(u)\nu(du), \quad \varphi \in \mathcal{B},$$

with a locally finite measure ν. For the phase space $\mathcal{B}$, we fix some $p \geqslant 1$, set $g(u) := \exp(\gamma u)$ and define

$$\begin{aligned}\mathcal{B} &= \mathbb{C}^d \times L^p(g) \\ &:= \{(u,\varphi) \mid \varphi : (-\infty,0] \to \mathbb{C}^d,\ u = \varphi(0),\ \int_{-\infty}^{0} |\varphi(s)|^p\, g(s)ds < \infty\},\end{aligned}$$

$$\|\varphi\|_{\mathcal{B}} = |\varphi(0)| + \left(\int_{-\infty}^{0} |\varphi(s)|^p\, g(s)ds\right)^{1/p},$$

where the constant γ is chosen, such that the operator L is continuous on $\mathcal{B}$. This phase space satisfies the conditions (A), (B) and (C) and the constants which are relevant for the application of Theorem 5.1 turn out to be $\beta = -\gamma/p$ and $N(t) = 1+\gamma^{-p}(1-e^{-\gamma t})^{1/p}$, for details see Theorem 1.3.8 in [5]. Hence, the parameter s has to be chosen larger than $\max\{0, -\gamma/p\}$. By setting $\theta = \kappa+\max\{0, s'\}$ an exponential growth of the function h is possible for every parameter κ less than $\min\{s - s', s\}$. To obtain the partition of the history of the solution in a bounded part and in a dominating part, we set $s = 0$. This choice is only possible, if the parameter γ is positive, which implies by the inequality (2.6) that the measure ν has to be finite. If we now assume that $h(t) = \mathcal{O}(\exp(\varepsilon t))$ holds for increasing values of t and for every $\varepsilon > 0$, Theorem 4.2 implies, that the projection $\|\pi_P \hat{x}_P\|_{\hat{\mathcal{B}}}$ grows exponentially fast. Moreover, Theorem 4.4 yields that the projection $\|\pi_Q \hat{x}\|_{\hat{\mathcal{B}}}$ is bounded. Consequently, if the function h in the equation (1.1) grows too fast or if the history is over-weighted by the operator L, the asymptotic behavior of the projections of the solution x_t does not differ as explicitly as above.

The result of Theorem 5.1 also allows to consider the case of a finite delay of positive length ρ under various phase spaces and norms. The phase space $\mathbb{C}^d \times L^p(g)$ with $g(u) = \exp(\gamma u)$ can be used as a phase space for differential equations with a finite delay if we set the function g to zero outside the interval $[-\rho, 0]$. Even for arbitrary functions g which are set to zero outside the interval $[-\rho, 0]$ and fulfill some conditions mentioned in [5] the space $\mathbb{C}^d \times L^p(g)$ can be chosen as an phase space for equations with a finite delay. In particular, for $p = 2$ the Hilbert space $\mathbb{C}^d \times L^2(g)$ is often encountered in control theory. Then Theorem 5.1 describes the long time behavior of the solution of the delay equation (1.1) in terms of the Hilbert space norm.

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