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Notes on an Endogenous Growth Model with two Capital Stocks I: The Deterministic Case

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Abstract

This paper extends the class of AK models with an explicit solution to the case where there are two capital goods in the model. This extension holds, even if an external effect in the use of human capital in goods production is assumed.

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1 Introduction

In the neoclassical growth theory, it is well known that a closed-form solution exists for the AK model with logarithmic preferences, Cobb Douglas production, and full depreciation of capital. The expression closed-form solution means to know the explicit form of the value function. This paper extends this class of explicit solutions to AK models with two capital goods. We assume a goods production sector with a Cobb Douglas technology using physical and human capital. The produced good can be consumed or invested for future production. Furthermore, we assume an educational sector producing next period's human capital stock. The production of new human capital is linear in existing human capital. We will argue that there still exists a closed-form solution for this wider class of models. This property holds even if an external effect in the use of human capital in the goods production is assumed. The paper is organized as follows: In the second section I will briefly describe the model and then characterize the decentralized as well as the centralized solutions by the usual Euler equations. Section three introduces an iterative method for finding a candidate for the value function and a criterion for verifying this guess. The main idea is summarized in a verification theorem. Then this method is applied in order to find and verify the value functions of the representative agent and the social planner respectively. The fourth section concludes. The appendix provides some proofs omitted for convenience in the main text.

2 The Model

Consider a closed economy populated by an infinite number of homogenous agents, who produce a single homogenous good and then can be consumed or invested. The representative agent has logarithmic preferences over consumption c_t and is maximizing her discounted lifetime utility

$$\max_{\{u_t, c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t), \quad (1)$$

where $\beta \in (0, 1)$ is the subjective discount factor and u_t is a second choice variable. The interpretation of u_t is postponed for a second. The agent enters every period with predetermined endowments of human and physical capital, h_t and k_t , respectively. There are two sectors in the economy. The goods sector describes the accumulation of physical capital. Next period's stock of physical capital is the difference between output, on the one hand, and consumption on the other, i.e. we assume that there is full physical capital depreciation. Output is produced using a Cobb-Douglas technology in physical and human capital:

$$y_t = A_t k_t^\alpha (u_t h_t)^{1-\alpha} h_{a,t}^\gamma. \quad (2)$$

The state variable A_t denotes the total factor productivity. Here we assume that A_t is constant, i.e.

$$A_t = A > 0 \quad \forall t \geq 0. \quad (3)$$

We therefore consider the following law of motion for physical capital:

$$k_{t+1} = y_t - c_t, \quad \forall t \geq 0, k_0 > 0. \quad (4)$$

The variable u_t describes the allocation of human capital to goods production. A very intuitive way to think about u_t is to assume a normalized time budget of one unit in every period. Then u_t can be interpreted as the fraction of time spent in goods production. Of course, $u_t \in [0, 1]$ must

hold. The variable $h_{a,t}$ stands for the average level of human capital per agent. This variable is due to an external effect and therefore responsible for the difference in the market and social planner solution to the model. Because the agents are homogenous the following symmetry

$$h_{a,t} = h_t, \quad \forall t \geq 0 \quad (5)$$

holds in equilibrium. The educational sector describes the accumulation of human capital. The production of next period's human capital is due to a linear technology and we again assume that there is full depreciation of human capital, i.e.

$$h_{t+1} = B_t (1 - u_t) h_t, \quad \forall t \geq 0, h_0 > 0. \quad (6)$$

Here, we can interpret $1 - u_t$ as the fraction of time spent in the educational sector. The variable B_t is the marginal and average productivity in this sector and is also assumed to be constant. That is

$$B_t = B > 0 \quad \forall t \geq 0. \quad (7)$$

The case where A_t and B_t follow stochastic processes is treated in another paper. The following two subsections describe the decentralized and the centralized solutions to the model using standard dynamic programming methods.

2.1 The Decentralized Solution

In the decentralized case, we assume a representative agent with rational behavior. The agent knows that her stock of human capital equals the average level of human capital in the economy. Furthermore, she knows that the external effects of human capital in the goods production, captured by the term $h_{a,t}^\gamma$, may increase her and all the other agents wealth. But here, in the decentralized case, the market mechanism prevents a coordination of agents' actions. This can be understood as a Nash game producing the prisoner's dilemma. Although the external effect of the economy's average human capital stock in period t is not exploited, the whole path of $h_{a,t}$ is predictable and is therefore treated as given.

We start the analysis of the decentralized economy with the definition of the value function as the solution to the representative agent's problem:

$$V(k_t, h_t; h_{a,t}) \equiv \max_{\{u_s, c_s\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^s \ln(c_s) \quad \text{s.t. (2), (3), (4), (6), and (7).}$$

The corresponding Bellman equation is given by

$$V(k_t, h_t; h_{a,t}) \equiv \max_{u_t, c_t} \{ \ln(c_t) + \beta [V(k_{t+1}, h_{t+1})] \} \quad (8)$$

The first order conditions for the optimal choices of consumption and the optimal allocation of human capital between the two sectors are

$$c_t: \quad \frac{1}{c_t} = \beta \frac{\partial V_{t+1}}{\partial k_{t+1}}, \quad (9)$$

$$u_t: \quad \beta \frac{\partial V_{t+1}}{\partial k_{t+1}} \frac{(1-\alpha)y_t}{u_t} = \beta \frac{\partial V_{t+1}}{\partial h_{t+1}} B h_t \Leftrightarrow \frac{\partial V_{t+1}}{\partial k_{t+1}} \frac{(1-\alpha)y_t}{h_t} = \frac{\partial V_{t+1}}{\partial h_{t+1}} B u_t, \quad (10)$$

where V_{t+1} is a shortcut for $V(k_t, h_t; h_{a,t})$. Equation (9) is very standard and characterizes the effect of shifting one unit of today's output from consumption to investment. Today's marginal

change in utility should equal the discounted marginal change in tomorrow's wealth with respect to tomorrow's capital stock. Equation (10) states that the weighted marginal change in wealth with respect to physical capital equals the weighted marginal change in wealth with respect to human capital. The first weight is the marginal product of human capital in goods production for a certain choice of u_t and the second is the potential marginal product of human capital when the same fraction of human capital is allocated to the educational sector. Note that the last equation can be written as follows

$$u_t^{-\alpha} = \frac{\frac{\partial V_{t+1}}{\partial h_{t+1}} B}{\frac{\partial V_{t+1}}{\partial k_{t+1}} (1-\alpha) A h_{a,t}^\gamma} \left(\frac{k_t}{h_t} \right)^{-\alpha} \Leftrightarrow u_t = \left(\frac{\frac{\partial V_{t+1}}{\partial k_{t+1}} (1-\alpha) A h_{a,t}^\gamma}{\frac{\partial V_{t+1}}{\partial h_{t+1}} B} \right)^{\frac{1}{\alpha}} \frac{k_t}{h_t}.$$

Using the envelope property of the optimal decision rules,

$$c_t^* = c(k_t, h_t) \quad \text{and} \quad u_t^* = u(k_t, h_t), \quad (11)$$

leads us to the following envelope condition for the stock of physical capital:

$$\frac{\partial V_t}{\partial k_t} = \frac{1}{c_t^*} \frac{\partial c}{\partial k_t} - \beta \frac{\partial V_{t+1}}{\partial h_{t+1}} B h_t \frac{\partial u}{\partial k_t} + \beta \frac{\partial V_{t+1}}{\partial k_{t+1}} \left(\frac{\alpha y_t}{k_t} + \frac{(1-\alpha) y_t}{u_t} \frac{\partial u}{\partial k_t} - \frac{\partial c}{\partial k_t} \right)$$

Substituting in the first order conditions for the optimal consumption (9) and for the optimal allocation of human capital (10) and simplifying gives

$$\frac{\partial V_t}{\partial k_t} = \beta \frac{\partial V_{t+1}}{\partial k_{t+1}} \frac{\alpha y_t}{k_t} \Leftrightarrow \frac{\partial V_t}{\partial k_t} \frac{k_t}{\alpha y_t} = \beta \frac{\partial V_{t+1}}{\partial k_{t+1}}. \quad (12)$$

Notice that the right-hand sides of (9) and (12) are the same. We therefore can equate the left hand sides:

$$\frac{\partial V_t}{\partial k_t} \frac{k_t}{\alpha y_t} = \frac{1}{c_t} \Leftrightarrow \frac{\partial V_t}{\partial k_t} = \frac{1}{c_t} \frac{\alpha y_t}{k_t}.$$

Of course, this relationship holds along the whole path such that we finally arrive at the Euler equation in consumption

$$\frac{1}{c_t} = \frac{\beta}{c_{t+1}} \frac{\alpha y_{t+1}}{k_{t+1}}. \quad (13)$$

The envelope condition for the stock of human capital is given by:

$$\frac{\partial V_t}{\partial h_t} = \frac{1}{c_t^*} \frac{\partial c}{\partial h_t} + \beta \frac{\partial V_{t+1}}{\partial h_{t+1}} \left(B(1-u_t) - B_t h_t \frac{\partial u}{\partial h_t} \right) + \beta \frac{\partial V_{t+1}}{\partial k_{t+1}} \left(\frac{(1-\alpha) y_t}{h_t} + \frac{(1-\alpha) y_t}{u_t} \frac{\partial u}{\partial h_t} - \frac{\partial c}{\partial h_t} \right).$$

By using the first order conditions (9) and (10) we can replace the last term and end up with the following relationship

$$\frac{\partial V_t}{\partial h_t} = \beta \frac{\partial V_{t+1}}{\partial h_{t+1}} B \Leftrightarrow \frac{\partial V_t}{\partial h_t} \frac{B h_t}{B} = \beta \frac{\partial V_{t+1}}{\partial h_{t+1}} B h_t. \quad (14)$$

Comparing the right hand sides of (10) and (14) gives

$$\frac{\partial V_t}{\partial h_t} \frac{B h_t}{B} = \beta \frac{\partial V_{t+1}}{\partial k_{t+1}} \frac{(1-\alpha) y_t}{u_t} \Leftrightarrow \frac{\partial V_t}{\partial h_t} = \beta \frac{\partial V_{t+1}}{\partial k_{t+1}} \frac{B(1-\alpha) y_t}{B h_t u_t}.$$

Notice that we can insert the first order condition (9) on the right-hand side of this expression. Furthermore, we know that these equations hold along the whole path, such that we get the Euler equation for the optimal human capital allocation:

$$u_t = \left(\frac{\alpha h_{t+1} u_{t+1}}{k_{t+1}} \frac{A h_{a,t}^\gamma}{B} \right)^{\frac{1}{\alpha}} \frac{k_t}{h_t} \quad (15)$$

The two Euler equations (13) and (15) are necessary for a policy to attain the optimum. Together with the following transversality conditions they are also sufficient:

$$\lim_{T \rightarrow \infty} \beta^T \frac{1}{c_T} \frac{\alpha y_T}{k_T} k_T = 0 \quad \text{and} \quad \lim_{T \rightarrow \infty} \beta^T \frac{1}{c_T} \frac{(1-\alpha) y_T}{u_T h_T} h_T = 0 \quad (16)$$

Notice that the first fraction in both conditions is the derivative of the utility function and the second fraction is the derivative of the goods sector production function with respect to the inputs of physical and human capital. To be more precise, the last derivative is taken with respect to the fraction of human capital that is allocated to the goods sector: $u_t h_t$. These derivatives are multiplied by the respective state variable. The transversality conditions tell us that the discounted marginal utility of an additional unit of "last period's" capital stocks is equal to zero. These requirements rule out Ponzi games.

2.2 The Social Planner's Solution

So far we considered a representative agent, who did not exploit the external effect of the economy's average human capital stock. Now we assume that there is a social planner taking this effect into account, i.e. we consider the following maximization problem:

$$V(k_t, h_t) \equiv \max_{\{u_t, c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t) \quad \text{s.t.} \quad (2), (3), (4), (5), (6), \text{ and } (7).$$

Notice that the value function of the social planner is not restricted to a given path of $h_{a,t}$. Because the first order conditions of the planning problem remain the same as before, we turn immediately to the envelope conditions. It turns out that the envelope property of the optimal decision rules and the necessary conditions (9) and (10) lead us to the same relationship for the stock of physical capital as before:

$$\frac{\partial V_t}{\partial k_t} = \beta \frac{\partial V_{t+1}}{\partial k_{t+1}} \frac{\alpha y_t}{k_t} \Leftrightarrow \frac{\partial V_t}{\partial k_t} \frac{k_t}{\alpha y_t} = \beta \frac{\partial V_{t+1}}{\partial k_{t+1}}. \quad (17)$$

Notice that the right-hand sides of (9) and (17) are the same. Equating the left-hand sides, we obtain:

$$\frac{\partial V_t}{\partial k_t} \frac{k_t}{\alpha y_t} = \frac{1}{c_t} \Leftrightarrow \frac{\partial V_t}{\partial k_t} = \frac{1}{c_t} \frac{\alpha y_t}{k_t}.$$

This relationship holds along the whole path such that we finally get the Euler equation in consumption:

$$\frac{1}{c_t} = \frac{\beta}{c_{t+1}} \frac{\alpha y_{t+1}}{k_{t+1}}. \quad (18)$$

But the envelope condition for the stock of human capital is slightly different:

$$\frac{\partial V_t}{\partial h_t} = \frac{1}{c_t^*} \frac{\partial c}{\partial h_t} + \beta \frac{\partial V_{t+1}}{\partial h_{t+1}} \left(B(1-u_t) - Bh_t \frac{\partial u}{\partial h_t} \right) + \beta \frac{\partial V_{t+1}}{\partial k_{t+1}} \left(\frac{(1-\alpha+\gamma)y_t}{h_t} + \frac{(1-\alpha)y_t}{u_t} \frac{\partial u}{\partial h_t} - \frac{\partial c}{\partial h_t} \right)$$

We can see that the planner takes the external effect of human capital into account. Therefore, the marginal social gain of exploiting this effect in goods production occurs in the second term on the right-hand side. Inserting the first order conditions (9) and (10) together with some transformations, we arrive at the following relationship:

$$\frac{\partial V_t}{\partial h_t} = \beta \frac{\partial V_{t+1}}{\partial h_{t+1}} \frac{B(1-\alpha+\gamma u_t)}{1-\alpha} \Leftrightarrow \frac{\partial V_t}{\partial h_t} \frac{(1-\alpha)h_t}{1-\alpha+\gamma u_t} = \beta \frac{\partial V_{t+1}}{\partial h_{t+1}} Bh_t. \quad (19)$$

Comparing the right-hand sides of (10) and (19) and replacing $\beta \frac{\partial V_{t+1}}{\partial k_{t+1}}$ with the first order condition in consumption (9) yields the Euler equation for the optimal allocation of human capital between the two sectors:

$$u_t = \left(\frac{\alpha u_{t+1} h_{t+1}}{k_{t+1}} \frac{(1-\alpha)A}{(1-\alpha+\gamma u_{t+1})B} \right)^{\frac{1}{\alpha}} \frac{k_t}{h_t}. \quad (20)$$

Again we need two transversality conditions in order to establish the sufficiency of the two Euler equations (18) and (20):

$$\lim_{T \rightarrow \infty} \beta^T \frac{1}{c_T} \frac{\alpha y_T}{k_T} k_T = 0 \quad \text{and} \quad \lim_{T \rightarrow \infty} \beta^T \frac{1}{c_T} \left[\frac{(1-\alpha)y_T}{u_T h_T} + \frac{\gamma y_T}{h_T} \right] h_T = 0. \quad (21)$$

Since the social planner exploits the external effect of human capital, the derivative of the production function with respect to human capital looks different. The derivative of the production function with respect to human capital is the sum of the private marginal return of $u_t h_t$ plus the marginal social gain of the average stock of human capital h_t . The interpretation of these conditions is slightly different from the decentral case. The planner cannot play Ponzi games but he has to obey his budget constraint. This is done by (21).

3 Finding the Value Functions by Iteration

In this section we will use an iterative method to find the value functions that attain the suprema of the two optimization problems considered in the previous section. This is done by applying a more general result published by Strauch (1966), whose treatment is adapted from the Stokey and Lucas (1989) textbook, chapter 4. The latter source states that under certain conditions a solution to the Bellman equation is necessary and sufficient even in the unbounded returns case. The main idea is summarized in a verification theorem. In order to state the theorem, I have to introduce some preliminary concepts of dynamic programming.

Let X be any set, $\Gamma : X \rightarrow X$ the correspondence describing the feasibility constraints, $F : (x, \Gamma(x)) \rightarrow \mathbb{R}$ the one period return function, and β the discount factor. We assume that $\Gamma(x)$ is nonempty, for all $x \in X$, such that we will call any sequence $\{x_{t+1}\}_{t=0}^{\infty}$ a plan. Given $x_0 \in X$, let $\Pi(x_0)$ be the set of all plans that are feasible from x_0 :

$$\Pi(x_0) := \{ \{x_t\}_{t=0}^{\infty} \mid x_{t+1} \in \Pi(x_t), t \in \mathbb{N}_0 \}.$$

For each $n \in \mathbb{N}_0$ we define $u_n : \Pi(x_0) \rightarrow \mathbb{R}$ by

$$u_n(\tilde{x}) := \sum_{t=0}^n \beta^t F(x_t, x_{t+1}), \text{ where } \tilde{x} \in \Pi(x_0).$$

Then $u_n(\tilde{x})$ is a partial sum of the discounted returns in periods 0 through n from the feasible plan \tilde{x} . We assume that for all $x_0 \in X$ and $\tilde{x} \in \Pi(x_0)$,

$$\lim_{n \rightarrow \infty} \sum_{t=0}^n \beta^t F(x_t, x_{t+1}) \quad (22)$$

exists. Under this assumption we can also define $u : \Pi(x_0) \rightarrow \mathbb{R}$ as

$$u(\tilde{x}) := \lim_{n \rightarrow \infty} u_n(\tilde{x}).$$

We can then define the supremum function $v^* : X \rightarrow \mathbb{R}$ as

$$v^*(x_0) := \sup_{\tilde{x} \in \Pi(x_0)} u(\tilde{x}).$$

Theorem 1 *Verification of the supremum function under unbounded returns*

Let $X, \Gamma, F, \beta, \Pi, u,$ and v^* be defined as above. Suppose there is a function $\hat{v} : X \rightarrow \mathbb{R}$ such that

1. $T\hat{v} \leq \hat{v}$;
2. $\lim_{n \rightarrow \infty} \beta^n \hat{v}(x_n) \leq 0$, for all $x_0 \in X$, all $\tilde{x} \in \Pi(x_0)$;
3. $u(\tilde{x}) \leq \hat{v}(x_0)$, for all $x_0 \in X$, all $\tilde{x} \in \Pi(x_0)$.

If the function $v : X \rightarrow \mathbb{R}$ defined by

$$v(x) := \lim_{n \rightarrow \infty} (T^n \hat{v})(x)$$

is a fixed point of T , then $v = v^*$.

Proof. See Stokey and Lucas (1989), page 93. ■

In the following subsections we will apply this theorem to the two economies considered. First, we turn to the decentralized market economy, then we will examine the centralized social planner's case. In order to write down the two applications of the theorem in a clear and comprehensible way, we will state claims at various points. The proofs of these claims can be found in the appendix.

3.1 The decentralized economy

The optimization problem can be rewritten so that in every period t the states h_t and k_t are given and period's states h_{t+1} and k_{t+1} have to be chosen, i.e. we want to replace the variables c_t and u_t . Equation (4) can be solved for c_t and the resulting expression can be substituted in the utility function. Similarly, we solve equation (6) for $u_t h_t$ and insert the result into the production function. In terms of the state variables, the maximization problem is now given by

$$\max_{\{k_{t+1}, h_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \ln \left[A k_t^\alpha \left(h_t - \frac{1}{B} h_{t+1} \right)^{1-\alpha} h_{a,t}^\gamma - k_{t+1} \right]$$

such that

$$\begin{aligned} h_t &= h_{a,t}, \\ 0 &< h_{t+1} < B h_t, \\ 0 &< k_{t+1} < A k_t^\alpha h_t^{1-\alpha} h_{a,t}^\gamma. \end{aligned}$$

We have argued in section 2 that the representative agent does not exploit the external effect, because the market mechanism prevents agents from coordinating their actions. However, the path of $h_{a,t}$ is predictable and the representative agent treats this path as given.

Let us now turn to the policy correspondence Γ , which is given by:

$$\Gamma(h_0, k_0; h_{a,0}) = \left\{ (h_t, k_t) \left| Ak_t^\alpha \left(h_t - \frac{h_{t+1}}{B} \right)^{1-\alpha} h_{a,t}^\gamma - k_{t+1} \in \mathbb{R}_{++}; k_0, h_0 > 0; h_{a,t} = h_t \text{ all } t \geq 0 \right. \right\}.$$

First notice that Γ is non-empty. This follows because the sequence

$$(h_{t+1}, k_{t+1}) = \left(\frac{1}{2} B h_t, \frac{1}{2} A k_t^\alpha h_t^{1-\alpha} h_{a,t}^\gamma \right)$$

is always feasible, hence $\Gamma \neq \emptyset$. Next we have to prove that the total discounted returns are bounded. The first step in this proof is to show that the sequence of human capital stocks is bounded¹:

$$\ln h_{t+1} < (t+1) \ln B + \ln h_0, \quad \forall t \geq 0$$

$$\ln h_{a,t+1} < (t+1) \ln B + \ln h_{a,0}, \quad \forall t \geq 0.$$

Using this result, we learn that the sequence of physical capital stocks is also bounded:

$$\ln k_{t+1} < \frac{1-\alpha^{t+1}}{1-\alpha} \ln A + (1-\alpha+\gamma) \left(\frac{t}{1-\alpha} + \frac{\alpha^{t+1}-\alpha}{(1-\alpha)^2} \right) \ln B + \frac{1-\alpha^{t+1}}{1-\alpha} \gamma \ln h_{a,0} + (1-\alpha^{t+1}) \ln h_0 + \alpha^{t+1} \ln k_0,$$

for all $t \geq 0$. Hence for any pair (k_0, h_0) and for any feasible path $\{k_t, h_t\}$, the sequence of one period utility returns $F(k_t, h_t, k_{t+1}, h_{t+1})$ satisfies

$$\begin{aligned} & F(k_t, h_t, k_{t+1}, h_{t+1}) \\ & \leq F(k_t, h_t, 0, 0) \\ & < \frac{1-\alpha^{t+1}}{1-\alpha} \ln A + (1-\alpha+\gamma) \left(\frac{t-\alpha}{1-\alpha} + \frac{\alpha^{t+1}-\alpha^2}{(1-\alpha)^2} \right) \ln B + \frac{1-\alpha^{t+1}}{1-\alpha} \gamma \ln h_{a,0} + (1-\alpha^{t+1}) \ln h_0 + \alpha^{t+1} \ln k_0. \end{aligned}$$

Then for any feasible sequence $\{k_t, h_t\}_{t=0}^\infty$

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{t=0}^n \beta^t \ln \left[Ak_t^\alpha \left(h_t - \frac{1}{B} h_{t+1} \right)^{1-\alpha} h_{a,t}^\gamma - k_{t+1} \right] \tag{23} \\ & < \frac{\ln A}{(1-\beta)(1-\alpha\beta)} + \frac{(1-\alpha+\gamma)\beta \ln B}{(1-\beta)^2(1-\alpha\beta)} + \frac{\alpha \ln k_0}{1-\alpha\beta} + \frac{(1-\alpha) \ln h_0}{(1-\beta)(1-\alpha\beta)} + \frac{\gamma \ln h_{a,0}}{(1-\beta)(1-\alpha\beta)}. \end{aligned}$$

Hence assumption (22) holds. Following the verification theorem we need a function that is an upper bound for the supremum function v_m^* . Since (23) implies that for all $k > 0$ and $h_a = h > 0$

$$v_m^*(k, h; h_a) < \phi_{AB} + \frac{\alpha \ln k_0}{1-\alpha\beta} + \frac{(1-\alpha) \ln h_0}{(1-\beta)(1-\alpha\beta)} + \frac{\gamma \ln h_{a,0}}{(1-\beta)(1-\alpha\beta)},$$

¹Note that we have used the fact that $\sum_{s=0}^t s \alpha^s = \alpha \frac{1-\alpha^{t+1}}{(1-\alpha)^2} - \frac{\alpha^{t+1} t}{1-\alpha}$.

where

$$\phi_{AB} \equiv \frac{\ln A}{(1-\beta)(1-\alpha\beta)} + \frac{(1-\alpha+\gamma)\beta \ln B}{(1-\beta)^2(1-\alpha\beta)}, \quad (24)$$

we consider the following function \widehat{v}_m as the upper bound:

$$\widehat{v}_m(k_t, h_t; h_{t,a}) \equiv \phi_{AB} + \frac{\alpha \ln k_t}{1-\alpha\beta} + \frac{(1-\alpha) \ln h_t}{(1-\beta)(1-\alpha\beta)} + \frac{\gamma \ln h_{a,t}}{(1-\beta)(1-\alpha\beta)}.$$

Furthermore, we define the operator T as follows

$$(Tf)(k_t, h_t; h_{t,a}) = \max_{\{k_{t+1}, h_{t+1}\}} \left\{ \ln \left[Ak_t^\alpha \left(h_t - \frac{1}{B} h_{t+1} \right)^{1-\alpha} h_{a,t}^\gamma - k_{t+1} \right] + \beta f(k_{t+1}, h_{t+1}; h_{t+1,a}) \right\}.$$

We apply this operator to $\widehat{v}_m(k, h)$, i.e. we consider the mapping

$$(T\widehat{v}_m)(k_t, h_t; h_{t,a}) = \max_{\{k_{t+1}, h_{t+1}\}} \left\{ \ln \left[Ak_t^\alpha \left(h_t - \frac{1}{B} h_{t+1} \right)^{1-\alpha} h_{a,t}^\gamma - k_{t+1} \right] + \beta \widehat{v}_m(k_{t+1}, h_{t+1}; h_{t+1,a}) \right\}. \quad (25)$$

This implies the following first order conditions

$$h_{t+1}^* = \beta B h_t, \quad (26)$$

$$k_{t+1}^* = \alpha\beta(1-\beta)^{1-\alpha} Ak_t^\alpha h_t^{1-\alpha} h_{a,t}^\gamma. \quad (27)$$

Claim 2 *The first order conditions (26) and (27) are sufficient for a decentralized maximum.*

Note, that (26) implies for the average stock of human capital:

$$h_{a,t+1} = \beta B h_{a,t},$$

Substituting $h_{a,t+1}$ together with the maximizing values h_{t+1}^* and k_{t+1}^* into the mapping (25) and simplifying, we obtain:

$$(T\widehat{v}_m)(k_t, h_t; h_{a,t}) = \phi_{AB} + \phi + \frac{\gamma \ln h_{a,t}}{(1-\beta)(1-\alpha\beta)} + \frac{\alpha \ln k_t}{1-\alpha\beta} + \frac{(1-\alpha) \ln h_t}{(1-\beta)(1-\alpha\beta)}$$

where

$$\phi \equiv \ln[1-\alpha\beta] + \frac{(1-\alpha) \ln[1-\beta]}{1-\alpha\beta} + \frac{\alpha\beta \ln \alpha}{1-\alpha\beta} + \frac{(1-\alpha\beta+\gamma)\beta \ln \beta}{(1-\beta)(1-\alpha\beta)}. \quad (28)$$

Notice that the restrictions on the parameters α and β imply that ϕ is strictly negative and $T\widehat{v}_m - \widehat{v}_m \leq 0$ holds, i.e.

$$T\widehat{v}_m(k_t, h_t) - \widehat{v}_m(k_t, h_t) = \phi < 0.$$

Iterating the operation gives $(T^2\widehat{v}_m)$:

$$\begin{aligned} (T^2\widehat{v}_m)(k_t, h_t; h_{a,t}) &= \ln \left[Ak_t^\alpha \left(h_t - \frac{1}{B} h_{t+1} \right)^{1-\alpha} h_{a,t}^\gamma - k_{t+1} \right] + \beta \phi_{AB} + \beta \phi \\ &\quad + \frac{\beta \gamma \ln h_{a,t+1}}{(1-\beta)(1-\alpha\beta)} + \frac{\alpha\beta \ln k_{t+1}}{1-\alpha\beta} + \frac{(1-\alpha)\beta \ln h_{t+1}}{(1-\beta)(1-\alpha\beta)} \end{aligned}$$

Again equations (26) and (27) are necessary and sufficient conditions for a maximum. Inserting h_{t+1}^* and k_{t+1}^* and collecting terms finally yields:

$$(T^2\hat{v})(k_t, h_t; h_{a,t}) = (1 + \beta)\phi + \phi_{AB} + \frac{\gamma \ln h_{a,t}}{(1 - \beta)(1 - \alpha\beta)} + \frac{\alpha \ln k_t}{1 - \alpha\beta} + \frac{(1 - \alpha) \ln h_t}{(1 - \beta)(1 - \alpha\beta)}$$

Obviously, $T^2\hat{v}_m - T\hat{v}_m \leq 0$ holds. Continuing the iteration procedure, one can show that the n -th iteration is given by:

$$(T^n\hat{v}_m)(k_t, h_t; h_{a,t}) = (1 + \dots + \beta^{n-1})\phi + \phi_{AB} + \frac{\gamma \ln h_{a,t}}{(1 - \beta)(1 - \alpha\beta)} + \frac{\alpha \ln k_t}{1 - \alpha\beta} + \frac{(1 - \alpha) \ln h_t}{(1 - \beta)(1 - \alpha\beta)}.$$

Hence, $T^{n+1}\hat{v}_m - T^n\hat{v}_m \leq 0$ holds for every n . And for $\lim_{n \rightarrow \infty} (T^n\hat{v}_m)(x)$ we have

$$\lim_{n \rightarrow \infty} (T^n\hat{v}_m)(x) = \frac{\phi}{1 - \beta} + \phi_{AB} + \frac{\gamma \ln h_{a,t}}{(1 - \beta)(1 - \alpha\beta)} + \frac{\alpha \ln k_t}{1 - \alpha\beta} + \frac{(1 - \alpha) \ln h_t}{(1 - \beta)(1 - \alpha\beta)} = v_m^*(x). \quad (29)$$

Indeed, it turns out that

$$(Tv_m^*)(k_t, h_t) = v_m^*(k_t, h_t)$$

holds. Therefore

$$(k_{t+1}, h_{t+1}) = \left(\alpha\beta(1 - \beta)^{1-\alpha} Ak_t^\alpha h_t^{1-\alpha} h_{a,t}^\gamma, \beta B h_t \right), \quad \forall t \geq 0$$

is the utility-maximizing sequence of capital stocks in the decentralized economy.

3.2 The centralized economy

As in the decentralized case, we can express the planner's maximization problem in terms of the state variables only:

$$\max_{\{k_{t+1}, h_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \ln \left[Ak_t^\alpha \left(h_t - \frac{1}{B} h_{t+1} \right)^{1-\alpha} h_t^\gamma - k_{t+1} \right]$$

such that

$$\begin{aligned} 0 &< h_{t+1} < B h_t, \\ 0 &< k_{t+1} < Ak_t^\alpha h_t^{1-\alpha+\gamma}. \end{aligned}$$

Note that we have dropped the index a . As mentioned above, we assume that the planner takes the external effect into account and knows that the economy's average human capital level is equal to the representative agent's level of human capital.

Since we follow the same strategy as in the decentralized case, we want to find a function that is an upper bound for the supremum function. Again our first step is to prove the boundedness of the sequences of the two capital stocks. After some algebra, we get:

$$\begin{aligned} \ln h_{t+1} &< (t + 1) \ln B + \ln h_0, \quad \forall t \geq 0 \\ \ln k_{t+1} &< \alpha^{t+1} \ln k_0 + \frac{1 - \alpha^{t+1}}{1 - \alpha} [\ln A + (1 - \alpha + \gamma) \ln h_0] + (1 - \alpha + \gamma) \left(\frac{t}{1 - \alpha} + \frac{\alpha^{t+1} - \alpha}{(1 - \alpha)^2} \right) \ln B. \end{aligned}$$

Hence, for any pair (k_0, h_0) and for any feasible path $\{k_t, h_t\}$, the sequence of one period returns $F(k_t, h_t, k_{t+1}, h_{t+1})$ satisfies

$$\begin{aligned} & F(k_t, h_t, k_{t+1}, h_{t+1}) \\ & \leq F(k_t, h_t, 0, 0) \\ & \leq \alpha^{t+1} \ln k_0 + \frac{1-\alpha^{t+1}}{1-\alpha} \ln A + \frac{1-\alpha^{t+1}}{1-\alpha} (1-\alpha+\gamma) \ln h_0 + (1-\alpha+\gamma) \left(\frac{t}{1-\alpha} - \frac{\alpha(1-\alpha^t)}{(1-\alpha)^2} \right) \ln B. \end{aligned}$$

Then for any feasible sequence, it must be that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{t=0}^n \beta^t \ln \left[Ak_t^\alpha \left(h_t - \frac{1}{B} h_{t+1} \right)^{1-\alpha} h_{\alpha,t}^\gamma - k_{t+1} \right] \\ & \leq \frac{\ln A}{(1-\beta)(1-\alpha\beta)} + \frac{(1-\alpha+\gamma)\beta \ln B}{(1-\beta)^2(1-\alpha\beta)} + \frac{\alpha \ln k_0}{1-\alpha\beta} + \frac{(1-\alpha+\gamma) \ln h_0}{(1-\beta)(1-\alpha\beta)}. \end{aligned} \quad (30)$$

Hence assumption (22) holds. Next we have to define the function \hat{v}_p that is an upper bound of the supremum function v_p^* . Condition (30) implies for all $k > 0$, $h > 0$, that

$$v_p^*(k, h) \leq \phi_{AB} + \frac{\alpha \ln k}{1-\alpha\beta} + \frac{(1-\alpha+\gamma) \ln h}{(1-\beta)(1-\alpha\beta)}$$

holds, where ϕ_{AB} was defined in the previous subsection (24). We define the upper bound for the supremum function \hat{v}_p of the social planner's problem as follows:

$$\hat{v}_p(k_t, h_t) \equiv \phi_{AB} + \frac{\alpha \ln k_t}{1-\alpha\beta} + \frac{(1-\alpha+\gamma) \ln h_t}{(1-\beta)(1-\alpha\beta)}.$$

The operator T is defined as follows

$$(Tf)(k_t, h_t) = \max_{\{k_{t+1}, h_{t+1}\}} \left\{ \ln \left[Ak_t^\alpha \left(h_t - \frac{1}{B} h_{t+1} \right)^{1-\alpha} h_t^\gamma - k_{t+1} \right] + \beta f(k_{t+1}, h_{t+1}) \right\}.$$

Applying this operator to $\hat{v}_p(k, h)$, i.e.

$$(T\hat{v}_p)(k_t, h_t) = \max_{\{k_{t+1}, h_{t+1}\}} \left\{ \ln \left[Ak_t^\alpha \left(h_t - \frac{1}{B} h_{t+1} \right)^{1-\alpha} h_t^\gamma - k_{t+1} \right] + \beta \hat{v}_p(k_{t+1}, h_{t+1}) \right\} \quad (31)$$

gives rise to the following first order conditions:

$$h_{t+1}^* = \frac{\beta(1-\alpha+\gamma)}{1-\alpha+\gamma\beta} B h_t, \quad (32)$$

$$k_{t+1}^* = \alpha\beta Ak_t^\alpha h_t^{1-\alpha+\gamma} \left(\frac{(1-\alpha)(1-\beta)}{1-\alpha+\gamma\beta} \right)^{1-\alpha}. \quad (33)$$

Claim 3 *The first order conditions (32) and (33) are sufficient for a centralized maximum.*

After some algebra, the insertion of (32) and (33) into the mapping (31) yields

$$(T\hat{v}_p)(k_t, h_t) = \phi_{AB} + \phi + \varphi + \frac{\alpha \ln k_t}{1-\alpha\beta} + \frac{(1-\alpha+\gamma) \ln h_t}{(1-\beta)(1-\alpha\beta)},$$

where

$$\varphi \equiv \frac{(1-\alpha)\ln[1-\alpha]}{1-\alpha\beta} + \frac{(1-\alpha+\gamma)\beta\ln[1-\alpha+\gamma]}{(1-\beta)(1-\alpha\beta)} - \frac{(1-\alpha+\beta\gamma)\ln[1-\alpha+\gamma\beta]}{(1-\beta)(1-\alpha\beta)}.$$

Since we have to ensure that $T\hat{v}_p \leq \hat{v}_p$ holds, we have to know the sign of $\phi + \varphi$. Like in the decentralized case, all terms are negative. However, the minus sign in front of the last term forces us to investigate $\phi + \varphi$ more thoroughly, which we do in detail in the appendix. There we show that $\phi + \varphi$ must be negative:

Claim 4 *Given $\alpha \in (0, 1)$, $\beta \in (0, 1)$, and $\gamma \in [0, \alpha]$ the condition $T\hat{v}_p \leq \hat{v}_p$ holds.*

Now we apply the operator T a second time. Notice that the first order conditions are the same as before. Now we obtain:

$$(T^2\hat{v}_p)(k_t, h_t) = \phi_{AB} + (1+\beta)(\phi + \varphi) + \frac{\alpha \ln k_t}{1-\alpha\beta} + \frac{(1-\alpha+\gamma)\ln h_t}{(1-\beta)(1-\alpha\beta)}.$$

And for $(T^3\hat{v}_p)$, we get:

$$(T^3\hat{v}_p)(k_t, h_t) = \phi_{AB} + (1+\beta+\beta^2)(\phi + \varphi) + \frac{\alpha \ln k_t}{1-\alpha\beta} + \frac{(1-\alpha+\gamma)\ln h_t}{(1-\beta)(1-\alpha\beta)}.$$

In general, we find that the following holds:

$$(T^n\hat{v}_p)(k_t, h_t) = \phi_{AB} + (1+\beta^1 + \dots + \beta^{n-1})(\phi + \varphi) + \frac{\alpha \ln k_t}{1-\alpha\beta} + \frac{(1-\alpha+\gamma)\ln h_t}{(1-\beta)(1-\alpha\beta)},$$

so that

$$\lim_{n \rightarrow \infty} (T^n\hat{v}_p)(k_t, h_t) = \phi_{AB} + \frac{\phi + \varphi}{1-\beta} + \frac{\alpha \ln k_t}{1-\alpha\beta} + \frac{(1-\alpha+\gamma)\ln h_t}{(1-\beta)(1-\alpha\beta)} = v_p^*(k_t, h_t). \quad (34)$$

Indeed, it turns out that

$$(Tv_p^*)(k_t, h_t) = v_p^*(k_t, h_t)$$

holds. Therefore

$$(k_{t+1}, h_{t+1}) = \left(\frac{\alpha\beta(1-\alpha)^{1-\alpha}(1-\beta)^{1-\alpha}Ak_t^\alpha h_t^{1-\alpha+\gamma}}{(1-\alpha+\gamma\beta)^{1-\alpha}}, \frac{\beta(1-\alpha+\gamma)Bh_t}{1-\alpha+\gamma\beta} \right), \quad \forall t \geq 0$$

is the utility maximizing sequence of capital stocks in the centralized economy.

4 Summary and Concluding Remarks

We have proven that the functions (29) and (34) are the value functions of the representative agent and the social planner, respectively. In the decentralized case, we can use the value function (29) and the first order condition (9) in order to find the optimal level of consumption:

$$c_{m,t} = (1-\alpha\beta)y_t. \quad (35)$$

This result is the typical consumption rule for the standard AK model with logarithmic preferences, Cobb Douglas technology, and full depreciation of physical capital. It is easy to check that this

result does fit the Euler equation in consumption (13). Similarly, we can use (29) and the first order condition for the optimal human capital allocation (10):

$$u_{m,t} = 1 - \beta \quad (36)$$

We have found that the optimal way to shift human capital between the two production sectors is to hold u_t constant, once we have found the optimal allocation. Similar to the consumption rule, it can be shown that this policy rule fulfills the Euler equation (15). Furthermore, the restriction $u_{m,t} = [0, 1]$ holds. In order to ensure the sufficiency of these rules, it remains to be shown that the two transversality conditions in (16) are met, i.e. for physical capital:

$$\lim_{T \rightarrow \infty} \beta^T \frac{1}{(1 - \alpha\beta) y_T} \frac{\alpha y_T}{k_T} k_T = \lim_{T \rightarrow \infty} \frac{\alpha \beta^T}{(1 - \alpha\beta)} = 0,$$

and for human capital:

$$\lim_{T \rightarrow \infty} \frac{\beta^T}{(1 - \alpha\beta) y_T} \frac{(1 - \alpha) y_T}{(1 - \beta)} = \lim_{T \rightarrow \infty} \frac{\beta^T (1 - \alpha)}{(1 - \alpha\beta) (1 - \beta)} = 0.$$

Thus the policy rules (35) and (36) of the representative agent are necessary and sufficient for a utility maximizing path. In the centralized case we apply the social planners value function (34) to the first order condition (9). This leads us to the same consumption rule as before:

$$c_{p,t} = (1 - \alpha\beta) y_t. \quad (37)$$

But for the optimal allocation of human capital we find a slightly different value:

$$u_{p,t} = \frac{(1 - \alpha) (1 - \beta)}{1 - \alpha + \beta\gamma} \quad (38)$$

Just as in the previous case, the allocation of human capital in the centralized economy is constant over time. Furthermore, the optimal u_t is a little bit smaller than in the decentralized case, although $u_{p,t} \in [0, 1]$ still holds. Hence, the path of human capital in the centralized economy lies above the human capital path in the decentralized economy given the same initial stocks of capital. The two policy functions (37) and (38) satisfy the Euler equations (18) and (20) for the centralized case. Finally, we check the transversality conditions (21):

$$\lim_{T \rightarrow \infty} \beta^T \frac{1}{c_T} \frac{\alpha y_T}{k_T} k_T = \lim_{T \rightarrow \infty} \frac{\beta^T \alpha y_T}{(1 - \alpha\beta) y_T} = 0,$$

$$\lim_{T \rightarrow \infty} \frac{\beta^T}{c_T} \left[\frac{(1 - \alpha) y_T}{u_T h_T} + \frac{\gamma y_T}{h_T} \right] h_T = \lim_{T \rightarrow \infty} \frac{\beta^T (1 - \alpha + \gamma)}{(1 - \alpha\beta) (1 - \beta)} = 0.$$

This shows the sufficiency of our results in the centralized case.

5 Appendix

Before we prove the two claims that the first order conditions, (26) and (27) for the decentralized economy and (32) and (33) for the centralized economy, are sufficient to derive the maximizing policies, we want to state a very useful criterion for sufficiency. Exploiting the fact that we consider a 2×2 Hessian matrix, we can easily check for negative definiteness. The idea is stated in the following theorem.

Theorem 5 The derived values for h_{t+1}^* and k_{t+1}^* describe the utility-maximizing policy if the Hessian matrix has a positive determinant and the upper left entry is negative.

Proof of Theorem 5 Let the symmetric matrix $H(k_{t+1}^*, h_{t+1}^*)$ be given by

$$H(k_{t+1}^*, h_{t+1}^*) = \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

where $a < 0$. The corresponding quadratic form is given by

$$\begin{aligned} & \begin{bmatrix} k & h \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} k \\ h \end{bmatrix} \\ &= ak^2 + 2bkh + ch^2 \\ &= a \left(k + \frac{b}{a}h \right)^2 + \frac{(ac - b^2)}{a}h^2. \end{aligned}$$

Hence, if $a < 0$ and $\det H = ac - b^2 > 0$, then the quadratic form is negative definite and (k_{t+1}^*, h_{t+1}^*) determines the maximum.

Proof of Claim 2 The second order derivatives are given by:

$$\begin{aligned} \frac{\partial(T\hat{v}_m)(k_t, h_t; h_{a,t})}{\partial k_{t+1} \partial k_{t+1}} &= \frac{-1}{\left(Ak_t^\alpha \left(h_t - \frac{1}{B}h_{t+1} \right)^{1-\alpha} h_{a,t}^\gamma - k_{t+1} \right)^2} - \frac{\alpha\beta}{(1-\alpha\beta)k_{t+1}^2} \\ \frac{\partial(T\hat{v}_m)(k_t, h_t; h_{a,t})}{\partial k_{t+1} \partial h_{t+1}} &= \frac{\partial(T\hat{v})(k_t, h_t; h_{a,t})}{\partial h_{t+1} \partial k_{t+1}} = \frac{-\frac{1}{B}(1-\alpha)Ak_t^\alpha \left(h_t - \frac{1}{B}h_{t+1} \right)^{-\alpha} h_{a,t}^\gamma}{\left(Ak_t^\alpha \left(h_t - \frac{1}{B}h_{t+1} \right)^{1-\alpha} h_{a,t}^\gamma - k_{t+1} \right)^2} \\ \frac{\partial(T\hat{v}_m)(k_t, h_t; h_{a,t})}{\partial h_{t+1} \partial h_{t+1}} &= \frac{-(1-\alpha)A^2k_t^{2\alpha} \left(h_t - \frac{1}{B}h_{t+1} \right)^{-2\alpha} h_{a,t}^{2\gamma}}{B^2 \left(Ak_t^\alpha \left(h_t - \frac{1}{B}h_{t+1} \right)^{1-\alpha} h_{a,t}^\gamma - k_{t+1} \right)^2} \\ &\quad + \frac{\alpha(1-\alpha)Ak_t^\alpha \left(h_t - \frac{1}{B}h_{t+1} \right)^{-\alpha-1} h_{a,t}^\gamma k_{t+1}}{B^2 \left(Ak_t^\alpha \left(h_t - \frac{1}{B}h_{t+1} \right)^{1-\alpha} h_{a,t}^\gamma - k_{t+1} \right)^2} - \frac{\beta(1-\alpha)}{(1-\beta)(1-\alpha\beta)h_{t+1}^2}. \end{aligned}$$

Substituting (26) and (27) into the expressions above:

$$\begin{aligned} \frac{\partial(T\hat{v}_m)(k_t, h_t; h_{a,t})}{\partial k_{t+1} \partial k_{t+1}} &= \frac{-1}{\alpha\beta(1-\alpha\beta)^2(1-\beta)^{2-2\alpha}A^2k_t^{2\alpha}h_t^{2-2\alpha}h_{a,t}^{2\gamma}} \\ \frac{\partial(T\hat{v}_m)(k_t, h_t; h_{a,t})}{\partial k_{t+1} \partial h_{t+1}} &= \frac{-(1-\alpha)}{(1-\alpha\beta)^2(1-\beta)^{2-\alpha}Bh_tAk_t^\alpha h_t^{1-\alpha}h_{a,t}^\gamma} \\ \frac{\partial(T\hat{v}_m)(k_t, h_t; h_{a,t})}{\partial h_{t+1} \partial h_{t+1}} &= \frac{(1-\alpha)(\alpha^2\beta^2 - 1 + \alpha\beta - \alpha\beta^2)}{(1-\alpha\beta)^2\beta B^2h_t^2(1-\beta)^2}. \end{aligned}$$

Hence the Hessian matrix H is given by:

$$\begin{aligned} H &= \begin{bmatrix} \frac{-1}{\alpha\beta} \left(\frac{1}{(1-\alpha\beta)(1-\beta)^{1-\alpha}Ak_t^\alpha h_t^{1-\alpha}h_{a,t}^\gamma} \right)^2 & \frac{-(1-\alpha)}{(1-\alpha\beta)^2(1-\beta)^{2-\alpha}Bh_tAk_t^\alpha h_t^{1-\alpha}h_{a,t}^\gamma} \\ \frac{-(1-\alpha)}{(1-\alpha\beta)^2(1-\beta)^{2-\alpha}Bh_tAk_t^\alpha h_t^{1-\alpha}h_{a,t}^\gamma} & \frac{(1-\alpha)(\alpha^2\beta^2 - 1 + \alpha\beta - \alpha\beta^2)}{(1-\alpha\beta)^2\beta B^2h_t^2(1-\beta)^2} \end{bmatrix} \\ &= \frac{1}{(1-\alpha\beta)^2(1-\beta)^2} \begin{bmatrix} \frac{-1}{\alpha\beta(1-\beta)^{-2\alpha}} \frac{1}{\left(Ak_t^\alpha h_t^{1-\alpha}h_{a,t}^\gamma \right)^2} & \frac{-(1-\alpha)}{(1-\beta)^{-\alpha}} \frac{1}{Bh_tAk_t^\alpha h_t^{1-\alpha}h_{a,t}^\gamma} \\ \frac{-(1-\alpha)}{(1-\beta)^{-\alpha}} \frac{1}{Bh_tAk_t^\alpha h_t^{1-\alpha}h_{a,t}^\gamma} & \frac{(1-\alpha)(\alpha^2\beta^2 - 1 + \alpha\beta - \alpha\beta^2)}{\beta B^2h_t^2} \end{bmatrix}. \end{aligned}$$

It is easy to see that the upper left entry of this matrix is negative. The determinant of the Hessian is obviously positive if the following inequality holds:

$$\frac{-(1-\alpha)(\alpha^2\beta^2 - 1 + \alpha\beta - \alpha\beta^2)}{\alpha\beta(1-\beta)^{-2\alpha}\beta} > \frac{-(1-\alpha)^2}{(1-\beta)^{-2\alpha}}.$$

Given the above set of parameters, we know that

$$0 < 1 - \alpha\beta + 2\alpha\beta^2(1-\alpha)$$

holds. Then

$$\alpha^2\beta^2 - 1 + \alpha\beta - \alpha\beta^2 < \alpha\beta^2(1-\alpha)$$

must also hold. Dividing by $\alpha\beta^2(1-\beta)^{-2\alpha} > 0$ gives

$$\frac{\alpha^2\beta^2 - 1 + \alpha\beta - \alpha\beta^2}{\alpha\beta^2(1-\beta)^{-2\alpha}} < \frac{(1-\alpha)}{(1-\beta)^{-2\alpha}}.$$

Multiplying both sides of this inequality by $-(1-\alpha) < 0$ gives the desired result. Hence $\det H > 0$ holds and our findings (26) and (27) are the maximizing values for tomorrow's capital stocks. ■

Proof of Claim 3 Again we apply Theorem 5 in order to show that these results are sufficient for a maximum. The second order derivatives are given by:

$$\begin{aligned} \frac{\partial(T\hat{v}_p)(k_t, h_t)}{\partial k_{t+1}\partial k_{t+1}} &= \frac{-1}{\left(Ak_t^\alpha \left(h_t - \frac{1}{B}h_{t+1}\right)^{1-\alpha} h_t^\gamma - k_{t+1}\right)^2} - \frac{\alpha\beta}{(1-\alpha\beta)(k_{t+1})^2} \\ \frac{\partial(T\hat{v}_p)(k_t, h_t)}{\partial k_{t+1}\partial h_{t+1}} &= \frac{\partial(T\hat{v})(k_t, h_t)}{\partial h_{t+1}\partial k_{t+1}} = \frac{\left((1-\alpha)Ak_t^\alpha \left(h_t - \frac{1}{B}h_{t+1}\right)^{-\alpha} \left(-\frac{1}{B}\right)h_t^\gamma\right)}{\left(Ak_t^\alpha \left(h_t - \frac{1}{B}h_{t+1}\right)^{1-\alpha} h_t^\gamma - k_{t+1}\right)^2} \\ \frac{\partial(T\hat{v}_p)(k_t, h_t)}{\partial h_{t+1}\partial h_{t+1}} &= \frac{(\alpha^2\beta - 1)(1-\alpha + \gamma\beta)^2}{B^2(1-\alpha\beta)^2(1-\alpha)(1-\beta)^2 h_t^2} - \frac{(1-\alpha + \gamma\beta)^2}{B^2(1-\alpha\beta)(1-\beta)(1-\alpha + \gamma)\beta h_t^2}. \end{aligned}$$

Substituting in (32) and (33) gives

$$\begin{aligned} \frac{\partial(T\hat{v}_p)(k_t, h_t)}{\partial k_{t+1}\partial k_{t+1}} &= \frac{-(1-\alpha + \gamma\beta)^{2-2\alpha}}{(1-\alpha\beta)^2 \alpha\beta \left(Ak_t^\alpha h_t^{1-\alpha+\gamma} (1-\alpha)^{1-\alpha} (1-\beta)^{1-\alpha}\right)^2} \\ \frac{\partial(T\hat{v}_p)(k_t, h_t)}{\partial k_{t+1}\partial h_{t+1}} &= \frac{-(1-\alpha + \gamma\beta)^{2-\alpha}}{(1-\alpha\beta)^2 (1-\beta)^{2-\alpha} B h_t A k_t^\alpha h_t^{1-\alpha+\gamma} (1-\alpha)^{1-\alpha}} \\ \frac{\partial(T\hat{v}_p)(k_t, h_t)}{\partial h_{t+1}\partial h_{t+1}} &= \frac{(\alpha^2\beta - 1)(1-\alpha + \gamma\beta)^2}{(1-\alpha\beta)^2 (1-\alpha)(1-\beta)^2 B^2 h_t^2} - \frac{(1-\alpha + \gamma\beta)^2}{(1-\beta)(1-\alpha\beta)(1-\alpha + \gamma)\beta B^2 h_t^2}. \end{aligned}$$

Hence the Hessian matrix H is given by:

$$H = \frac{(1-\alpha + \gamma\beta)^2}{(1-\beta)(1-\alpha\beta)} \begin{bmatrix} \frac{-(1-\alpha + \gamma\beta)^{-2\alpha}}{(1-\alpha\beta)\alpha\beta(1-\alpha)^{2-2\alpha}(1-\beta)^{1-2\alpha}} \frac{1}{(Ak_t^\alpha h_t^{1-\alpha+\gamma})^2} & \frac{-(1-\alpha + \gamma\beta)^{-\alpha}}{(1-\alpha)^{1-\alpha}(1-\alpha\beta)(1-\beta)^{1-\alpha}} \frac{1}{B h_t A k_t^\alpha h_t^{1-\alpha+\gamma}} \\ \frac{-(1-\alpha + \gamma\beta)^{-\alpha}}{(1-\alpha)^{1-\alpha}(1-\alpha\beta)(1-\beta)^{1-\alpha}} \frac{1}{B h_t A k_t^\alpha h_t^{1-\alpha+\gamma}} & \left(\frac{\alpha^2\beta - 1}{(1-\alpha\beta)(1-\alpha)(1-\beta)} - \frac{1}{(1-\alpha + \gamma)\beta}\right) \frac{1}{(B h_t)^2} \end{bmatrix}.$$

The upper left entry of this matrix is obviously negative. The determinant of the Hessian is positive if

$$\frac{-(1-\alpha+\gamma\beta)^{-2\alpha}}{(1-\alpha\beta)\alpha\beta(1-\alpha)^{2-2\alpha}(1-\beta)^{1-2\alpha}} \frac{\alpha^2\beta-1}{(1-\alpha\beta)(1-\alpha)(1-\beta)} > \frac{-(1-\alpha+\gamma\beta)^{-2\alpha}}{(1-\alpha\beta)^2(1-\alpha)^{2-2\alpha}(1-\beta)^{2-2\alpha}} \quad (39)$$

holds. We know that

$$1 > \alpha\beta$$

holds. Subtracting $\alpha^2\beta$ and dividing by $\alpha\beta(1-\alpha) > 0$ gives

$$\frac{1-\alpha^2\beta}{\alpha\beta(1-\alpha)} > 1.$$

Multiplying both sides of this inequality by $\frac{(1-\alpha+\gamma\beta)^{-2\alpha}}{(1-\alpha\beta)^2(1-\alpha)^{2-2\alpha}(1-\beta)^{2-2\alpha}} > 0$ gives:

$$\frac{(1-\alpha^2\beta)(1-\alpha+\gamma\beta)^{-2\alpha}}{\alpha\beta(1-\alpha)(1-\alpha\beta)^2(1-\alpha)^{2-2\alpha}(1-\beta)^{2-2\alpha}} > \frac{(1-\alpha+\gamma\beta)^{-2\alpha}}{(1-\alpha\beta)^2(1-\alpha)^{2-2\alpha}(1-\beta)^{2-2\alpha}} > 0.$$

This expression can be rewritten to arrive at (39) and we therefore have the desired result. ■

Proof of Claim 4 The condition $T\hat{v} \leq \hat{v}$ is equivalent to $\phi + \varphi \leq 0$. Since (28) implies that $\left[\phi - \frac{\beta\gamma \ln \beta}{(1-\alpha\beta)(1-\beta)}\right]$ is negative (28) it is sufficient to show that

$$\frac{\beta\gamma \ln \beta}{(1-\alpha\beta)(1-\beta)} + \frac{(1-\alpha) \ln [1-\alpha]}{1-\alpha\beta} + \frac{(1-\alpha+\gamma)\beta \ln [1-\alpha+\gamma]}{(1-\beta)(1-\alpha\beta)} - \frac{(1-\alpha+\beta\gamma) \ln [1-\alpha+\gamma\beta]}{(1-\beta)(1-\alpha\beta)} \leq 0 \quad (40)$$

holds. Given the above parameter space the concavity of the \ln function implies

$$(1-\beta) \ln [1-\alpha] + \beta \ln [1-\alpha+\gamma] - \ln [1-\alpha+\gamma\beta] \leq 0. \quad (41)$$

Multiplying both sides of this inequality by $\frac{1-\alpha}{(1-\beta)(1-\alpha\beta)} > 0$ gives:

$$\frac{(1-\alpha) \ln [1-\alpha]}{1-\alpha\beta} + \frac{(1-\alpha)\beta \ln [1-\alpha+\gamma]}{(1-\beta)(1-\alpha\beta)} - \frac{(1-\alpha) \ln [1-\alpha+\gamma\beta]}{(1-\beta)(1-\alpha\beta)} \leq 0. \quad (42)$$

Because we know that

$$(1-\alpha+\gamma)\beta < 1-\alpha+\gamma\beta$$

holds, we can use the fact that the logarithmic function is strictly increasing:

$$\ln [1-\alpha+\gamma] + \ln \beta < \ln [1-\alpha+\gamma\beta]. \quad (43)$$

Subtracting the left hand side of (43) and multiplying both sides of the inequality by $\frac{\beta\gamma}{(1-\alpha\beta)(1-\beta)} \geq 0$ gives:

$$\frac{\beta\gamma \ln [1-\alpha+\gamma]}{(1-\alpha\beta)(1-\beta)} + \frac{\beta\gamma \ln \beta}{(1-\alpha\beta)(1-\beta)} - \frac{\beta\gamma \ln [1-\alpha+\gamma\beta]}{(1-\alpha\beta)(1-\beta)} \leq 0. \quad (44)$$

Notice that the inequality is no longer strict, as we allow for $\gamma = 0$. Adding the two inequalities (42) and (44) gives the desired result stated in (40). This proves that $\phi + \varphi$ is negative. ■

References

- [1] Lucas, Robert E. (1988): "On The Mechanics Of Economic Development", *Journal of Monetary Economics* 22(1), 3-42.
- [2] Stokey, Nancy L. and Robert E. Lucas with Edward C. Prescott (1989): *Recursive Methods in Economic Dynamics*, Cambridge (Massachusetts) and London (England): Harvard University Press.
- [3] Strauch, Ralph E. (1966): "Negative Dynamic Programming", *Annals of Mathematical Statistics* 37, 871-890.
- [4] Uzawa, Hirofumi (1965): "Optimum Technical Change In An Aggregate Model Of Economic Growth", *International Economic Review* 6(1), 18-31.