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Intersection bounds: estimation and inference

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INTERSECTION BOUNDS: ESTIMATION AND INFERENCE

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ABSTRACT. We develop a practical and novel method for inference on intersection bounds, namely bounds defined by either the infimum or supremum of a parametric or nonparametric function, or equivalently, the value of a linear programming problem with a potentially infinite constraint set. Our approach is especially convenient for models comprised of a continuum of inequalities that are separable in parameters, and also applies to models with inequalities that are non-separable in parameters. Since analog estimators for intersection bounds can be severely biased in finite samples, routinely underestimating the size of the identified set, we also offer a median-bias-corrected estimator of such bounds as a natural by-product of our inferential procedures. We develop theory for large sample inference based on the strong approximation of a sequence of series or kernel-based empirical processes by a sequence of “penultimate” Gaussian processes. These penultimate processes are generally not weakly convergent, and thus non-Donsker. Our theoretical results establish that we can nonetheless perform asymptotically valid inference based on these processes. Our construction also provides new adaptive inequality/moment selection methods. We provide conditions for the use of nonparametric kernel and series estimators, including a novel result that establishes strong approximation for any general series estimator admitting linearization, which may be of independent interest.

KEY WORDS. Bound analysis, conditional moments, partial identification, strong approximation, infinite dimensional constraints, linear programming, concentration inequalities, anti-concentration inequalities, non-Donsker empirical process methods, moderate deviations, adaptive moment selection.

JEL SUBJECT CLASSIFICATION. C12, C13, C14. **AMS SUBJECT CLASSIFICATION.** 62G05, 62G15, 62G32.

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1. INTRODUCTION

This paper develops a practical and novel method for estimation and inference on intersection bounds. Such bounds arise in settings where the parameter of interest, denoted θ^* , is known to lie within the bounds $[\theta^l(v), \theta^u(v)]$ for each v in some set $\mathcal{V} \subseteq \mathbb{R}^d$, which may be uncountably infinite. The identification region for θ^* is then

$$\Theta_I = \bigcap_{v \in \mathcal{V}} [\theta^l(v), \theta^u(v)] = \left[\sup_{v \in \mathcal{V}} \theta^l(v), \inf_{v \in \mathcal{V}} \theta^u(v) \right]. \quad (1.1)$$

Intersection bounds stem naturally from exclusion restrictions (Manski (2003)) and appear in numerous applied and theoretical examples.¹ A leading case is that where the bounding functions are conditional expectations with continuous conditioning variables, yielding conditional moment inequalities. More generally, the methods of this paper apply to any estimator for the value of a linear programming problem with an infinite dimensional constraint set.

This paper covers both parametric and non-parametric estimators of bounding functions $\theta^l(\cdot)$ and $\theta^u(\cdot)$. We provide formal justification for parametric, series, and kernel-type estimators via asymptotic theory based on the strong approximation of a sequence of empirical processes by a sequence of Gaussian processes. This includes an important new result on strong approximation for series estimators that applies to any estimator that admits a linear approximation, essentially providing a functional central limit theorem for series estimators for the first time in the literature. For each of these estimation methods, the paper provides

- (i) confidence regions that achieve a desired asymptotic level,
- (ii) novel adaptive inequality selection (AIS) needed to construct sharp critical values, which in some cases result in confidence regions with exact asymptotic size,²
- (iii) convergence rates for the boundary points of these regions,
- (iv) a characterization of local alternatives against which the associated tests have non-trivial power,
- (v) half-median-unbiased estimators of the intersection bounds.

¹Examples include monotone instrumental variables and the returns to schooling (Manski and Pepper (2000)), English auctions (Haile and Tamer (2003)), the returns to language skills (Gonzalez (2005)), set identification with Tobin regressors (Chernozhukov, Rigobon, and Stoker (2010)), endogeneity with discrete outcomes (Chesher (2010)), changes in the distribution of wages (Blundell, Gosling, Ichimura, and Meghir (2007)), the study of disability and employment (Kreider and Pepper (2007)), estimation of income poverty measures (Nicoletti, Foliano, and Peracchi (2011)), unemployment compensation reform (Lee and Wilke (2009)), and set identification with imperfect instruments (Nevo and Rosen (2010)).

² The previous literature, e.g. Chernozhukov, Hong, and Tamer (2007) and contemporaneous papers, such as Andrews and Shi (2009), use “non-adaptive” cutoffs such as $C\sqrt{\log n}$. Ideally C should depend on the problem at hand and so careful calibration might be required in practice. Our new AIS procedure provides data-driven, adaptive cutoffs, which do not require calibration.

Moreover, our paper also extends inferential theory based on empirical processes in Donsker settings to non-Donsker cases. The empirical processes arising in our problems do not converge weakly to a Gaussian process, but can be strongly approximated by a sequence of “penultimate” Gaussian processes, which we use directly for inference without resorting to further approximations, such as extreme value approximations as in Bickel and Rosenblatt (1973). These new methods may be of independent interest for a variety of other problems.

Our results also apply to settings where a parameter of interest, say μ , is characterized by intersection bounds of the form (1.1) on an auxiliary function $\theta(\mu)$. Then the bounding functions have the representation

$$\theta^l(v) := \theta^l(v; \mu) \text{ and } \theta^u(v) := \theta^u(v; \mu), \quad (1.2)$$

and thus inference statements for $\theta^* := \theta(\mu)$ bounded by $\theta^l(\cdot)$ and $\theta^u(\cdot)$ can be translated to inference statements for the parameter μ . This includes cases where the bounding functions are a collection of conditional moment functions indexed by μ . When the auxiliary function is additively separable in μ , the relation between the two is simply a location shift. When the auxiliary function is nonseparable in μ inference statements on θ^* still translate to inference statements on μ , though the functional relation between the two is more complex.

This paper overcomes significant complications for estimation of and inference on intersection bounds. First, because the bound estimates are suprema and infima of parametric or nonparametric estimators, closed-form characterization of their asymptotic distributions are typically unavailable or difficult to establish. As a consequence, researchers have often used the canonical bootstrap for inference, yet the recent literature indicates that the canonical bootstrap is not generally consistent in such settings, see e.g. Andrews and Han (2009), Bugni (2010), and Canay (2010).³ Second, since sample analogs of the bounds of Θ_I are the suprema and infima of estimated bounding functions, they have substantial finite sample bias, and estimated bounds tend to be much tighter than the population bounds. This has been noted by Manski and Pepper (2000, 2009), and some heuristic bias adjustments have been proposed by Haile and Tamer (2003) and Kreider and Pepper (2007).

We solve the problem of estimation and inference for intersection bounds by proposing bias-corrected estimators of the upper and lower bounds, as well as confidence intervals. Specifically, our approach employs a precision-correction to the estimated bounding functions $v \mapsto \hat{\theta}^l(v)$ and $v \mapsto \hat{\theta}^u(v)$ before applying the supremum and infimum operators.

³The recent papers Andrews and Shi (2009) and Kim (2009) provide justification for subsampling procedures for the statistics they employ for inference with conditional moment inequalities. We discuss these papers further in our literature review below.

We adjust the estimated bounding functions for their precision by adding to each of them an appropriate critical value times their pointwise standard error. Then, depending on the choice of the critical value, the intersection of these precision-adjusted bounds provides (i) confidence sets for either the identified set Θ_I or the true parameter value θ^* , or (ii) bias-corrected estimators for the lower and upper bounds. Our bias-corrected estimators are half-median-unbiased in the sense that the upper bound estimator $\widehat{\theta}^u$ exceeds θ^u and the lower bound estimator $\widehat{\theta}^l$ falls below θ^l each with probability at least one half asymptotically. Note that achieving full unbiasedness is impossible in general, as shown by Hirano and Porter (2009), which motivates the half-unbiasedness property. Bound estimators with this property are also proposed by Andrews and Shi (2009), henceforth AS. An attractive feature of our approach is that the only difference in the construction of our estimators and confidence intervals is the choice of a critical value. Thus, practitioners need not implement two entirely different methods to construct estimators and confidence bands with desirable properties.

This paper contributes to a growing literature on inference on set-identified parameters bounded by inequality restrictions. The prior literature has focused primarily on models with a finite number of unconditional inequality restrictions. Some examples include Andrews and Jia (2008), Andrews and Guggenberger (2009), Andrews and Soares (2010), Beresteanu and Molinari (2008), Bugni (2010), Canay (2010), Chernozhukov, Hong, and Tamer (2007), Galichon and Henry (2009), Romano and Shaikh (2008), Romano and Shaikh (2010), and Rosen (2008), among others. We contribute to this literature by considering inference with a continuum of inequalities. Contemporaneous and independently written research on conditional moment inequalities includes AS, Kim (2009), and Menzel (2009). Our approach differs from all of these. Whereas we treat the problem with fundamentally nonparametric methods, AS provide inferential statistics that transform the model's conditional restrictions to unconditional ones through the use of instrument functions. Thus our approach is similar in spirit to that of Haerdle and Mammen (1993) while the approach of AS parallels that of Bierens (1982) for testing a parametric specification against a nonparametric alternative. As such, these approaches are complementary, each with their relative merits, as we describe in further detail below. In sum, AS provide results on the uniform asymptotic coverage properties of their confidence sets, asymptotic power properties, and half-median-unbiased estimation of parameter bounds. Kim (2009) proposes an inferential method related to that of AS, but where data dependent indicator functions play the role of instrument functions. Menzel (2009) considers problems where the number of moment

inequalities defining the identified set is large relative to the sample size. He provides results on the use of a subset of such restrictions in any finite sample, where the number of restrictions employed grows with the sample size, and examines the sensitivity of estimation and inference methods to the rate with which the number of moments used grows with the sample size.

The classes of models to which our approach and others in the recent literature apply have considerable overlap, most notably in models comprised of conditional moment inequalities, equivalently models whose bounding functions are conditional moment functions. Relative to other approaches, our approach is especially convenient for inference in parametric and non-parametric models with a continuum of inequalities that are separable in parameters. Our explicit use of nonparametric estimation of bounding functions renders our method applicable in settings where the bounding functions depend on exogenous covariates in addition to the variable V , i.e. where the function $\theta(x)$ at a point x is the object of interest, with

$$\sup_{v \in \mathcal{V}} \theta^l(x, v) \leq \theta(x) \leq \inf_{v \in \mathcal{V}} \theta^u(x, v).$$

When the functions $\theta^l(x, v)$ and $\theta^u(x, v)$ are nonparametrically specified, these can be estimated by either the series or kernel-type estimators we study in Section 4. At present most other approaches do not appear to immediately apply when we are interested in $\theta(x)$ at a point x , when covariates X are continuously distributed, with the exception of the recent work by Fan and Park (2011) in the context of IV and MIV bounds.⁴

To better understand the comparison between our point and interval estimators and those of AS when both are applicable, consider as a simple example the case where $\theta^* \leq E[Y|V]$ almost surely, so that the upper bound on θ^* is given by $\theta_0 = \min_{v \in \mathcal{V}} E[Y|V = v]$ over some region \mathcal{V} . The upper bound θ_0 is a nonparametric functional and can in general only be estimated at a nonparametric rate. That is, one can not construct point or interval estimators that converge to θ_0 at superefficient rates, i.e. rates that exceed the optimal nonparametric rate for estimating $\theta(v) := \theta^u(v) = E[Y|V = v]$.⁵ Our procedure delivers point and interval estimators that can converge to θ_0 at this rate, up to an undersmoothing factor. However, there exist point and interval estimators that can achieve faster (superefficient) convergence

⁴The complication is that inclusion of additional covariates in a nonparametric framework requires a method for localization of the bounding function around the point x . With some non-trivial work and under appropriate conditions, the other approaches can likely be adapted to this context.

⁵Suppose for example that $V_0 = \arg \min_{v \in \mathcal{V}} \theta(v)$ is singleton, with $\theta_0 = \theta(v)$ for some $v \in \mathcal{V}$. Then θ_0 is a nonparametric function evaluated at a single point, which cannot be estimated faster than the optimal nonparametric rate. Lower bounds on the rates of convergence in nonparametric models are characterized e.g. by Stone (1982) and Tsybakov (2009). Having a uniformly super-efficient procedure would contradict these lower bounds.

rates at *some* values of the nuisance parameter $\theta(\cdot)$. In particular, if the bounding function $\theta(\cdot)$ happens to be flat on the contact set $V_0 = \{v \in \mathcal{V} : \theta(v) = \theta_0\}$, meaning that V_0 is a set of positive Lebesgue measure, then the point and interval estimator of AS can achieve the convergence rate of $n^{-1/2}$. As a consequence, their procedure for testing $\theta_{na} \leq \theta_0$ against $\theta_{na} > \theta_0$, where $\theta_{na} = \theta_0 + C/\sqrt{n}$ for $C > 0$, has non-trivial asymptotic power, while our procedure does not. If, however, $\theta(\cdot)$ is not flat on V_0 , then the testing procedure of AS no longer has power against the aforementioned $n^{-1/2}$ alternatives, and results in point and interval estimators that converge to θ_0 at a sub-optimal rate.⁶ In contrast, our procedure delivers point and interval estimators that can converge at nearly the optimal rate, and hence can provide better power in these cases. Note that in applications both flat and non-flat cases are important.⁷ Therefore, we believe that both testing procedures are useful. For further comparisons, we refer the reader to our Monte-Carlo section and to Supplemental Appendices J and K, which confirm these points both analytically and numerically.

There have also been some more recent additions to the literature on conditional moment inequalities. Armstrong (2011b) and Chetverikov (2011) both propose interesting and important approaches to estimation and inference based on conditional moment inequalities, respectively. The proposals can be seen as introducing full studentization in the procedure of AS, which fundamentally changes its behavior. The resulting procedures use a collection of fully studentized nonparametric estimators for inference, which brings them much closer to the approach of the present paper. In Armstrong (2011b) and Chetverikov (2011) the implicit nonparametric estimators are locally constant, with an adaptively chosen bandwidth. In contrast, our approach does not rely on locally constant estimators, allowing for the use of local polynomials, higher-order kernels, and series. Thus our approach is specifically geared towards smooth cases, where $\theta^u(\cdot)$ and $\theta^l(\cdot)$ are continuously differentiable of order $s \geq 1$. In these cases it results in more precise estimates of the bounding functions and hence higher power. On the other hand, in non-smooth cases, $0 < s \leq 1$, the procedures of Armstrong (2011b) and Chetverikov (2011) automatically adapt to deliver optimal estimation and testing procedures, respectively, and so can perform somewhat better

⁶With regard to confidence intervals/interval estimators, we mean here that the upper bound of the confidence interval does not converge at this rate.

⁷Note also that the latter case can be justified as generic if e.g. one takes $\theta(\cdot)$ as a random draw from the Sobolev ball equipped with the Gaussian (Wiener) measure.

than our approach.⁸ Other recent papers include those of Armstrong (2011a) and Ponomareva (2010). To compare to their approaches in the context of the previous one-sided example with $\theta_0 = \min_{v \in \mathcal{V}} E[Y|V = v]$, suppose that the bounding function is uniquely minimized at a single point V_0 and is locally quadratic. In such cases these papers propose to employ the usual extremum approach for inference, and the resulting inference is asymptotically exact. Armstrong (2011a) also considers a more general case where the contact set $V_0 = \arg \min_{v \in \mathcal{V}} E[Y|V = v]$ is finite. Note that when V_0 is singleton or a finite set, our simulation-based approach will automatically achieve asymptotic exactness under some regularity conditions on smoothing parameters and is in fact first-order equivalent to the extremum approach when V_0 is singleton. However, our approach does not rely on the bounding function being uniquely minimized and locally quadratic, or finite V_0 for its validity.

Plan of the Paper. We organize the paper as follows. In section 2, we motivate the analysis with examples and provide an informal overview of our results. In section 3 we provide a formal treatment of our method under high level conditions. In section 4 we provide conditions and theorems for validity for both parametric and nonparametric estimators. We provide several examples demonstrating the use of primitive conditions for parametric, series, and kernel estimators to verify the conditions of section 3. This includes sufficient conditions for the application of each of these estimators to models comprised of conditional moment inequalities. In section 5 we illustrate the performance of our method in Monte Carlo experiments, which we compare to that of AS in terms of coverage frequency and power. Our method performs well in these experiments, and we find that our approach and that of AS perform favorably in different models, depending on the shape of the bounding function. Section 6 concludes. In Appendix A we provide a step-by-step implementation guide for our method. In Appendices B - F we provide proofs and establish strong approximation results for both series and kernel estimators. An on-line supplement contains five additional appendices. The first of these, Appendix G provides proofs omitted from the main text in order to abide by space constraints.⁹ Appendix H provides additional details on the use of primitive conditions to verify an asymptotic linear expansion needed for strong approximation of series estimators and Appendix I gives some detailed arguments

⁸Note that for locally constant or any sign-preserving estimation of bounding functions, there is no need to undersmooth, since the approximation bias is conservatively signed. Our inference algorithm still applies to nonparametric estimates of bounding functions without undersmoothing, although our theory requires some minor modifications to handle this case. We do not formally pursue this case, and focus on smooth cases, and so we rely on undersmoothing.

⁹Specifically, Appendix G contains the proofs of Lemmas 2, 4, 7, and 8, and Theorems 8 and 9.

omitted from the main text for local polynomial estimation of conditional moment inequalities. Appendix J provides local asymptotic power analysis that supports the findings of our Monte Carlo experiments. Appendix K provides further Monte Carlo evidence.

Notation. For any two reals a and b , $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. $Q_p(X)$ denotes the p -th quantile of random variable X . We use $\text{wp} \rightarrow 1$ as shorthand for “with probability approaching one as $n \rightarrow \infty$.” To denote probability statements conditional on observed data, we write statements conditional on \mathcal{D}_n . \mathbb{E}_n and \mathbb{P}_n denote the sample mean and empirical measure, respectively. That is, given i.i.d. random vectors X_1, \dots, X_n , we have $\mathbb{E}_n f = \int f d\mathbb{P}_n = n^{-1} \sum_{i=1}^n f(X_i)$. In addition, let $\mathbb{G}_n f = \sqrt{n}(\mathbb{E}_n - E)f = n^{-1/2} \sum_{i=1}^n [f(X_i) - Ef(X)]$. The notation $a_n \lesssim b_n$ means that $a_n \leq Cb_n$ for all n ; $X_n \lesssim_{\mathbb{P}_n} c_n$ abbreviates $X_n = O_{\mathbb{P}_n}(c_n)$. $X_n \rightarrow_{\mathbb{P}_n} \infty$ means that for any constant $C > 0$, $\mathbb{P}_n(X_n < C) \rightarrow 0$. We write $\text{diam}(V)$ to denote the diameter of V in the Euclidian metric. $\|\cdot\|$ denotes the Euclidean norm, and for any two sets A, B in Euclidean space, $d_H(A, B)$ denotes the Hausdorff pseudo-distance between A and B with respect to the Euclidean norm. C stands for a generic positive constant, which may be different in different places, unless stated otherwise. For a set V and an element v in Euclidean space, let $d(v, V) := \inf_{v' \in V} \|v - v'\|$. For a function $p(v)$, let $\text{lip}(p)$ denote the Lipschitz coefficient, that is $\text{lip}(p) := L$ such that $\|p(v_1) - p(v_2)\| \leq L\|v_1 - v_2\|$ for all v_1 and v_2 in the domain of $p(v)$.

2. MOTIVATING EXAMPLES AND INFORMAL OVERVIEW OF RESULTS

In this section we briefly describe three examples of intersection bounds from the literature and provide an informal overview of our results.

Example A: Treatment Effects and Instrumental Variables. In the analysis of treatment response, the ability to uniquely identify the distribution of potential outcomes is typically lacking without either experimental data or strong assumptions. This owes to the fact that for each individual unit of observation, only the outcome from the received treatment is observed; the counterfactual outcome that would have occurred given a different treatment is not known. Although we focus here on treatment effects, similar issues are present in other areas of economics. In the analysis of markets, for example, observed equilibrium outcomes reveal quantity demanded at the observed price, but do not reveal what demand would have been at other prices.

Suppose only that the support of the outcome space is known, $Y \in [0, 1]$, but no other assumptions are made regarding the distribution of counterfactual outcomes. Manski

(1989, 1990) provide worst-case bounds on mean treatment outcomes for any treatment t conditional on observables $(X, V) = (x, v)$,

$$\theta^l(x, v) \leq E[Y(t) | X = x, V = v] \leq \theta^u(x, v),$$

where the bounds are

$$\begin{aligned} \theta^l(x, v) &:= E[Y \cdot 1\{Z = t\} | X = x, V = v], \\ \theta^u(x, v) &:= E[Y \cdot 1\{Z = t\} + 1\{Z \neq t\} | X = x, V = v], \end{aligned}$$

where Z is the observed treatment. If V is an instrument satisfying $E[Y(t) | X, V] = E[Y(t) | X]$, then for any fixed x sharp bounds on $\theta^* := \theta^*(x) := E[Y(t) | X = x]$ are given by

$$\sup_{v \in \mathcal{V}} \theta^l(x, v) \leq \theta^*(x) \leq \inf_{v \in \mathcal{V}} \theta^u(x, v),$$

for any $\mathcal{V} \subseteq \text{support}(V | X = x)$, where the subset \mathcal{V} will be taken as known for estimation purposes. Similarly, bounds implied by restrictions such as monotone treatment response, monotone treatment selection, and monotone instrumental variables, as in Manski (1997) and Manski and Pepper (2000), also take the form of intersection bounds.

Example B: Bounding Distributions to Account for Selection. Similar analysis applies to inference on distributions whose observations are censored due to selection. This approach is used by Blundell, Gosling, Ichimura, and Meghir (2007) to study changes in male and female wages. The starting point of their analysis is that the cumulative distribution $F(w|x, v)$ of wages W at any point w , conditional on observables $(X, V) = (x, v)$ must satisfy the worst case bounds

$$\theta^l(x, v) \leq F(w|x, v) \leq \theta^u(x, v), \tag{2.1}$$

where D is an indicator of employment, and hence observability of W , so that

$$\begin{aligned} \theta^l(x, v) &:= E[D \cdot 1\{W \leq w\} | X = x, V = v], \\ \theta^u(x, v) &:= E[D \cdot 1\{W \leq w\} + (1 - D) | X = x, V = v]. \end{aligned}$$

This relation is used to bound quantiles of conditional wage distributions. Additional restrictions motivated by economic theory are then used to tighten the bounds.

One such restriction is an exclusion restriction of the continuous variable out-of-work income, V . They consider the use of V as either an excluded or monotone instrument.

The former restriction implies bounds on the parameter $\theta^* := F(w|x)$,

$$\sup_{v \in \mathcal{V}} \theta^l(x, v) \leq F(w|x) \leq \inf_{v \in \mathcal{V}} \theta^u(x, v), \quad (2.2)$$

for any $\mathcal{V} \subseteq \text{support}(V|X = x)$, while the weaker monotonicity restriction implies the following bounds on $\theta^* := F(w|x, v_0)$ for any v_0 in $\text{support}(V|X = x)$,

$$\sup_{v \in \mathcal{V}_l} \theta^l(x, v) \leq F(w|x, v_0) \leq \inf_{v \in \mathcal{V}_u} \theta^u(x, v), \quad (2.3)$$

where $\mathcal{V}_l = \{v \in \mathcal{V} : v \leq v_0\}$ and $\mathcal{V}_u = \{v \in \mathcal{V} : v \geq v_0\}$.

Example C: (Conditional) Conditional Moment Inequalities. Our inferential method can also be used for pointwise inference on parameters restricted by (possibly conditional) conditional moment inequalities. Such restrictions arise naturally in empirical work in industrial organization, see for example Pakes, Porter, Ho, and Ishii (2005) and Berry and Tamer (2007).

To illustrate, consider the restriction

$$E[m_j(X, \mu_0) | Z = z] \geq 0 \text{ for all } j = 1, \dots, J \text{ and } z \in \mathcal{Z}_j. \quad (2.4)$$

where each $m_j(\cdot, \cdot)$, $j = 1, \dots, J$ is a real-valued function, (X, Z) are observables, and μ_0 is the parameter of interest. Note that this parameter can be dependent on some particular covariate value. For instance, we may be interested in a subgroup of the population with $\tilde{Z}_1 = \tilde{z}_1$, where \tilde{Z}_1 denotes a subvector of Z . In this case, $\mu_0 = \mu_0(z)$ depends on z , and $\mathcal{Z}_j \subseteq \text{support}(Z | \tilde{Z}_1 = \tilde{z}_1)$ for $j = 1, \dots, J$. Note also that regions \mathcal{Z}_j can depend on the inequality j as in (2.3) of the previous example, and that the previous two examples can in fact be cast as special cases of this one.

Suppose that we would like to test (2.4) at level α for the conjectured parameter value $\mu_0 = \mu$ against an unrestricted alternative. To see how our framework can be used to test this hypothesis, define

$$v = (z, j), \quad \mathcal{V} := \{(z, j) : z \in \mathcal{Z}_j, j \in \{1, \dots, J\}\} \text{ and } \theta(\mu, v) := E[m_j(X, \mu) | Z = z]$$

and $\hat{\theta}(\mu, v)$ a consistent estimator. Under some continuity conditions this is equivalent to a test of $\theta_0(\mu) := \inf_{v \in \mathcal{V}} \theta(\mu, v) \geq 0$ against $\inf_{v \in \mathcal{V}} \theta(\mu, v) < 0$. Our method for inference delivers a statistic

$$\hat{\theta}_\alpha(\mu) = \inf_{v \in \mathcal{V}} \left[\hat{\theta}(\mu, v) + \hat{k} \cdot s(\mu, v) \right]$$

such that $\lim_{n \rightarrow \infty} P(\theta_0(\mu) \geq \widehat{\theta}_\alpha(\mu)) \leq \alpha$. Here, $s(\mu, v)$ is the standard error of $\widehat{\theta}(\mu, v)$ and \widehat{k} is an estimated critical value, as we describe below. If $\widehat{\theta}_\alpha(\mu) < 0$, we reject the null hypothesis, while if $\widehat{\theta}_\alpha(\mu) \geq 0$, we do not.

Informal Overview of Results. We now provide an informal description of our method for estimation and inference. Consider an upper bound θ_0 on θ^* of the form

$$\theta^* \leq \theta_0 := \inf_{v \in \mathcal{V}} \theta(v), \quad (2.5)$$

where $v \mapsto \theta(v)$ is a bounding function, and \mathcal{V} is the set over which the infimum is taken. We focus on describing our method for the upper bound (2.5), as the lower bound is entirely symmetric. In fact, any combination of upper and lower bounds can be combined into upper bounds on an auxiliary function of θ^* of the form (2.5), and this can be used for inference on θ^* , as we describe in Section A.¹⁰

What are good *estimators* and *confidence regions* for the bound θ_0 ? A natural idea is to base estimation and inference on the sample analog: $\inf_{v \in \mathcal{V}} \widehat{\theta}(v)$. However, this estimator does not perform well in practice. First, the analog estimator tends to be downward biased in finite samples. As discussed in the introduction, this will typically result in bound estimates that are much narrower than those in the population, see e.g. Manski and Pepper (2000) and Manski and Pepper (2009) for more on this point. Second, inference must appropriately take account of sampling error of the estimator $\widehat{\theta}(v)$ across all values of v . Indeed, different levels of precision of $\widehat{\theta}(v)$ at different points can severely distort the perception of the minimum of the bounding function $\theta(v)$. Figure 1 illustrates these problems geometrically. The solid curve is the true bounding function $v \mapsto \theta(v)$, and the dash-dotted thick curve is its estimate $v \mapsto \widehat{\theta}(v)$. The remaining dashed curves represent eight additional potential realizations of the estimator, illustrating its precision. In particular, we see that the precision of the estimator is much lower on the right side than on the left. A naïve sample analog estimate for θ_0 is provided by the minimum of the dash-dotted curve, but this estimate can in fact be quite far away from θ_0 . This large deviation from the true value arises from both the lower precision of the estimated curve on the right side of the figure and from the downward bias created by taking the minimum of the estimated curve.

¹⁰Alternatively, one can combine one-sided intervals for lower and upper bounds for inference on the identified set Θ_I using Bonferroni's inequality, or for inference on θ^* using the method described in Chernozhukov, Lee, and Rosen (2009) Section 3.7, which is a slight generalization of methods previously developed by Imbens and Manski (2004) and Stoye (2009).

To overcome these problems, we propose a *precision-corrected* estimate of θ_0 :

$$\widehat{\theta}(p) := \min_{v \in \mathcal{V}} [\widehat{\theta}(v) + k(p) \cdot s(v)], \quad (2.6)$$

where $s(v)$ is the standard error of $\widehat{\theta}(v)$, and $k(p)$ is a critical value, the selection of which is described below. That is, our estimator $\widehat{\theta}(p)$ minimizes the *precision-corrected curve* given by $\widehat{\theta}(v)$ plus critical value $k(p)$ times the pointwise standard error $s(v)$. Figure 2 shows a precision-corrected curve as a dashed curve with a particular choice of critical value k . In this figure, we see that the minimizer of the precision-corrected curve can indeed be much closer to θ_0 than the sample analog $\inf_{v \in \mathcal{V}} \widehat{\theta}(v)$. Although this illustration is schematic in nature, it conveys geometrically why our approach can ameliorate the downward bias. In what follows, we provide both theoretical and Monte-Carlo evidence that further supports this point.

The main input in the selection of our critical value $k(p)$ for the estimator $\widehat{\theta}(p)$ in (2.6) above is the standardized process

$$Z_n(v) = \frac{\theta(v) - \widehat{\theta}(v)}{\sigma(v)},$$

where $\sigma(v)/s(v) \rightarrow 1$ uniformly in v . Generally, the finite-sample distribution of the process Z_n is unknown, but we can approximate it uniformly by a sequence of Gaussian processes Z_n^* such that for an appropriate sequence of constants \bar{a}_n

$$\bar{a}_n \sup_{v \in \mathcal{V}} |Z_n(v) - Z_n^*(v)| = o_p(1). \quad (2.7)$$

For any compact set V , used throughout to denote a generic compact subset of \mathcal{V} , we then approximate the quantiles of $\sup_{v \in V} Z_n^*(v)$ either by analytical methods based on asymptotic approximations, or by simulation. We then use the p -quantile of this statistic, $k_{n,V}(p)$, in place of $k(p)$ in (2.6). We show that in general simulated critical values provide sharper inference, and therefore advocate their use.

The estimated critical value $k_{n,V}(p)$ is monotone in V . For the estimator in (2.6) to exceed θ_0 with probability no less than p asymptotically, we require that $\text{wp} \rightarrow 1$ the set V contains the argmin set

$$V_0 := \arg \min_{v \in \mathcal{V}} \theta(v).$$

A simple way to achieve this is to use $V = \mathcal{V}$, which leads to asymptotically valid but conservative inference. We thus propose the use of a preliminary estimator \widehat{V}_n for V_0 in the construction of $k_{n,V}(p)$ above, and verify its validity. The estimator \widehat{V}_n is constructed using a novel adaptive inequality selection procedure. Note that because the critical value

$k_{n,V}(p)$ is monotone in V , this yields a critical value no larger than those based on $V = \mathcal{V}$. In section 3.5 we provide conditions for consistency and rates of convergence for the set estimate \widehat{V}_n , and in section 3.6 we provide conditions whereby simulation-based selection of the critical value results in asymptotically exact inference.

At an abstract level our method does not distinguish parametric estimators of $\theta(v)$ from nonparametric estimators; however, details of the analysis and regularity conditions are quite distinct. In all cases, we employ strong approximation analysis to approximate the quantiles of $\sup_{v \in V} Z_n(v)$, and we verify our conditions separately for each case. The formal definition of strong approximation is provided in Appendix B.

3. ESTIMATION AND INFERENCE THEORY UNDER GENERAL CONDITIONS

3.1. Basic Framework. In this and subsequent sections we allow the model and the probability measure to depend on n . Formally, we work with a probability space (A, \mathcal{A}, P_n) throughout. This approach is conventionally used in asymptotic statistics to ensure robustness of statistical conclusions with respect to perturbations in P_n . It guarantees the validity of our inference procedure under any sequence of probability laws P_n that obey our conditions, including the case with fixed P . We thus generalize our notation in this section to allow model parameters to depend on n .

The basic setting is as follows:

Condition C. 1 (Setting). *There is a non-empty compact set $\mathcal{V} \subset \mathcal{K} \subset \mathbb{R}^d$, where \mathcal{V} can depend on n , and \mathcal{K} is a bounded fixed set, independent of n . There is a continuous real valued function $v \mapsto \theta_n(v)$. There is an estimator $v \mapsto \widehat{\theta}_n(v)$ of this function, which is an a.s. continuous stochastic process. There is a continuous function $v \mapsto \sigma_n(v)$ representing non-stochastic normalizing factors bounded by $\bar{\sigma}_n := \sup_{v \in \mathcal{V}} \sigma_n(v)$, and there is an estimator $v \mapsto s_n(v)$ of these factors, which is an a.s. continuous stochastic process, bounded above by $\bar{s}_n := \sup_{v \in \mathcal{V}} s_n(v)$.*

We are interested in constructing point estimators and one-sided interval estimators for

$$\theta_{n0} = \inf_{v \in \mathcal{V}} \theta_n(v).$$

The main input in this construction is the standardized process

$$Z_n(v) = \frac{\theta_n(v) - \widehat{\theta}_n(v)}{\sigma_n(v)}.$$

In the following we require that this process can be approximated by a standardized Gaussian process in the metric space $\ell^\infty(\mathcal{V})$ of bounded functions mapping \mathcal{V} to \mathbb{R} , which can be simulated for inference.

Condition C. 2 (Strong Approximation). *(a) Z_n is strongly approximated by a sequence of penultimate Gaussian processes Z_n^* having zero mean and a.s. continuous sample paths:*

$$\sup_{v \in \mathcal{V}} |Z_n(v) - Z_n^*(v)| = o_{\mathbb{P}_n}(\delta_n),$$

where $E_{\mathbb{P}_n}[Z_n^*(v)]^2 = 1$ for each $v \in \mathcal{V}$, where $\delta_n = o(\bar{a}_n^{-1})$ for the sequence of constants \bar{a}_n defined in Condition C.3 below. *(b) Moreover, for simulation purposes, there is a process Z_n^* , whose distribution is zero-mean Gaussian conditional on the data \mathcal{D}_n and such that $E_{\mathbb{P}_n}[Z_n^*(v) \mid \mathcal{D}_n]^2 = 1$ for each $v \in \mathcal{V}$, that can approximate an identical copy \bar{Z}_n^* of Z_n^* , where \bar{Z}_n^* is independent of \mathcal{D}_n , namely there is $o(\delta_n)$ term such that*

$$\mathbb{P}_n \left[\sup_{v \in \mathcal{V}} |\bar{Z}_n^*(v) - Z_n^*(v)| > o(\delta_n) \mid \mathcal{D}_n \right] = o_{\mathbb{P}_n}(1/\ell_n)$$

for some $\ell_n \rightarrow \infty$ chosen below.

For convenience we refer to Appendix B, where the definition of strong approximation is recalled. The penultimate process Z_n^* is often called a coupling, and we construct such couplings for parametric and nonparametric estimators under both high-level and primitive conditions. It is quite convenient to work with Z_n^* , since we can rely on the fine properties of Gaussian processes. Note that Z_n^* depend on n and generally do not converge weakly to a fixed Gaussian process, and therefore they are not asymptotically Donsker. Nonetheless we shall perform either analytical or simulation-based inference based on these processes.

Our next condition captures the so-called *concentration* properties of Gaussian processes:

Condition C. 3 (Concentration). *For all n sufficiently large and for any compact, non-empty $V \subseteq \mathcal{V}$, there is a normalizing factor $a_n(V)$ satisfying*

$$1 \leq a_n(V) \leq a_n(\mathcal{V}) =: \bar{a}_n, \quad a_n(V) \text{ is increasing in } V,$$

such that

$$\mathcal{E}_n(V) := a_n(V) \left(\sup_{v \in V} Z_n^*(v) - a_n(V) \right)$$

obeys

$$\mathbb{P}_n[\mathcal{E}_n(V) \geq x] \leq \mathbb{P}[\mathcal{E} \geq x], \tag{3.1}$$

where \mathcal{E} is a random variable with continuous distribution function such that $P(\mathcal{E} > x) \leq \exp(-x/\eta)$ for some $\eta > 0$.

The *concentration* condition will be verified in our applications by appealing to the Talagrand-Samorodnitsky inequality for the concentration of the suprema of Gaussian processes, which is sharper than the classical concentration inequalities. These concentration properties play a key role in our analysis, as they determine the uniform speed of convergence $\bar{a}_n \bar{\sigma}_n$ of the estimator $\widehat{\theta}_n(p)$ to θ_{n0} . In particular this property implies that for any compact $V_n \subseteq \mathcal{V}$ $E_{P_n}[\sup_{v \in V_n} Z_n^*(v)] \lesssim \bar{a}_n$. As there is concentration, there is an opposite force, called *anti-concentration*, which implies that under C.2(a) and C.3 for any $\delta_n = o(1/\bar{a}_n)$ we have

$$\sup_{x \in \mathbb{R}} P_n \left(\left| \sup_{v \in V_n} Z_n^*(v) - x \right| \leq \delta_n \right) \rightarrow 0. \quad (3.2)$$

This follows from a generic anti-concentration inequality derived in Chernozhukov and Kato (2011), quoted in Appendix C for convenience. Anti-concentration simplifies the construction of our confidence intervals. Finally, the exponential tail property of \mathcal{E} plays an important role in the construction of our adaptive inequality selector, introduced below, since it allows us to bound moderate deviations of one-sided estimation noise $\sup_{v \in V} Z_n^*(v)$.

Our next assumption requires uniform consistency as well as suitable estimates of σ_n :

Condition C.4 (Uniform Consistency). *We have that*

$$(a) \bar{a}_n \bar{\sigma}_n = o(1) \quad \text{and} \quad (b) \sup_{v \in \mathcal{V}} \left| \frac{s_n(v)}{\sigma_n(v)} - 1 \right| = o_{P_n} \left(\frac{\delta_n}{\bar{a}_n + \ell \ell_n} \right),$$

where $\ell \ell_n \nearrow \infty$ is a sequence of constants defined below.

In what follows we let

$$\ell_n := \log n, \quad \text{and} \quad \ell \ell_n := \log \ell_n,$$

but it should be noted that ℓ_n can be replaced by other slowly increasing sequences.

3.2. The Inference and Estimation Strategy. For any compact subset $V \subseteq \mathcal{V}$ and $\gamma \in (0, 1)$, define:

$$\kappa_{n,V}(\gamma) := Q_\gamma \left(\sup_{v \in V} Z_n^*(v) \right).$$

Given this notation, the following result is a key observation that helps us set up inference.

Lemma 1 (Inference Concentrates on a Neighborhood V_n of V_0). *Under C.1-C.4*

$$P_n \left(\sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \widehat{\theta}_n(v)}{s_n(v)} \leq x \right) \geq P_n \left(\sup_{v \in V_n} Z_n^*(v) \leq x \right) - o(1),$$

uniformly in $x \in [0, \infty)$, where

$$V_n := \{v \in \mathcal{V} : \theta_n(v) \leq \theta_{n0} + \kappa_n \sigma_n(v)\}, \quad \text{for } \kappa_n := \kappa_{n,\mathcal{V}}(\gamma'_n), \quad (3.3)$$

where γ'_n is any sequence such that $\gamma'_n \nearrow 1$ with $\kappa_n/(\bar{a}_n + \ell\ell_n) \lesssim 1$.

Thus, with probability converging to one, the inferential process concentrates on a neighborhood of V_0 given by V_n . The “size” of the neighborhood is determined by κ_n , a high quantile of $\sup_{v \in \mathcal{V}} Z_n^*(v)$, which summarizes the maximal one-sided estimation error over \mathcal{V} . We use this to construct half-median-unbiased estimators for θ_{n0} as well as one-sided interval estimators for θ_{n0} with correct asymptotic level, based on analytical and simulation methods for obtaining critical values proposed below.

Remark 1 (Sharp Concentration of Inference). In general, it is not possible for the inferential processes to concentrate on smaller subsets than V_n . However, as shown, in Section 3.6, in some special cases, e.g. when V_0 is a well-identified singleton, the inference process will in fact concentrate on V_0 . In this case our simulation-based construction will automatically adapt to deliver median-unbiased estimators for θ_{n0} as well as one-sided interval estimators for θ_{n0} with correct asymptotic size. Indeed, in the special but extremely important case of V_0 being singleton we can achieve

$$\Pr_n \left(\sup_{v \in V_n} Z_n^*(v) > x \right) = \Pr(N(0, 1) > x) - o(1),$$

under some regularity conditions. In this case, our simulation-based procedure will automatically produce a critical value that approaches the p -th quantile of the standard normal, delivering asymptotically exact inference. ■

Definition 1 (Generic Interval and Point Estimators). Let $p \geq 1/2$, then our interval estimator takes the form:

$$\hat{\theta}_{n0}(p) = \inf_{v \in \mathcal{V}} \left[\hat{\theta}_n(v) + k_{n, \hat{V}_n}(p) s_n(v) \right], \quad (3.4)$$

where the half-median unbiased estimator corresponds to $p = 1/2$. This construction relies on the principal critical value $k_{n, \hat{V}_n}(p)$, which depends on a preliminary set estimator:

$$\hat{V}_n = \left\{ v \in \mathcal{V} : \hat{\theta}_n(v) \leq \min_{\tilde{v} \in \mathcal{V}} \left(\hat{\theta}_n(\tilde{v}) + k_{n, \mathcal{V}}(\gamma_n) s_n(\tilde{v}) \right) + 2k_{n, \mathcal{V}}(\gamma_n) s_n(v) \right\}, \quad (3.5)$$

which in turn depends on the auxiliary critical value $k_{n, \mathcal{V}}(\gamma_n)$, where we set $\gamma_n := 1 - .1/\ell_n \nearrow 1$. These critical values are constructed below using either the analytical or simulation method.

The main idea is to construct simulated or analytical critical values so that $\text{wp} \rightarrow 1$,

$$k_{n, \widehat{V}_n}(p) \geq \kappa_{n, V_n}(p - o(1)), \quad (3.6)$$

$$k_{n, \mathcal{V}}(\gamma_n) \geq \kappa_{n, \mathcal{V}}(\gamma'_n), \quad (3.7)$$

where $\gamma'_n = \gamma_n - o(1) \nearrow 1$. As a consequence, we show in Theorems 1 and 2 below that

$$\mathbb{P}_n \left\{ \theta_{n0} \leq \widehat{\theta}_{n0}(p) \right\} \geq p - o(1), \quad (3.8)$$

for any fixed $1/2 \leq p < 1$. The construction relies on the new set estimator \widehat{V}_n , which we call an *adaptive inequality selector* (AIS), since it uses the problem-dependent cutoff $k_{n, \mathcal{V}}(\gamma_n)$, which is a bound on a high quantile of $\sup_{v \in \mathcal{V}} Z_n^*(v)$. The analysis therefore must take into account the moderate deviations (tail behavior) of the latter.

Before proceeding to the details of its construction, we note that the argument for establishing the coverage results and analyzing power properties of the procedure depends crucially on the following result proven in Lemma 2 below:

$$\mathbb{P}_n \left\{ V_n \subseteq \widehat{V}_n \subseteq \overline{V}_n \right\} \rightarrow 1,$$

where

$$\overline{V}_n := \{v \in \mathcal{V} : \theta_n(v) \leq \theta_{n0} + \bar{\kappa}_n \sigma_n(v)\}, \text{ for } \bar{\kappa}_n := 4(\bar{a}_n + \eta \ell \ell_n / \bar{a}_n). \quad (3.9)$$

Thus, the preliminary set estimator \widehat{V}_n is sandwiched between two deterministic sequences of sets, facilitating the analysis of its impact on the convergence of $\widehat{\theta}_{n0}(p)$ to θ_{n0} .

3.3. Analytical Method and Its Theory. Our first construction is quite simple and demonstrates the main – though not the finest – points. This construction uses the majorizing variable \mathcal{E} appearing in C.3.

Definition 2 (Analytical Method for Critical Values). *For any compact set V and any $p \in (0, 1)$, we set*

$$k_{n, V}(p) = a_n(V) + c(p)/a_n(V), \quad (3.10)$$

where $c(p) = Q_p(\mathcal{E})$ is the p -th quantile of the majorizing variable \mathcal{E} defined in C.3, where we require that $V \mapsto k_{n, V}(p)$ is monotone in V .

The first main result is as follows.

Theorem 1 (Analytical Inference, Estimation, Power under C.1-C.4). *Suppose C.1-C.4 hold. Consider the interval estimator given in Definition 1 with critical value function given in Definition 2. Then, for a given $p \in [1/2, 1)$,*

1. The interval estimator has asymptotic level p :

$$\mathbb{P}_n \left\{ \theta_{n0} \leq \widehat{\theta}_{n0}(p) \right\} \geq p - o(1).$$

2. The estimation risk is bounded by, $w_p \rightarrow 1$ under \mathbb{P}_n ,

$$\left| \widehat{\theta}_{n0}(p) - \theta_{n0} \right| \leq 4\bar{\sigma}_n \left(a_n(\bar{V}_n) + \frac{O_{\mathbb{P}_n}(1)}{a_n(\bar{V}_n)} \right) \lesssim_{\mathbb{P}_n} \bar{\sigma}_n \bar{a}_n.$$

3. Hence, any, possibly data-dependent, alternative $\theta_{na} > \theta_{n0}$ such that

$$\theta_{na} \geq \theta_{n0} + 4\bar{\sigma}_n \left(a_n(\bar{V}_n) + \frac{\mu_n}{a_n(\bar{V}_n)} \right), \quad \mu_n \rightarrow_{\mathbb{P}_n} \infty,$$

is rejected with probability converging to 1 under \mathbb{P}_n .

Thus, $(-\infty, \widehat{\theta}_{n0}(p)]$ is a valid one-sided interval estimator for θ_{n0} . Moreover, $\widehat{\theta}_{n0}(1/2)$ is a half-median-unbiased estimator for θ_{n0} in the sense that

$$\lim_{n \rightarrow \infty} \mathbb{P}_n \left[\theta_{n0} \leq \widehat{\theta}_{n0}(1/2) \right] \geq 1/2.$$

The rate of convergence of $\widehat{\theta}_{n0}(p)$ to θ_{n0} is bounded above by the uniform rate $\bar{\sigma}_n \bar{a}_n$ for estimation of the bounding function $v \mapsto \theta_n(v)$. This implies that the test of $H_0 : \theta_{n0} = \theta_{na}$ that rejects if $\theta_{na} > \widehat{\theta}_{n0}(p)$ asymptotically rejects all local alternatives that are more distant than $\bar{\sigma}_n \bar{a}_n$, including fixed alternatives as a special case. In Section 4 below we show that in parametric cases this results in power against $n^{-1/2}$ local alternatives. For kernel-type estimators of bounding functions the rate $\bar{a}_n \bar{\sigma}_n$ is proportional to $(\log n)^c / \sqrt{nh^d}$ where c is some positive constant and h is the bandwidth, assuming some undersmoothing is done. For example, if the bounding function is s -times differentiable, σ_n can be made close to $(\log n/n)^{s/(2s+d)}$ apart from some undersmoothing factor by considering a local polynomial estimator, see Stone (1982). Similarly, for series estimators $\bar{a}_n \bar{\sigma}_n$ is proportional to $(\log n)^c \sqrt{K/n}$ where c is some positive constant, and $K \rightarrow \infty$ is the number of series terms. For both series and kernel-type estimators we show below that \bar{a}_n can be bounded by $\sqrt{\log n}$.

3.4. Simulation-Based Construction and Its Theory. Our main and preferred approach is based on the simple idea of simulating quantiles of relevant statistics.

Definition 3 (Simulation Method for Critical Values). *For any compact set $V \subseteq \mathcal{V}$, we set*

$$k_{n,V}(p) = Q_p \left(\sup_{v \in V} Z_n^*(v) \mid \mathcal{D}_n \right). \quad (3.11)$$

We have the following result for simulation inference, analogous to that obtained for analytical inference.

Theorem 2 (Simulation Inference, Estimation, Power under C.1-C.4). *Suppose C.1-C.4 hold. Consider the interval estimator given in Definition 1 with the critical value function specified in Definition 3. Then, for a given $p \in [1/2, 1)$,*

1. *The interval estimator has asymptotic level p :*

$$\mathbb{P}_n \left\{ \theta_{n0} \leq \widehat{\theta}_{n0}(p) \right\} \geq p - o(1).$$

2. *The estimation risk is bounded by, $w_p \rightarrow 1$ under \mathbb{P}_n ,*

$$\left| \widehat{\theta}_{n0}(p) - \theta_{n0} \right| \leq 4\bar{\sigma}_n \left(a_n(\bar{V}_n) + \frac{O_{\mathbb{P}_n}(1)}{a_n(\bar{V}_n)} \right) \lesssim_{\mathbb{P}_n} \bar{\sigma}_n \bar{a}_n.$$

3. *Any, possibly data-dependent, alternative $\theta_{na} > \theta_{n0}$ such that*

$$\theta_{na} \geq \theta_{n0} + 4\bar{\sigma}_n \left(a_n(\bar{V}_n) + \frac{\mu_n}{a_n(\bar{V}_n)} \right), \quad \mu_n \rightarrow_{\mathbb{P}_n} \infty,$$

is rejected with probability converging to 1 under \mathbb{P}_n .

3.5. Properties of the Set Estimator \widehat{V}_n . In this section we establish some containment properties for the estimator \widehat{V}_n . Moreover, these containment properties imply a useful rate result under the following condition:

Condition V (Degree of Identifiability for V_0). *There exist constants $\rho_n > 0$ and $c_n > 0$, possibly dependent on n , and a positive constant δ , independent of n , such that*

$$\theta_n(v) - \theta_{n0} \geq (c_n d(v, V_0))^{\rho_n} \wedge \delta, \quad \forall v \in \mathcal{V}. \quad (3.12)$$

We say $(c_n, 1/\rho_n)$ characterize the degree of identifiability of V_0 , as these parameters determine the rate at which V_0 can be consistently estimated. Note that if $V_0 = \mathcal{V}$, then this condition holds with $c_n = \infty$ and $\rho_n = 1$, where we adopt the convention that $0 \cdot \infty = 0$.

We have the following result, whose first part we use in the proof of Theorems 1 and 2 above, and whose second part we use below in the proof of Theorem 3.

Lemma 2 (Estimation of V_n and V_0). *Suppose C.1-C.4 hold.*

1. *(Containment). Then $w_p \rightarrow 1$, for either analytical or simulation methods,*

$$V_n \subseteq \widehat{V}_n \subseteq \bar{V}_n,$$

for V_n defined in (3.3) with $\gamma'_n = \gamma_n - o(1)$, and \bar{V}_n defined in (3.9).

2. (Rate) If also Condition V holds and $\bar{\kappa}_n \bar{\sigma}_n \rightarrow 0$, then $wp \rightarrow 1$

$$\begin{aligned} d_H(\widehat{V}_n, V_0) &\leq d_H(\widehat{V}_n, V_n) + d_H(V_n, V_0) \\ &\leq d_H(\bar{V}_n, V_n) + d_H(V_n, V_0) \leq r_n := 2(\bar{\kappa}_n \bar{\sigma}_n)^{1/\rho_n} / c_n. \end{aligned}$$

3.6. Automatic Sharpness of Simulation Construction. When the penultimate process Z_n^* does not lose equicontinuity too fast, and V_0 is sufficiently well-identified, our simulation-based inference procedure becomes sharp in the sense of not only achieving the right level but in fact automatically achieving the right size. In such cases we typically have some small improvements in the rates of convergence of the estimators. The most important case covered is that where V_0 is singleton¹¹ (or a finite collection of points) and θ_n is locally quadratic, i.e. $\rho_n \geq 2$ and $c_n \geq c > 0$ for all n . These sharp situations occur when the inferential process concentrates on V_0 and not just on the neighborhood V_n , in the sense described below. For this to happen we impose the following condition.

Condition S (Equicontinuity radii are not smaller than r_n). Under Condition V holding, the scaled penultimate process $\bar{a}_n Z_n^*$ has an equicontinuity radius φ_n that is no smaller than $r_n := 2(\bar{\kappa}_n \bar{\sigma}_n)^{1/\rho_n} / c_n$:

$$\sup_{\|v-v'\| \leq \varphi_n} \bar{a}_n |Z_n^*(v) - Z_n^*(v')| = o_{\mathbb{P}_n}(1), \quad r_n \leq \varphi_n.$$

When Z_n^* is Donsker, i.e. asymptotically equicontinuous, this condition holds automatically, since in this case $\bar{a}_n \propto 1$, and for any $o(1)$ term, equicontinuity radii obey $\varphi_n = o(1)$, so that consistency $r_n = o(1)$ is sufficient. When Z_n^* is not Donsker, its finite-sample equicontinuity properties decay as $n \rightarrow \infty$, with radii φ_n characterizing the decay. However, as long as φ_n is not smaller than r_n , we have just enough finite-sample equicontinuity left to achieve the following result.

Lemma 3 (Inference Sometimes Concentrates on V_0). Suppose C.1-C.4, S, and V hold. Then for any $\gamma_n \nearrow 1$,

$$\mathbb{P}_n \left(\sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \widehat{\theta}_n(v)}{s_n(v)} \leq x \right) = \mathbb{P}_n \left(\sup_{v \in V_0} Z_n^*(v) \leq x \right) + o(1).$$

Under the stated conditions, our inference and estimation procedures *automatically* become sharp in terms of size and rates.

¹¹This case is generic in the sense that if one draws $\theta_n(\cdot)$ from the Sobolev ball equipped with the Wiener measure, then V_0 is singleton with probability one.

Theorem 3 (Sharpness of Simulation Inference). *Suppose C.1-C.4, S, and V hold. Consider the interval estimator given in Definition 1 with the critical value function specified in Definition 3. Then, for a given $p \in [1/2, 1)$,*

1. *The interval estimator has asymptotic size p :*

$$\mathbb{P}_n \left\{ \theta_{n0} \leq \widehat{\theta}_{n0}(p) \right\} = p + o(1).$$

2. *Its estimation risk is bounded by, $w_p \rightarrow 1$ under \mathbb{P}_n ,*

$$\left| \widehat{\theta}_{n0}(p) - \theta_{n0} \right| \leq 4\bar{\sigma}_n \left(a_n(V_0) + \frac{O_{\mathbb{P}_n}(1)}{a_n(V_0)} \right) \lesssim_{\mathbb{P}_n} \bar{\sigma}_n a_n(V_0).$$

3. *Any, possibly data-dependent, alternative $\theta_{na} > \theta_{n0}$ such that*

$$\theta_{na} \geq \theta_{n0} + 4\bar{\sigma}_n \left(a_n(V_0) + \frac{\mu_n}{a_n(V_0)} \right), \quad \mu_n \rightarrow_{\mathbb{P}_n} \infty,$$

is rejected with probability converging to 1 under \mathbb{P}_n .

4. INFERENCE ON INTERSECTION BOUNDS IN LEADING CASES

4.1. Parametric estimation of bounding function. We now show that the above conditions apply to various parametric estimation methods for $v \mapsto \theta_n(v)$. This is an important practical, and indeed tractable, case. The required conditions are formally stated below, and cover standard parametric estimators of bounding functions such as least squares, quantile regression, and other estimators.

Condition P (Finite-Dimensional Bounding Function). *We have that (i) $\theta_n(v) := \theta_n(v, \gamma_n)$, where $\mathcal{V} \times \mathcal{G} \mapsto \theta_n(v, \gamma)$ is a known function parameterized by finite-dimensional vector $\gamma \in \mathcal{G}$, where \mathcal{V} is a compact subset of \mathbb{R}^d and \mathcal{G} is a subset of \mathbb{R}^k , where the sets do not depend on n . (ii) The function $(v, \gamma) \mapsto p_n(v, \gamma) := \partial \theta_n(v, \gamma) / \partial \gamma$ is uniformly Lipschitz with Lipschitz coefficient $L_n \leq L$, where L is a finite constant that does not depend on n . (iii) An estimator $\widehat{\gamma}_n$ is available such that*

$$\Omega_n^{-1/2} \sqrt{n}(\widehat{\gamma}_n - \gamma_n) = \mathcal{N}_k + o_{\mathbb{P}_n}(1), \quad \mathcal{N}_k \stackrel{d}{=} N(0, I_k),$$

(iv) $\|p_n(v, \gamma_n)\|$ is bounded away from zero, uniformly in v and n . The eigenvalues of Ω_n are bounded from above and away from zero, uniformly in n . (v) There is also a consistent estimator $\widehat{\Omega}_n$ such that $\|\widehat{\Omega}_n - \Omega_n\| = O_{\mathbb{P}_n}(n^{-b})$ for some constant $b > 0$, independent of n .

Example 1 (A Saturated Model). As a simple, but relevant example we consider the following model. Suppose that v takes on a finite set of values, denoted $1, \dots, k$, so that $\theta_n(v, \gamma) = \sum_{j=1}^k \gamma_j \mathbf{1}(v = j)$. Suppose first that $\mathbb{P}_n = \mathbb{P}$ is fixed, so that $\gamma_n = \gamma_0$, a fixed value.

Condition (ii) and the boundedness requirement of (iv) follow from $\partial\theta_n(v, \gamma)/\partial\gamma_j = 1(v = j)$ for each $j = 1, \dots, k$. Condition (v) applies to many estimators. Then if the estimator $\hat{\gamma}$ satisfies $\Omega^{-1/2}\sqrt{n}(\hat{\gamma} - \gamma_0) \rightarrow_d N(0, I_k)$ where Ω is positive definite, the strong approximation in condition (iii) follows from Skorohod's Theorem and Lemma 9.¹² Suppose next that P_n and the true value $\gamma_n = (\gamma_{n1}, \dots, \gamma_{nk})'$ change with n . Then if

$$\Omega_n^{-1/2}\sqrt{n}(\hat{\gamma}_n - \gamma_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^n u_{i,n} + o_{P_n}(1),$$

with $\{u_{i,n}\}$ i.i.d. with mean zero, for each n , and $E\|u_{i,n}\|^{2+\delta}$ bounded uniformly in n for some $\delta > 0$, then $\Omega_n^{-1/2}\sqrt{n}(\hat{\gamma}_n - \gamma_n) \rightarrow_d N(0, I_k)$, then again condition (iii) follows from Skorohod's theorem and Lemma 9. \blacksquare

Lemma 4 (P and V imply C.1-C.4, S). *Condition P implies Conditions C.1-C.4 and S, where, for $p_n(v, \gamma) := \frac{\partial\theta_n(v, \gamma)}{\partial\gamma}$,*

$$\begin{aligned} Z_n(v) &= \frac{\theta_n(v) - \hat{\theta}_n(v)}{\sigma_n(v)}, \quad Z_n^*(v) = \frac{p_n(v, \gamma_n)' \Omega_n^{1/2}}{\|p_n(v, \gamma_n)' \Omega_n^{1/2}\|} \mathcal{N}_k, \quad Z_n^\star(v) = \frac{p_n(v, \hat{\gamma}_n)' \hat{\Omega}_n^{1/2}}{\|p_n(v, \hat{\gamma}_n)' \hat{\Omega}_n^{1/2}\|} \mathcal{N}_k, \\ \sigma_n(v) &= \|n^{-1/2} p_n(v, \gamma_n)' \Omega_n^{1/2}\|, \quad s_n(v) = \|n^{-1/2} p_n(v, \hat{\gamma}_n)' \hat{\Omega}_n^{1/2}\|, \quad \delta_n = o(1), \\ \bar{a}_n &\lesssim 1, \quad \bar{\sigma}_n \lesssim \sqrt{1/n}, \quad a_n(V) = \left(2\sqrt{\log\{C(1 + C'L_n \text{diam}(V))^d\}} \right) \vee (1 + \sqrt{d}), \end{aligned}$$

for some positive constants C and C' , and $P[\mathcal{E} > x] = \exp(-x/2)$. Furthermore, if also Condition V holds and $c_n^{-1}(\ell_n/\sqrt{n})^{1/\rho_n} = o(1)$, then Condition S holds.

The following is an immediate consequence of Lemma 4 and Theorems 1, 2, and 3.

Theorem 4 (Estimation and Inference with Parametrically Estimated Bounding Functions). *Suppose Condition P holds and consider the interval estimator $\hat{\theta}_{n0}(p)$ given in Definition 1 with simulation-based critical values specified in Definition 3 for the simulation process Z_n^\star specified above. (1) Then (i) $P_n[\theta_{n0} \leq \hat{\theta}_{n0}(p)] \geq p - o(1)$, (ii) $|\theta_{n0} - \hat{\theta}_{n0}(p)| = O_{P_n}(\sqrt{1/n})$, (iii) $P_n(\theta_{n0} + \mu_n\sqrt{1/n} \geq \hat{\theta}_{n0}(p)) \rightarrow 1$ for any $\mu_n \rightarrow_{P_n} \infty$. (2) If Condition V holds with $c_n \geq c > 0$ and $\rho_n \leq \rho < \infty$, then $P_n[\theta_{n0} \leq \hat{\theta}_{n0}(p)] = p + o(1)$.*

The next example generalizes the simple saturated example of Example 1 to a more substantive example. This example also offers a natural means of transition to the next section, which deals with series estimation, which could merely be viewed as parametric estimation with parameters of increasing dimension and vanishing approximation errors.

¹²See Theorem 1.10.3 of van der Vaart and Wellner (1996) on page 58 and the subsequent historical discussion attributing the earliest such results to Skorohod (1956), later generalized by Wichura and Dudley.

Example 2 (Linear Bounding Function). Suppose that $\theta_n(v, \gamma_n) = p_n(v)' \gamma_n$, where $p_n(v)' \gamma : \mathcal{V} \times \mathcal{G} \mapsto \mathbb{R}$. Suppose that (a) $v \mapsto p_n(v)$ is Lipschitz with Lipschitz coefficient $L_n \leq L$, for all n , with the first component equal to 1, (b) there is an estimator available that is asymptotically linear

$$\Omega_n^{-1/2} \sqrt{n}(\hat{\gamma}_n - \gamma_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^n u_{i,n} + o_{P_n}(1),$$

with $\{u_{i,n}\}$ i.i.d. with mean zero, for each n , and $E\|u_{i,n}\|^{2+\delta}$ bounded uniformly in n for some $\delta > 0$, and (c) Ω_n has eigenvalues bounded away from zero and from above. These conditions imply Condition P(i)-(iv). Indeed, (i),(ii), and (iv) hold immediately, while (iii) follows from the Lindeberg-Feller CLT, which implies that under P_n

$$\Omega_n^{-1/2} \sqrt{n}(\hat{\gamma}_n - \gamma_n) \rightarrow_d N(0, I_k),$$

and the strong approximation follows by the Skorohod representation and Lemma 9 by suitably enriching the probability space if needed. Note that if $\theta_n(v, \gamma_n)$ is the conditional expectation of Y_i given $V_i = v$, then $\hat{\gamma}_n$ can be obtained by the mean regression of Y_i on $p_n(V_i)$, $i = 1, \dots, n$; if $\theta_n(v, \gamma_n)$ is the conditional u -quantile of Y_i given $V_i = v$, then $\hat{\gamma}_n$ can be obtained by the u -quantile regression of Y_i on $p_n(V_i)$, $i = 1, \dots, n$. Regularity conditions that imply the ones stated above can be found in e.g. White (1984) and Koenker (2005). Finally estimators of Ω_n depend on the estimator of γ_n ; for mean regression the standard estimator is the Eicker-Huber-White estimator, and for quantile regression the standard estimator is Powell's estimator, see Powell (1984). For brevity we do not restate sufficient conditions for Condition P(v), but these are readily available for common estimators. ■

Example 3 (Conditional Moment Inequalities). This is a generalization of the previous example where now the bounding function is the minimum of J conditional mean functions. Referring to the conditional moment inequality setting specified in Section 2, suppose we have an i.i.d. sample of (X_i, Z_i) , $i = 1, \dots, n$, with $\text{support}(Z_i) = \mathcal{Z} \subseteq [0, 1]^d$. Let $v = (z, j)$, where j denotes the enumeration index for the conditional moment inequality, $j \in \{1, \dots, J\}$, and suppose $\mathcal{V} \subseteq \mathcal{Z} \times \{1, \dots, J\}$. The parameters J and d do not depend on n . Hence

$$\theta_{n0} = \min_{v \in \mathcal{V}} \theta_n(v) = \min_{(z,j) \in \mathcal{V}} \theta_n(z, j).$$

Suppose that $\theta_n(v) = E_{P_n}[m(X, \mu, j)|z] = b(z)' \gamma_n(j)$, for $b : \mathcal{Z} \mapsto \mathbb{R}^m$, denoting some transformation of z , with m independent of n , and where $\gamma_n(j)$ are the population regression coefficients in the regression of $Y(j) := m(X, \mu, j)$ on $b(Z)$, $j = 1, \dots, J$, respectively, under P_n . Suppose that the first $J_0/2$ pairs correspond to moment inequalities generated from

moment equalities so that $\theta_n(j) = -\theta_n(j-1)$, $j = 2, 4, \dots, J_0$, and so these functions are replicas of each other up to sign; also note that $\gamma_n(j) = -\gamma_n(j-1)$, $j = 2, 4, \dots, J_0$. Then we can rewrite

$$\begin{aligned}\theta_n(v) &= E_{P_n}[m(X, \mu, j)|Z = z] = b(z, j)' \gamma_n(j) := p_n(v)' \beta_n, \\ \beta_n &= (\gamma_n(j)', j \in \mathcal{J})', \quad \mathcal{J} := \{2, 4, \dots, J_0, J_0 + 1, J_0 + 2, \dots, J\}',\end{aligned}$$

where β_n is a vector of regression coefficients, and $p_n(v)$ a $K = \dim(\beta_n)$ -vector defined by the relation above, i.e. $p_n(z, j) = [0'_m, \dots, 0'_m, (-1)^{j+1} b'_m(z), 0'_m, \dots, 0'_m]'$ with $b'_m(z)$ appearing in the j -th block for $1 \leq j \leq J_0$; $p_n(z, j) = [0'_m, \dots, 0'_m, b'_m(z), 0'_m, \dots, 0'_m]'$ with $b(z)$ appearing in the j -th block for $J_0 + 1 \leq j \leq J$, where 0_m is an m -dimensional vector of zeroes.¹³ Note that this removal of duplicated regressions is done to simplify the technical arguments; it is not needed in practical implementation, where duplication is allowed.

We impose the following conditions:

- (a) $b(z)$ includes constant 1, (b) $z \mapsto b(z)$ has Lipschitz coefficient bounded above by L , (c) for $Y_i = (Y_i(j), j \in \mathcal{J})'$ and for $\epsilon_i := Y_i - E_{P_n}[Y_i|Z_i]$, the eigenvalues of $E_{P_n}[\epsilon_i \epsilon_i' | Z_i = z]$ are bounded away from zero and from above, uniformly in $z \in \mathcal{Z}$ and n ; (d) $Q = E_{P_n}[b(Z_i) b(Z_i)']$ has eigenvalues bounded away from zero and from above, uniformly in n , and (e) $E_{P_n} \|b(Z_i)\|^4$ and $E_{P_n} \|\epsilon_i\|^4$ are bounded from above uniformly in n .

Then it follows from e.g. by White (1984) that for $\hat{\gamma}_n(j)$ denoting the ordinary least square estimator obtained by regressing $Y_i(j), i = 1, \dots, n$, on $b(Z_i), i = 1, \dots, n$,

$$\sqrt{n}(\hat{\gamma}_n(j) - \gamma_n(j)) = Q^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n b(Z_i) \epsilon_i(j) + o_{P_n}(1), \quad j \in \mathcal{J},$$

so that

$$\sqrt{n}(\hat{\beta}_n - \beta_n) = (I_{|\mathcal{J}|} \otimes Q)^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n \underbrace{(I_{|\mathcal{J}|} \otimes b(Z_i))}_{u_i} \epsilon_i + o_{P_n}(1).$$

By conditions (c) and (d) $E_{P_n}[u_i u_i']$ and Q have eigenvalues bounded away from zero and from above, so the same is true of $\Omega_n = (I_{|\mathcal{J}|} \otimes Q)^{-1} E_{P_n}[u_i u_i'] (I_{|\mathcal{J}|} \otimes Q)^{-1}$. These conditions verify condition P(i),(ii),(iv). Application of the Lindeberg-Feller CLT, Skorohod's theorem, and Lemma 9 verifies Condition P(iii). By the argument given in Chapter VI of White

¹³Note the absence of $\gamma_n(j)$ for odd j up to J_0 in the definition of the coefficient vector β_n . This is required to enable non-singularity of $E_{P_n}[\epsilon_i \epsilon_i' | Z_i = z]$. Imposing non-singularity simplifies the proofs, and is not needed for practical implementation, as the removal of these indices is not required for estimation and inference, and the estimated variance matrix $\hat{\Omega}_n$ can be allowed to be singular.

(1984), Condition P(v) holds for the standard analog estimator for Ω_n :

$$\hat{\Omega}_n = (I_{|\mathcal{J}|} \otimes \hat{Q})^{-1} \mathbb{E}_n[\hat{u}_i \hat{u}_i'] (I_{|\mathcal{J}|} \otimes \hat{Q})^{-1},$$

where $\hat{Q} = \mathbb{E}_n[b(Z_i)b(Z_i)']$ and $\hat{u}_i = (I_{|\mathcal{J}|} \otimes b(Z_i))\hat{\epsilon}_i$, with $\hat{\epsilon}_i(j) = Y_i(j) - b(Z_i)'\hat{\gamma}_n(j)$, and $\hat{\epsilon}_i = (\hat{\epsilon}_i(j), j \in \mathcal{J})'$. \blacksquare

4.2. Nonparametric Estimation of $\theta_n(v)$ via Series. Series estimation is effectively like parametric estimation, but the dimension of the estimated parameter tends to infinity and bias arises due to approximation based on a finite number of basis functions. If we select the number of terms in the series expansion so that the estimation error is of larger magnitude than the approximation error, i.e. if we undersmooth, then the analysis closely mimics the parametric case.

Condition NS. *The function $v \mapsto \theta_n(v)$ is continuous in v . The series estimator $\hat{\theta}_n(v)$ has the form $\hat{\theta}_n(v) = p_n(v)'\hat{\beta}_n$, where $p_n(v) := (p_{n,1}(v), \dots, p_{n,K_n}(v))'$ is a collection of K_n continuous series functions mapping $\mathcal{V} \subset K \subset \mathbb{R}^d$ to \mathbb{R}^{K_n} , and $\hat{\beta}_n$ is a K_n -vector of coefficient estimates, and K is a fixed compact set. Furthermore,*

NS.1 (a) *The estimator satisfies the following linearization and strong approximation condition:*

$$\frac{\hat{\theta}_n(v) - \theta_n(v)}{\|p_n(v)'\Omega_n^{1/2}\|/\sqrt{n}} = \frac{p_n(v)'\Omega_n^{1/2}}{\|p_n(v)'\Omega_n^{1/2}\|} \mathcal{N}_n + R_n(v),$$

where

$$\mathcal{N}_n =_d N(0, I_{K_n}), \quad \sup_{v \in \mathcal{V}} |R_n(v)| = o_{P_n}(1/\log n).$$

(b) *The matrices Ω_n are positive definite, with eigenvalues bounded from above and away from zero, uniformly in n . Moreover, there are sequences of constants ζ_n and ζ'_n such that $1 \leq \zeta'_n \lesssim \|p_n(v)\| \leq \zeta_n$ uniformly for all $v \in \mathcal{V}$ and $\sqrt{\zeta_n^2 \log n/n} \rightarrow 0$, and $\|p_n(v) - p_n(v')\|/\zeta'_n \leq L_n \|v - v'\|$ for all $v, v' \in \mathcal{V}$, where $\log L_n \lesssim \log n$, uniformly in n .*

NS.2 *There exists $\hat{\Omega}_n$ such that $\|\hat{\Omega}_n - \Omega_n\| = O_{P_n}(n^{-b})$, where $b > 0$ is a constant.*

Condition NS is not primitive, but reflects the function-wise large sample normality of series estimators. It requires that the studentized nonparametric process is approximated by a sequence of Gaussian processes, which take a very simple intuitive form, rather than by a fixed single Gaussian process. Indeed, the latter would be impossible in non-parametric settings, since the sequence of Gaussian processes is not asymptotically tight. Note also that the condition implicitly requires that some undersmoothing takes place so that the approximation error is negligible relative to the sampling error. We provide primitive conditions

that imply condition NS.1 in three examples presented below. In particular, we show that the asymptotic linearization for $\widehat{\beta}_n - \beta_n$, which is available from the literature on series regression, e.g. from Andrews (1991) and Newey (1997), and the use of Yurinskii's coupling Yurinskii (1977) imply condition NS.1. This result could be of independent interest, although we only provide sufficient conditions for the strong approximation to hold.

Note that under condition NS, the uniform rate of convergence of $\widehat{\theta}_n(v)$ to $\theta_n(v)$ is given by $\sqrt{\zeta_n^2/n}\sqrt{\log n} \rightarrow 0$, where $\zeta_n \propto \sqrt{K_n}$ for standard series terms such as B-splines or trigonometric series.

Lemma 5 (NS implies C.1-C.4). *Condition NS implies Conditions C.1-C.4 with*

$$\begin{aligned} Z_n(v) &= \frac{\theta_n(v) - \widehat{\theta}_n(v)}{\sigma_n(v)}, \quad Z_n^*(v) = \frac{p_n(v)' \Omega_n^{1/2}}{\|p_n(v)' \Omega_n^{1/2}\|} \mathcal{N}_n, \quad Z_n^*(v) = \frac{p_n(v)' \widehat{\Omega}_n^{1/2}}{\|p_n(v)' \widehat{\Omega}_n^{1/2}\|} \mathcal{N}_n, \\ \sigma_n(v) &= \|n^{-1/2} p_n(v)' \Omega_n^{1/2}\|, \quad s_n(v) = \|n^{-1/2} p_n(v)' \widehat{\Omega}_n^{1/2}\|, \quad \delta_n = 1/\log n, \\ \bar{a}_n &\lesssim \sqrt{\log n}, \quad \bar{\sigma}_n \lesssim \sqrt{\zeta_n^2/n}, \quad a_n(V) = \left(2\sqrt{\log\{C(1 + C' L_n \text{diam}(V))^d\}} \right) \vee (1 + \sqrt{d}), \end{aligned}$$

for some constants C and C' , where $\text{diam}(V)$ denotes the diameter of the set V , and $P[\mathcal{E} > x] = \exp(-x/2)$.

Remark 2. Lemma 5 verifies the main conditions C.1-C.4. These conditions enable construction of simulated or analytical critical values. For the latter, the p -th quantile of \mathcal{E} is given by $c(p) = -2 \log(1 - p)$, so we can set

$$k_{n,V}(p) = a_n(V) - 2 \log(1 - p)/a_n(V), \quad (4.1)$$

where

$$a_n(V) = \left(2\sqrt{\log\{\ell_n (1 + \ell_n L_n \text{diam}(V))^d\}} \right), \quad (4.2)$$

is a feasible scaling factor which bounds the scaling factor in the statement of Lemma 5, at least for all large n . Here, all unknown constants have been replaced by slowly growing numbers ℓ_n such that $\ell_n > C \vee C'$ for all large n . Note also that $V \mapsto k_{n,V}(p)$ is monotone in V for all sufficiently large n , as required in the analytical construction given in Definition 2. A sharper analytical approach can be based on Hotelling's tube method; for details we refer to Chernozhukov, Lee, and Rosen (2009). That approach is tractable for the case of $d = 1$ but does not immediately extend to $d > 1$. Note that the simulation-based approach is effectively a numeric version of the exact version of the tube formula, and is less conservative than using simplified tube formulas. ■

Lemma 6 (Condition NS implies S in some cases). *Suppose Condition NS holds. Then, (1) The radius φ_n of equicontinuity of Z_n^* obeys:*

$$\varphi_n \leq o(1) \cdot \left(\frac{1}{L_n \sqrt{\log n}} \right),$$

for any $o(1)$ term. (2) If Condition V holds and

$$\left(\sqrt{\frac{\zeta_n^2}{n} \log n} \right)^{1/\rho_n} c_n^{-1} = o\left(\frac{1}{L_n \sqrt{\log n}} \right), \quad (4.3)$$

then Condition S holds. (3) If V_0 is singleton and (4.3) holds, $\rho_n \leq 2$, and $c_n \geq c > 0$, for all n , $\zeta_n \lesssim \sqrt{K_n}$ and $L_n \lesssim K_n$, we have $a_n(V_0) \propto 1$ and this condition reduces to

$$K_n^5 \log^3 n/n \rightarrow 0.$$

The following is an immediate consequence of Lemmas 5 and 6 and Theorems 1, 2, and 3.

Theorem 5 (Estimation and Inference with Series-Estimated Bounding Functions). *Suppose Condition NS holds and consider the interval estimator $\hat{\theta}_{n0}(p)$ given in Definition 1 with either analytical critical value $c(p) = -2 \log(1 - p)$, or simulation-based critical values from Definition 3 for the simulation process Z_n^* above. (1) Then (i) $\mathbb{P}_n[\theta_{n0} \leq \hat{\theta}_{n0}(p)] \geq p - o(1)$, (ii) $|\theta_{n0} - \hat{\theta}_{n0}(p)| = O_{\mathbb{P}_n}(\sqrt{\log n} \sqrt{\zeta_n^2/n})$, (iii) $\mathbb{P}_n(\theta_{n0} + \mu_n \sqrt{\log n} \sqrt{\zeta_n^2/n} \geq \hat{\theta}_{n0}(p)) \rightarrow 1$ for any $\mu_n \rightarrow_{\mathbb{P}_n} \infty$. (2) Moreover, for the simulation-based critical values, if Condition V and relation (4.3) hold, then (i) $\mathbb{P}_n[\theta_{n0} \leq \hat{\theta}_{n0}(p)] = p - o(1)$, (ii) $|\theta_{n0} - \hat{\theta}_{n0}(p)| = O_{\mathbb{P}_n}(\sqrt{\zeta_n^2/n})$, (iii) $\mathbb{P}_n(\theta_{n0} + \mu_n \sqrt{\zeta_n^2/n} \geq \hat{\theta}_{n0}(p)) \rightarrow 1$ for any $\mu_n \rightarrow_{\mathbb{P}_n} \infty$.*

We next present some examples with primitive conditions that imply Condition NS.

Example 4 (Bounding Function is Conditional Quantile). Suppose that $\theta_n(v) := Q_{Y_i|V_i}[\tau|v]$ is the τ -th conditional quantile of Y_i given V_i under \mathbb{P}_n , assumed to be a continuous function in v . Suppose we estimate $\theta_n(v)$ with a series estimator. There is an i.i.d. sample $(Y_i, V_i), i = 1, \dots, n$, with $\text{support}(V_i) \subseteq [0, 1]^d$ for each n , defined on a probability space equipped with probability measure \mathbb{P}_n . Suppose that the intersection region of interest is $\mathcal{V} \subseteq \text{support}(V_i)$. Here the index d does not depend on n , but all other parameters, unless stated otherwise, can depend on n . Then $\theta_n(v) = p_n(v)' \beta_n + a_n(v)$, where $p_n : [0, 1]^d \mapsto \mathbb{R}^{K_n}$ are the series functions, β_n is the quantile regression coefficient in the population, $a_n(v)$ is the approximation error, and K_n is the number of series terms that depends on n . Let C be a positive constant.

We impose the following technical conditions to verify NS.1 and NS.2:

Uniformly in n , (i) p_n are either b-splines of a fixed order or trigonometric series terms or any other terms $p_n = (p_{n1}, \dots, p_{nK_n})$ with $\|p_n(v)\| \lesssim \zeta_n = \sqrt{K_n}$ and $\max_{1 \leq l \leq K_n} |p_{nl}(v)| \leq C$ for all $v \in \text{support}(V_i)$, $\|p_n(v)\| \gtrsim \zeta'_n \geq 1$ for all $v \in \mathcal{V}$, and $\log \text{lip}(p_n) \lesssim \log K_n$, (ii) the mapping $v \mapsto \theta_n(v)$ is sufficiently smooth, namely $\sup_{v \in \mathcal{V}} |a_n(v)| \lesssim K_n^{-s}$, for some $s > 0$, (iii) $\lim_{n \rightarrow \infty} (\log n)^c K_n^{-s+1} = 0$ and $\lim_{n \rightarrow \infty} (\log n)^c \sqrt{n} K_n / \zeta'_n = 0$, for each $c > 0$, (iv) eigenvalues of $Q_n = E_{P_n}[p_n(V_i)p_n(V_i)']$ are bounded away from zero and from above, (v) $f_{Y_i|V_i}(\theta_n(v)|v)$ is bounded uniformly over $v \in \mathcal{V}$ away from zero and from above, (vi) $\lim_{n \rightarrow \infty} K_n^5 (\log n)^c / n = 0$ for each $c > 0$, and (vii) the restriction on the bandwidth sequence in Powell's estimator \hat{J}_n of $J_n = E_{P_n}[f_{Y_i|V_i}(\theta_n(V_i)|V_i)p_n(V_i)p_n(V_i)']$ specified in Belloni, Chernozhukov, and Fernandez-Val (2011) holds.

Suppose that we use the standard quantile regression estimator

$$\hat{\beta}_n = \arg \min_{b \in \mathbb{R}^{K_n}} \mathbb{E}_n[\rho_\tau(Y_i - p_n(V_i)'b)],$$

so that $\hat{\theta}_n(v) = p_n(v)' \hat{\beta}$ for $\rho_\tau(u) = (\tau - 1(u < 0))u$. Then by Belloni, Chernozhukov, and Fernandez-Val (2011), under conditions (i)-(vi), the following asymptotically linear representation holds:

$$\sqrt{n}(\hat{\beta}_n - \beta_n) = J_n^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n \underbrace{p_n(Z_i)}_{u_i} \epsilon_i + o_{P_n} \left(\frac{1}{\log n} \right),$$

for $\epsilon_i = (\tau - 1(w_i \leq \tau))$, where $(w_i, i = 1, \dots, n)$ are i.i.d. uniform, independent of $(V_i, i = 1, \dots, n)$, and $J_n = E_{P_n}[f_{Y_i|V_i}(\theta_n(V_i)|V_i)p_n(V_i)p_n(V_i)']$. Note that by conditions (iv) and (v) $E_{P_n}[u_i u_i'] = \tau(1 - \tau)Q_n$, for $Q_n = E_{P_n}[p_n(V_i)p_n(V_i)']$, and J_n have eigenvalues bounded away from zero and from above uniformly in n , and so the same is also true of $\Omega_n = \tau(1 - \tau)J_n^{-1}Q_nJ_n^{-1}$. Given other restrictions imposed in condition (i), Condition NS.1(b) is verified. Next using condition (iv) and boundedness of $\max_{1 \leq l \leq K_n} \sup_{v \in \mathcal{V}} |p_{nl}(v)|$ under condition (i) we verify the strong approximation required in NS.1(a) by invoking Theorem 7, namely its Corollary 2 stated in Appendix E. The latter results are based on Yurinskii's coupling. To verify Condition NS.2, consider the plug-in estimator $\hat{\Omega}_n = \hat{J}_n^{-1} \hat{Q}_n \hat{J}_n^{-1}$, where \hat{J}_n is the Powell's estimator for J_n , and $\hat{Q}_n = \mathbb{E}_n[p_n(V_i)p_n(V_i)']$. Then by Belloni, Chernozhukov, and Fernandez-Val (2011) under condition (vii) $\|\hat{\Omega}_n - \Omega_n\| = O_{P_n}(1/\log n)$.

■

Example 5 (Bounding Function is Conditional Mean). Now suppose that $\theta_n(v) = E_{P_n}[Y_i|V_i = v]$, assumed to be a continuous function with respect to $v \in \text{support}(V_i)$, and

the intersection region is $\mathcal{V} \subseteq \text{support}(V_i)$. Suppose we are using the series approach to approximating and estimating $\theta_n(v)$. There is an i.i.d. sample $(Y_i, V_i), i = 1, \dots, n$, with $\text{support}(V_i) \subseteq [0, 1]^d$ for each n . Here d does not depend on n , but all other parameters, unless stated otherwise, can depend on n . Then we have $\theta_n(v) = p_n(v)' \beta_n + a_n(v)$, for $p_n : [0, 1]^d \mapsto \mathbb{R}^{K_n}$ representing the series functions; β_n is the coefficient of the best least squares approximation to $\theta_n(v)$ in the population, and $a_n(v)$ is the approximation error. The number of series terms K_n depends on n .

We impose the following technical conditions:

Uniformly in n , (i) p_n are either b-splines of a fixed order or trigonometric series terms or any other series terms $p_n = (p_{n1}, \dots, p_{nK_n})$ with $\|p_n(v)\| \lesssim \zeta_n = \sqrt{K_n}$ and $\max_{1 \leq l \leq K_n} |p_{nl}(v)| \leq C$ for all $v \in \text{support}(V_i)$, $\|p_n(v)\| \gtrsim \zeta'_n \geq 1$ for all $v \in \mathcal{V}$, and $\log \text{lip}(p_n) \lesssim \log K_n$, (ii) the mapping $v \mapsto \theta_n(v)$ is sufficiently smooth, namely $\sup_{v \in \mathcal{V}} |a_n(v)| \lesssim K_n^{-s}$, for some $s > 0$, (iii) $\lim_{n \rightarrow \infty} (\log n)^c \sqrt{n} K_n^{-s} = 0$ for each $c > 0$,¹⁴ (iv) for $\epsilon_i = Y_i - E_{P_n}[Y_i|V_i]$, $E_{P_n}[\epsilon_i^2|V_i = v]$ is bounded away from zero uniformly in $v \in \text{support}(V_i)$, and (v) eigenvalues of $Q_n = E_{P_n}[p_n(V_i)p_n(V_i)']$ are bounded away from zero and from above, and (vi) $E_{P_n}[|\epsilon_i|^4|V_i = v]$ is bounded from above uniformly in $v \in \text{support}(V_i)$, (vii) $\lim_{n \rightarrow \infty} (\log n)^c K_n^5/n = 0$ for each $c > 0$.

We use the standard least squares estimator

$$\hat{\beta}_n = \mathbb{E}_n[p_n(V_i)p_n(V_i)']^{-1} \mathbb{E}_n[p_n(V_i)Y_i],$$

so that $\hat{\theta}_n(v) = p_n(v)' \hat{\beta}_n$. Then by Newey (1997), under conditions implied by (i)-(vii), we have the following asymptotically linear representation:

$$\sqrt{n}(\hat{\beta}_n - \beta_n) = Q_n^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n \underbrace{p_n(Z_i)\epsilon_i}_{u_i} + o_{P_n}(1/\log n).$$

For details, see Supplementary Appendix H. Note that $E_{P_n}(u_i u_i')$ and Q_n have eigenvalues bounded away from zero and from above uniformly in n , and so the same is also true of $\Omega_n = Q_n^{-1} E_{P_n}(u_i u_i') Q_n^{-1}$. Thus, under condition (i), Condition NS.1(a) is verified. Next under condition (vi) and since $\max_{1 \leq j \leq K_n} \sup_v |p_{nj}(v)|$ is bounded by condition (i), the strong approximation condition NS.1(a) now follows from invoking Theorem 7 in Appendix E. Finally, Newey (1997) verifies that NS.2 holds for the standard analog estimator $\hat{\Omega}_n =$

¹⁴This condition, which is based on Newey (1997) can be relaxed to $(\log n)^c K_n^{-s+1} \rightarrow 0$ and $(\log n)^c \sqrt{n} K_n^{-s} / \zeta'_n \rightarrow 0$, using the recent results of Belloni, Chen, and Chernozhukov (2010) for least squares series estimators.

$\hat{Q}_n^{-1} \mathbb{E}_n(\hat{u}_i \hat{u}_i') \hat{Q}_n^{-1}$ for $\hat{u}_i = p_n(V_i)(Y_i - \hat{\theta}_n(V_i))$ and $\hat{Q}_n = \mathbb{E}_n[p_n(V_i)p_n(V_i)']$ under conditions that are implied by those above.

Finally, note that if we had $\epsilon_i \sim N(0, \sigma^2(V_i))$, conditional on V_i , we could establish Condition NS.1 with a much weaker growth restriction than (vii). Thus, while our use of Yurinskii's coupling provides concrete sufficient conditions for strong approximation, the function-wise large sample normality is likely to hold even under weaker conditions in many situations. \blacksquare

Example 6 (Bounding Function from Conditional Moment Inequalities). Consider now Example C of Section 2, which is in fact a slight generalization of the previous example, where now the bounding function is the minimum of J conditional mean functions. Suppose we have an i.i.d. sample of $(X_i, Z_i), i = 1, \dots, n$, with $\text{support}(Z_i) = \mathcal{Z} \subseteq [0, 1]^d$, defined on a probability space equipped with probability measure P_n . Let $v = (z, j)$, where j denotes the enumeration index for the conditional moment inequality, $j \in \{1, \dots, J\}$, and $\mathcal{V} \subseteq \mathcal{Z} \times \{1, \dots, J\}$. The parameters J and d do not depend on n . Hence

$$\theta_{n0} = \min_{v \in \mathcal{V}} \theta_n(v),$$

for $\theta_n(v) = E_{P_n}[m(X_i, \mu, j) | Z_i = z]$, assumed to be a continuous function with respect to $z \in \mathcal{Z}$. Suppose that we use the series approach to approximate and estimate $\theta_n(z, j)$ for each j . Then $E_{P_n}[m(x, \mu, j) | z] = b_n(z)' \gamma_n(j) + a_n(z, j)$, for $b_n : [0, 1]^d \mapsto \mathbb{R}^{m_n}$ denoting a m_n -vector of series functions; $\gamma_n(j)$ is the coefficient of the best least squares approximation to $E_{P_n}[m(x, \mu, j) | z]$ in the population, and $a_n(z, j)$ is the approximation error. Let \mathcal{J} be a subset of $\{1, \dots, J\}$ as defined as in the parametric Example 3 (to handle inequalities associated with equalities).

We impose the following conditions:

Uniformly in n , (i) b_n are either b-splines of a fixed order or trigonometric series terms or any other terms $b_n = (b_{n1}, \dots, b_{nK_n})$ with $\|b_n(v)\| \lesssim \zeta_n = \sqrt{K}$ and $\max_{1 \leq l \leq K_n} |b_{nl}(v)| \leq C$ for all $v \in \text{support}(V_i)$, $\|b_n(v)\| \gtrsim \zeta'_n \geq 1$ for all $v \in \mathcal{V}$, and $\log \text{lip } p_n \lesssim \log K_n$; (ii) the mapping $z \mapsto \theta_n(z, j)$ is sufficiently smooth, namely $\sup_{z \in \mathcal{Z}} |a_n(z, j)| \lesssim m_n^{-s}$, for some $s > 0$, for all $j \in \mathcal{J}$; (iii) $\lim_{n \rightarrow \infty} (\log n)^c \sqrt{nm_n^{-s}} = 0$ for each $c > 0$; ¹⁵ (iv) for $Y(j) := m(x, \mu, j)$ and $Y_i := (Y_i(j), j \in \mathcal{J})'$ and $U_i := Y_i - E_{P_n}[Y_i | Z_i]$, the eigenvalues of $E_{P_n}[\epsilon_i \epsilon_i' | Z_i = z]$ are bounded away from zero, uniformly in $z \in \mathcal{Z}$; (v) eigenvalues of $Q_n = E_{P_n}[b_n(Z_i)b_n(Z_i)']$ are bounded away from zero and

¹⁵See the previous footnote on a possible relaxation of this condition.

from above; (vi) $E_{P_n}[\|\epsilon_i\|^4 \mid Z_i = z]$ is bounded above, uniformly in $z \in \mathcal{Z}$; and (vii) $\lim_{n \rightarrow \infty} m_n^5 (\log n)^c / n = 0$ for each $c > 0$.

The above construction implies $\theta_n(v) = b_n(z)' \gamma_n(j) + a_n(z, j) =: p_n(v)' \beta_n + a_n(v)$, for $\beta_n = (\gamma_n'(j), j \in \mathcal{J})'$. Consider the standard least squares estimator $\hat{\beta}_n = (\hat{\gamma}_n'(j), j \in \mathcal{J})'$ consisting of $|\mathcal{J}|$ least square estimators, where $\hat{\gamma}_n(j) = \mathbb{E}_n[b_n(Z_i) b_n(Z_i)']^{-1} \mathbb{E}_n[b_n(Z_i) Y_i(j)]$. Then it follows from Newey (1997) that for $Q_n = E_{P_n}[b_n(Z_i) b_n(Z_i)']^{-1}$

$$\sqrt{n}(\hat{\gamma}_n(j) - \gamma_n(j)) = \frac{1}{\sqrt{n}} \sum_{i=1}^n Q_n^{-1} b_n(Z_i) \epsilon_i(j) + o_{P_n}(1/\log n), \quad j \in \mathcal{J},$$

so that

$$\sqrt{n}(\hat{\beta}_n - \beta_n) = (I_{|\mathcal{J}|} \otimes Q_n)^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n \underbrace{(I_{|\mathcal{J}|} \otimes b_n(Z_i))}_{u_i} \epsilon_i + o_{P_n}(1/\log n).$$

By conditions (iv), (v), and (vi) $E_{P_n}[u_i u_i']$ and Q_n have eigenvalues bounded away from zero and from above, so the same is true of $\Omega_n = (I_{|\mathcal{J}|} \otimes Q_n)^{-1} E_{P_n}[u_i u_i'] (I_{|\mathcal{J}|} \otimes Q_n)^{-1}$. This and condition (i) imply that Condition NS.1(b) holds. Application of Theorem 7, based on Yurinskii's coupling, verifies Condition NS.1(a). Finally, Condition NS.2 holds for the standard plug-in estimator for Ω_n , by the same argument as given in the proof of Theorem 2 of Newey (1997). ■

4.3. Nonparametric Estimation of $\theta(v)$ via local methods. In this section we provide conditions under which kernel-type estimators satisfy Conditions C.1-C.4. These conditions cover both standard kernel estimators as well as local polynomial estimators.

Condition NK. Let $v = (z, j)$ and $\mathcal{V} \subseteq \mathcal{Z} \times \{1, \dots, J\}$, where \mathcal{Z} is a compact convex set that does not depend on n . The estimator $v \mapsto \hat{\theta}_n(v)$ and the function $v \mapsto \theta_n(v)$ are continuous in v . In what follows, let e_j denote the J -vector with j th element one and all other elements zero. Suppose that (U, Z) is a $(J + d)$ -dimensional random vector, where U is a generalized residual such that $E[U|Z] = 0$ a.s. and Z is a covariate; the density f_n of Z is continuous and bounded away from zero and from above on \mathcal{Z} , uniformly in n ; and the support of U is bounded uniformly in n . \mathbf{K} is a twice continuously differentiable, possibly higher-order, product kernel function with support on $[-1, 1]^d$, $\int \mathbf{K}(u) du = 1$; and h_n is a sequence of bandwidths such that $h_n \rightarrow 0$ and $nh_n^d \rightarrow \infty$ at a polynomial rate in n .

NK.1 We have that uniformly in $v \in \mathcal{V}$,

$$(nh_n^d)^{1/2}(\hat{\theta}_n(v) - \theta_n(v)) = \mathbb{B}_n(g_v) + o_{P_n}(\delta_n), \quad g_v(U, Z) := \frac{e_j' U}{(h_n^d)^{1/2} f_n(z)} \mathbf{K}\left(\frac{z - Z}{h_n}\right),$$

where \mathbb{B}_n is a P_n -Brownian bridge such that $v \mapsto \mathbb{B}_n(g_v)$ has continuous paths over \mathcal{V} . Moreover, the latter process can be approximated via the Gaussian multiplier method, namely there exists sequences $o(\delta_n)$ and $o(1/\ell_n)$ such that

$$P_n \left(\sup_{v \in \mathcal{V}} |\mathbb{G}_n^o(g_v) - \bar{\mathbb{B}}_n(g_v)| > o(\delta_n) \middle| \mathcal{D}_n \right) = o_{P_n}(1/\ell_n),$$

for some independent copy $v \mapsto \bar{\mathbb{B}}_n(g_v)$ of the process $v \mapsto \mathbb{B}_n(g_v)$. Here, $\mathbb{G}_n^o(g_v) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \eta_i g_v(U_i, Z_i)$, where η_i are i.i.d. $N(0, 1)$, independent of the data \mathcal{D}_n and of $\{(U_i, Z_i)\}_{i=1}^n$, which are i.i.d. copies of (U, Z) . Covariates $\{Z_i\}_{i=1}^n$ are part of the data.

NK.2 There exists an estimator $z \mapsto \hat{f}_n(z)$, having continuous sample paths, such that $\sup_{z \in \mathcal{Z}} |\hat{f}_n(z) - f_n(z)| = O_{P_n}(n^{-b})$, and there are estimators \hat{U}_i of generalized residuals such that $\max_{1 \leq i \leq n} \|\hat{U}_i - U_i\| = O_{P_n}(n^{-\tilde{b}})$ for some constants $b > 0$ and $\tilde{b} > 0$.

Condition NK.1 is a high-level condition that captures the large sample Gaussianity of the entire estimated function where estimation is done via a kernel or local method. Under some mild regularity conditions, specifically those stated in Appendix F, NK.1 follows from the Rio-Massart coupling and from the Bahadur expansion holding uniformly in $v \in \mathcal{V}$:

$$(nh_n^d)^{1/2}(\hat{\theta}_n(v) - \theta_n(v)) = \mathbb{G}_n(g_v) + o_{P_n}(\delta_n).$$

Uniform Bahadur expansions have been established for a variety of local estimators, see e.g. Masry (1996) and Kong, Linton, and Xia (2010), including higher-order kernel and local polynomial estimators. It is possible to use more primitive sufficient conditions stated in the Appendix F based on the Rio-Massart coupling, but these conditions are merely sufficient and other primitive conditions may also be adequate. Our general argument, however, relies only on validity of Condition NK.1.

For simulation purposes, we define

$$\begin{aligned} \mathbb{G}_n^o(\hat{g}_v) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \eta_i \hat{g}_v(U_i, Z_i), \quad \eta_i \text{ i.i.d. } N(0, 1), \text{ independent of the data } \mathcal{D}_n, \\ \hat{g}_v(U_i, Z_i) &= \frac{e_j' \hat{U}_i}{(h_n^d)^{1/2} \hat{f}_n(z)} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right). \end{aligned}$$

Lemma 7 (Condition NK implies C.1-C.4). *Condition NK implies C.1-C.4 with $v = (z, j) \in \mathcal{V} \subseteq \mathcal{Z} \times \{1, \dots, J\}$,*

$$\begin{aligned} Z_n(v) &= \frac{\theta_n(v) - \hat{\theta}_n(v)}{\sigma_n(v)}, \quad Z_n^*(v) = \frac{\mathbb{B}_n(g_v)}{\sqrt{E_{\mathbb{P}_n}[g_v^2]}}, \quad Z_n^\star(v) = \frac{\mathbb{G}_n^\circ(\hat{g}_v)}{\sqrt{\mathbb{E}_n[\hat{g}_v^2]}}, \\ \sigma_n^2(v) &= E_{\mathbb{P}_n}[g_v^2]/(nh_n^d), \quad s_n^2(v) = \mathbb{E}_n[\hat{g}_v^2]/(nh_n^d), \quad \delta_n = 1/\log n, \\ \bar{a}_n &\lesssim \sqrt{\log n}, \quad \bar{\sigma}_n \lesssim \sqrt{1/(nh^d)}, \quad \text{and} \\ a_n(V) &= \left(2\sqrt{\log\{C(1 + C'(1 + h_n^{-1})\text{diam}(V))^d\}} \right) \vee (1 + \sqrt{d}), \end{aligned}$$

for some constants C and C' , where $\text{diam}(V)$ denotes the diameter of the set V . Moreover, $P[\mathcal{E} > x] = \exp(-x/2)$.

Remark 3. Lemma 7 verifies the main conditions C.1-C.4. These conditions enable construction of either simulated or analytical critical values. For the latter, the p -th quantile of \mathcal{E} is given by $c(p) = -2\log(1 - p)$, so we can set

$$k_{n,V}(p) = a_n(V) - 2\log(1 - p)/a_n(V), \quad (4.4)$$

where

$$a_n(V) = \left(2\sqrt{\log\{\ell_n (1 + \ell_n(1 + h_n^{-1})\text{diam}(V))^d\}} \right), \quad (4.5)$$

is a feasible version of the scaling factor, in which unknown constants have been replaced by the slowly growing sequence ℓ_n . Note that $V \mapsto k_{n,V}(p)$ is monotone in V for large n , as required in the analytical construction given in Definition 2. A sharper analytical approach can be based on Hotelling's tube method or on the use of extreme value theory. For details of the extreme value approach, we refer the reader to Chernozhukov, Lee, and Rosen (2009). Note that the simulation-based approach is effectively a numeric version of the exact version of the tube formula, and is less conservative than using simplified tube formulas. In Chernozhukov, Lee, and Rosen (2009) we established that inference based on the extreme value theory achieves the correct asymptotic size, but the asymptotic approximation is accurate only when sets V are "large", and does not seem to provide an accurate approximation when V is small. Moreover, it often requires a very large sample size for accuracy even in the case where V is large. \blacksquare

Lemma 8 (Condition NK implies S in some cases). *Suppose Condition NK holds. Then (1) The radius φ_n of equicontinuity of Z_n^* obeys:*

$$\varphi_n \leq o(1) \cdot \left(\frac{h_n}{\sqrt{\log n}} \right),$$

for any $o(1)$ term. (2) If Condition V holds and

$$\left(\sqrt{\frac{\log n}{nh^d} \log n}\right)^{1/\rho_n} c_n^{-1} = o\left(\frac{h_n}{\sqrt{\log n}}\right), \quad (4.6)$$

then Condition S holds.

The following is an immediate consequence of Lemmas 7 and 8 and Theorems 1, 2, and 3.

Theorem 6 (Estimation and Inference for Bounding Functions Using Local Methods). *Suppose Condition NK holds and consider the interval estimator $\widehat{\theta}_{n0}(p)$ given in Definition 1 with either analytical critical values specified in Remark 3 or simulation-based critical values given in Definition 3 for the simulation process Z_n^* specified above. (1) Then (i) $\mathbb{P}_n[\theta_{n0} \leq \widehat{\theta}_{n0}(p)] \geq p - o(1)$, (ii) $|\theta_{n0} - \widehat{\theta}_{n0}(p)| = O_{\mathbb{P}_n}\left(\sqrt{\log n/(nh_n^d)}\right)$, (iii) $\mathbb{P}_n(\theta_{n0} + \mu_n \sqrt{\log n/(nh_n^d)} \geq \widehat{\theta}_{n0}(p)) \rightarrow 1$ for any $\mu_n \rightarrow_{\mathbb{P}_n} \infty$. (2) Moreover, for simulation-based critical values, if condition V and (4.6) hold, then (i) $\mathbb{P}_n[\theta_{n0} \leq \widehat{\theta}_{n0}(p)] = p - o(1)$, (ii) $|\theta_{n0} - \widehat{\theta}_{n0}(p)| = O_{\mathbb{P}_n}(\sqrt{1/(nh_n^d)})$, (iii) $\mathbb{P}_n(\theta_{n0} + \mu_n \sqrt{1/(nh_n^d)} \geq \widehat{\theta}_{n0}(p)) \rightarrow 1$ for any $\mu_n \rightarrow_{\mathbb{P}_n} \infty$.*

We next present a leading example in which Condition NK holds under primitive conditions. We provide only one example for brevity, but more examples can be covered as in Section 4.2.

Example 7 (Bounding Function from Conditional Moment Inequalities). Suppose that we have an i.i.d. sample of $(X_i, Z_i), i = 1, \dots, n$ defined on the probability space $(A, \mathcal{A}, \mathbb{P})$, where we take \mathbb{P} fixed in this example. Suppose that $\text{support}(Z_i) = \mathcal{Z} \subseteq [0, 1]^d$, and

$$\theta_{n0} = \min_{v \in \mathcal{V}} \theta_n(v),$$

for $\theta_n(v) = E_{\mathbb{P}}[m(X_i, \mu, j) | Z_i = z]$, $v = (z, j)$, where $\mathcal{V} \subseteq \mathcal{Z} \times \{1, \dots, J\}$ be the set of interest. Suppose the first J_0 functions correspond to equalities treated as inequalities, so that $m(X_i, \mu, j) = -m(X_i, \mu, j + 1)$, for $j \in \mathcal{J}_0 = \{1, 3, \dots, J_0 - 1\}$. Hence $\theta_n(z, j) = -\theta_n(z, j + 1)$ for $j \in \mathcal{J}_0$, and we only need to estimate functions $\theta_n(z, j)$ with the index $j \in \mathcal{J} := \mathcal{J}_0 \cup \{J_0 + 1, J_0 + 2, \dots, J\}$. Suppose we use the local polynomial approach to approximating and estimating $\theta_n(z, j)$. For $u \equiv (u_1, \dots, u_d)$, a d -dimensional vector of nonnegative integers, let $[u] = u_1 + \dots + u_d$. Let A_p be the set of all d -dimensional vectors u such that $[u] \leq p$ for some integer $p \geq 0$ and let $|A_p|$ denote the number of elements in

A_p . For $z \in \mathbb{R}^d$ with $u \in A_p$, let $z^u = \prod_{i=1}^d z_i^{u_i}$. Now define

$$\mathbf{p}(b, z) = \sum_{u \in A_p} b_u z^u, \quad (4.7)$$

where $b = (b_u)_{u \in A_p}$ is a vector of dimension $|A_p|$. For each $v = (z, j)$ and $Y_i(j) := m(X_i, \mu, j)$, define

$$S_n(b) := \sum_{i=1}^n \left[Y_i(j) - \mathbf{p} \left(b, \frac{Z_i - z}{h_n} \right) \right]^2 K_{h_n}(Z_i - z),$$

where $K_h(u) := K(u/h)$, $K(\cdot)$ is a d -dimensional kernel function and h_n is a sequence of bandwidths. The local polynomial estimator $\hat{\theta}_n(v)$ of the regression function is the first element of $\hat{b}(z, j) := \arg \min_{b \in \mathbb{R}^{|A_p|}} S_n(b)$.

We impose the following conditions:

- (i) for each $j \in \mathcal{J}$, $\theta(z, j)$ is $(p+1)$ times continuously differentiable with respect to $z \in \mathcal{Z}$, where \mathcal{Z} is convex. (ii) the probability density function f of Z_i is bounded above and bounded below from zero with continuous derivatives on \mathcal{Z} ; (iii) for $Y_i(j) := m(X_i, \mu, j)$, $Y_i := (Y_i(j), j \in \mathcal{J})'$, and $U_i := Y_i - E_{\mathbb{P}}[Y_i|Z_i]$; and U_i is a bounded random vector; (iv) for each j , the conditional on Z_i density of U_i exists and is uniformly bounded from above and below, or, more generally, condition R stated in Appendix F holds; (v) $K(\cdot)$ has support on $[-1, 1]^d$, is twice continuously differentiable, $\int u K(u) du = 0$, and $\int K(u) du = 1$; (vi) $h_n \rightarrow 0$, $nh_n^{d+|\mathcal{J}|+1} \rightarrow \infty$, and $nh_n^{d+2(p+1)} \rightarrow 0$ at polynomial rates in n .

These conditions are imposed to verify Assumptions A1-A7 in Kong, Linton, and Xia (2010). Details of verification are given in Supplementary Appendix I. Note that $p > |\mathcal{J}|/2 - 1$ is necessary to satisfy bandwidth conditions in (vi). Conditions (i)-(vi) above are sufficient conditions to check Assumptions A1-A7 in Kong, Linton, and Xia (2010). The assumption that U_i is bounded is technical and is made to simplify exposition and proofs.

Let $\delta_n = 1/\log n$. Then it follows from Corollary 1 and Lemmas 8 and 10 of Kong, Linton, and Xia (2010) that

$$\hat{\theta}_n(z, j) - \theta(z, j) = \frac{1}{nh_n^d f(z)} \mathbf{e}'_1 S_p^{-1} \sum_{i=1}^n (e'_j U_i) K_h(Z_i - z) \mathbf{u}_p \left(\frac{Z_i - z}{h_n} \right) + B_n(z, j) + R_n(z, j), \quad (4.8)$$

where \mathbf{e}_1 is an $|A_p| \times 1$ vector whose first element is one and all others are zeros, S_p is an $|A_p| \times |A_p|$ matrix such that $S_p = \{ \int z^u (z^v)' du : u \in A_p, v \in A_p \}$, $\mathbf{u}_p(z)$ is an $|A_p| \times 1$ vector

such that $\mathbf{u}_p(z) = \{z^u : u \in A_p\}$,

$$B_n(z, j) = O(h_n^{p+1}) \text{ and } R_n(z, j) = o_P\left(\frac{\delta_n}{(nh_n^d)^{1/2}}\right),$$

uniformly in $(z, j) \in \mathcal{Z} \times \{1, \dots, J\}$. The exact form of $B_n(z, j)$ is given in equation (12) of Kong, Linton, and Xia (2010). The result that $B_n(z, j) = O(h_n^{p+1})$ uniformly in (z, j) follows from the standard argument based on Taylor expansion given in Fan and Gijbels (1996), Kong, Linton, and Xia (2010), or Masry (1996). The condition that $nh_n^{d+2(p+1)} \rightarrow 0$ at a polynomial rate in n corresponds to the undersmoothing condition.

Now set $\mathbf{K}(z/h) \equiv \mathbf{e}'_1 S_p^{-1} K_h(z) \mathbf{u}_p(z/h)$, which is a kernel of order $(p+1)$ (See section 3.2.2 of Fan and Gijbels (1996)). Let

$$g_v(U, Z) := \frac{e'_j U}{(h_n^d)^{1/2} f(z)} \mathbf{K}\left(\frac{Z-z}{h_n}\right).$$

Then it follows from (I.1) that uniformly in $v \in \mathcal{V}$

$$(nh_n^d)^{1/2}(\hat{\theta}_n(z, j) - \theta_n(z, j)) = \mathbb{G}_n(g_v) + o_P(\delta_n).$$

Application of Theorems 8 and 9 in Appendix F, based on the Rio-Massart coupling, verifies condition NK.1 (a) and NK.1 (b). Finally, Condition NK.2 holds if we take $\hat{f}_n(z)$ to be the standard kernel density estimator with kernel K and let $e'_j \hat{U}_i = Y_i(j) - \hat{\theta}_n(z, j)$. ■

5. MONTE CARLO EXPERIMENTS

In this section we present the results of some Monte Carlo experiments that illustrate the finite-sample performance of our method. We consider a Monte Carlo design with bounding function of the form

$$\theta(v) := L\phi(v), \tag{5.1}$$

where L is a constant and $\phi(\cdot)$ is the standard normal density function. Throughout the Monte Carlo experiments, the parameter of interest is $\theta_0 = \sup_{v \in \mathcal{V}} \theta(v)$.

5.1. Data-Generating Processes. Here we consider four Monte Carlo designs for the sake of illustration.¹⁶ In the first Monte Carlo design, labeled DGP1, the bounding function is completely flat so that $V_0 = \mathcal{V}$. In the second design, DGP2, the bounding function is non-flat, but smooth in a neighborhood of its maximizer, which is unique so that V_0 is singleton. In DGP3 and DGP4, the bounding function is also non-flat and smooth in a neighborhood of its (unique) maximizer, though relatively peaked. Illustrations of the bounding functions

¹⁶We consider four additional designs in Section K of the on-line supplement.

for all DGPs are provided at the end of our on-line supplement. Of course, in practice the shape of the bounding function is unknown, and the inference and estimation methods we consider do not make use of this information. As we describe in more detail below, we evaluate the finite sample performance of our approach in terms of coverage probability for the true point θ_0 and coverage for a false parameter value θ that is close to but below θ_0 . We compare the performance of our approach to that of the Cramer Von-Mises statistic proposed by AS. DGP1 and DGP2 in particular serve to effectively illustrate the relative advantages of both procedures as we describe below. Neither approach dominates.

For all DGPs we generated 1000 independent samples from the following model:

$$V_i \sim \text{Unif}[-2, 2], U_i = \min\{\max\{-3, \sigma\tilde{U}_i\}, 3\}, \text{ and } Y_i = L\phi(V_i) + U_i,$$

where $\tilde{U}_i \sim N(0, 1)$ and L and σ are constants. We set these constants in the following way:

$$\text{DGP1: } L = 0 \text{ and } \sigma = 0.1; \quad \text{DGP2: } L = 1 \text{ and } \sigma = 0.1;$$

$$\text{DGP3: } L = 5 \text{ and } \sigma = 0.1; \quad \text{DGP4: } L = 5 \text{ and } \sigma = 0.01.$$

We considered sample sizes $n = 500$ and $n = 1000$, and we implemented both series and kernel-type estimators to estimate the bounding function $\theta(v)$ in (K.1). We set \mathcal{V} to be an interval between the 5% and 95% sample quantiles of V_i 's in order to avoid undue influence of outliers at the boundary of the support of V_i . For both types of estimators, we computed critical values via simulation as described in Appendix A, and we implemented our method with both the conservative but simple, non-stochastic choice $\hat{V} = \mathcal{V}$ and the set estimate $\hat{V} = \hat{V}_n$ described in Section 3.2.

5.2. Series Estimation. For basis functions we use cubic B-splines and polynomials with knots equally spaced over the sample quantiles of V_i . The number $K = K_n$ of approximating functions was obtained by the following simple rule-of-thumb:

$$K = \underline{\hat{K}}, \quad \hat{K} := \hat{K}_{cv} \times n^{-1/5} \times n^{2/7}, \tag{5.2}$$

where \underline{a} is defined as the largest integer that is smaller than or equal to a , and \hat{K}_{cv} is the minimizer of the leave-one-out least squares cross validation score from the set $\{5, 6, 7, 8, 9\}$ for the B-splines and $\{3, 4, 5, 6\}$ for polynomials. If $\theta(v)$ is twice continuously differentiable, then a cross-validated K has the form $K \propto n^{1/5}$ asymptotically. Hence, the multiplicative factor $n^{-1/5} \times n^{2/7}$ in (5.2) ensures that the bias is asymptotically negligible from under-smoothing.

5.3. Kernel-Type Estimation. We use local linear smoothing since it is known to behave better at the boundaries of the support than the standard kernel method. We used the kernel function $K(s) = \frac{15}{16}(1 - s^2)^2 1(|s| \leq 1)$ and the rule of thumb bandwidth:

$$h = \widehat{h}_{ROT} \times \widehat{s}_v \times n^{1/5} \times n^{-2/7}, \quad (5.3)$$

where \widehat{s}_v is the square root of the sample variance of the V_i , and \widehat{h}_{ROT} is the rule-of-thumb bandwidth for estimation of $\theta(v)$ with studentized V , as prescribed in Section 4.2 of Fan and Gijbels (1996). The exact form of \widehat{h}_{ROT} is

$$\widehat{h}_{ROT} = 2.036 \left[\frac{\tilde{\sigma}^2 \int w_0(v) dv}{n^{-1} \sum_{i=1}^n \left\{ \tilde{\theta}_l^{(2)}(\tilde{V}_i) \right\}^2 w_0(\tilde{V}_i)} \right]^{1/5} n^{-1/5},$$

where \tilde{V}_i 's are studentized V_i 's, $\tilde{\theta}_l^{(2)}(\cdot)$ is the second-order derivative of the global quartic parametric fit of $\theta_l(v)$ with studentized V_i , $\tilde{\sigma}^2$ is the simple average of squared residuals from the parametric fit, $w_0(\cdot)$ is a uniform weight function that has value 1 for any \tilde{V}_i that is between the 10th and 90th sample quantiles of \tilde{V}_i . Again, the factor $n^{1/5} \times n^{-2/7}$ is multiplied in (5.3) to ensure that the bias is asymptotically negligible due to under-smoothing.

5.4. Simulation Results. To evaluate the relative performance of our inference method, we also implemented one of the inference methods proposed by AS, specifically their Cramér-von Mises-type (CvM) statistic with both plug-in asymptotic (PA/Asy) and asymptotic generalized moment selection (GMS/Asy) critical values. For instrument functions we used countable hypercubes and the S -function of AS Section 3.¹⁷ We set the weight function and tuning parameters for the CvM statistic exactly as in AS (see AS Section 9). These values performed well in their simulations, but our Monte Carlo design differs from theirs, and alternative choices of tuning parameters could perform more or less favorably in our design. We did not examine sensitivity to the choice of tuning parameters for the CvM statistic.

The coverage probability (CP) of confidence intervals with nominal level 95% is evaluated for the true lower bound θ_0 , and false coverage probability (FCP) is reported at $\theta = \theta_0 - 0.02$. There were 1,000 replications for each experiment. Tables 1, 2, and 3 summarize the results. CLR and AS refer to our inference method and that of AS, respectively.

We first consider the performance of our method for DGP1. In terms of coverage for θ_0 both series estimators and the local linear estimator perform reasonably well, with series estimation via B-splines performing best. The polynomial series and local linear estimators

¹⁷All three S -functions in AS Section 3 are equivalent in our design, since there is a single conditional moment inequality.

perform somewhat better in terms of false coverage probabilities, which decrease with the sample size for all estimators. The argmax set V_0 is the entire set \mathcal{V} , and our set estimator \widehat{V}_n detects this. Turning to DGP2 we see that coverage for θ_0 is in all cases roughly .98 to .99. There is non-trivial power against the false parameter θ in all cases, with the series estimators giving the lowest false coverage probabilities. For DGP3 the bounding function is relatively peaked compared to the smooth but non-flat bounding function of DGP2. Consequently the average endpoints of the preliminary set estimator \widehat{V}_n become more concentrated around 0, the maximizer of the bounding function. Performance in terms of coverage probabilities improves in nearly all cases, with the series estimators performing significantly better when $n = 1000$ and $V = \widehat{V}_n$ is used. With DGP4 the bounding function remains as in DGP3, but now with the variance of Y_i decreased by a factor of 100, the same as would occur by increasing the sample size at least by a factor of 100. The result is that the bounding function is more accurately estimated at every point. Moreover, the set estimator \widehat{V}_n is now a much smaller interval around 0. Coverage frequencies for θ_0 do not change much relative to DGP3, but false coverage probabilities drop to 0. Note that in DGPs 2-4, our method performs better when V_0 is estimated in that it makes the coverage probability more accurate and the false coverage probability smaller. DGPs 3-4 serve to illustrate the convergence of our set estimator \widehat{V}_n when the bounding function is peaked and precisely estimated, respectively.

In Table 2 we report the results of using the CvM statistic of AS to perform inference. For DGP1 with a flat bounding function the CvM statistic with both the PA/Asy and GMS/Asy performs well. Coverage frequencies for θ_0 were close to the nominal level, closer than our method using polynomial series or local linear regression, although not quite as close as when we use B-splines. The CvM statistic has a lower false coverage probability than the CLR confidence intervals in this case, although at a sample size of 1000 the difference is not large. For DGP2 the bounding function is non-flat but smooth in a neighborhood of V_0 and the situation is much different. For both PA/Asy and GMS/Asy critical values with the CvM statistic, coverage frequencies for θ_0 were 1. Our confidence intervals also over-covered in this case, with coverage frequencies of roughly .98 to .99. Moreover, the CvM statistic has low power against the false parameter θ , with coverage 1 with PA/Asy and coverage .977 and .933 with sample size 500 and 1000, respectively using GMS/Asy critical values. For DGP3 and DGP4 both critical values for the CvM statistic gave coverage for θ_0 and the false parameter θ equal to one. Thus under DGPs 2,3, and 4 our confidence intervals perform better by both measures. However, overall neither approach dominates.

Thus, in our Monte Carlo experiments the CvM statistic exhibits better power when the bounding function is flat, while our confidence intervals exhibit better power when the bounding function is non-flat. AS establish that the CvM statistic has power against some $n^{-1/2}$ local alternatives under conditions that are satisfied under DGP1, but that do not hold when the bounding function has a unique minimum.¹⁸ We have established local asymptotic power for nonparametric estimators of polynomial order less distant than $n^{-1/2}$ that apply whether the bounding function is flat or non-flat. Our Monte Carlo results accord with these findings.¹⁹ In the on-line supplement, we present further supporting Monte Carlo evidence and local asymptotic power analysis to show why our method performs better than the AS method in non-flat cases.

In Table 4 we report computation times for our Monte Carlo experiments.²⁰ The fastest performance in terms of total simulation time was achieved with the CvM statistic of AS, which took 24 minutes to execute a total of 16,000 replications. Simulations using our approach with B-spline series, polynomial series, and local linear polynomials took roughly 73, 62, and 397 minutes, respectively. Based on these times the table shows for each statistic the average time for a single test, and the relative performance of each method to that obtained using the CvM statistic.

In practice one will not perform Monte Carlo experiments but will rather be interested in computing a single confidence region for the parameter of interest. When the bounding function is separable our approach offers the advantage that the critical value does not vary with the parameter value being tested. As a result, we can compute a confidence region in the same amount of time it takes to compute a single test. On the other hand, to construct a confidence region based on the CvM statistic, one must compute the statistic and its associated critical value at a large number of points in the parameter space, where the number of points required will depend on the size of the parameter space and the degree of precision desired. If however the bounding function is not separable in the parameter of interest, then both approaches use parameter-dependent critical values.

¹⁸Specifically Assumptions LA3 and LA3' of AS Theorem 4 do not hold when the sequence of models has a fixed bounding function with a unique minimum. As they discuss after the statement of Assumptions LA3 and LA3', in such cases GMS and plug-in asymptotic tests have trivial power against $n^{-1/2}$ local alternatives.

¹⁹We did not do CP-correction in our reported results. Our conclusion will remain valid even with CP-correction as in AS, since our method performs better in DGP2-DGP4 where we have over-coverage.

²⁰These were computation times based on our implementation. Generally speaking, performance time for both methods will depend on the choice of tuning parameters and the efficiency of one's code. More efficient implementation times for both methods may be possible.

6. CONCLUSION

In this paper we provided a novel method for inference on intersection bounds. Bounds of this form are common in the recent literature, but two issues have posed difficulties for valid asymptotic inference and bias-corrected estimation. First, the application of the supremum and infimum operators to boundary estimates results in finite-sample bias. Second, unequal sampling error of estimated bounding functions complicates inference. We overcame these difficulties by applying a precision-correction to the estimated bounding functions before taking their intersection. We employed strong approximation to justify the magnitude of the correction in order to achieve the correct asymptotic size. As a by-product, we proposed a bias-corrected estimator for intersection bounds based on an asymptotic median adjustment. We provided formal conditions that justified our approach in both parametric and nonparametric settings, the latter using either kernel or series estimators.

At least two of our results may be of independent interest beyond the scope of inference on intersection bounds. First, our result on the strong approximation of series estimators is new. This essentially provides a functional central limit theorem for any series estimator that admits a linear asymptotic expansion, and is applicable quite generally. Second, our method for inference applies to any value that can be defined as a linear programming problem with either finite or infinite dimensional constraint set. Estimators of this form can arise in a variety of contexts, including, but not limited to intersection bounds. We therefore anticipate that although our motivation lay in inference on intersection bounds, our results may have further application.

APPENDIX A. IMPLEMENTATION ALGORITHMS

In this section we lay out steps for implementation. We begin with parametric bounding functions, and then cover nonparametric cases. While the basic steps are similar, some adjustments are necessary when moving from parametric to nonparametric cases. The end goal in each case is to obtain estimators $\hat{\theta}_{n0}(p)$ that provide bias-corrected estimates or the endpoints of confidence intervals depending on the chosen value of p , e.g. $p = 1/2$ or $p = 1 - \alpha$. As in the main text, we focus here on the upper bound. If instead $\hat{\theta}_{n0}(p)$ were the lower bound for θ^* , given by the supremum of a bounding function, the same algorithm could be applied to perform inference on $-\theta^*$, bounded above by the infimum of the negative of the original bounding function, and then any inference statement for $-\theta^*$ could trivially be transformed to inference statements for θ^* . Indeed, any set of lower and upper bounds can be similarly transformed to a collection of upper bounds, and the above algorithm

applied to perform inference on θ^* , e.g. according to the methods laid out for inference on parameters bounded by conditional moment inequalities in Section 3.²¹ Alternatively, if one wishes to perform inference on the identified set in such circumstance one can use the intersection of upper and lower one-sided intervals each based on $\tilde{p} = (1 + p)/2$ as an asymptotic level- p confidence set for Θ_I , which is valid by Bonferroni's inequality.²²

A.1. Parametric Estimators. We start by considering implementation when the bounding function is estimated parametrically, i.e. where Condition P holds. We provide a simple approach that relies on simulation from the multivariate normal distribution.

Algorithm 1 (Implementation for Parametric Case). (1) Set $\tilde{\gamma}_n \equiv 1 - .1/\log n$. Simulate a large number R of draws from $\mathcal{N}(0, I_K)$, denoted Z_1, \dots, Z_R , where $K = \dim(\gamma_n)$ and I_K is the identity matrix, where γ_n is the parameter of interest. (2) Compute $\hat{\Omega}_n$, a consistent estimator for the asymptotic variance of $\sqrt{n}(\hat{\gamma}_n - \gamma_n)$. (3) For each $v \in \mathcal{V}$, compute $\hat{g}(v) = \partial\theta_n(v, \hat{\gamma}_n)/\partial\gamma_n \cdot \hat{\Omega}_n^{1/2}$, and $s_n(v) = \|\hat{g}(v)\|/\sqrt{n}$. (4) Compute $k_{n,\mathcal{V}}(\tilde{\gamma}_n) = \gamma_n - \text{quantile of } \{\sup_{v \in \mathcal{V}} (\hat{g}(v)' Z_r / \|\hat{g}(v)\|), r = 1, \dots, R\}$, and

$$\hat{V}_n = \{v \in \mathcal{V} : \hat{\theta}_n(v) \leq \min_{v \in \mathcal{V}} (\hat{\theta}_n(v) + k_{n,\mathcal{V}}(\tilde{\gamma}_n)s_n(v)) + 2k_{n,\mathcal{V}}(\tilde{\gamma}_n)s_n(v)\},$$

(5) Compute $k_{n,\hat{V}_n}(p) = p - \text{quantile of } \{\sup_{v \in \hat{V}_n} (\hat{g}(v)' Z_r / \|\hat{g}(v)\|), r = 1, \dots, R\}$, and set $\hat{\theta}_{n0}(p) = \inf_{v \in \mathcal{V}} [\hat{\theta}_n(v) + k_{n,\hat{V}_n}(p) \|\hat{g}(v)\|/\sqrt{n}]$.

Remark 4. (1) An important special case is when the support of v is finite, as in Example 1 of Section 4.1, so that $\mathcal{V} = \{1, \dots, J\}$. In this case the algorithm applies with $\theta_n(v, \gamma_n) = \sum_{j=1}^J 1[v=j]\gamma_{nj}$, i.e. where for each j , $\theta_n(j, \gamma_n) = \gamma_{nj}$ and $\hat{g}(v) = (1[v=1], \dots, 1[v=J]) \cdot \hat{\Omega}_n^{1/2}$. (2) The above algorithm applies when the bounding function is separable in the parameter of interest. When the bounding function is non-separable in this parameter, say μ where $\theta(v) := \theta(\mu, v)$, it can be used to test the hypothesis that any given μ is in the identified set as described in Example C in Section 2. That is, for any fixed μ and any chosen $\alpha \in [1/2, 1)$ it can be used to produce a critical value $\hat{\theta}_{1-\alpha}$ such that for the true parameter value μ ,

$$P_n \{ \inf_{v \in \mathcal{V}} \theta(\mu, v) \geq \hat{\theta}_{1-\alpha} \} \leq \alpha + o(1).$$

²¹For example if we have $\theta_n^l(z) \leq \theta_n^* \leq \theta_n^u(z)$ for all $z \in \mathcal{Z}$, then we can equivalently write $\min_{z \in \mathcal{Z}} \min_{j=1,2} g_n(\theta_n^*, z, j) \geq 0$, where $g_n(\theta_n^*, z, 1) = \theta_n^u(z) - \theta_n^*$ and $g_n(\theta_n^*, z, 2) = \theta_n^* - \theta_n^l(z)$. Then we can apply our method through use of the auxiliary function $g_n(\theta_n, z, j)$, in similar fashion as in Example C with multiple conditional moment inequalities.

²²In an earlier version of this paper, Chernozhukov, Lee, and Rosen (2009), we provided a different method for inference on a parameter with both lower and upper bounding functions, which can also be used for valid inference on θ^* .

A confidence set for μ can then be formed by inverting this test. This is done by first carrying out step (1), computing all components of $\widehat{\Omega}_n$ in step (2) that are not dependent upon μ , and then performing the rest of step (2) and steps (3)-(5) at every μ in some set of points approximating the parameter space. For example, in the context of Example 3, conditional moment inequalities, we had

$$\widehat{\Omega}_n = (I_{|\mathcal{J}|} \otimes \widehat{Q})^{-1} \mathbb{E}_n[\widehat{u}_i \widehat{u}_i'] (I_{|\mathcal{J}|} \otimes \widehat{Q})^{-1},$$

where the matrix \widehat{Q} did not depend on the model parameter, so need not be re-computed for every iteration of step (2). (3) Note that objective function approaches to inference with set identification construct confidence sets through the inversion of tests in both non-separable and separable cases. Similarly to our procedure, in the non-separable case this requires computing a test statistic and critical value at each of a large grid of points approximating the parameter space. In the separable case our approach produces a critical value that is not parameter-dependent, so that the steps above need only be carried out once to produce the desired confidence set. ■

A.2. Series Estimators. In practice, implementation with a series estimator does not substantially differ from the parametric case.

Algorithm 2 (Implementation for Series Case). *Perform Steps (1)-(5) as in Algorithm 1, except now in step (2) compute $\widehat{\Omega}_n$, a consistent estimate of the large sample variance of $\sqrt{n}(\widehat{\beta}_n - \beta_n)$, and in step (3) $\widehat{g}(v) = p_n(v)' \widehat{\Omega}_n^{1/2}$.*

Remark 5. (1) If desired one can bypass simulation of the stochastic process by instead employing the analytical critical value in step 4, $k_{n,V}(p) = a_n(V) - 2 \log(1-p)/a_n(V)$ from Remark 2 in Section 4.2. This is convenient because it does not involve simulation, though it requires computation of $a_n(\widehat{V}_n) = 2\sqrt{\log\{\ell_n(1 + \ell_n L_n \text{diam}(\widehat{V}_n))^d\}}$. Moreover, it could be too conservative in some applications. Thus, we recommend using simulation, unless the computational cost is too high. (2) Note that the algorithm can be used for inference when the bounding function is non-separable in a parameter of interest exactly as described in the parametric case. Again, in step (2) computational efficiency can be increased by computing components of $\widehat{\Omega}_n$ that do not vary across iterations once only. In Example 6 for instance, conditional moment inequalities, a consistent estimator for \widehat{Q}_n , a component of $\widehat{\Omega}_n$, will not vary across iterations and thus need be computed only once. ■

A.3. Kernel Estimators. For kernel estimation the steps are also similar.

Algorithm 3 (Implementation for Kernel Case). *(1) Set $\gamma_n \equiv 1 - .1/\log n$. Simulate $R \times n$ times independent draws from $N(0,1)$, denoted by $\{\eta_{ir} : i = 1, \dots, n, r = 1, \dots, R\}$, where*

n is the sample size and R is the number of simulation repetitions. (2) For each $v \in \mathcal{V}$ and $r = 1, \dots, R$, compute $\mathbb{G}_n^o(\hat{g}_v; r) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \eta_{ir} \hat{g}_v(U_i, Z_i)$, where $\hat{g}_v(U_i, Z_i)$ is defined in Section 4.3, that is

$$\hat{g}_v(U_i, Z_i) = \frac{e'_j \hat{U}_i}{(h_n^d)^{1/2} \hat{f}_n(z)} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right).$$

Let $s_n^2(v) = \mathbb{E}_n[\hat{g}_v^2]/(nh_n^d)$ and $\mathbb{E}_n[\hat{g}_v^2] = n^{-1} \sum_{i=1}^n \hat{g}_v^2(U_i, Z_i)$. Here, \hat{U}_i is the kernel-type regression residual and $\hat{f}_n(z)$ is the kernel density estimator of density of Z_i . (3) Compute $k_{n,\mathcal{V}}(\gamma_n) = \gamma_n - \text{quantile of } \left\{ \sup_{v \in \mathcal{V}} \mathbb{G}_n^o(\hat{g}_v; r) / \sqrt{\mathbb{E}_n[\hat{g}_v^2]}, r = 1, \dots, R \right\}$, and $\hat{V}_n = \{v \in \mathcal{V} : \hat{\theta}_n(v) \leq \min_{v \in \mathcal{V}} (\hat{\theta}_n(v) + k_{n,\mathcal{V}}(\gamma_n) s_n(v)) + 2k_{n,\mathcal{V}}(\gamma_n) s_n(v)\}$, (4) Compute $k_{n,\hat{V}_n}(p) = p - \text{quantile of } \left\{ \sup_{v \in \hat{V}_n} \mathbb{G}_n^o(\hat{g}_v; r) / \sqrt{\mathbb{E}_n[\hat{g}_v^2]}, r = 1, \dots, R \right\}$, and set $\hat{\theta}_{n0}(p) = \inf_{v \in \mathcal{V}} [\hat{\theta}(v) + k_{n,\hat{V}_n}(p) s_n(v)]$.

Remark 6. (1) The researcher also has the option of employing an analytical approximation in place of simulation if desired. This can be done by using $k_{n,V}(p) = a_n(V) - 2 \log(1 - p)/a_n(V)$ from Remark 3, but requires computation of

$$a_n(\hat{V}_n) = 2 \sqrt{\log\{\ell_n(1 + \ell_n(1 + h_n^{-1}) \text{diam}(\hat{V}_n)^d)\}}.$$

This approximation could be too conservative in some applications, and thus we recommend using simulation, unless the computational cost is too high. (2) In the case where the bounding function is non-separable in a parameter of interest, but is nonparametrically estimated, a confidence interval for this parameter can be constructed as described in the parametric case above, where step(1) is carried out once and steps (2)-(4) are executed iteratively on a set of parameter values approximating the parameter space. However, the bandwidth, $\hat{f}_n(z)$, and $\mathbf{K} \left(\frac{z - Z_i}{h_n} \right)$, each Z_i , do not vary across iterations and thus only need to be computed once. ■

APPENDIX B. DEFINITION OF STRONG APPROXIMATION

The following definitions are used extensively.

Definition 4 (Strong approximation). *Suppose that for each n there are random variables Z_n and Z'_n defined on a probability space $(A, \mathcal{A}, \mathbb{P}_n)$ and taking values in the separable metric space (S, d_S) . We say that $Z_n =_d Z'_n + o_{\mathbb{P}_n}(\delta_n)$, for $\delta_n \rightarrow 0$, if there are identically distributed copies of Z_n and Z'_n , denoted \bar{Z}_n and \bar{Z}'_n , defined on $(A, \mathcal{A}, \mathbb{P}_n)$ (suitably enriched if needed), such that*

$$d_S(\bar{Z}_n, \bar{Z}'_n) = o_{\mathbb{P}_n}(\delta_n).$$

Note that copies \bar{Z}_n and \bar{Z}'_n can be always defined on (A, \mathcal{A}, P_n) by suitably enriching this space by taking product probability spaces. It turns out that for the Polish spaces, this definition implies the following stronger, and much more convenient, form.

Lemma 9 (A Convenient Implication for Polish Spaces via Dudley and Philipp). *Suppose that (S, d_S) is Polish, i.e. complete, separable metric space, and (A, \mathcal{A}, P_n) has been suitably enriched. Suppose that Definition 4 holds, then there is also an identical copy Z_n^* of Z'_n such that $Z_n = Z_n^* + o_{P_n}(\delta_n)$, that is,*

$$d_S(Z_n, Z_n^*) = o_{P_n}(\delta_n)$$

Proof. We start with the original probability space (A', \mathcal{A}', P'_n) that can carry Z_n and (\bar{Z}_n, \bar{Z}'_n) . In order to apply Lemma 2.11 of Dudley and Philipp (1983), we need to carry a standard uniform random variable $U \sim U(0, 1)$ that is independent of Z_n . To guarantee this we can always consider $U \sim U(0, 1)$ on the standard space $([0, 1], \mathcal{F}, \lambda)$, where \mathcal{F} is the Borel sigma algebra on $[0, 1]$ and λ is the usual Lebesgue measure, and then enrich the original space (A', \mathcal{A}', P'_n) by creating formally a new space (A, \mathcal{A}, P_n) as the product of (A', \mathcal{A}', P'_n) and $([0, 1], \mathcal{F}, \lambda)$. Then using Polishness of (S, d_S) , given the joint law of (\bar{Z}_n, \bar{Z}'_n) , we can apply Lemma 2.11 of Dudley and Philipp (1983) to construct Z_n^* such that (Z_n, Z_n^*) has the same law as (\bar{Z}_n, \bar{Z}'_n) , so that $d_S(\bar{Z}_n, \bar{Z}'_n) = o_{P_n}(\delta_n)$ implies $d_S(Z_n, Z_n^*) = o_{P_n}(\delta_n)$. ■

Since in all of our cases the relevant metric spaces are either the space of continuous functions defined on a compact set equipped with the uniform metric or finite-dimensional Euclidian spaces, which are all Polish spaces, we can use Lemma 9 throughout the paper. Using this implication of strong approximation makes our proofs slightly simpler.

APPENDIX C. PROOFS FOR SECTION 3

C.1. Some Useful Facts and Lemmas. A useful result in our case is the anti-concentration inequality derived in Chernozhukov and Kato (2011).

Lemma 10 (Anti-Concentration Inequality, Chernozhukov and Kato (2011)). *Let $X = (X_t)_{t \in T}$ be a separable Gaussian process indexed by a semimetric space T such that $E_P[X_t] = 0$ and $E_P[X_t^2] = 1$ for all $t \in T$. Then*

$$\sup_{x \in \mathbb{R}} P \left(\left| \sup_{t \in T} X_t - x \right| \leq \epsilon \right) \leq C \epsilon \left(E_P \left[\sup_{t \in T} X_t \right] \vee 1 \right), \quad \forall \epsilon > 0, \quad (\text{C.1})$$

where C is an absolute constant.

An immediate consequence of this lemma is the following result:

Corollary 1 (Anti-concentration for $\sup_{v \in V_n} Z_n^*(v)$). *Let V_n be any sequence of compact non-empty subsets in \mathcal{V} . Then under condition C.2-C.3, we have that for $\delta_n \rightarrow 0$ such that $\delta_n = o(1/\bar{a}_n)$*

$$\sup_{x \in \mathbb{R}} \mathbb{P}_n \left(\left| \sup_{v \in V_n} Z_n^*(v) - x \right| \leq \delta_n \right) = o(1).$$

Proof. Continuity in Condition C.2 implies separability of Z_n^* . Condition C.4 implies that $E_{\mathbb{P}_n}[\sup_{v \in V_n} Z_n^*(v)] \leq E_{\mathbb{P}_n}[\sup_{v \in \mathcal{V}} Z_n^*(v)] \leq K\bar{a}_n$ for some constant K that depends only on η , so that

$$\sup_{x \in \mathbb{R}} \mathbb{P}_n \left(\left| \sup_{v \in \mathcal{V}} Z_n^*(v) - x \right| \leq \delta_n \right) \leq C\delta_n[K\bar{a}_n \vee 1] = o(1). \quad \blacksquare$$

Lemma 11 (Closeness in Conditional Probability Implies Closeness of Conditional Quantiles Unconditionally). *Let X_n and Y_n be random variables and \mathcal{D}_n be a random vector. Let $F_{X_n}(x | \mathcal{D}_n)$ and $F_{Y_n}(y | \mathcal{D}_n)$ denote the conditional distribution functions, and $F_{X_n}^{-1}(p | \mathcal{D}_n)$ and $F_{Y_n}^{-1}(p | \mathcal{D}_n)$ denote the corresponding conditional quantile functions. If $\mathbb{P}_n(|X_n - Y_n| > \xi_n | \mathcal{D}_n) = o_{\mathbb{P}_n}(\tau_n)$ for some sequence $\tau_n \searrow 0$, then with unconditional probability \mathbb{P}_n converging to one, for some $\varepsilon_n = o(\tau_n)$,*

$$F_{X_n}^{-1}(p | \mathcal{D}_n) \leq F_{Y_n}^{-1}(p - \varepsilon_n | \mathcal{D}_n) + \xi_n \text{ and } F_{Y_n}^{-1}(p | \mathcal{D}_n) \leq F_{X_n}^{-1}(p - \varepsilon_n | \mathcal{D}_n) + \xi_n, \forall p \in (\varepsilon_n, 1 - \varepsilon_n).$$

Proof. We have that for some $\varepsilon_n = o(\tau_n)$, $\mathbb{P}_n[\mathbb{P}_n\{|X_n - Y_n| > \xi_n | \mathcal{D}_n\} \leq \varepsilon_n] \rightarrow 1$, that is, there is a set Ω_n such that $\mathbb{P}_n(\Omega_n) \rightarrow 1$ such that $\mathbb{P}_n\{|X_n - Y_n| > \xi_n | \mathcal{D}_n\} \leq \varepsilon_n$ for all $\mathcal{D}_n \in \Omega_n$. So, for all $\mathcal{D}_n \in \Omega_n$

$$F_{X_n}(x | \mathcal{D}_n) + \varepsilon_n \geq F_{Y_n + \xi_n}(x | \mathcal{D}_n) \text{ and } F_{Y_n}(x | \mathcal{D}_n) + \varepsilon_n \geq F_{X_n + \xi_n}(x | \mathcal{D}_n), \forall x \in \mathbb{R},$$

which implies the inequality stated in the lemma, by definition of the conditional quantile function and equivariance of quantiles to location shifts. \blacksquare

C.2. Proof of Lemma 1. (Concentration of Inference on V_n .) Step 1. Letting

$$A_n := \sup_{v \in V_n} Z_n(v), \quad B_n := \sup_{v \in \mathcal{V}} Z_n(v), \quad R_n := \left(\sup_{v \in \mathcal{V}} |Z_n(v)| + \kappa_n \right) \sup_{v \in \mathcal{V}} \left| \frac{\sigma_n(v)}{s_n(v)} - 1 \right|,$$

$$A_n^* := \sup_{v \in V_n} Z_n^*(v), \quad B_n^* := \sup_{v \in \mathcal{V}} Z_n^*(v), \quad R_n^* := \left(\sup_{v \in \mathcal{V}} |Z_n^*(v)| + \kappa_n \right) \sup_{v \in \mathcal{V}} \left| \frac{\sigma_n(v)}{s_n(v)} - 1 \right|,$$

we obtain

$$\begin{aligned}
\sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \widehat{\theta}_n(v)}{s_n(v)} &= \sup_{v \in \mathcal{V}} \left\{ \frac{\theta_{n0} - \theta_n(v)}{s_n(v)} + Z_n(v) \frac{\sigma_n(v)}{s_n(v)} \right\} \\
&= \sup_{v \in V_n} \left\{ \frac{(\theta_{n0} - \theta_n(v))}{s_n(v)} + Z_n(v) \frac{\sigma_n(v)}{s_n(v)} \right\} \vee \sup_{v \notin V_n} \left\{ \frac{(\theta_{n0} - \theta_n(v))}{s_n(v)} + Z_n(v) \frac{\sigma_n(v)}{s_n(v)} \right\} \\
&\leq_{(1)} \sup_{v \in V_n} \left\{ Z_n(v) \frac{\sigma_n(v)}{s_n(v)} \right\} \vee \sup_{v \notin V_n} \left\{ \frac{-\kappa_n \sigma_n(v)}{s_n(v)} + Z_n(v) \frac{\sigma_n(v)}{s_n(v)} \right\} \\
&\leq A_n \vee (B_n - \kappa_n) + 2R_n \leq_{(2)} A_n^* \vee (B_n^* - \kappa_n) + 2R_n^* + o_{P_n}(\delta_n),
\end{aligned}$$

where in (1) we used that $\theta_n(v) \geq \theta_{n0}$ and $\theta_{n0} - \theta_n(v) \leq -\kappa_n \sigma_n(v)$ outside V_n , and in (2) we used C.2. Next, since we assumed in the statement of the lemma that $\kappa_n \lesssim \bar{a}_n + \ell \ell_n$, and by C.4: $R_n^* = O_{P_n}(\bar{a}_n + \bar{a}_n + \ell \ell_n) o_{P_n}(\delta_n / (\bar{a}_n + \ell \ell_n)) = o_{P_n}(\delta_n)$. Therefore, there is a deterministic term $o(\delta_n)$ such that $P_n(2R_n^* + o_{P_n}(\delta_n) > o(\delta_n)) = o(1)$.²³

Hence uniformly in $x \in [0, \infty)$

$$\begin{aligned}
P_n \left(\sup_{v \in \mathcal{V}} \frac{(\theta_{n0} - \widehat{\theta}_n(v))}{s_n(v)} > x \right) &\leq P_n(A_n^* + o(\delta_n) > x) + P_n(B_n^* - \kappa_n + o(\delta_n) > 0) + o(1) \\
&\leq P_n(A_n^* > x) + P_n(B_n^* - \kappa_n > 0) + o(1) \leq P_n(A_n^* > x) + (1 - \gamma'_n) + o(1),
\end{aligned}$$

where the last two inequalities follow by Corollary 1 and by $\kappa_n = Q_{\gamma'_n}(B_n^*)$.

Step 2. To complete the proof, we must show that there is $\gamma'_n \nearrow 1$ such that $\kappa_n \lesssim \bar{a}_n + \ell \ell_n$. Let $1 - \gamma'_n \searrow 0$ such that $1 - \gamma'_n \geq C/\ell_n$. It suffices to show that

$$\kappa_n \leq \left(\bar{a}_n + \frac{c(\gamma'_n)}{\bar{a}_n} \right) \leq \left(\bar{a}_n + \frac{\eta \ell \ell_n + \eta \log C^{-1}}{\bar{a}_n} \right) \lesssim \bar{a}_n + \ell \ell_n, \quad (\text{C.2})$$

where $c(\gamma'_n) = Q_{\gamma'_n}(\mathcal{E})$. To show the first inequality in (C.2) note

$$P_n \left(\sup_{v \in \mathcal{V}} Z_n^*(v) \leq (\bar{a}_n + c(\gamma'_n)/\bar{a}_n) \right) =_{(1)} P_n(\mathcal{E}_n(\mathcal{V}) \leq c(\gamma'_n)) \geq_{(2)} P_n(\mathcal{E} \leq c(\gamma'_n)) = \gamma'_n,$$

where (1) holds by definition of $\mathcal{E}_n(\mathcal{V})$ and (2) by C.3. To show the second inequality in (C.2) note that by C.3 $P(\mathcal{E} > t) \leq \exp(-t\eta^{-1})$, for some constant $\eta > 0$, so that $c(\gamma'_n) \leq -\eta \log(1 - \gamma'_n) \leq \eta \ell \ell_n + \eta \log C^{-1}$. \blacksquare

²³ Throughout the paper we use the elementary fact: If $X_n = o_{P_n}(\Delta_n)$, for some $\Delta_n \searrow 0$, then there is $o(\Delta_n)$ term such that $P_n\{|X_n| > o(\Delta_n)\} \rightarrow 0$.

C.3. Proof of Theorem 1 (Analytical Construction). Part 1.(Level) Observe that

$$\begin{aligned}
\mathbb{P}_n \left(\theta_{n0} \leq \widehat{\theta}_{n0}(p) \right) &= \mathbb{P}_n \left(\sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \widehat{\theta}_n(v)}{s_n(v)} \leq k_{n, \widehat{V}_n}(p) \right) \\
&\geq_{(1)} \mathbb{P}_n \left(\sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \widehat{\theta}_n(v)}{s_n(v)} \leq k_{n, V_n}(p) \right) - \mathbb{P}_n \left(V_n \not\subseteq \widehat{V}_n \right) \\
&\geq_{(2)} \mathbb{P}_n \left(\sup_{v \in \widehat{V}_n} Z_n^*(v) \leq k_{n, V_n}(p) \right) - o(1) \\
&= \mathbb{P}_n \left(\mathcal{E}_n(V_n) \leq c(p) - o(1) \right) - o(1) \geq_{(3)} \mathbb{P}_n \left(\mathcal{E} \leq c(p) - o(1) \right) - o(1) =_{(4)} p - o(1),
\end{aligned}$$

where (1) follows by monotonicity of $V \mapsto k_{n, V}(p) = a_n(V) + c(p)/a_n(V)$ holding by assumption, (2) by Lemma 1, by $\mathbb{P}_n \left(V_n \not\subseteq \widehat{V}_n \right) = o(1)$ holding by Lemma 2, and also by the fact that the critical value $k_{n, V_n}(p) \geq 0$ is non-stochastic, and (3) and (4) by the existence of majorizing rv \mathcal{E} with a continuous distribution function (see C.3).

Part 2.(Estimation Risk) We have that under \mathbb{P}_n

$$\begin{aligned}
\left| \widehat{\theta}_{n0}(p) - \theta_{n0} \right| &= \left| \inf_{v \in \mathcal{V}} \left[\widehat{\theta}_n(v) + k_{n, \widehat{V}_n}(p) s_n(v) \right] - \theta_{n0} \right| \\
&= \left| \sup_{v \in \mathcal{V}} \left(\left[\frac{\theta_{n0} - \widehat{\theta}_n(v)}{s_n(v)} + k_{n, \widehat{V}_n}(p) \right] \sigma_n(v) \frac{s_n(v)}{\sigma_n(v)} \right) \right| \\
&\leq_{(1)} \left(\left| \sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \widehat{\theta}_n(v)}{s_n(v)} \right| + k_{n, \widehat{V}_n}(p) \right) \bar{\sigma}_n \left(1 + o_{\mathbb{P}_n} \left(\frac{\delta_n}{\bar{a}_n + \ell \ell_n} \right) \right) \\
&\leq_{(2)} \left(\left| \sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \widehat{\theta}_n(v)}{\sigma_n(v)} \right| + k_{n, \widehat{V}_n}(p) \right) \bar{\sigma}_n \left(1 + o_{\mathbb{P}_n} \left(\frac{\delta_n}{\bar{a}_n + \ell \ell_n} \right) \right)^2 \\
&\leq_{(3)} \left(\sup_{v \in \widehat{V}_n} |Z_n^*(v)| + o_{\mathbb{P}_n}(\delta_n) + k_{n, \widehat{V}_n}(p) \right) \bar{\sigma}_n \left(1 + o_{\mathbb{P}_n} \left(\frac{\delta_n}{\bar{a}_n + \ell \ell_n} \right) \right)^2 \text{ wp} \rightarrow 1 \\
&\leq_{(4)} \left(\sup_{v \in \widehat{V}_n} |Z_n^*(v)| + o_{\mathbb{P}_n}(\delta_n) + k_{n, \widehat{V}_n}(p) \right) \bar{\sigma}_n \left(1 + o_{\mathbb{P}_n} \left(\frac{\delta_n}{\bar{a}_n + \ell \ell_n} \right) \right)^2 \text{ wp} \rightarrow 1 \\
&\leq_{(5)} 3 \left| a_n(\widehat{V}_n) + \frac{O_{\mathbb{P}_n}(1)}{a_n(\widehat{V}_n)} + o_{\mathbb{P}_n}(\delta_n) \right| \bar{\sigma}_n \left(1 + o_{\mathbb{P}_n} \left(\frac{\delta_n}{\bar{a}_n + \ell \ell_n} \right) \right)^2 \text{ wp} \rightarrow 1 \\
&\leq_{(6)} 4 \left| a_n(\widehat{V}_n) + \frac{O_{\mathbb{P}_n}(1)}{a_n(\widehat{V}_n)} \right| \bar{\sigma}_n \text{ wp} \rightarrow 1,
\end{aligned}$$

where (1) holds by C.4 and the triangle inequality; (2) holds by C.4; (3) follows because $\text{wp} \rightarrow 1$, for some $o(\delta_n)$

$$\sup_{v \in \widehat{V}_0} Z_n^*(v) - o(\delta_n) \leq_{(a)} \sup_{v \in \widehat{V}_0} Z_n(v) \leq_{(b)} \sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \widehat{\theta}_n(v)}{\sigma_n(v)} \leq_{(c)} \left(\sup_{v \in \widehat{V}_n} Z_n^*(v) \right) \vee 0 + o(\delta_n),$$

where (a) is by C.2, (b) by definition of Z_n , while (c) by the proof of Lemma 1, so that

$$\left| \sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \hat{\theta}_n(v)}{\sigma_n(v)} \right| \leq \sup_{v \in V_n} |Z_n^*(v)| + o_{\mathbb{P}_n}(\delta_n);$$

(4) follows by Lemma 2 which implies $V_n \subseteq \hat{V}_n \subseteq \bar{V}_n$ wp $\rightarrow 1$, so that

$$k_{n, \hat{V}_n}(p) \leq k_{n, \bar{V}_n}(p) = a_n(\bar{V}_n) + \frac{c(p)}{a_n(\bar{V}_n)},$$

Condition C.3 gives (5). Inequality (6) follows because $a_n(\bar{V}_n) \geq 1$, $\bar{a}_n \geq 1$, and $\delta_n = o(1)$; this inequality is the claim that we needed to prove.

Part 3. We have that

$$\theta_{na} - \theta_{n0} \geq 4\bar{\sigma}_n \left(a_n(\bar{V}_n) + \frac{\mu_n}{a_n(\bar{V}_n)} \right) > \hat{\theta}_{n0}(p) - \theta_{n0} \text{ wp } \rightarrow 1,$$

with the last inequality occurring by Part 2 since $\mu_n \rightarrow_{\mathbb{P}_n} \infty$. ■

C.4. Proof of Theorem 2 (Simulation Construction). Part 1. (Level Consistency)

Let us compare critical values

$$k_{n, V_n}(p) = Q_p \left(\sup_{v \in V_n} Z_n^*(v) \mid \mathcal{D}_n \right) \text{ and } \kappa_{n, V_n}(p) = Q_p \left(\sup_{v \in V_n} \bar{Z}_n^*(v) \right).$$

The former is data-dependent while the latter is deterministic. Note that $k_{n, V_n}(p) \geq 0$ by C.2(b) for $p \geq 1/2$. By C.2 wp $\rightarrow 1$ for some deterministic term $o(\delta_n)$,

$$\mathbb{P}_n \left(\left| \sup_{v \in V_n} Z_n^*(v) - \sup_{v \in V_n} \bar{Z}_n^*(v) \right| > o(\delta_n) \mid \mathcal{D}_n \right) = o_{\mathbb{P}_n}(1),$$

which implies by Lemma 11 that for some $\varepsilon_n \searrow 0$, wp $\rightarrow 1$

$$k_{n, V_n}(p) \geq (\kappa_{n, V_n}(p - \varepsilon_n) - o(\delta_n))_+ \text{ for all } p \in [1/2, 1 - \varepsilon_n]. \quad (\text{C.3})$$

The result follows analogously to the proof in Part 1 of Theorem 1, namely:

$$\begin{aligned}
\mathbb{P}_n \left(\theta_{n0} \leq \widehat{\theta}_{n0}(p) \right) &= \mathbb{P}_n \left(\sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \widehat{\theta}_n(v)}{s_n(v)} \leq k_{n, \widehat{V}_n}(p) \right) \\
&\geq_{(1)} \mathbb{P}_n \left(\sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \widehat{\theta}_n(v)}{s_n(v)} \leq k_{n, V_n}(p) \right) - o(1) \\
&\geq_{(2)} \mathbb{P}_n \left(\sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \widehat{\theta}_n(v)}{s_n(v)} \leq (\kappa_{n, V_n}(p - \varepsilon_n) - o(\delta_n))_+ \right) - o(1) \\
&\geq_{(3)} \mathbb{P}_n \left(\sup_{v \in V_n} Z_n^*(v) \leq (\kappa_{n, V_n}(p - \varepsilon_n) - o(\delta_n))_+ \right) - o(1) \\
&\geq \mathbb{P}_n \left(\sup_{v \in V_n} Z_n^*(v) \leq \kappa_{n, V_n}(p - \varepsilon_n) - o(\delta_n) \right) - o(1) \geq_{(4)} p - \varepsilon_n - o(1) = p - o(1),
\end{aligned}$$

where (1) follows by monotonicity of $V \mapsto k_{n, V}$ holding by construction and by $\mathbb{P}_n \left(V_n \not\subseteq \widehat{V}_n \right) = o(1)$ shown in Lemma 2, (2) holds by the comparison of quantiles in equation (C.3), (3) by Lemma 1. (4) holds by anti-concentration Corollary 1.

Parts 2 & 3.(Estimation Risk and Power) By Lemma 2 $\text{wp} \rightarrow 1$, $\widehat{V}_n \subseteq \overline{V}_n$, so that $k_{n, \widehat{V}_n}(p) \leq k_{n, \overline{V}_n}(p)$. By C.2 for some deterministic term $o(\delta_n)$,

$$\mathbb{P}_n \left(\left| \sup_{v \in \overline{V}_n} Z_n^*(v) - \sup_{v \in \widehat{V}_n} \bar{Z}_n^*(v) \right| > o(\delta_n) \mid \mathcal{D}_n \right) = o_{\mathbb{P}_n}(1/\ell_n), \quad (\text{C.4})$$

which implies by Lemma 11 that for some $\varepsilon_n \searrow 0$, $\text{wp} \rightarrow 1$, for all $p \in (\varepsilon_n, 1 - \varepsilon_n)$

$$k_{n, \overline{V}_n}(p) \leq \kappa_{n, \overline{V}_n}(p + \varepsilon_n) + o(\delta_n) \quad (\text{C.5})$$

where the terms $o(\delta_n)$ are different in different places. By C.3, for any fixed $p \in (0, 1)$,

$$\kappa_{\overline{V}_n}(p + \varepsilon_n) \leq a_n(\overline{V}_n) + c(p + \varepsilon_n)/a_n(\overline{V}_n) = a_n(\overline{V}_n) + O(1)/a_n(\overline{V}_n).$$

Thus, combining inequalities above and $o(\delta_n) = o(\bar{a}_n^{-1}) = o(a_n^{-1}(\overline{V}_n))$ by C.2, $\text{wp} \rightarrow 1$,

$$k_{n, \widehat{V}_n}(p) \leq a_n(\overline{V}_n) + O(1)/a_n(\overline{V}_n).$$

Now Parts 2 and 3 follow as in the Proof of Parts 2 and 3 of Theorem 1 using this bound on the simulated critical value instead of the bound on the analytical critical value. \blacksquare

C.5. Proof of Lemma 3 (Concentration on V_0). By S and V

$$\left| \sup_{v \in V_n} Z_n^*(v) - \sup_{v \in V_0} Z_n^*(v) \right| \leq \sup_{\|v-v'\| \leq O_{\mathbb{P}_n}(r_n)} |Z_n^*(v) - Z_n^*(v')| = \sup_{v \in V_0} Z_n^*(v) + o_{\mathbb{P}_n}(\delta_n). \quad (\text{C.6})$$

Conclude similarly to the proof of Lemma 1, using anti-concentration Corollary 1

$$\mathbb{P}_n \left(\sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \widehat{\theta}_n(v)}{s_n(v)} \leq x \right) \geq \mathbb{P}_n \left(\sup_{v \in \widehat{V}_0} Z_n^*(v) + o(\delta_n) \leq x \right) - o(1) \geq \mathbb{P}_n \left(\sup_{v \in \widehat{V}_0} Z_n^*(v) \leq x \right) - o(1)$$

This gives a lower bound. Similarly, using C.3 and C.4 and anti-concentration Corollary 1

$$\begin{aligned} \mathbb{P}_n \left(\sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \widehat{\theta}_n(v)}{s_n(v)} \leq x \right) &\leq \mathbb{P}_n \left(\sup_{v \in \widehat{V}_0} Z_n(v) \frac{\sigma_n(v)}{s_n(v)} \leq x \right) \\ &\leq \mathbb{P}_n \left(\sup_{v \in \widehat{V}_0} Z_n^*(v) - o(\delta_n) \leq x \right) + o(1) \leq \mathbb{P}_n \left(\sup_{v \in \widehat{V}_0} Z_n^*(v) \leq x \right) + o(1) \end{aligned}$$

where $o(\cdot)$ terms above are different in different places, and the first inequality follows from

$$\sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \widehat{\theta}_n(v)}{s_n(v)} \geq \sup_{v \in \widehat{V}_0} \frac{\theta_{n0} - \widehat{\theta}_n(v)}{s_n(v)} = \sup_{v \in \widehat{V}_0} Z_n(v) \frac{\sigma_n(v)}{s_n(v)}.$$

This gives the upper bound. ■

C.6. Proof of Theorem 3 (When Simulation Inference Becomes Sharp). Part 1.

(Size) By Lemma 2 wp $\rightarrow 1$, $\widehat{V}_n \subseteq \overline{V}_n$, so that $k_{n, \widehat{V}_n}(p) \leq k_{n, \overline{V}_n}(p)$ wp $\rightarrow 1$. So let us compare critical values

$$k_{n, \overline{V}_n}(p) = Q_p \left(\sup_{v \in \overline{V}_n} Z_n^*(v) \mid \mathcal{D}_n \right) \text{ and } \kappa_{n, V_0}(p) = Q_p \left(\sup_{v \in V_0} \bar{Z}_n^*(v) \right).$$

The former is data-dependent while the latter is deterministic. Recall that by C.2 wp $\rightarrow 1$ we have (C.4). By V $d_H(\overline{V}_n, V_0) \leq r_n$, and so by S, we have for some $o(\delta_n)$,

$$\mathbb{P}_n \left(\left| \sup_{v \in \overline{V}_n} \bar{Z}_n^*(v) - \sup_{v \in V_0} \bar{Z}_n^*(v) \right| > o(\delta_n) \mid \mathcal{D}_n \right) = o_{\mathbb{P}_n}(1).$$

Combining (C.4) and this relation, we obtain that for some $o(\delta_n)$,

$$\mathbb{P}_n \left(\left| \sup_{v \in \overline{V}_n} Z_n^*(v) - \sup_{v \in V_0} \bar{Z}_n^*(v) \right| > o(\delta_n) \mid \mathcal{D}_n \right) = o_{\mathbb{P}_n}(1).$$

This implies by Lemma 11 that for some $\varepsilon_n \searrow 0$, and any $p \in (\varepsilon_n, 1 - \varepsilon_n)$, wp $\rightarrow 1$,

$$k_{n, \widehat{V}_n}(p) \leq k_{n, \overline{V}_n}(p) \leq \kappa_{n, V_0}(p + \varepsilon_n) + o(\delta_n). \tag{C.7}$$

Hence, for any fixed p ,

$$\begin{aligned}
\mathbb{P}_n \left(\theta_{n0} \leq \widehat{\theta}_{n0}(p) \right) &= \mathbb{P}_n \left(\sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \widehat{\theta}_n(v)}{s_n(v)} \leq k_{n, \widehat{V}_n}(p) \right) \\
&\leq_{(1)} \mathbb{P}_n \left(\sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \widehat{\theta}_n(v)}{s_n(v)} \leq \kappa_{n, V_0}(p + \varepsilon_n) + o(\delta_n) \right) + o(1) \\
&\leq_{(2)} \mathbb{P}_n \left(\sup_{v \in V_0} Z_n^*(v) \leq \kappa_{n, V_0}(p + \varepsilon_n) + o(\delta_n) \right) + o(1) \leq_{(3)} p + \varepsilon_n + o(1) = p + o(1),
\end{aligned}$$

where (1) is by the quantile comparison (C.7), (2) is by Lemma 3, and (3) is by anti-concentration Corollary 1. Combining this with the lower bound of Theorem 2, we have the result.

Parts 2 & 3.(Estimation Risk and Power) We have that by C.3

$$\kappa_{n, V_0}(p + \varepsilon_n) \leq a_n(V_0) + c(p + \varepsilon_n)/a_n(V_0) = a_n(V_0) + O(1)/a_n(V_0).$$

Hence combining this with equation (C.7) we have $\text{wp} \rightarrow 1$

$$k_{n, \widehat{V}_n}(p) \leq a_n(V_0) + O(1)/a_n(V_0) + o(\bar{a}_n^{-1}) = a_n(V_0) + O(1)/a_n(V_0),$$

where $o(\delta_n) = o(\bar{a}_n^{-1}) = o(a_n^{-1}(V_0))$ by C.2. Then Parts 2 and 3 follow identically to the Proof of Parts 2 and 3 of Theorem 1 using this bound on the simulated critical value instead of the bound on the analytical critical value. \blacksquare

APPENDIX D. PROOFS FOR SECTION 4

D.1. Tools and Auxiliary Lemmas. We shall heavily rely on the Talagrand-Samorodnitsky Inequality, which was obtained by Talagrand sharpening earlier results by Samorodnitsky. Here it is restated from van der Vaart and Wellner (1996) Proposition A.2.7, page 442:

Talagrand-Samorodnitsky Inequality: Let X be a separable zero-mean Gaussian process indexed by a set T . Suppose that for some $\Gamma > \sigma(X) = \sup_{t \in T} \sigma(X_t)$, $0 < \epsilon_0 \leq \sigma(X)$,

$$N(\varepsilon, T, \rho) \leq \left(\frac{\Gamma}{\varepsilon} \right)^\nu, \text{ for } 0 < \varepsilon < \epsilon_0,$$

where $N(\varepsilon, T, \rho)$ is the covering number of T by ε -balls w.r.t. the standard deviation metric $\rho(t, t') = \sigma(X_t - X_{t'})$. Then there exists an universal constant D such that for every $\lambda \geq \sigma^2(X)(1 + \sqrt{\nu})/\epsilon_0$ we have

$$P \left(\sup_{t \in T} X_t > \lambda \right) \leq \left(\frac{D\Gamma\lambda}{\sqrt{\nu}\sigma^2(X)} \right)^\nu (1 - \Phi(\lambda/\sigma(X))), \quad (\text{D.1})$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function.

The following lemma is an application of this inequality that we use:

Lemma 12 (Concentration Inequality via Talagrand-Samorodnitsky). *Let Z_n be a separable zero-mean Gaussian process indexed by a set V such that $\sup_{v \in V} \sigma(Z_n(v)) = 1$. Suppose that for some $\Gamma_n(V) > 1$, and $d \geq 1$*

$$N(\varepsilon, V, \rho) \leq \left(\frac{\Gamma_n(V)}{\varepsilon} \right)^d, \quad \text{for } 0 < \varepsilon < 1,$$

where $N(\varepsilon, V, \rho)$ is the covering number of V by ε -balls w.r.t. the standard deviation metric $\rho(v, v') = \sigma(Z_n(v) - Z_n(v'))$. Then for

$$a_n(V) = (2\sqrt{\log L_n(V)}) \vee (1 + \sqrt{d}), \quad L_n(V) := C'_n \left(\frac{\Gamma_n(V)}{\sqrt{d}} \right)^d,$$

where for D denoting Talagrand's constant in (D.1), and C'_n such that

$$C'_n \geq D^d C_d \frac{1}{\sqrt{2\pi}}, \quad C_d := \max_{\lambda \geq 0} \lambda^{d-1} e^{-\lambda^2/4},$$

we have for $z \geq 0$

$$P \left(a_n(V) \left(\sup_{v \in V} Z_n(v) - a_n(V) \right) > z \right) \leq \exp \left(-\frac{z}{2} - \frac{z^2}{4a_n^2(V)} \right) \leq \exp(-z/2).$$

Proof. We apply the TS inequality by setting $t = v$, $X = Z$, $\sigma(X) = 1$, $\epsilon_0 = 1$, $\nu = d$, with $\lambda \geq (1 + \sqrt{d})$, so that

$$\begin{aligned} P \left(\sup_{v \in V} Z_n(v) > \lambda \right) &\leq \left(\frac{D\kappa_n(V)\lambda}{\sqrt{d}} \right)^d (1 - \Phi(\lambda)) \\ &\leq \left(\frac{D\Gamma_n(V)\lambda}{\sqrt{d}} \right)^d \frac{1}{\sqrt{2\pi}} \frac{1}{\lambda} e^{-\lambda^2/2} \leq L_n(V) e^{-\lambda^2/4}. \end{aligned}$$

Setting for $z \geq 0$, $\lambda = \frac{z}{a_n(V)} + a_n(V) \geq (1 + \sqrt{d})$, we obtain

$$L_n(V) \exp \left(-\frac{\lambda^2}{4} \right) \leq \exp \left(-\frac{z}{2} - \frac{z^2}{4a_n^2(V)} \right).$$

■

The following lemma is an immediate consequence of Corollary 2.2.8 of van der Vaart and Wellner (1996).

Lemma 13 (Maximal Inequality for a Gaussian Process). *Let X be a separable zero-mean Gaussian process indexed by a set T . Then for every $\delta > 0$*

$$E \sup_{\rho(s,t) \leq \delta} |X_s - X_t| \lesssim \int_0^\delta \sqrt{\log N(\varepsilon, T, \rho)} d\varepsilon, \quad E \sup_{t \in T} |X_t| \leq \sigma(X) + \int_0^{2\sigma(X)} \sqrt{\log N(\varepsilon, T, \rho)} d\varepsilon,$$

where $\sigma(X) = \sup_{t \in T} \sigma(X_t)$, and $N(\varepsilon, T, \rho)$ is the covering number of T with respect to the semi-metric $\rho(s, t) = \sigma(X_s - X_t)$.

Proof. The first conclusion follows from Corollary 2.2.8 of van der Vaart and Wellner (1996) since covering and packing numbers are related by $N(\varepsilon, T, \rho) \leq D(\varepsilon, T, \rho) \leq N(\varepsilon/2, T, \rho)$. The second conclusion follows from the special case of the first conclusion: for any $t_0 \in T$, $E \sup_{t \in T} |X_t| \leq E|X_{t_0}| + \int_0^{\text{diam}(T)} \sqrt{\log N(\varepsilon, T, \rho)} d\varepsilon \leq \sigma(X) + \int_0^{2\sigma(X)} \sqrt{\log N(\varepsilon, T, \rho)} d\varepsilon$. ■

D.2. Proof of Lemma 5. Step 1. Verification of C.1. This condition holds by inspection, in view of continuity of $v \mapsto p_n(v)$ and by Ω_n and $\widehat{\Omega}_n$ being positive definite.

Step 2. Verification of C.2 Set $\delta_n = 1/\log n$. Condition NS.1 directly assumes C.2(a).

In order to show C.2(b), we employ the maximal inequality stated in Lemma 13. Set $X_t = Z_n^*(v) - Z_n^*(v)$, $t = v$, $T = \mathcal{V}$ and note that for some absolute constant C , conditional on \mathcal{D}_n ,

$$N(\varepsilon, T, \rho) \leq \left(\frac{1 + C\Upsilon_n \text{diam}(T)}{\varepsilon} \right)^d, \quad 0 < \varepsilon < 1,$$

since $\sigma(X_t - X_{t'}) \lesssim \Upsilon_n \|t - t'\|$, $T \subset \mathbb{R}^d$, where Υ_n is an upper bound on the Lipschitz constant of the function

$$v \mapsto \frac{p_n(v)' \Omega_n^{1/2}}{\|p_n(v)' \Omega_n^{1/2}\|} - \frac{p_n(v)' \widehat{\Omega}_n^{1/2}}{\|p_n(v)' \widehat{\Omega}_n^{1/2}\|},$$

where $\text{diam}(T)$ is the diameter of set T under the Euclidian metric. Using inequality (G.6) we can bound

$$\Upsilon_n \leq 2L_n \frac{\lambda_{\max}(\Omega_n^{1/2})}{\lambda_{\min}(\Omega_n^{1/2})} + 2L_n \frac{\lambda_{\max}(\widehat{\Omega}_n^{1/2})}{\lambda_{\min}(\widehat{\Omega}_n^{1/2})} = O_{P_n}(L_n),$$

where L_n is the constant defined in NS.1, and by assumption $\log L_n \lesssim \log n$. Here we use the fact the eigenvalues of Ω_n and $\widehat{\Omega}_n$ are bounded away from zero and from above by NS.1 and NS.2. Therefore, $\log N(\varepsilon, T, \rho) \lesssim \log n + \log(1/\varepsilon)$.

Using (G.6) again,

$$\begin{aligned} \sigma(X) &\lesssim \sup_{v \in \mathcal{V}} \left\| \frac{p_n(v)' \Omega_n^{1/2}}{\|p_n(v)' \Omega_n^{1/2}\|} - \frac{p_n(v)' \widehat{\Omega}_n^{1/2}}{\|p_n(v)' \widehat{\Omega}_n^{1/2}\|} \right\| \\ &\leq \sup_{v \in \mathcal{V}} 2 \frac{\|p_n(v)' (\widehat{\Omega}_n^{1/2} - \Omega_n^{1/2})\|}{\|p_n(v)' \Omega_n^{1/2}\|} \leq \sup_{v \in \mathcal{V}} 2 \frac{\|p_n(v)' \Omega_n^{1/2} (\Omega_n^{1/2} \widehat{\Omega}_n^{-1/2} - I)\|}{\|p_n(v)' \Omega_n^{1/2}\|} \\ &\leq \|\Omega_n^{1/2} \widehat{\Omega}_n^{-1/2} - I\| \leq \|\Omega_n^{-1/2}\| \|\widehat{\Omega}_n^{1/2} - \Omega_n^{1/2}\| = O_{P_n}(n^{-b}) \end{aligned}$$

for some constant $b > 0$, where we have used that the eigenvalues of Ω_n and $\widehat{\Omega}_n$ are bounded away from zero and from above under NS.1 and NS.2, and the assumption $\|\widehat{\Omega}_n - \Omega_n\| = O_{P_n}(n^{-b})$. Hence

$$E \left(\sup_{t \in T} |X_t| \mid \mathcal{D}_n \right) \lesssim \sigma(X) + \int_0^{2\sigma(X)} \sqrt{\log(n/\varepsilon)} d\varepsilon = O_{P_n}(n^{-b} \sqrt{\log n}).$$

Hence for each $C > 0$

$$P_n \left(\sup_{v \in \mathcal{V}} |Z_n^*(v) - Z_n^*(\tilde{v})| > C\delta_n \mid \mathcal{D}_n \right) \lesssim \frac{1}{C\delta_n} O_{P_n}(n^{-b} \sqrt{\log n}) = o_{P_n}(1/\ell_n),$$

which verifies C.2(b).

Step 3. Verification of C.3. We shall employ Lemma 12, which has the required notation in place. We only need to compute an upper bound on the covering numbers $N(\varepsilon, V, \rho)$ for the process Z_n^* . We have that

$$\begin{aligned} \sigma(Z_n^*(v) - Z_n^*(\tilde{v})) &\leq \left\| \frac{p_n(v)' \Omega_n^{1/2}}{\|p_n(v)' \Omega_n^{1/2}\|} - \frac{p_n(\tilde{v})' \Omega_n^{1/2}}{\|p_n(\tilde{v})' \Omega_n^{1/2}\|} \right\| \leq 2 \left\| \frac{(p_n(v) - p_n(\tilde{v}))' \Omega_n^{1/2}}{\|p_n(v)' \Omega_n^{1/2}\|} \right\| \\ &\leq 2L_n \frac{\lambda_{\max}(\Omega_n^{1/2})}{\lambda_{\min}(\Omega_n^{1/2})} \|v - \tilde{v}\| \leq CL_n \|v - \tilde{v}\|, \end{aligned}$$

where C is some constant that does not depend on n , by the eigenvalues of Ω_n bounded away from zero and from above. Hence it follows that

$$N(\varepsilon, V, \rho) \leq \left(\frac{1 + CL_n \text{diam}(V)}{\varepsilon} \right)^d, \quad 0 < \varepsilon < 1,$$

where the diameter of V is measured by the Euclidian metric. Condition C.3 now follow by Lemma 12, with $a_n(V) = (2\sqrt{\log L_n(V)}) \vee (1 + \sqrt{d})$, $L_n(V) = C' (1 + CL_n \text{diam}(V))^d$. where C' is a constant from Lemma 12.

Step 4. Verification of C.4. Under Condition NS, we have that

$$a_n(V) \leq \bar{a}_n := a_n(\mathcal{V}) \lesssim \sqrt{\log \ell_n + \log n} \lesssim \sqrt{\log n},$$

so that C.4(a) follows if $\sqrt{\log n} \sqrt{\xi^2(K)/n} \rightarrow 0$.

To verify C.4(b) note that uniformly in $v \in \mathcal{V}$,

$$\begin{aligned} \left| \frac{\|p_n(v)' \widehat{\Omega}_n^{1/2}\|}{\|p_n(v)' \Omega_n^{1/2}\|} - 1 \right| &\leq \left| \frac{\|p_n(v)' \widehat{\Omega}_n^{1/2}\| - \|p_n(v)' \Omega_n^{1/2}\|}{\|p_n(v)' \Omega_n^{1/2}\|} \right| \\ &\leq \frac{\|p_n(v)' (\widehat{\Omega}_n^{1/2} - \Omega_n^{1/2})\|}{\|p_n(v)' \Omega_n^{1/2}\|} \leq \frac{\|p_n(v)' \Omega_n^{1/2} (\Omega_n^{-1/2} \widehat{\Omega}_n^{-1/2} - I)\|}{\|p_n(v)' \Omega_n^{1/2}\|} \\ &\leq \|\Omega_n^{1/2} \widehat{\Omega}_n^{-1/2} - I\| \leq \|\Omega_n^{-1/2}\| \|\widehat{\Omega}_n^{1/2} - \Omega_n^{1/2}\| = o_{\mathbb{P}_n}(\delta_n / \bar{a}_n), \end{aligned}$$

by $\|\widehat{\Omega}_n^{1/2} - \Omega_n^{1/2}\| = O_{\mathbb{P}_n}(n^{-b})$ and $\|\Omega_n^{-1/2}\|$ bounded, both implied by the assumptions. ■

D.3. Proof of Lemma 6. To show claim (1), we need to establish that for $\varphi_n = o(1) \cdot \left(\frac{1}{L_n \sqrt{\log n}}\right)$, with any $o(1)$ term, we have that $\sup_{\|v - \tilde{v}\| \leq \varphi_n} |Z_n^*(v) - Z_n^*(\tilde{v})| = o_{\mathbb{P}_n}(1)$.

Consider the stochastic process $X = \{Z_n(v), v \in \mathcal{V}\}$. We shall use the standard maximal inequality stated in Lemma 13. From the proof of Lemma 5 we have $\sigma(Z_n^*(v) - Z_n^*(\tilde{v})) \leq CL_n \|v - \tilde{v}\|$, where C is some constant that does not depend on n , and $\log N(\varepsilon, V, \rho) \lesssim \log n + \log(1/\varepsilon)$. Since $\|v - \tilde{v}\| \leq \varphi_n \implies \sigma(Z_n^*(v) - Z_n^*(\tilde{v})) \leq C \frac{o(1)}{\sqrt{\log n}}$ we have

$$E \sup_{\|v - \tilde{v}\| \leq \varphi_n} |X_s - X_t| \lesssim \int_0^C \frac{o(1)}{\sqrt{\log n}} \sqrt{\log(n/\varepsilon)} d\varepsilon \lesssim \frac{o(1)}{\sqrt{\log n}} \sqrt{\log n} = o(1).$$

Hence the conclusion follows from Markov's Inequality.

Under Condition V by Lemma 2 $r_n \lesssim \left(\sqrt{\log n \frac{\xi^2(K)}{n}}\right)^{1/\rho_n} c_n^{-1}$, so $r_n = o(\varphi_n)$ if

$$\left(\sqrt{\log n \frac{\xi^2(K)}{n}}\right)^{1/\rho_n} c_n^{-1} = o\left(\frac{1}{L_n \sqrt{\log n}}\right). \quad (\text{D.2})$$

Thus, Condition S holds. The remainder of the lemma follows by direct calculation. ■

APPENDIX E. STRONG APPROXIMATION FOR ASYMPTOTICALLY LINEAR SERIES ESTIMATORS

Here we establish strong approximation for series estimators considered in Section 4.2.

Theorem 7 (Strong Approximation For Asymptotically Linear Series Estimators). *Let $(A, \mathcal{A}, \mathbb{P}_n)$ be the probability space for each n , and let $n \rightarrow \infty$. Let $\delta_n \rightarrow 0$ be a sequence of constants converging to 0 at no faster than a polynomial rate in n . Assume (a) the series estimator has the form $\widehat{\theta}_n(v) = p_n(v)' \widehat{\beta}_n$, where $p_n(v) := (p_{n,1}(v), \dots, p_{n,K_n}(v))$ is a collection of K_n -dimensional approximating functions such that $K_n \rightarrow \infty$ and $\widehat{\beta}_n$ is a K_n -vector of estimates; (b) The estimator $\widehat{\beta}_n$ satisfies an asymptotically linear representation*

around some K_n -dimensional vector β_n

$$\Omega_n^{-1/2} \sqrt{n}(\widehat{\beta}_n - \beta_n) = n^{-1/2} \sum_{i=1}^n u_{i,n} + r_n, \quad \|r_n\| = o_{\mathbb{P}_n}(\delta_n), \quad (\text{E.1})$$

$$u_{i,n}, i = 1, \dots, n \text{ are independent with } E_{\mathbb{P}_n}[u_{i,n}] = 0, E_{\mathbb{P}_n}[u_{i,n}u'_{i,n}] = I_{K_n}, \text{ and} \quad (\text{E.2})$$

$$\Delta_n = \sum_{i=1}^n E\|u_{i,n}\|^3/n^{3/2} \text{ such that } K_n\Delta_n/\delta_n^3 \rightarrow 0, \quad (\text{E.3})$$

where Ω_n is a sequence of $K_n \times K_n$ invertible matrices. (c) The function $\theta_n(v)$ admits the approximation $\theta_n(v) = p_n(v)' \beta_n + A_n(v)$, where the approximation error $A_n(v)$ satisfies $\sup_{v \in \mathcal{V}} \sqrt{n}|A_n(v)|/\|g_n(v)\| = o(\delta_n)$, for $g_n(v) := p_n(v)' \Omega_n^{1/2}$. Then we can find a random normal vector $\mathcal{N}_n =_d \mathcal{N}(0, I_{K_n})$ such that $\|\Omega_n^{-1/2} \sqrt{n}(\widehat{\beta}_n - \beta_n) - \mathcal{N}_n\| = o_{\mathbb{P}_n}(\delta_n)$ and

$$\sup_{v \in \mathcal{V}} \left| \frac{\sqrt{n}(\widehat{\theta}_n(v) - \theta_n(v))}{\|g_n(v)\|} - \frac{g_n(v)}{\|g_n(v)\|} \mathcal{N}_n \right| = o_{\mathbb{P}_n}(\delta_n).$$

The following corollary is used in examples in the main text.

Corollary 2 (A Leading Case of Influence Function). *Suppose the conditions of Theorem 7 hold with $u_{i,n} := \Omega_n^{-1/2} Q_n^{-1} p_n(V_i) \epsilon_i$, where (V_i, ϵ_i) are i.i.d. with $E_{\mathbb{P}_n}[\epsilon_i p_n(V_i)] = 0$, $S_n := E_{\mathbb{P}_n}[\epsilon_i^2 p_n(V_i) p_n(V_i)']$, and $\Omega_n := Q_n^{-1} S_n (Q_n^{-1})'$, where Q_n^{-1} is some non-random invertible matrix, not necessarily symmetric, and eigenvalues of S_n^{-1} are bounded above by τ_n ; $E_{\mathbb{P}_n}[\|\epsilon_i\|^3]$ is bounded. Then, the key growth restriction on the number of series terms $K_n \Delta_n / \delta_n^3 \rightarrow 0$ holds if for $C_n := \sup_{v \in \mathcal{V}} \max_j |p_{nj}(v)|$*

$$\tau_n^3 C_n^6 K_n^5 / (n \delta_n^6) \rightarrow 0,$$

or, more generally, if $\tau_n^{3/2} \delta_n^{-3} K_n^{3/2} \max_{v \in \mathcal{V}} \sum_{j=1}^{K_n} |p_{nj}(v)|^3 / n^{1/2} \rightarrow 0$.

Remark 7 (Applicability). In the paper $\delta_n = 1/\log n$. Sufficient conditions for linear approximation (b) follow from results in the literature on series estimation, e.g. Andrews (1991), Newey (1995), and Newey (1997), and Belloni, Chernozhukov, and Fernandez-Val (2011). See also Chen (2007) and references therein for a general overview of sieve estimation and recent developments. The main text provides several examples, including mean and quantile regression, with primitive conditions that provide sufficient conditions for the linear approximation. ■

E.1. Proof of Theorem 7 and Corollary 2. The first step of our proof uses Yurinskii's (1977) coupling. For completeness we now state the formal result from Pollard (2002), page 244.

Yurinskii's Coupling: Consider a sufficiently rich probability space (A, \mathcal{A}, P) . Let ξ_1, \dots, ξ_n be independent K_n -vectors with $E\xi_i = 0$ for each i , and $\Delta := \sum_i E\|\xi_i\|^3$ finite. Let $S = \xi_1 + \dots + \xi_n$. For each $\delta > 0$ there exists a random vector T with $N(0, \text{var}(S))$ distribution such that

$$P\{\|S - T\| > 3\delta\} \leq C_0 B \left(1 + \frac{|\log(1/B)|}{K_n}\right) \text{ where } B := \Delta K_n \delta^{-3},$$

for some universal constant C_0 .

The proof has two steps: in the first, we couple the estimator $\sqrt{n}(\widehat{\beta}_n - \beta_n)$ with the normal vector; in the second, we establish the strong approximation.

STEP 1. In order to apply the coupling, consider

$$\sum_{i=1}^n \xi_i, \quad \xi_i = u_{i,n}/\sqrt{n} \sim (0, I_{K_n}/n),$$

Then we have that $\sum_{i=1}^n E\|\xi_i\|^3 = \Delta_n$. Therefore, by Yurinskii's coupling,

$$P_n \left\{ \left\| \sum_{i=1}^n \xi_i - \mathcal{N}_n \right\| \geq 3\delta_n \right\} \lesssim K_n \Delta_n / \delta_n^3 \rightarrow 0,$$

Combining this with the assumption on the linearization error r_n , we obtain

$$\begin{aligned} \|\Omega_n^{-1/2} \sqrt{n}(\widehat{\beta}_n - \beta_n) - \mathcal{N}_n\| &\leq \left\| \sum_{i=1}^n \xi_i - \mathcal{N}_n \right\| + \|\Omega_n^{-1/2} \sqrt{n}(\widehat{\beta}_n - \beta_n) - \sum_{i=1}^n \xi_i\| \\ &= o_{P_n}(\delta_n) + r_n = o_{P_n}(\delta_n). \end{aligned}$$

STEP 2. Using the result of Step 1 and that

$$\frac{\sqrt{n} p_n(v)'(\widehat{\beta}_n - \beta_n)}{\|g_n(v)\|} = \frac{\sqrt{n} g_n(v)' \Omega_n^{-1/2} (\widehat{\beta}_n - \beta_n)}{\|g_n(v)\|},$$

we conclude that

$$\begin{aligned} |S_n(v)| &:= \left| \frac{\sqrt{n} g_n(v)' \Omega_n^{-1/2} (\widehat{\beta}_n - \beta_n)}{\|g_n(v)\|} - \frac{g_n(v)' \mathcal{N}_n}{\|g_n(v)\|} \right| \\ &\leq \left\| \sqrt{n} \Omega_n^{-1/2} (\widehat{\beta}_n - \beta_n) - \mathcal{N}_n \right\| = o_{P_n}(\delta_n), \end{aligned} \tag{E.4}$$

uniformly in $v \in \mathcal{V}$. Finally,

$$\begin{aligned}
& \sup_{v \in \mathcal{V}} \left| \frac{\sqrt{n}(\widehat{\theta}_n(v) - \theta_n(v))}{\|g_n(v)\|} - \frac{g_n(v)' \mathcal{N}_n}{\|g_n(v)\|} \right| \\
& \leq \sup_{v \in \mathcal{V}} \left| \frac{\sqrt{n}(\widehat{\theta}_n(v) - \theta_n(v))}{\|g_n(v)\|} - \frac{\sqrt{n}g_n(v)' \Omega_n^{-1/2}(\widehat{\beta}_n - \beta_n)}{\|g_n(v)\|} \right| \\
& \quad + \sup_{v \in \mathcal{V}} \left| \frac{\sqrt{n}g_n(v)' \Omega_n^{-1/2}(\widehat{\beta}_n - \beta_n)}{\|g_n(v)\|} - \frac{g_n(v)' \mathcal{N}_n}{\|g_n(v)\|} \right| \\
& = \sup_{v \in \mathcal{V}} |\sqrt{n}A_n(v)/\|g_n(v)\|| + \sup_{v \in \mathcal{V}} |S_n(v)| = o(\delta_n) + o_{\mathbb{P}_n}(\delta_n),
\end{aligned}$$

using the assumption on the approximation error $A_n(v) = \theta(v) - p_n(v)' \beta_n$ and (E.4). This proves the theorem.

STEP 3. To show the corollary note that

$$\begin{aligned}
E_{\mathbb{P}_n} \|u_{i,n}\|^3 & \leq \text{maxeig}(S_n^{-1})^{3/2} \cdot E_{\mathbb{P}_n} \|p_n(V_i) \epsilon_i\|^3 = \tau_n^{3/2} \cdot K_n^{3/2} E_{\mathbb{P}_n} \left(\epsilon_i^2 \frac{1}{K_n} \sum_{j=1}^{K_n} p_{nj}(V_i)^2 \right)^{3/2} \\
& \leq \tau_n^{3/2} \cdot K_n^{3/2} \max_{v \in \mathcal{V}} \frac{1}{K_n} \sum_{j=1}^{K_n} |p_{nj}(v)|^3 E_{\mathbb{P}_n} |\epsilon_i|^3 \\
& \leq \tau_n^{3/2} \cdot K_n^{3/2} \max_j \sup_{v \in \mathcal{V}} |p_{nj}(v)|^3 E_{\mathbb{P}_n} |\epsilon_i|^3 \lesssim \tau_n^{3/2} K_n^{3/2} C_n^3,
\end{aligned}$$

using the boundedness assumptions stated in the corollary. More generally, we have that $E_{\mathbb{P}_n} \|u_{i,n}\|^3 \lesssim \tau_n^{3/2} K_n^{3/2} \max_{v \in \mathcal{V}} \frac{1}{K_n} \sum_{j=1}^{K_n} |p_{nj}(v)|^3 E_{\mathbb{P}_n} |\epsilon_i|^3$. This implies the corollary. \blacksquare

APPENDIX F. STRONG APPROXIMATION FOR LOCAL METHODS

In establishing strong approximation for kernel-type estimators we use the following result, Theorem 1.1 in Rio (1994), which builds on the earlier results of Massart (1989). The proofs for all subsequent results in this section, which are novel to this paper, are provided in Appendix G of the supplementary material.

F.1. Rio-Massart Coupling. Consider a sufficiently rich probability space $(A, \mathcal{A}, \mathbb{P})$. If not, then we can always enrich the original space by taking the product with $[0, 1]$ equipped with the uniform measure over Borel sets of $[0, 1]$. Consider a suitably measurable, namely image admissible Suslin, function class \mathcal{F} containing functions $f : I^d \rightarrow I$ for $I = (-1, 1)$. A function class \mathcal{F} is of uniformly bounded variation of at most $K(\mathcal{F})$ if

$$TV(\mathcal{F}) := \sup_{f \in \mathcal{F}} \sup_{g \in \mathcal{D}_c(I^d)} \left(\int_{\mathbb{R}^d} f(x) \operatorname{div} g(x) / \|g\|_{\infty} dx \right) \leq K(\mathcal{F}),$$

where $\mathcal{D}_c(I^d)$ is the space of C^∞ functions taking values in \mathbb{R}^d with compact support included in I^d , and where $\operatorname{div}g(x)$ is the divergence of $g(x)$. Suppose the function class \mathcal{F} obeys the following uniform L_1 covering condition

$$\sup_Q N(\epsilon, \mathcal{F}, L_1(Q)) \leq C(\mathcal{F})\epsilon^{d(\mathcal{F})},$$

where sup is taken over probability measures with finite support, and $N(\epsilon, \mathcal{F}, L_1(Q))$ is the covering number under the $L_1(Q)$ norm on \mathcal{F} . Let X_1, \dots, X_n be an i.i.d. sample on the probability space (A, \mathcal{A}, P) from density f_X with support on I^d , bounded from above and away from zero. Let P_X be the measure induced by f_X . Then there exists a P_X -Brownian Bridge \mathbb{B}_n with a.s. continuous paths with respect to the $L^1(P_X)$ metric such that for any positive $t \geq C \log n$,

$$\mathbb{P} \left(\sqrt{n} \sup_{f \in \mathcal{F}} |\mathbb{G}_n(f) - \mathbb{B}_n(f)| \geq C \sqrt{tn^{\frac{d-1}{d}} K(\mathcal{F})} + Ct\sqrt{\log n} \right) \leq e^{-t},$$

where constant C depends only on d , $C(\mathcal{F})$, and $d(\mathcal{F})$.

F.2. Strong Approximation for Kernel-Type Estimators. We shall use the following technical condition in what follows.

Condition R. *The random $(J+d)$ -vector (U_i, Z_i) obeys $U_i = (U_{i,1}, \dots, U_{i,J}) = \varphi_n(X_{i,1})$, and $Z_i = \tilde{\varphi}_n(X_{2i})$, where $X_i = (X'_{1i}, X'_{2i})'$ is a $(d_1 + d)$ -vector with $1 \leq d_1 \leq J$, which has density bounded away from zero by \underline{f} and above by \bar{f} on the support I^{d_1+d} , where $\varphi_n : I^{d_1} \mapsto I^J$ and $\sum_{l=1}^{d_1} \int_{I^{d_1}} |D_{x_{1l}} \varphi_n(x_1)| dx_1 \leq B$, where $D_{x_{1l}} \varphi_n(x_1)$ denotes the weak derivative with respect to the l -th component of x_1 , and $\tilde{\varphi}_n : I^d \mapsto I^d$ is continuously differentiable such that $\max_{k \leq d} \sup_{x_2} |\partial \tilde{\varphi}_n(x_2) / \partial x_{2k}| \leq B$ and $|\det \partial \tilde{\varphi}_n(x_2) / \partial x_2| \geq c > 0$, where $\partial \tilde{\varphi}_n(x_2) / \partial x_{2k}$ denotes the partial derivative with respect to the k -th component of x_2 . The constants $J, B, \underline{f}, \bar{f}, c$ and vector dimensions do not depend on n . ($|\cdot|$ denotes ℓ_1 norm.)*

A simple example of (U_i, Z_i) satisfying this condition is given in Corollary 3 below.

Theorem 8 (Strong Approximation for Local Estimators). *Consider a suitably enriched probability space (A, \mathcal{A}, P_n) for each n . Let $n \rightarrow \infty$. Assume the following conditions hold for each n : (a) There are n i.i.d. $(J+d)$ -dimensional random vectors of the form (U_i, Z_i) that obey Condition R, and the density f_n of Z is bounded from above and away from zero on the set \mathcal{Z} , uniformly in n . (b) Let $v = (z, j)$ and $\mathcal{V} = \mathcal{Z} \times \{1, \dots, J\}$, where $\mathcal{Z} \subseteq I^d$. The kernel estimator $v \mapsto \hat{\theta}_n(v)$ of some target function $v \mapsto \theta_n(v)$ has an asymptotic linear*

expansion uniformly in $v \in \mathcal{V}$

$$(nh_n^d)^{1/2}(\widehat{\theta}_n(v) - \theta_n(v)) = \mathbb{G}_n(g_v) + o_{\mathbb{P}_n}(\delta_n), \quad g_v(U_i, Z_i) := \frac{1}{(h_n^d)^{1/2} f_n(z)} e'_j U_i \mathbf{K} \left(\frac{z - Z_i}{h_n} \right),$$

where $e'_j U_i \equiv U_{ij}$, \mathbf{K} is twice continuously differentiable product kernel function with support on I^d , $\int \mathbf{K}(u) du = 1$, and h_n is a sequence of bandwidths that converges to zero, (c) for a given $\delta_n \searrow 0$, the bandwidth sequence obeys: $(n^{-1/(d+d_1)} h_n^{-1} \log n)^{1/2} + (nh_n^d)^{-1/2} \log^{3/2} n = o(\delta_n)$. Then there exists a sequence of centered \mathbb{P}_n -Gaussian Bridges \mathbb{B}_n such that

$$\sup_{v \in \mathcal{V}} |(nh_n^d)^{1/2}(\widehat{\theta}_n(v) - \theta_n(v)) - \mathbb{B}_n(g_v)| = o_{\mathbb{P}_n}(\delta_n).$$

Moreover, the paths of $v \mapsto \mathbb{B}_n(g_v)$ can be chosen to be continuous a.s.

Remark 8. Conditions (a) and (b) cover standard conditions in the literature, imposing a uniform Bahadur expansion for kernel-type estimators, which have been shown in Masry (1996) and Kong, Linton, and Xia (2010) for kernel mean regression estimators and also local polynomial estimators under fairly general conditions. Implicit in the expansion above is that the asymptotic bias is negligible, which can be achieved by the standard procedure of undersmoothing, i.e. choosing the bandwidth to be smaller than the rate-optimal bandwidths.

Corollary 3 (A Simple Leading Case for Moment Inequalities Application). *Suppose that (U_i, Z_i) has bounded support, which we then take to be a subset of I^{J+d} without loss of generality. Suppose that $U_i = (U_{ij}, j = 1, \dots, J)$ where for the first $J_0/2$ pairs of terms, we have $U_{ij} = -U_{i,j+1}, j = 1, 3, \dots, J_0 - 1$. Let $\mathcal{J} = \{1, 3, \dots, J_0 - 1, J_0 + 1, J_0 + 2, \dots\}$. Suppose that $(U_{ij}, Z_i, j \in \mathcal{J})$ have joint density bounded from above and below by some constants \bar{f} and \underline{f} . Suppose these constants and d, J , and $d_1 = |\mathcal{J}|$ do not depend on n . Then Condition R holds, and the conclusions of Theorem 8 then hold under the additional conditions imposed in the theorem.*

Note that Condition R allows for much more general error terms and regressors. For example, it allows error terms U_i not to have a density at all, and Z_i need only have density bounded from above.

The next theorem shows that the Brownian bridge $\mathbb{B}_n(g_v)$ can be approximately simulated via the Gaussian multiplier method. That is, consider the following symmetrized process

$$\mathbb{G}_n^o(g_v) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i g_v(U_i, Z_i) = \mathbb{G}_n(\xi g_v), \quad (\text{F.1})$$

where ξ_1, \dots, ξ_n are i.i.d $N(0, 1)$, independent of the data \mathcal{D}_n and of $\{(U_i, Z_i)\}_{i=1}^n$, which are i.i.d. copies of (U, Z) . Conditional on the data this is a Gaussian process with a covariance function which is a consistent estimate of the covariance function of $v \mapsto \mathbb{B}_n(g_v)$. The theorem below shows that the uniform distance between $\mathbb{B}_n(g_v)$ and $\mathbb{G}_n^o(g_v)$ is small with an explicit probability bound. Note that if the function class $\{g_v, v \in \mathcal{V}\}$ were Donsker, then such a result would follow from the multiplier functional central limit theorem. In our case, this function class is not Donsker, so we require a different argument.

Theorem 9 (Multiplier Method for Kernel Processes). *Consider a suitably enriched probability space (A, \mathcal{A}, P_n) for each n . Let $n \rightarrow \infty$. Assume the following conditions hold for each n : (a) There are n i.i.d. $(J + d)$ -dimensional random vectors of the form (U_i, Z_i) that obey Condition R, and the density f_n of Z is bounded from above and away from zero on the set \mathcal{Z} , uniformly in n . (b) Let $v = (z, j)$ and $\mathcal{V} = \mathcal{Z} \times \{1, \dots, J\}$, where $\mathcal{Z} \subseteq I^d$. Let*

$$g_v(U_i, Z_i) := \frac{1}{(h_n^d)^{1/2} f_n(z)} e_j' U_i \mathbf{K} \left(\frac{z - Z_i}{h_n} \right),$$

where $e_j' U_i \equiv U_{ij}$, \mathbf{K} is a twice continuously differentiable product kernel function with support on I^d , $\int \mathbf{K}(u) du = 1$, and h_n is a sequence of bandwidths that converges to zero, (c) for a given $\delta_n \searrow 0$, the following holds: $\log n (n^{(\frac{-1}{d+d_1+1})} h_n^{-1})^{1/2} + (n h_n^d)^{-1/2} \log^2 n = o(\delta_n)$. Then there is an independent copy of $\bar{\mathbb{B}}_n$ of the P_n -Gaussian Bridge \mathbb{B}_n appearing in Theorem 8 such that

$$P_n \left(\sup_{v \in \mathcal{V}} |\bar{\mathbb{B}}_n(g_v) - \mathbb{G}_n^o(g_v)| \geq o(\delta_n) \right) \lesssim 1/n,$$

for some $o(\delta_n)$ term.

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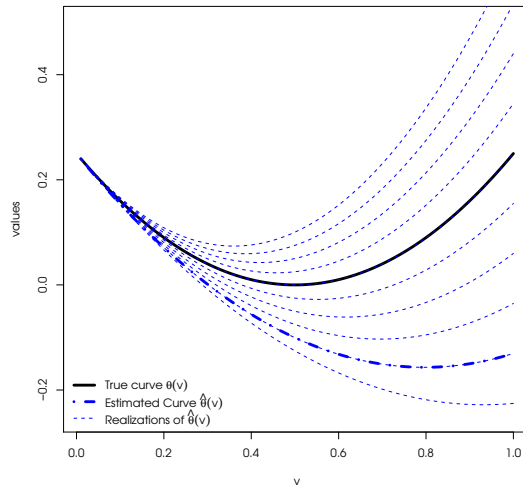


FIGURE 1. This figure illustrates how variation in the precision of the analog estimator at different points may impede inference. The solid curve is the true bounding function $\theta(v)$, while the dash-dot curve is a single realization of its estimator, $\hat{\theta}(v)$. The lighter dashed curves depict eight additional representative realizations of the estimator, illustrating its precision at different values of v . The minimum of the estimator $\hat{\theta}(v)$ is indeed quite far from the minimum of $\theta(v)$, making the empirical upper bound unduly tight.

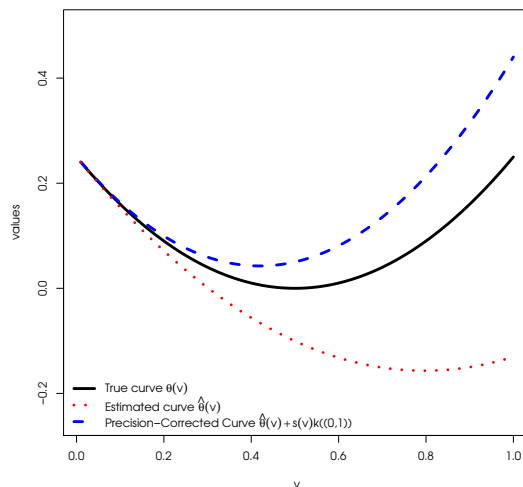


FIGURE 2. This figure depicts a precision-corrected curve (dashed curve) that adjusts the boundary estimate $\hat{\theta}(v)$ (dotted curve) by an amount proportional to its point-wise standard error. The minimum of the precision-corrected curve is closer to the minimum of the true curve (solid) than the minimum of $\hat{\theta}(v)$, removing the downward bias.

TABLE 1. Results for Monte Carlo Experiments (Series Estimation using B-splines)

DGP	Sample Size	Critical Value	Ave. Smoothing Parameter	Cov. Prob.	False Cov. Prob.	Ave. Argmax Set	
						Min.	Max.
CLR with Series Estimation using B-splines							
Estimating V_n ?							
1	500	No	8.872	0.954	0.150	-1.800	1.792
1	500	Yes	8.872	0.954	0.150	-1.800	1.792
1	1000	No	9.692	0.951	0.013	-1.801	1.797
1	1000	Yes	9.692	0.951	0.013	-1.801	1.797
2	500	No	8.963	0.993	0.774	-1.800	1.792
2	500	Yes	8.963	0.984	0.658	-0.744	0.743
2	1000	No	9.778	0.997	0.609	-1.801	1.797
2	1000	Yes	9.778	0.982	0.458	-0.656	0.659
3	500	No	10.516	0.995	0.897	-1.800	1.792
3	500	Yes	10.516	0.982	0.756	-0.335	0.336
3	1000	No	12.187	0.994	0.682	-1.801	1.797
3	1000	Yes	12.187	0.973	0.454	-0.282	0.282
4	500	No	14.872	0.996	0.000	-1.800	1.792
4	500	Yes	14.872	0.970	0.000	-0.104	0.104
4	1000	No	15.994	0.998	0.000	-1.801	1.797
4	1000	Yes	15.994	0.977	0.000	-0.089	0.089

TABLE 2. Results for Monte Carlo Experiments (AS)

DGP	Sample Size	Critical Value	Cov. Prob.	False Cov. Prob.
AS with CvM (Cramér-von Mises-type statistic)				
1	500	PA/Asy	0.959	0.007
1	500	GMS/Asy	0.955	0.007
1	1000	PA/Asy	0.958	0.000
1	1000	GMS/Asy	0.954	0.000
2	500	PA/Asy	1.000	1.000
2	500	GMS/Asy	1.000	0.977
2	1000	PA/Asy	1.000	1.000
2	1000	GMS/Asy	1.000	0.933
3	500	PA/Asy	1.000	1.000
3	500	GMS/Asy	1.000	1.000
3	1000	PA/Asy	1.000	1.000
3	1000	GMS/Asy	1.000	1.000
4	500	PA/Asy	1.000	1.000
4	500	GMS/Asy	1.000	1.000
4	1000	PA/Asy	1.000	1.000
4	1000	GMS/Asy	1.000	1.000

TABLE 3. Results for Monte Carlo Experiments (Other Estimation Methods)

DGP	Sample Size	Critical Value	Ave. Smoothing Parameter	Cov. Prob.	False Cov. Prob.	Ave. Argmax Set Min.	Max.
CLR with Series Estimation using Polynomials							
Estimating V_n ?							
1	500	No	5.524	0.954	0.086	-1.800	1.792
1	500	Yes	5.524	0.954	0.086	-1.800	1.792
1	1000	No	5.646	0.937	0.003	-1.801	1.797
1	1000	Yes	5.646	0.937	0.003	-1.801	1.797
2	500	No	8.340	0.995	0.744	-1.800	1.792
2	500	Yes	8.340	0.989	0.602	-0.724	0.724
2	1000	No	9.161	0.996	0.527	-1.801	1.797
2	1000	Yes	9.161	0.977	0.378	-0.619	0.620
3	500	No	8.350	0.998	0.809	-1.800	1.792
3	500	Yes	8.350	0.989	0.612	-0.300	0.301
3	1000	No	9.155	0.996	0.560	-1.801	1.797
3	1000	Yes	9.155	0.959	0.299	-0.253	0.252
4	500	No	8.254	1.000	0.000	-1.800	1.792
4	500	Yes	8.254	0.999	0.000	-0.081	0.081
4	1000	No	9.167	0.998	0.000	-1.801	1.797
4	1000	Yes	9.167	0.981	0.000	-0.069	0.069
CLR with Local Linear Estimation							
Estimating V_n ?							
1	500	No	0.606	0.923	0.064	-1.799	1.792
1	500	Yes	0.606	0.923	0.064	-1.799	1.792
1	1000	No	0.576	0.936	0.003	-1.801	1.796
1	1000	Yes	0.576	0.936	0.003	-1.801	1.796
2	500	No	0.264	0.995	0.871	-1.799	1.792
2	500	Yes	0.264	0.989	0.808	-0.890	0.892
2	1000	No	0.218	0.996	0.779	-1.801	1.796
2	1000	Yes	0.218	0.990	0.675	-0.776	0.776
3	500	No	0.140	0.995	0.943	-1.799	1.792
3	500	Yes	0.140	0.986	0.876	-0.426	0.424
3	1000	No	0.116	0.992	0.907	-1.801	1.796
3	1000	Yes	0.116	0.986	0.816	-0.380	0.377
4	500	No	0.078	0.991	0.000	-1.799	1.792
4	500	Yes	0.078	0.981	0.000	-0.142	0.142
4	1000	No	0.064	0.997	0.000	-1.801	1.796
4	1000	Yes	0.064	0.991	0.000	-0.127	0.127

TABLE 4. Computation Times of Monte Carlo Experiments

	AS	Series (B-splines)	Series (Polynomials)	Local Linear
Total minutes for simulations	24.00	73.17	61.93	396.95
Average seconds for each test	0.09	0.27	0.23	1.49
Relative Ratio relative to AS	1.00	3.05	2.58	16.54

ON-LINE SUPPLEMENT TO “INTERSECTION BOUNDS: ESTIMATION AND INFERENCE”

VICTOR CHERNOZHUKOV, SOKBAE LEE, AND ADAM M. ROSEN

ABSTRACT. This supplement includes appendices to Chernozhukov, Lee, and Rosen (2011) not included in the main text. Appendix G provides proofs omitted from the main text in order to abide by space constraints. Appendix H illustrates the use of primitive conditions of Example 5 (series estimation of a conditional mean function) in the main text to verify an asymptotic linear representation needed for application of Theorem 7, which delivers strong approximation for series estimators. Appendix I provides some detailed arguments omitted from the main text for Example 7 (local polynomial estimation for conditional moment inequalities). Appendix J compares the local asymptotic power of our inference method to that of Andrews and Shi (2009) in some examples. Appendix K presents the results of some additional Monte Carlo experiments.

KEY WORDS. Bound analysis, conditional moments, partial identification, strong approximation, infinite dimensional constraints, linear programming, concentration inequalities, anti-concentration inequalities, non-Donsker empirical process methods, moderate deviations, adaptive moment selection.

JEL SUBJECT CLASSIFICATION. C12, C13, C14. AMS SUBJECT CLASSIFICATION. 62G05, 62G15, 62G32.

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APPENDIX G. PROOFS OMITTED FROM THE MAIN TEXT.

G.1. Proof of Lemma 2 (Estimation of V_n). There is a single proof for both analytical and simulation methods, but it is convenient for clarity to split the first step of the proof into separate cases. There are four steps in total.

Step 1a. (Bounds on $k_{n,\mathcal{V}}(\gamma_n)$ in Analytical Case) We have that for some constant $\eta > 0$

$$k_{n,\mathcal{V}}(\gamma_n) := \left(\bar{a}_n + \frac{c(\gamma_n)}{\bar{a}_n} \right),$$

$$\kappa_n := \kappa_n(\gamma'_n) := Q_{\gamma'_n} \left(\sup_{v \in \mathcal{V}} Z_n^*(v) \right), \quad \bar{\kappa}_n := 4 \left(\bar{a}_n + \frac{\eta \ell \ell_n}{\bar{a}_n} \right).$$

The claim of this step is that given the sequence γ_n we have for all large n :

$$k_{n,\mathcal{V}}(\gamma_n) \geq \kappa_n(\gamma_n) \tag{G.1}$$

$$3k_{n,\mathcal{V}}(\gamma_n) < \bar{\kappa}_n \tag{G.2}$$

Inequality (G.2) follows from (C.2) in step 2 of the proof of Lemma 1; (G.1) follows immediately from Condition C.3 (with γ_n in place of γ'_n).

Step 1b. (Bounds on $\kappa_{n,\mathcal{V}}(\gamma_n)$ in Simulation Case) We have

$$k_{n,\mathcal{V}}(\gamma_n) := Q_{\gamma_n} \left(\sup_{v \in \mathcal{V}} Z_n^*(v) \mid \mathcal{D}_n \right),$$

$$\kappa_n = \kappa_n(\gamma'_n) := Q_{\gamma'_n} \left(\sup_{v \in \mathcal{V}} \bar{Z}_n^*(v) \right), \quad \bar{\kappa}_n := 4 \left(\bar{a}_n + \frac{\eta \ell \ell_n}{\bar{a}_n} \right).$$

The claim of this step is that given γ_n there is $\gamma'_n = \gamma_n - o(1)$ such that, $\text{wp} \rightarrow 1$

$$k_{n,\mathcal{V}}(\gamma_n) \geq \kappa_n(\gamma'_n) \tag{G.3}$$

$$3k_{n,\mathcal{V}}(\gamma_n) < \bar{\kappa}_n \tag{G.4}$$

To show inequality (G.3), note that by C.2 and Lemma 11 $\text{wp} \rightarrow 1$

$$\kappa_{n,\mathcal{V}}(\gamma_n + o(1/\ell_n)) + o(\delta_n) \geq k_{n,\mathcal{V}}(\gamma_n) \geq \kappa_{n,\mathcal{V}}(\gamma_n - o(1/\ell_n)) - o(\delta_n). \tag{G.5}$$

Hence (G.3) follows from

$$\begin{aligned} & \mathbb{P}_n \left(\sup_{v \in \mathcal{V}} \bar{Z}_n^*(v) \leq x \right) \Big|_{x=k_{n,\mathcal{V}}(\gamma_n)} \\ & \geq_{(1)} \mathbb{P}_n \left(\sup_{v \in \mathcal{V}} \bar{Z}_n^*(v) \leq \kappa_{n,\mathcal{V}}(\gamma_n - o(1/\ell_n)) - o(\delta_n) \right) - o(1) \text{ wp} \rightarrow 1 \\ & \geq_{(2)} \mathbb{P}_n \left(\sup_{v \in \mathcal{V}} \bar{Z}_n^*(v) \leq \kappa_{n,\mathcal{V}}(\gamma_n - o(1/\ell_n)) \right) - o(1) = \gamma_n - o(1/\ell_n) - o(1) =: \gamma'_n, \end{aligned}$$

where (1) holds by (G.5) and by C.2 (b) and (2) holds by anti-concentration Corollary 1.

To show inequality (G.4) note that by C.2(b) and Lemma 11 we have $\text{wp} \rightarrow 1$

$$\kappa_{n,\mathcal{V}}(\gamma_n + o(1/\ell_n)) + o(\delta_n) \leq \bar{a}_n + \frac{c(\gamma_n + o(1/\ell_n))}{\bar{a}_n} + o(\delta_n) \leq \bar{a}_n + \frac{\eta\ell_n + \eta \log 10}{\bar{a}_n} + o(\delta_n),$$

where the last inequality relies on

$$c(\gamma_n + o(1/\ell_n)) \leq -\eta \log((1 - \gamma_n - o(1/\ell_n))) = \eta o(\ell_n) + \eta \log 10,$$

holding for large n by C.3. From this we deduce (G.4).

Step 2. (Lower Containment) We have that for all $v \in V_n$,

$$\begin{aligned} A_n(v) &:= \hat{\theta}_n(v) - \theta_{n0} - \inf_{v \in \mathcal{V}} \left(\hat{\theta}_n(v) + k_{n,\mathcal{V}} s_n(v) \right) \\ &\leq -Z_n(v) \sigma_n(v) + \kappa_n \sigma_n(v) + \sup_{v \in \mathcal{V}} \{ \theta_{n0} - \hat{\theta}_n(v) - k_{n,\mathcal{V}}(\gamma_n) s_n(v) \} := B_n(v) \end{aligned}$$

since $\theta_n(v) \leq \theta_{n0} + \kappa_n \sigma_n(v)$, $\forall v \in V_n$ and $\hat{\theta}_n(v) - \theta_n(v) = -Z_n(v) \sigma_n(v)$. Therefore,

$$\begin{aligned} \mathbb{P}_n \{ V_n \subseteq \hat{V}_n \} &= \mathbb{P}_n \{ A_n(v) \leq 2k_{n,\mathcal{V}}(\gamma_n) s_n(v), \forall v \in V_n \} \\ &\geq \mathbb{P}_n \{ B_n(v) \leq 2k_{n,\mathcal{V}}(\gamma_n) s_n(v), \forall v \in V_n \} \\ &\geq \mathbb{P}_n \{ -Z_n(v) \sigma_n(v) \leq 2k_{n,\mathcal{V}}(\gamma_n) s_n(v) - \kappa_n \sigma_n(v), \forall v \in V_n \} \\ &= \mathbb{P}_n \left\{ \sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \hat{\theta}_n(v)}{s_n(v)} \geq k_{n,\mathcal{V}}(\gamma_n) \right\} \\ &:= a - b = \gamma'_n - o(1) = 1 - o(1), \end{aligned}$$

where $b = o(1)$ follows similarly to the proof of Theorems 1 (analytical case) and Theorem 2 (simulation case), using the observation that $k_{n,\mathcal{V}}(\gamma_n) \geq k_{n,V_n}(\gamma_n)$, and $a = o(1)$ follows from the following argument:

$$\begin{aligned} a &\stackrel{(1)}{\geq} \mathbb{P}_n \left(\sup_{v \in V_n} -Z_n(v) \leq 2k_{n,\mathcal{V}}(\gamma_n) [1 - o_{\mathbb{P}_n}(\delta_n / (\bar{a}_n + \ell \ell_n))] - \kappa_n \right) \\ &\stackrel{(2)}{\geq} \mathbb{P}_n \left(\sup_{v \in V_n} -Z_n^*(v) \leq 2k_{n,\mathcal{V}}(\gamma_n) - \kappa_n - o_{\mathbb{P}_n}(\delta_n) \right) - o(1) \\ &\stackrel{(3)}{\geq} \mathbb{P}_n \left(\sup_{v \in V_n} -Z_n^*(v) \leq \kappa_n - o_{\mathbb{P}_n}(\delta_n) \right) - o(1) \\ &\stackrel{(4)}{\geq} \mathbb{P}_n \left(\sup_{v \in V_n} -Z_n^*(v) \leq \kappa_{n,V_n}(\gamma'_n) - o(\delta_n) \right) - o(1) \\ &\stackrel{(5)}{\geq} \gamma'_n - o(1) = 1 - o(1), \end{aligned}$$

where terms $o(\delta_n)$ are different in different places; where (1) follows by C.4, (2) is by C.2 and by Step 1, namely by $k_{n,\mathcal{V}}(\gamma'_n) \leq \bar{\kappa}_n \lesssim \bar{a}_n + \ell\ell_n$ wp $\rightarrow 1$, (3) follows by Step 1, (4) follows by monotonicity of $V \mapsto \kappa_{n,\mathcal{V}}(\gamma'_n)$ and $V_n \subseteq \mathcal{V}$, (5) follows by the anti-concentration Corollary 1.

Step 3. (Upper Containment). We have that for all $v \notin \bar{V}_n$,

$$\begin{aligned} A_n(v) &:= \hat{\theta}_n(v) - \theta_{n0} - \inf_{v \in \mathcal{V}} \left(\hat{\theta}_n(v) + k_{n,\mathcal{V}} s_n(v) \right) \\ &> -Z_n(v) \sigma_n(v) + \bar{\kappa}_n \sigma_n(v) + \sup_{v \in \mathcal{V}} \{ \theta_{n0} - \hat{\theta}_n(v) - k_{n,\mathcal{V}}(\gamma_n) s_n(v) \} := C_n(v), \end{aligned}$$

since $\theta_n(v) > \theta_{n0} + \bar{\kappa}_n \sigma_n(v)$, $\forall v \notin \bar{V}_n$, $\hat{\theta}_n(v) - \theta_n(v) = -Z_n(v) \sigma_n(v)$. Hence

$$\begin{aligned} \mathbb{P}_n \left(\widehat{V}_n \not\subseteq \bar{V}_n \right) &= \mathbb{P}_n \{ A_n(v) \leq 2k_{n,\mathcal{V}}(\gamma_n) s_n(v), \exists v \notin \bar{V}_n \} \\ &\leq \mathbb{P}_n \{ C_n(v) \leq 2k_{n,\mathcal{V}}(\gamma_n) s_n(v), \exists v \notin \bar{V}_n \} \\ &\leq \mathbb{P}_n \{ -Z_n(v) \sigma_n(v) < 2k_{n,\mathcal{V}}(\gamma_n) s_n(v) - \bar{\kappa}_n \sigma_n(v), \exists v \notin \bar{V}_n \} \\ &+ \mathbb{P}_n \left\{ \sup_{v \in \mathcal{V}} \frac{\theta_{n0} - \hat{\theta}_n(v)}{s_n(v)} \geq k_{n,\mathcal{V}}(\gamma_n), \exists v \notin \bar{V}_n \right\} \\ &=: c - b \leq (1 - \gamma'_n) + o(1) = o(1), \end{aligned}$$

where $b = o(1)$ from the Step 2, and $c \leq (1 - \gamma'_n) + o(1)$ follows from the following:

$$\begin{aligned} c &\leq_{(1)} \mathbb{P}_n \left(-Z_n(v) < 2k_{n,\mathcal{V}}(\gamma_n) [1 + o_{\mathbb{P}_n}(\delta_n / (\bar{a}_n + \ell\ell_n))] - \bar{\kappa}_n, \exists v \in \mathcal{V} \right) \\ &\leq_{(2)} \mathbb{P}_n \left(-Z_n^*(v) < 2k_{n,\mathcal{V}}(\gamma_n) - \bar{\kappa}_n + o(\delta_n), \exists v \in \mathcal{V} \right) + o(1) \\ &\leq_{(3)} \mathbb{P}_n \left(\sup_{v \in \mathcal{V}} Z_n^*(v) > k_{n,\mathcal{V}}(\gamma_n) - o(\delta_n) \right) + o(1) \\ &\leq_{(4)} \mathbb{P}_n \left(\sup_{v \in \mathcal{V}} Z_n^*(v) > \kappa_n - o(\delta_n) \right) + o(1) \leq_{(5)} (1 - \gamma'_n) + o(1), \end{aligned}$$

where (1) follows by C.4, (2) follows by C.2 and Step 1, namely by $k_{n,\mathcal{V}}(\gamma'_n) \leq \bar{\kappa}_n \lesssim \bar{a}_n + \ell\ell_n$ wp $\rightarrow 1$, (3) follows by Step 1, (4) holds by Step 1, (5) holds by the definition of κ_n and the anti-concentration Corollary 1.

Step 4. (Rate). We have that wp $\rightarrow 1$

$$d_H(\widehat{V}_n, V_0) \leq_{(1)} d_H(\widehat{V}_n, V_n) + d_H(V_n, V_0) \leq_{(2)} 2d_H(\bar{V}_n, V_0) \leq_{(3)} (\bar{\sigma}_n \bar{\kappa}_n)^{1/\rho_n} / c_n$$

where (1) holds by the triangle inequality, (2) follows by the containment $V_0 \subseteq V_n \subseteq \widehat{V}_n \subseteq \bar{V}_n$ holding wp $\rightarrow 1$, and (3) follows from $\bar{\kappa}_n \bar{\sigma}_n \rightarrow 0$ holding by assumption, and from the

following relation holding by Condition V:

$$\begin{aligned}
d_H(\bar{V}_n, V_0) &= \sup_{v \in V_n} d(v, V_0) \leq \sup\{d(v, V_0) : \theta_n(v) - \theta_{n0} \leq \bar{\kappa}_n \sigma_n(v)\} \\
&\leq \sup\{d(v, V_0) : (c_n d(v, V_0))^{\rho_n} \wedge \delta \leq \bar{\kappa}_n \bar{\sigma}_n\} \\
&\leq \sup\{t : (c_n t)^{\rho_n} \wedge \delta \leq \bar{\kappa}_n \bar{\sigma}_n\} \leq c_n^{-1} (\bar{\kappa}_n \bar{\sigma}_n)^{1/\rho_n} \text{ for all } 0 \leq \bar{\kappa}_n \bar{\sigma}_n \leq \delta.
\end{aligned}$$

■

G.2. Proof of Lemma 4. Step 1. Verification of C.1. This condition holds by inspection in view of continuity of $v \mapsto p_n(v, \gamma_n)$ and $v \mapsto p_n(v, \hat{\gamma})$ implied by Condition P(ii) and by Ω_n and $\hat{\Omega}_n$ being positive definite.

Step 2. Verification of C.2. Part (a). By Condition P, uniformly in $v \in \mathcal{V}$,

$$\begin{aligned}
Z_n(v) &= \frac{p_n(v, \gamma_n^*(v))'}{\|p_n(v, \gamma_n)' \Omega_n^{1/2}\|} \sqrt{n} (\hat{\gamma}_n - \gamma_n) \\
&= \frac{p_n(v, \gamma_n)'}{\|p_n(v, \gamma_n)' \Omega_n^{1/2}\|} \sqrt{n} (\hat{\gamma}_n - \gamma_n) + \frac{L_n \sqrt{n} \|\hat{\gamma}_n - \gamma_n\|^2}{\min_{v \in \mathcal{V}} \|p_n(v, \gamma_n)\|} \frac{\lambda_{\max}(\Omega_n^{1/2})}{\lambda_{\min}(\Omega_n^{1/2})} \\
&= \frac{p_n(v, \gamma_n)' \Omega_n^{1/2}}{\|p_n(v, \gamma_n)' \Omega_n^{1/2}\|} \mathcal{N}_k + o_{P_n}(\delta'_n) + O_{P_n}(n^{-1/2}).
\end{aligned}$$

Part (b). First note using the inequality

$$\left\| \frac{a}{\|a\|} - \frac{b}{\|b\|} \right\| \leq \left(2 \frac{\|a - b\|}{\|a\|} \right) \wedge \left(2 \frac{\|a - b\|}{\|b\|} \right), \quad (\text{G.6})$$

we have

$$\begin{aligned}
M_n &= \left\| \frac{p_n(v, \gamma_n)' \Omega_n^{1/2}}{\|p_n(v, \gamma_n)' \Omega_n^{1/2}\|} - \frac{p_n(v, \hat{\gamma}_n)' \hat{\Omega}_n^{1/2}}{\|p_n(v, \hat{\gamma}_n)' \hat{\Omega}_n^{1/2}\|} \right\| \leq 2 \frac{\|p_n(v, \gamma_n)' \Omega_n^{1/2} - p_n(v, \hat{\gamma}_n)' \hat{\Omega}_n^{1/2}\|}{\|p_n(v, \gamma_n)' \Omega_n^{1/2}\|} \\
&\leq 2 \frac{\|p_n(v, \gamma_n)' \Omega_n^{1/2} (I - \Omega_n^{-1/2} \hat{\Omega}_n^{1/2})\|}{\|p_n(v, \gamma_n)' \Omega_n^{1/2}\|} + 2 \frac{L_n \|\hat{\gamma}_n - \gamma_n\|}{\min_{v \in \mathcal{V}} \|p_n(v, \gamma_n)\|} \frac{\lambda_{\max}(\Omega_n^{1/2})}{\lambda_{\min}(\Omega_n^{1/2})} \\
&\leq 2 \|\Omega_n^{-1/2}\| \|\hat{\Omega}_n^{1/2} - \Omega_n^{1/2}\| + O_{P_n}(n^{-1/2}) \leq O_{P_n}(n^{-b}) + O_{P_n}(n^{-1/2}) = O_{P_n}(n^{-b}),
\end{aligned}$$

for some $b > 0$. We have that

$$E_{P_n} \left(\sup_{v \in \mathcal{V}} |Z_n^*(v) - Z_n^*(v)| \mid \mathcal{D}_n \right) \leq M_n E_{P_n} \|\mathcal{N}_k\| \leq M_n \sqrt{k}.$$

Hence for any $\delta_n'' \propto n^{-b'}$ with a constant $0 < b' < b$, we have by Markov's Inequality that

$$\mathbb{P}_n \left(\sup_{v \in \mathcal{V}} |Z_n^*(v) - Z_n^*(\tilde{v})| > \delta_n \ell_n \mid \mathcal{D}_n \right) \leq \frac{O_{\mathbb{P}_n}(n^{-b})}{\delta_n'' \ell_n} = o_{\mathbb{P}_n}(1/\ell_n).$$

Now select $\delta_n = \delta_n' \vee \delta_n''$.

Step 3. Verification of C.3. We shall employ Lemma 12, which has the required notation in place. We only need to compute an upper bound on the covering numbers $N(\varepsilon, V, \rho)$ for the process Z_n . We have that

$$\begin{aligned} \sigma(Z_n^*(v) - Z_n^*(\tilde{v})) &\leq \left\| \frac{p_n(v, \gamma_n)' \Omega_n^{1/2}}{\|p_n(v, \gamma_n)' \Omega_n^{1/2}\|} - \frac{p_n(\tilde{v}, \gamma_n)' \Omega_n^{1/2}}{\|p_n(\tilde{v}, \gamma_n)' \Omega_n^{1/2}\|} \right\| \\ &\leq 2 \left\| \frac{(p_n(v, \gamma_n) - p_n(\tilde{v}, \gamma_n))' \Omega_n^{1/2}}{\|p_n(v, \gamma_n)' \Omega_n^{1/2}\|} \right\| \leq 2 \frac{L_n}{\min_{v \in \mathcal{V}} \|p_n(v, \gamma_n)\|} \frac{\lambda_{\max}(\Omega_n^{1/2})}{\lambda_{\min}(\Omega_n^{1/2})} \|v - \tilde{v}\| \leq CL \|v - \tilde{v}\|, \end{aligned}$$

where C is some constant that does not depend on n , by the eigenvalues of Ω_n bounded away from zero and from above. Hence by the standard volumetric argument

$$N(\varepsilon, V, \rho) \leq \left(\frac{1 + CL \text{diam}(V)}{\varepsilon} \right)^d, \quad 0 < \varepsilon < 1,$$

where the diameter of V is measured by the Euclidian metric. Condition C.3 now follows by Lemma 12, with $a_n(V) = (2\sqrt{\log L_n(\bar{V})}) \vee (1 + \sqrt{d})$, $L_n(V) = C'(1 + CL \text{diam}(V))^d$, where C' is a constant from Lemma 12.

Step 4. Verification of C.4. Under Condition P, we have that $1 \leq a_n(V) \leq \bar{a}_n := a_n(\mathcal{V}) \lesssim 1$, so that C.4(a) follows since by Condition P

$$\bar{\sigma}_n = \sqrt{\max_{v \in \mathcal{V}} \|p_n(v, \gamma_n) \Omega_n^{1/2}\| / n} \leq \sqrt{\max_{v \in \mathcal{V}} \|p_n(v, \gamma_n)\| \|\Omega_n^{1/2}\| / n} \lesssim \sqrt{1/n}$$

To verify C.4(b) note that uniformly in $v \in \mathcal{V}$,

$$\begin{aligned} \left| \frac{\|p_n(v, \gamma_n)' \widehat{\Omega}_n^{1/2}\|}{\|p_n(v, \gamma_n)' \Omega_n^{1/2}\|} - 1 \right| &\leq \left| \frac{\|p_n(v, \gamma_n)' \widehat{\Omega}_n^{1/2}\| - \|p_n(v, \gamma_n)' \Omega_n^{1/2}\|}{\|p_n(v, \gamma_n)' \Omega_n^{1/2}\|} \right| \\ &\leq \frac{\|p_n(v, \gamma_n)' (\widehat{\Omega}_n^{1/2} - \Omega_n^{1/2})\|}{\|p_n(v, \gamma_n)' \Omega_n^{1/2}\|} \leq \frac{\|p_n(v, \gamma_n)' \Omega_n^{1/2} (\widehat{\Omega}_n^{1/2} \Omega_n^{-1/2} - I)\|}{\|p_n(v, \gamma_n)' \Omega_n^{1/2}\|} \\ &\leq \|\widehat{\Omega}_n^{1/2} \Omega_n^{-1/2} - I\| \leq \|\Omega_n^{-1/2}\| \|\widehat{\Omega}_n^{1/2} - \Omega_n^{1/2}\| = o_{\mathbb{P}_n}(\delta_n), \end{aligned}$$

since $\|\widehat{\Omega}_n^{1/2} - \Omega_n^{1/2}\| = O_{\mathbb{P}_n}(n^{-b})$ for some $b > 0$, and since $\|\Omega_n^{-1/2}\|$ is uniformly bounded, both implied by the assumptions.

Step 5. Verification of S. Then, since under Condition V with for large enough n , $r_n \lesssim c_n^{-1}(1/\sqrt{n})^{\ell_n/\rho_n} = o(1)$, we have that $r_n \leq \varphi_n$ for large n for some $\varphi_n = o(1)$. S then follows

by noting that for any positive $o(1)$ term, $\sup_{\|v-\bar{v}\|\leq o(1)} |Z_n(v) - Z_n(\bar{v})| \leq \Upsilon o(1) \|\mathcal{N}_k\| = o_{\mathbb{P}_n}(1)$. \blacksquare

G.3. Proof of Lemma 7. There are six steps, with the first four verifying conditions C.1-C.4, and the last two providing auxiliary calculations. Let $U_{ij} \equiv e'_j U_i$.

Step 1. Verification of C.1 and C.2. Condition C.1 holds by inspection, in view of continuity of $v \mapsto \hat{\theta}_n(v)$, $v \mapsto \theta_n(v)$, $v \mapsto \sigma_n(v)$, and $v \mapsto s_n(v)$. Condition C.2 is assumed directly.

Step 2. Verification of C.3. Note that

$$\frac{g_v(U_i, Z_i)}{\sigma_n(v) \sqrt{nh_n^d}} = \frac{\frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}}$$

We shall employ Lemma 12, which has the required notation in place. We only need to compute an upper bound on the covering numbers $N(\varepsilon, V, \rho)$ of V under the metric $\rho(v, \bar{v}) = \sigma(Z_n^*(v) - Z_n^*(\bar{v}))$. We have that for $v = (z, j)$ and $\bar{v} = (\bar{z}, j)$

$$\begin{aligned} \sigma(Z_n^*(v) - Z_n^*(\bar{v})) &\leq \Upsilon_n \|z - \bar{z}\|, \\ \Upsilon_n &:= \sup_{v \in \mathcal{V}, 1 \leq k \leq d} \left\| \nabla_{z_k} \frac{\frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}} \right\|_{\mathbb{P}_{n,2}}. \end{aligned}$$

We have that

$$\Upsilon_n \leq \sup_{v \in \mathcal{V}, 1 \leq k \leq d} \frac{\left\| \nabla_{z_k} \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}} + \frac{\left| \nabla_{z_k} \left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}} \right|}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}},$$

which is bounded by $C(1 + h_n^{-1})$ for large n by Step 6. Since J is finite, it follows that for all large $n > n_0$ for all non-empty subsets of $V \subseteq \mathcal{V}$,

$$N(\varepsilon, V, \rho) \leq \left(\frac{J^{1/d} (1 + C(1 + h_n^{-1}) \text{diam}(V))}{\varepsilon} \right)^d, \quad 0 < \varepsilon < 1.$$

Condition C.3 now follows for all $n > n_0$ by Lemma 12, with

$$a_n(V) = (2\sqrt{\log L_n(V)}) \vee (1 + \sqrt{d}), \quad L_n(V) = C' (1 + C(1 + h_n^{-1}) \text{diam}(V))^d,$$

where C' is a constant from Lemma 12.

Step 3. Verification of C.4. Under Condition NK, we have that

$$a_n(V) \leq \bar{a}_n := a_n(\mathcal{V}) \lesssim \sqrt{\log \ell_n + \log n} \lesssim \sqrt{\log n},$$

so that C.4(a) follows if $\sqrt{\log n / (nh_n^d)} \rightarrow 0$.

To verify C.4(b) note that

$$\left| \frac{s_n(v)}{\sigma_n(v)} - 1 \right| = \left| \underbrace{\left(\frac{f_n(z)}{\hat{f}_n(z)} \right)}_a \left(\underbrace{\frac{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) \hat{U}_{ij} \right\|_{\mathbb{P}_{n,2}}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}}}_{b/c} \right) - 1 \right|.$$

Since $|a(b/c) - 1| \leq 2|a - 1| + |(b - c)/c|$ when $|(b - c)/c| \leq 1$, the result follows from $|a - 1| = O_{\mathbb{P}_n}(n^{-b}) = o_p(\delta_n/\bar{a}_n)$ holding by NK.2 for some $b > 0$ and from

$$\begin{aligned} |(b - c)/c| &\leq \max_{1 \leq i \leq n} \|\hat{U}_i - U_i\| \frac{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) \right\|_{\mathbb{P}_{n,2}}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}} + \left| \frac{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}} - 1 \right|, \\ &\leq O_{\mathbb{P}_n}(n^{-b}) O_{\mathbb{P}_n}(1) + O_{\mathbb{P}_n} \left(\sqrt{\frac{\log n}{nh^d}} \right) = O_{\mathbb{P}_n}(n^{-b}) = o_{\mathbb{P}_n}(\delta_n/\bar{a}_n) \end{aligned}$$

for some $b > 0$ where we used NK.2, the results of Step 6, and the condition that $nh_n^d \rightarrow \infty$ at a polynomial rate.

Step 4. Verification of C.2. By NK.1 and $1 \lesssim E_{\mathbb{P}_n}[g_v^2] \lesssim 1$ uniformly in $v \in \mathcal{V}$ holding by Step 6 give

$$\sup_{v \in \mathcal{V}} \left| \frac{\mathbb{G}_n(g_v)}{E_{\mathbb{P}_n}[g_v^2]} - \frac{\mathbb{B}_n(g_v)}{E_{\mathbb{P}_n}[g_v^2]} \right| = O_{\mathbb{P}_n}(\delta_n),$$

where $v \mapsto \mathbb{B}_n(g_v)$ is zero-mean \mathbb{P}_n -Brownian bridge, with a.s. continuous sample paths. This and the condition on the remainder term in NK.1 in turn imply C.2(a).

To show C.2(b) we need to show that for any $C > 0$

$$\mathbb{P}_n \left(\sup_{v \in \mathcal{V}} \left| \frac{\mathbb{G}_n^o(\hat{g}_v)}{\mathbb{E}_n[\hat{g}_v^2]} - \frac{\bar{\mathbb{B}}_n(g_v)}{E_{\mathbb{P}_n}[g_v^2]} \right| > C\delta_n \mid \mathcal{D}_n \right) = o_{\mathbb{P}_n}(1/\ell_n),$$

where $\bar{\mathbb{B}}_n$ is a copy of \mathbb{B}_n , which is independent of the data. First, Condition NK.1 with the fact that $1 \lesssim E_{\mathbb{P}_n}[g_v^2] \lesssim 1$ uniformly in $v \in \mathcal{V}$ implies that

$$\mathbb{P}_n \left(\sup_{v \in \mathcal{V}} \left| \frac{\mathbb{G}_n^o(g_v)}{E_{\mathbb{P}_n}[g_v^2]} - \frac{\bar{\mathbb{B}}_n(g_v)}{E_{\mathbb{P}_n}[g_v^2]} \right| > C\delta_n \mid \mathcal{D}_n \right) = o_{\mathbb{P}_n}(1/\ell_n).$$

Therefore, in view of the triangle inequality and the union bound, it remains to show that

$$\mathbb{P}_n \left(\sup_{v \in \mathcal{V}} \left| \frac{\mathbb{G}_n^o(\hat{g}_v)}{\mathbb{E}_n[\hat{g}_v^2]} - \frac{\mathbb{G}_n^o(g_v)}{E_{\mathbb{P}_n}[g_v^2]} \right| > C\delta_n \mid \mathcal{D}_n \right) = o_{\mathbb{P}_n}(1/\ell_n). \quad (\text{G.7})$$

We have that

$$\sup_{v \in \mathcal{V}} \left| \frac{\mathbb{G}_n^o(\hat{g}_v)}{\mathbb{E}_n[\hat{g}_v^2]} - \frac{\mathbb{G}_n^o(g_v)}{E_{\mathbb{P}_n}[g_v^2]} \right| \leq \sup_{v \in \mathcal{V}} \left| \frac{\mathbb{G}_n^o(\hat{g}_v - g_v)}{E_{\mathbb{P}_n}[g_v^2]} \right| + \sup_{v \in \mathcal{V}} \left| \frac{\mathbb{G}_n^o(g_v)}{E_{\mathbb{P}_n}[g_v^2]} \right| \sup_{v \in \mathcal{V}} \left| \frac{\sigma_n(v)}{s_n(v)} - 1 \right|.$$

We observe that

$$\begin{aligned} & E_{\mathbb{P}_n} \left(\sup_{v \in \mathcal{V}} \left| \frac{\mathbb{G}_n^o(g_v)}{E_{\mathbb{P}_n}[g_v^2]} \right| \sup_{v \in \mathcal{V}} \left| \frac{\sigma_n(v)}{s_n(v)} - 1 \right| \mid \mathcal{D}_n \right) \\ &= E_{\mathbb{P}_n} \left(\sup_{v \in \mathcal{V}} \left| \frac{\mathbb{G}_n^o(g_v)}{E_{\mathbb{P}_n}[g_v^2]} \right| \mid \mathcal{D}_n \right) \sup_{v \in \mathcal{V}} \left| \frac{\sigma_n(v)}{s_n(v)} - 1 \right| = O_{\mathbb{P}_n} \left(\sqrt{\log nn^{-b}} \right) = O_{\mathbb{P}_n}(1/\ell_n), \end{aligned}$$

where the last equality follows from Steps 5 and Step 3. Also we note that

$$\begin{aligned} & E_{\mathbb{P}_n} \left(\sup_{v \in \mathcal{V}} \left| \frac{\mathbb{G}_n^o(\hat{g}_v - g_v)}{E_{\mathbb{P}_n}[g_v^2]} \right| \mid \mathcal{D}_n \right) \\ &\leq_{(1)} O_{\mathbb{P}_n}(\sqrt{\log n}) \sup_{v \in \mathcal{V}} \frac{\left\| \left(\frac{U_{ij}}{f_n(z)} - \frac{\hat{U}_{ij}}{\hat{f}_n(z)} \right) \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) \right\|_{\mathbb{P}_{n,2}}}{\frac{1}{f_n(z)} \left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}} \\ &\lesssim_{(2)} O_{\mathbb{P}_n}(\sqrt{\log n}) \sup_{v \in \mathcal{V}} \frac{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) (1 + |U_{ij}|) \right\|_{\mathbb{P}_{n,2}}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}} \left(\left| \frac{f_n(z)}{\hat{f}_n(z)} - 1 \right| \vee \max_{1 \leq i \leq n} \|\hat{U}_i - U_i\| \right) \\ &\leq_{(3)} O_{\mathbb{P}_n}(\sqrt{\log n}) O_{\mathbb{P}_n}(1) O_{\mathbb{P}_n}(n^{-b}) = o_{\mathbb{P}_n}(1/\ell_n), \end{aligned}$$

where (1) follows from Step 5, (2) by elementary inequalities, and (3) by Step 6 and NK.2.

It follows that (G.7) holds by Markov's Inequality.

Step 5. This step shows that

$$E_{\mathbb{P}_n} \left(\sup_{v \in \mathcal{V}} \left| \frac{\mathbb{G}_n^o(g_v)}{E_{\mathbb{P}_n}[g_v^2]} \right| \mid \mathcal{D}_n \right) = O_{\mathbb{P}_n}(\sqrt{\log n}) \quad (\text{G.8})$$

$$\begin{aligned} E_{\mathbb{P}_n} \left(\sup_{v \in \mathcal{V}} \left| \frac{\mathbb{G}_n^o(\hat{g}_v - g_v)}{E_{\mathbb{P}_n}[g_v^2]} \right| \mid \mathcal{D}_n \right) &\leq O_{\mathbb{P}_n}(\sqrt{\log n}) \times \\ &\times \sup_{v \in \mathcal{V}} \frac{\left\| \left(\frac{U_{ij}}{f_n(z)} - \frac{\hat{U}_{ij}}{\hat{f}_n(z)} \right) \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) \right\|_{\mathbb{P}_{n,2}}}{\frac{1}{f_n(z)} \left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}}. \quad (\text{G.9}) \end{aligned}$$

To show (G.8) we use Lemma 13 applied to $X_v = \frac{\mathbb{G}_n^o(g_v)}{E_{\mathbb{P}_n}[g_v^2]}$ conditional on \mathcal{D}_n . First, we compute

$$\sigma(X) = \sup_{v \in \mathcal{V}} E_{\mathbb{P}_n}(X_v^2 | \mathcal{D}_n) = \sup_{v \in \mathcal{V}} \frac{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}} = 1 + o_{\mathbb{P}_n}(1),$$

where the last equality holds by Step 6. Second, we observe that for $v = (z, j)$ and $\bar{v} = (\bar{z}, j)$

$$\sigma(X_v - X_{\bar{v}}) \leq \Upsilon_n \|z - \bar{z}\|, \quad \Upsilon_n := \sup_{v \in \mathcal{V}, 1 \leq k \leq d} \left\| \nabla_{z_k} \frac{\frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}} \right\|_{\mathbb{P}_{n,2}}.$$

We have that

$$\begin{aligned} \Upsilon_n &\leq \sup_{v \in \mathcal{V}, 1 \leq k \leq d} \frac{\left\| \nabla_{z_k} \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}} \\ &+ \frac{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}} \cdot \frac{\left| \nabla_{z_k} \left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}} \right|}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}}. \end{aligned}$$

which is bounded with probability converging to one by $C(h_n^{-1} + 1)$ for large n by Step 6 and NK.2. Since J is finite, it follows that for all large $n > n_0$, the covering number for \mathcal{V} under $\rho(v, \bar{v}) = \sigma(X_v - X_{\bar{v}})$ obeys with probability converging to 1,

$$N(\varepsilon, \mathcal{V}, \rho) \leq \left(\frac{J^{1/d} (1 + C(1 + h_n^{-1}) \text{diam}(\mathcal{V}))}{\varepsilon} \right)^d, \quad 0 < \varepsilon < \sigma(X),$$

Hence $\log N(\varepsilon, \mathcal{V}, \rho) \lesssim \log n + \log(1/\varepsilon)$. Hence by Lemma 13, we have that

$$E_{\mathbb{P}_n} \left(\sup_{v \in \mathcal{V}} |X_v| \mid \mathcal{D}_n \right) \leq \sigma(X) + \int_0^{2\sigma(X)} \sqrt{\log(n/\varepsilon)} d\varepsilon = O_{\mathbb{P}_n}(\sqrt{\log n}).$$

To show (G.9) we use Lemma 13 applied to $X_v = \frac{\mathbb{G}_n^o(\hat{g}_v - g_v)}{E_{\mathbb{P}_n}[g_v^2]}$ conditional on \mathcal{D}_n . First, we compute

$$\sigma(X) = \sup_{v \in \mathcal{V}} E_{\mathbb{P}_n}(X_v^2 | \mathcal{D}_n) = \sup_{v \in \mathcal{V}} \frac{\left\| \left(\frac{U_{ij}}{f_n(z)} - \frac{\hat{U}_{ij}}{\hat{f}_n(z)} \right) \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) \right\|_{\mathbb{P}_{n,2}}}{\frac{1}{f_n(z)} \left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z-Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}}.$$

Second, we observe that for $v = (z, j)$ and $\bar{v} = (\bar{z}, j)$

$$\sigma(X_v - X_{\bar{v}}) \leq (\Upsilon_n + \hat{\Upsilon}_n) \|z - \bar{z}\|,$$

where

$$\hat{\Upsilon}_n := \sup_{v \in \mathcal{V}, 1 \leq k \leq d} \left\| \frac{\nabla_{z_k} \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) \hat{U}_{ij}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}} \right\|_{\mathbb{P}_{n,2}},$$

and Υ_n is the same as defined above.

We have that

$$\begin{aligned} \hat{\Upsilon}_n &\leq \sup_{v \in \mathcal{V}, 1 \leq k \leq d} \frac{\left\| \nabla_{z_k} \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) \hat{U}_{ij} \right\|_{\mathbb{P}_{n,2}}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}} \\ &\quad + \frac{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) \hat{U}_{ij} \right\|_{\mathbb{P}_{n,2}}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}} \cdot \frac{\left| \nabla_{z_k} \left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}} \right|}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}} \end{aligned}$$

The first term is bounded by

$$\sup_{v \in \mathcal{V}, 1 \leq k \leq d} \frac{\left\| \nabla_{z_k} \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}} + \max_{1 \leq i \leq n} \|\hat{U}_i - U_i\| \sup_{v \in \mathcal{V}, 1 \leq k \leq d} \frac{\left\| \nabla_{z_k} \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) \right\|_{\mathbb{P}_{n,2}}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}}$$

which is bounded by $C(1 + h_n^{-1}) + O_{\mathbb{P}_n}(n^{-b})O_{\mathbb{P}_n}(1)$ for large n by Step 6 and NK.2. In the second term, the left term of the product is bounded by

$$\sup_{v \in \mathcal{V}, 1 \leq k \leq d} \frac{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}} + \max_{1 \leq i \leq n} \|\hat{U}_i - U_i\| \sup_{v \in \mathcal{V}, 1 \leq k \leq d} \frac{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) \right\|_{\mathbb{P}_{n,2}}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}}$$

which is bounded by $C(1 + o_{\mathbb{P}_n}(1)) + O_{\mathbb{P}_n}(n^{-b})O_{\mathbb{P}_n}(1)$ for large n by Step 6 and NK.2; the right term of the product is bounded by $C(1 + o_{\mathbb{P}_n}(1))$ by Step 6. Conclude that $\hat{\Upsilon}_n \leq C(1 + h_n^{-1})$ for some constant $C > 0$ with probability converging to one.

Since J is finite, it follows that for all large $n > n_0$, the covering number for \mathcal{V} under $\rho(v, \bar{v}) = \sigma(X_v - X_{\bar{v}})$ obeys with probability converging to 1,

$$N(\varepsilon, \mathcal{V}, \rho) \leq \left(\frac{J^{1/d}(1 + C(1 + h_n^{-1})\text{diam}(\mathcal{V}))}{\varepsilon} \right)^d, \quad 0 < \varepsilon < \sigma(X),$$

Hence

$$\log N(\varepsilon, \mathcal{V}, \rho) \lesssim \log n + \log(1/\varepsilon).$$

Hence by Lemma 13, we have that

$$E_{P_n} \left(\sup_{v \in \mathcal{V}} |X_v| \mid \mathcal{D}_n \right) \leq \sigma(X) + \int_0^{2\sigma(X)} \sqrt{\log(n/\varepsilon)} d\varepsilon = O_{P_n}(\sqrt{\log n})\sigma(X).$$

Step 6. The claim of this step are the following relations: uniformly in $v \in \mathcal{V}, 1 \leq k \leq d$

$$1 \lesssim \left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{P_{n,2}} \lesssim 1 \quad (\text{G.10})$$

$$1 \lesssim \left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) \right\|_{P_{n,2}} \lesssim 1 \quad (\text{G.11})$$

$$h_n^{-1} \lesssim \left\| \nabla_{z_k} \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{P_{n,2}} \lesssim h_n^{-1} \quad (\text{G.12})$$

$$h_n^{-1} \lesssim \left\| \nabla_{z_k} \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) \right\|_{P_{n,2}} \lesssim h_n^{-1} \quad (\text{G.13})$$

$$h_n^{-1} \lesssim \left| \nabla_{z_k} \left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{P_{n,2}} \right| \lesssim h_n^{-1} \quad (\text{G.14})$$

and

$$\frac{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{P_{n,2}}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{P_{n,2}}} = 1 + O_{P_n} \left(\sqrt{\frac{\log n}{nh_n^d}} \right) \quad (\text{G.15})$$

$$\frac{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) \right\|_{P_{n,2}}}{\left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) \right\|_{P_{n,2}}} = 1 + O_{P_n} \left(\sqrt{\frac{\log n}{nh_n^d}} \right) \quad (\text{G.16})$$

$$\frac{\left\| \nabla_{z_k} \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{P_{n,2}}}{\left\| \nabla_{z_k} \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{P_{n,2}}} = 1 + O_{P_n} \left(\sqrt{\frac{\log n}{nh_n^d}} \right) \quad (\text{G.17})$$

$$\frac{\left\| \nabla_{z_k} \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) \right\|_{P_{n,2}}}{\left\| \nabla_{z_k} \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) \right\|_{P_{n,2}}} = 1 + O_{P_n} \left(\sqrt{\frac{\log n}{nh_n^d}} \right) \quad (\text{G.18})$$

The proofs of (G.10)-(G.14) are all similar to one another, as are those of (G.15)-(G.18), and are standard in the kernel estimator literature. We therefore prove only (G.10) and

(G.15) to demonstrate the argument. To establish (G.10) we have

$$\begin{aligned} \left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}^2 &= \int \mathbf{K}^2((z - \bar{z})/h_n) E[U_{ij}^2 | \bar{z}] f_n(\bar{z}) d\bar{z} \\ &\leq_{(1)} \int \mathbf{K}^2((z - \bar{z})/h_n) C d\bar{z} \leq_{(2)} h_n^d \int \mathbf{K}^2(u) C, \end{aligned}$$

for some constant $0 < C < \infty$, where in (1) we use the assumption that $E[U_{ij}^2 | z]$ and $f_n(z)$ are bounded uniformly from above and in (2) we use the assumption that \mathcal{Z} is bounded away from the boundary of the support of Z_i by at least h_n . On the other hand,

$$\begin{aligned} \left\| \frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij} \right\|_{\mathbb{P}_{n,2}}^2 &= \int \mathbf{K}^2((z - \bar{z})/h_n) E[U_{ij}^2 | \bar{z}] f_n(\bar{z}) d\bar{z} \\ &\geq_{(1)} \int \mathbf{K}^2((z - \bar{z})/h_n) C d\bar{z} \geq_{(2)} h_n^d \int \mathbf{K}^2(u) C, \end{aligned}$$

for some constant $0 < C < \infty$, where in (1) we use the assumption that $E[U_{ij}^2 | z]$ and $f_n(z)$ are bounded away from zero uniformly in n , and in (2) we use the assumption that \mathcal{Z} is bounded away from the boundary of the support of Z_i by at least h_n .

Moving to (G.15), it suffices to show that uniformly in $v \in \mathcal{V}$,

$$\mathbb{E}_n \left(\left(\frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij}^2 \right)^2 \right) - E_{\mathbb{P}_n} \left(\left(\frac{1}{h_n^{d/2}} \mathbf{K} \left(\frac{z - Z_i}{h_n} \right) U_{ij}^2 \right)^2 \right) = O_{\mathbb{P}_n} \left(\sqrt{\frac{\log n}{nh_n^d}} \right),$$

or equivalently

$$\mathbb{E}_n \left(\mathbf{K}^2 \left(\frac{z - Z_i}{h_n} \right) U_{ij}^2 \right) - E_{\mathbb{P}_n} \left(\mathbf{K}^2 \left(\frac{z - Z_i}{h_n} \right) U_{ij}^2 \right) = O_{\mathbb{P}_n} \left(\sqrt{\frac{h_n^d \log n}{n}} \right). \quad (\text{G.19})$$

Given the boundedness of U_{ij} imposed by Condition R, this is in fact a standard result on local empirical processes, using Pollard's empirical process methods. Specifically, (G.19) follows by the application of Theorem 37 in chapter II of Pollard (1984). \blacksquare

G.4. Proof of Lemma 8. To show claim (1), we need to establish that for

$$\varphi_n = o(1) \cdot \left(\frac{h_n}{\sqrt{\log n}} \right),$$

for any $o(1)$ term, we have that

$$\sup_{\|v - v'\| \leq \varphi_n} |Z_n^*(v) - Z_n^*(v')| = o_{\mathbb{P}_n}(1).$$

Consider the stochastic process $X = \{Z_n(v), v \in \mathcal{V}\}$. We shall use the standard maximal inequality stated in Lemma 13. From the proof of Lemma 7 we have that for $v = (z, j)$ and $v' = (z', j)$, $\sigma(Z_n^*(v) - Z_n^*(v')) \leq C(1 + h_n^{-1})\|z - z'\|$, where C is some constant that does not depend on n , and $\log N(\varepsilon, V, \rho) \lesssim \log n + \log(1/\varepsilon)$. Since

$$\|v - v'\| \leq \varphi_n \implies \sigma(Z_n^*(v) - Z_n^*(v')) \leq C \frac{o(1)}{\sqrt{\log n}},$$

we have

$$E \sup_{\|v-v'\| \leq \varphi_n} |X_s - X_t| \lesssim \int_0^C \frac{o(1)}{\sqrt{\log n}} \sqrt{\log(n/\varepsilon)} d\varepsilon \lesssim \frac{o(1)}{\sqrt{\log n}} \sqrt{\log n} = o(1).$$

Hence the conclusion follows from Markov's Inequality.

Under Condition V by lemma 2

$$r_n \lesssim \left(\sqrt{\frac{\log n}{nh_n^d} \log n} \right)^{1/\rho_n} c_n^{-1},$$

so $r_n = o(\varphi_n)$ if

$$\left(\sqrt{\frac{\log n}{nh_n^d} \log n} \right)^{1/\rho_n} c_n^{-1} = o\left(\frac{h_n}{\sqrt{\log n}} \right).$$

Thus, Condition S is satisfied. ■

G.5. Proof of Theorem 8. To prove this theorem, we use the Rio-Massart coupling. First we note that

$$\mathcal{M} = \{h_n^{d/2} f_n(z) g_v(U_i, Z_i) = e_j' U_i \mathbf{K}((z - Z_i)/h_n), z \in \mathcal{Z}, j \in \{1, \dots, J\}\}$$

is the product of $\{e_j' U_i, j \in 1, \dots, J\}$ with covering number trivially bounded above by J and $\mathcal{K} := \{\mathbf{K}((z - Z_i)/h_n), z \in \mathcal{Z}\}$ obeys $\sup_Q N(\varepsilon, \mathcal{K}, L_1(Q)) \lesssim \varepsilon^{-\nu}$ for some finite constant ν ; see Lemma 4.1 of Rio (1994). Therefore, By Lemma A.1 in Ghosal, Sen, and van der Vaart (2000), we have that

$$\sup_Q N(\varepsilon, \mathcal{M}, L_1(Q)) \lesssim J(\varepsilon/2)^{-\nu} \lesssim \varepsilon^{-\nu}. \tag{G.20}$$

Next we bound, for $\mathbf{K}_l(u) = \partial \mathbf{K}(u) / \partial u_l$

$$\begin{aligned}
TV(\mathcal{M}) &\leq \sup_{f \in \mathcal{M}} \int |D_{(x'_1, x'_2)'} f(x_1, x_2)| dx_1 dx_2 \\
&\leq \sup_{v \in \mathcal{V}} \int_{I^d} \int_{I^{d_1}} \left(\sum_{l=1}^{d_1} |e'_j D_{x_{1l}} \varphi_n(x_1) \mathbf{K}((z - \tilde{\varphi}_n(x_2))/h_n)| \right. \\
&\quad \left. + \sum_{k=1}^d |e'_j \varphi_n(x_1) \nabla \mathbf{K}((z - \tilde{\varphi}_n(x_2))/h_n) h_n^{-1} \partial \tilde{\varphi}(x_2) / \partial x_{2k}| \right) dx_1 dx_2 \\
&\leq C \sup_{v \in \mathcal{V}} \int_{I^d} \left(|\mathbf{K}((z - \tilde{\varphi}_n(x_2))/h_n)| + h_n^{-1} |\mathbf{K}_l((z - \tilde{\varphi}_n(x_2))/h_n)| B \right) dx_2 \\
&\leq Ch_n^d + Ch_n^{-1} h_n^d \leq Ch_n^{d-1} =: K(\mathcal{M})
\end{aligned}$$

where C is a generic constant, possibly different in different places, and where we rely on

$$\int_{I^{d_1}} |D_{x_{1l}} \varphi_n(x_1)| dx_1 \leq B, \quad \sup_{x_1} |e'_j \varphi_n(x_1)| \leq 1, \quad \sup_{x_2} |\partial \tilde{\varphi}(x_2) / \partial x_{2k}| \leq B$$

as well as on

$$\int_{I^d} |\mathbf{K}((z - \tilde{\varphi}_n(x_2))/h_n)| dx_2 \leq Ch^d, \quad \int_{I^d} |\mathbf{K}_l((z - \tilde{\varphi}_n(x_2))/h_n)| dx_2 \leq Ch^d.$$

To see how the latter relationships holds, note that $Y = \tilde{\varphi}_n(v)$ when $v \sim U(I^d)$ has a density bounded uniformly from above: $f_Y(y) \lesssim 1/|\det \partial \tilde{\varphi}_n(v) / \partial v| \lesssim 1/c$. Moreover, the functions $|\mathbf{K}((z - y)/h_n)|$ and $|\mathbf{K}_l((z - y)/h_n)|$ are bounded above by some constant \bar{K} and are non-zero only over a y belonging to cube centered at z of volume $(2h)^d$. Hence

$$\int_{I^d} |\mathbf{K}((z - \tilde{\varphi}_n(x_2))/h_n)| dx_2 \leq \int_{I^d} |\mathbf{K}((z - y)/h_n)| f_Y(y) dy \leq \bar{K} (2h)^d (1/c) \leq Ch^d,$$

and similarly for the second term.

By the Rio-Massart coupling we have that for some constant C and $t \geq C \log n$:

$$\mathbb{P}_n \left(\sqrt{n} \sup_{f \in \mathcal{M}} |\mathbb{G}_n(f) - \mathbb{B}_n(f)| \geq C \sqrt{tn^{\frac{d+d_1-1}{d+d_1}} K(\mathcal{M})} + Ct \sqrt{\log n} \right) \leq e^{-t},$$

which implies that

$$\mathbb{P}_n \left(\sup_{v \in \mathcal{V}} |\mathbb{G}_n(g_v) - \mathbb{B}_n(g_v)| \geq n^{-1/2} C \sqrt{tn^{\frac{d+d_1-1}{d+d_1}} h_n^{d-1} h_n^{-d/2}} + n^{-1/2} h_n^{-d/2} Ct \sqrt{\log n} \right) \leq e^{-t},$$

which upon inserting $t = C \log n$ gives

$$\mathbb{P}_n \left(\sup_{v \in \mathcal{V}} |\mathbb{G}_n(g_v) - \mathbb{B}_n(g_v)| \geq C \left[n^{-1/2(d+d_1)} (h_n^{-1} \log n)^{1/2} + (nh_n^d)^{-1/2} \log^{3/2} n \right] \right) \lesssim 1/n.$$

This implies the required conclusion. Note that $g_v \mapsto \mathbb{B}_n(g_v)$ is continuous under the $L_1(f_X)$ metric by the Rio-Massart coupling, which implies continuity of $v \mapsto \mathbb{B}_n(g_v)$, since $v - v' \rightarrow 0$ implies $g_v - g_{v'} \rightarrow 0$ in the $L_1(f_X)$ metric. \blacksquare

G.6. Proof of Theorem 9. In what follows it is useful to keep in mind (F.1).

Step 1. Let $M = \sqrt{4 \log n}$ and $t = C \log n$ for some $C \geq 1$ in what follows. Consider the truncation mapping from \mathbb{R} to \mathbb{R} defined by $x \mapsto T_M(x) = \max(-M, \min(M, x))$. Consider the function class

$$\mathcal{G}_M = (T_M(\xi)/M) \times \mathcal{M},$$

for \mathcal{M} defined in the proof of Theorem 8 and $\xi \sim N(0, 1)$. Note that $\xi = \Phi^{-1}([1 + X_0]/2)$ where $X_0 \sim U(-1, 1)$. Note that \mathcal{G}_M has envelope 1. Next we bound, for $\mathbf{K}_l(u) := \partial \mathbf{K}(u)/\partial u_l$,

$$\begin{aligned} TV(\mathcal{G}_M) &\leq \sup_{f \in \mathcal{G}_M} \int |D_{(x_0, x'_1, x'_2)'} f(x_0, x_1, x_2)| \\ &\leq \sup_{v \in \mathcal{V}} \int_{I^d} \int_{I^{d_1}} \int_I \left(|D_{x_0} [T_M(\Phi^{-1}([1 + x_0]/2))/M] e'_j \varphi_n(x_1) \mathbf{K}((z - \tilde{\varphi}_n(x_2))/h_n)| \right. \\ &\quad + \sum_{l=1}^{d_1} |[T_M(\Phi^{-1}([1 + x_0]/2))/M] e'_j D_{x_{1l}} \varphi_n(x_1) \mathbf{K}((z - \tilde{\varphi}_n(x_2))/h_n)| \\ &\quad \left. + \sum_{k=1}^d |[T_M(\Phi^{-1}([1 + x_0]/2))/M] e'_j \varphi_n(x_1) \nabla \mathbf{K}((z - \tilde{\varphi}_n(x_2))/h_n) h_n^{-1} \partial \tilde{\varphi}(x_{2k})/\partial x_{2k}| \right) dx_0 dx_1 dx_2 \\ &\leq C h_n^d + C h_n^{-1} h_n^d \leq C h_n^{d-1} =: K(\mathcal{G}_M) \end{aligned}$$

where C is a generic constant, where we rely on the previous proof, and on

$$\int_I |D_{x_0} [T_M(\Phi^{-1}(x_0))/M]| dx_0 \leq C, \quad \sup_{x_0} |T_M(\Phi^{-1}(x_0))/M| \leq 1.$$

Note that \mathcal{G}_M is product of a single function $(T_M(\xi)/M)$ with function class \mathcal{M} , both having envelope 1. Hence by Lemma A.1 in Ghosal, Sen, and van der Vaart (2000), $\sup_Q N(\epsilon, \mathcal{G}_M, L_1(Q)) \lesssim 1 \cdot \sup_Q N(\epsilon/2, \mathcal{M}, L_1(Q))$, and from the preceding proof, $\sup_Q N(\epsilon, \mathcal{M}, L_1(Q)) \lesssim \epsilon^{-\nu}$. Hence $\sup_Q N(\epsilon, \mathcal{M} \cup \mathcal{G}_M, L_1(Q)) \lesssim \epsilon^{-\nu}$.

By the Rio-Massart coupling we have that for $t \geq \log n$, there exists a P_n -Brownian Bridge \mathbb{B}_n such that for some constant $C > 0$

$$P_n \left(\sqrt{n} \sup_{f \in \mathcal{M} \cup \mathcal{G}_M} |\mathbb{G}_n(f) - \mathbb{B}_n(f)| \geq C \sqrt{tn^{\frac{d+d_1}{d+d_1+1}} K(\mathcal{G}_M) \vee K(\mathcal{M})} + Ct \sqrt{\log n} \right) \leq e^{-t}, \quad (\text{G.21})$$

or for $R_n = CM \left(\sqrt{tn^{\frac{d+d_1}{d+d_1+1}} K(\mathcal{G}_M) \vee K(\mathcal{M})} + t\sqrt{\log n} \right)$

$$\mathbb{P}_n \left(n^{1/2} \sup_{f \in \mathcal{M} \cup (T_M(\xi) \times \mathcal{M})} |\mathbb{G}_n(f) - \mathbb{B}_n(f)| \geq R_n \right) \lesssim 1/n. \quad (\text{G.22})$$

Note $T_M(\xi) \times \mathcal{M}$ appears instead of \mathcal{G}_M , so we are rescaling the function class \mathcal{G}_M by M , and also take into account the impact of this rescaling in the bound R_n . Note that $d_1 + 1$ appears instead of d_1 , as compared to the previous proof, due to having an extra variable X_{0i} that generates the multiplier variable ξ_i .

Step 2. Here we verify that truncation by M has a negligible impact and the result above holds without truncation.

First, by the union bound and by $1 - \Phi(u) \leq \phi(u)/u$ we have that $\max_{i \leq n} |\xi_i| \leq M$ with probability at least $1 - 2/n$, so that

$$\mathbb{P}_n \left(\sqrt{n} \sup_{f \in \mathcal{M}} |\mathbb{G}_n(T_M(\xi)f) - \mathbb{G}_n(\xi f)| \neq 0 \right) \leq 2/n. \quad (\text{G.23})$$

Second, we note that the process \mathbb{B}_n can be extended outside $\mathcal{M} \cup (T_M(\xi)\mathcal{M})$ to the class $\mathcal{M} \cup \xi\mathcal{M}$. The latter class is pre-Gaussian (see Dudley (1999) for definition), as follows from the entropy bound calculated below, hence \mathbb{B}_n can be constructed to have a.s. continuous paths on $\mathcal{M} \cup \xi\mathcal{M}$.

Third, we want to show that for some constant $C > 0$

$$\mathbb{P}_n \left(\sup_{f \in \mathcal{M}} |\mathbb{B}_n(T_M(\xi)f) - \mathbb{B}_n(\xi f)| > \frac{C \log n}{\sqrt{n}} \right) \lesssim 1/n. \quad (\text{G.24})$$

Define $X_f = \mathbb{B}_n(T_M(\xi)f) - \mathbb{B}_n(\xi f) = \mathbb{B}_n(\xi 1(|\xi| > M)f)$. The covering number for this process by $L^2(\mathbb{P}_n)$ -balls of size ϵ is bounded above by a constant times $1/\epsilon$, for all $0 < \epsilon < 1$, since the function is a product of square-integrable function $\xi 1(|\xi| > M)$ and uniformly bounded functions $f \in \mathcal{M}$; the bound on covering numbers then follows by Lemma A.1 in Ghosal, Sen, and van der Vaart (2000) and from $\log N(\epsilon, \mathcal{M}, L_2(\mathbb{P}_n)) \lesssim \log(1/\epsilon)$, established in the proof of Theorem 8. Hence the class is pre-Gaussian. Moreover,

$$\sigma(X) = \sup_{v \in \mathcal{V}} \sigma(X_v) \lesssim \sqrt{M^2 \exp(-M^2/2)} \lesssim \sqrt{\log n/n},$$

which follows from elementary Gaussian tail bounds and integration by parts. We conclude by Lemma 13

$$E_{\mathbb{P}_n}(\sup_{f \in \mathcal{M}} |X_f|) \lesssim \sqrt{\log n/n}.$$

Moreover, by the Borell-Sudakov-Tsyrlson Gaussian concentration inequality

$$\mathbb{P}_n \left(\sup_{f \in \mathcal{M}} |X_f| - E_{\mathbb{P}_n}(\sup_{f \in \mathcal{M}} |X_f|) > \sqrt{2 \log n} \sigma(X) \right) \leq 1/n,$$

proving the claim (G.24).

Step 3. Putting (G.22)-(G.24) together we conclude that

$$\mathbb{P}_n \left(\sqrt{n} \sup_{f \in \mathcal{M} \cup (\xi \mathcal{M})} |\mathbb{G}_n(f) - \mathbb{B}_n(f)| \geq R'_n \right) \lesssim 1/n,$$

for $R'_n = CM \sqrt{\log n n^{\frac{d+d_1}{d+d_1+1}} K(\mathcal{G}_M) \vee K(\mathcal{M})} + CM \log n (\log n)^{1/2} + C \log n / \sqrt{n}$. Rescaling everything by $h_n^{-d/2}$ gives the conclusion that with \mathbb{P}_n -probability at most C'/n we have that

$$\sup_{v \in \mathcal{V}} |\mathbb{G}_n(g_v) - \mathbb{B}_n(g_v)| \leq n^{-1/2} R'_n h_n^{-d/2}, \quad \sqrt{n} \sup_{v \in \mathcal{V}} |\mathbb{G}_n(\xi g_v) - \mathbb{B}_n(\xi g_v)| \leq n^{-1/2} R'_n h_n^{-d/2}.$$

It is easy to check by covariance calculations that the processes $\{\mathbb{B}_n(g_v), v \in \mathcal{V}\}$ and $\{\bar{\mathbb{B}}_n(g_v), v \in \mathcal{V}\} := \{\mathbb{B}_n(\xi g_v), v \in \mathcal{V}\}$ are identically distributed and are independent from each other. Finally note that

$$n^{-1/2} R'_n h_n^{-d/2} \lesssim (\log n) n^{\frac{-1}{2(d+d_1+1)}} h_n^{-1/2} + (n h_n^d)^{-1/2} \log^2 n + (n^2 h_n^d)^{-1/2} (\log n) = o(\delta_n)$$

under the stated conditions on the bandwidth sequence. ■

APPENDIX H. ASYMPTOTIC LINEAR REPRESENTATION FOR SERIES ESTIMATOR OF A CONDITIONAL MEAN

In this section we use the primitive conditions set out in Example 5 of the main text to verify the required asymptotically linear representation for $\sqrt{n}(\hat{\beta}_n - \beta_n)$ using Newey (1997). This representation is also Condition (b) of Theorem 7. We now reproduce the imposed conditions from the example for clarity. We note that it is also possible to develop similar conditions for nonlinear estimators, see for example Theorem 1(d) of Horowitz and Mammen (2004).

We have that $\theta_n(v) = E_{\mathbb{P}_n}[Y_i | V_i = v]$, assumed to be a continuous function. There is an i.i.d. sample $(Y_i, V_i), i = 1, \dots, n$, with $\mathcal{V} \subseteq \text{support}(V_i) \subseteq [0, 1]^d$ for each n . Here d does not depend on n , but all other parameters, unless stated otherwise, can depend on n . Then we have $\theta_n(v) = p_n(v)' \beta_n + a_n(v)$, for $p_n : \mathcal{V} \mapsto \mathbb{R}^{K_n}$ representing the series functions; β_n is the coefficient of the best least squares approximation to $\theta_n(v)$ in the population, and $a_n(v)$ is the approximation error. The number of series terms K_n depends on n .

Recall that we have imposed the following technical conditions in the main text:

Uniformly in n , (i) p_n are either b-splines of a fixed order or trigonometric series terms or any other series terms $p_n = (p_{n1}, \dots, p_{nK_n})$ with $\|p_n(v)\| \lesssim \zeta_n = \sqrt{K_n}$ and $\max_{1 \leq l \leq K_n} |p_{nl}(v)| \leq C$ for all $v \in \text{support}(V_i)$, $\|p_n(v)\| \gtrsim \zeta'_n \geq 1$ for all $v \in \mathcal{V}$, and $\log \text{lip}(p_n) \lesssim \log K_n$, (ii) the mapping $v \mapsto \theta_n(v)$ is sufficiently smooth, namely $\sup_{v \in \mathcal{V}} |a_n(v)| \lesssim K_n^{-s}$, for some $s > 0$, (iii) $\lim_{n \rightarrow \infty} (\log n)^c \sqrt{n} K_n^{-s} = 0$ for each $c > 0$,²⁴ (iv) for $\epsilon_i = Y_i - E_{P_n}[Y_i|V_i]$, $E_{P_n}[\epsilon_i^2|V_i = v]$ is bounded away from zero uniformly in $v \in \text{support}(V_i)$, and (v) eigenvalues of $Q_n = E_{P_n}[p_n(V_i)p_n(V_i)']$ are bounded away from zero and from above, and (vi) $E_{P_n}[|\epsilon_i|^4|V_i = v]$ is bounded from above uniformly in $v \in \text{support}(V_i)$, (vii) $\lim_{n \rightarrow \infty} (\log n)^c K_n^5/n = 0$ for each $c > 0$.

We impose Condition (i) directly through the choice of basis functions. Condition (ii) is a standard condition on the error of the series approximation, and is the same as Assumption A3 of Newey (1997), also used by Chen (2007). Condition (v) is Assumption 2(i) of Newey (1997). The constant s will depend on the choice of basis functions. For example, if splines or power series are used, then $\alpha = s/d$, where s is the number of continuous derivatives of $\theta_n(v)$ and d is the dimension of v . Restrictions on K_n in conditions (iii) and (vii) require that $s > 5d/2$. Invoking Corollary 2 to Theorem 7, the constraint on the rate of growth in the number of series terms is satisfied if K_n is chosen to satisfy

$$\tau_n^{3/2} \delta_n^{-3} K_n^{3/2} \max_{v \in \mathcal{V}} \sum_{j=1}^{K_n} |p_{nj}(v)|^3 / n^{1/2} \rightarrow 0,$$

which holds under conditions (i) and (vii) in this example.

Define $S_n \equiv E[\epsilon_i^2 p_n(V_i) p_n(V_i)']$ and $\Omega_n \equiv Q_n^{-1} S_n Q_n^{-1}$. Arguments based on Newey (1997) give the following lemma, which verifies the linear expansion required in condition (b) of Theorem 7 with $\delta_n = 1/\log n$.

Lemma 14 (Asymptotically Linear Representation of Series Estimator). *Suppose conditions (i)-(vii) hold. Then we have the following asymptotically linear representation:*

$$\Omega_n^{-1/2} \sqrt{n} (\hat{\beta}_n - \beta_n) = \Omega_n^{-1/2} Q_n^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n p_n(Z_i) \epsilon_i + o_{P_n}(1/\log n).$$

²⁴This condition, which is based on Newey (1997) can be relaxed to $(\log n)^c K_n^{-s+1} \rightarrow 0$ and $(\log n)^c \sqrt{n} K_n^{-s} / \zeta'_n \rightarrow 0$, using the recent results of Belloni, Chen, and Chernozhukov (2010) for least squares series estimators.

Proof of Lemma 14. As in Newey (1997), we have the following representation: with probability approaching one,

$$\widehat{\beta}_n - \beta_n = n^{-1} \widehat{Q}_n^{-1} \sum_{i=1}^n p_n(V_i) \epsilon_i + \nu_n, \quad (\text{H.1})$$

where $\widehat{Q}_n \equiv \mathbb{E}_n[p_n(V_i)p_n(V_i)']$, $\epsilon_i \equiv Y_i - E_{P_n}[Y|V = V_i]$, $\nu_n \equiv n^{-1} \widehat{Q}_n^{-1} \sum_{i=1}^n p_n(V_i) a_n(V_i)$, and $a_n(v) \equiv \theta_n(v) - p_n(v)' \beta_n$. As shown in the proof of Theorem 1 of Newey (1997), we have $\|\nu_n\| = O_{\mathbb{P}}(K_n^{-\alpha})$. In addition, write

$$\bar{R}_n := \left[\widehat{Q}_n^{-1} - Q_n^{-1} \right] n^{-1} \sum_{i=1}^n p_n(V_i) \epsilon_i = Q_n^{-1} \left[Q_n - \widehat{Q}_n \right] n^{-1} \widehat{Q}_n^{-1} \sum_{i=1}^n p_n(V_i) \epsilon_i.$$

Then it follows from the proof of Theorem 1 of Newey (1997) that

$$\|\bar{R}_n\| = O(\xi(K_n)K_n/n),$$

where $\xi(K_n) \equiv \sup_v \|p_n(v)\| = \sqrt{K_n}$ by condition (i). Combining the results above gives

$$\widehat{\beta}_n - \beta_n = n^{-1} Q_n^{-1} \sum_{i=1}^n p_n(V_i) \epsilon_i + R_n, \quad (\text{H.2})$$

where the remainder term R_n satisfies

$$\|R_n\| = O\left(\frac{K_n^{3/2}}{n} + K_n^{-\alpha}\right).$$

Note that by condition (iv), eigenvalues of S_n^{-1} are bounded above. In other words, using the notation used in Corollary 2 in the main text, we have that $\tau_n \lesssim 1$. Then

$$\Omega_n^{-1/2} \sqrt{n} (\widehat{\beta}_n - \beta_n) = n^{-1/2} \sum_{i=1}^n u_{i,n} + r_n, \quad (\text{H.3})$$

where

$$u_{i,n} := \Omega_n^{-1/2} Q_n^{-1} p_n(V_i) \epsilon_i, \quad (\text{H.4})$$

and the new remainder term r_n satisfies

$$\|r_n\| = O\left[n^{1/2} \left(K_n^{3/2}/n + K_n^{-\alpha}\right)\right].$$

Therefore, $r_n = o_{\mathbb{P}_n}(1/\log n)$ if

$$(\log n) n^{1/2} \left(K_n^{3/2}/n + K_n^{-\alpha}\right) \rightarrow 0, \quad (\text{H.5})$$

which is satisfied under conditions (iii) and (vii). Therefore, we have proved the lemma. \blacksquare

APPENDIX I. ASYMPTOTIC LINEAR REPRESENTATION FOR LOCAL POLYNOMIAL
ESTIMATOR OF A CONDITIONAL MEAN

In this section we provide details of Example 7 that are omitted in the main text. Recall that we assume that $P_n = P$ is fixed in this example, and impose the following conditions:

- (i) for each $j \in \mathcal{J}$, $\theta(z, j)$ is $(p + 1)$ times continuously differentiable with respect to $z \in \mathcal{Z}$, where \mathcal{Z} is convex. (ii) the probability density function f of Z_i is bounded above and bounded below from zero with continuous derivatives on \mathcal{Z} ; (iii) for $Y_i(j) := m(X_i, \mu, j)$, $Y_i := (Y_i(j), j \in \mathcal{J})'$, and $U_i := Y_i - E_P[Y_i|Z_i]$; and U_i is a bounded random vector; (iv) for each j , the conditional on Z_i density of U_i exists and is uniformly bounded from above and below, or, more generally, condition R stated in Appendix F holds; (v) $K(\cdot)$ has support on $[-1, 1]^d$, is twice continuously differentiable, $\int uK(u)du = 0$, and $\int K(u)du = 1$; (vi) $h_n \rightarrow 0$, $nh_n^{d+|\mathcal{J}|+1} \rightarrow \infty$, and $nh_n^{d+2(p+1)} \rightarrow 0$ at polynomial rates in n .

Let $\mathbf{K}(z/h) \equiv \mathbf{e}'_1 S_p^{-1} K_h(z) \mathbf{u}_p(z/h)$, and let

$$g_v(U, Z) := \frac{e'_j U}{(h_n^d)^{1/2} f(z)} \mathbf{K} \left(\frac{Z - z}{h_n} \right).$$

Then results obtained in Kong, Linton, and Xia (2010) give the following lemma, which verifies the linear expansion required in condition (b) of Theorem 8 with $\delta_n = 1/\log n$.

Lemma 15 (Asymptotically Linear Representation of Local Polynomial Estimator). *Suppose conditions (i)-(vi) hold. Then we have the following asymptotically linear representation: uniformly in $v = (z, j) \in \mathcal{V} \subseteq \mathcal{Z} \times \mathcal{J}$,*

$$(nh_n^d)^{1/2} (\hat{\theta}_n(v) - \theta_n(v)) = \mathbb{G}_n(g_v) + o_P(1/\log n).$$

Proof of Lemma 15. We first verify Assumptions A1-A7 in Kong, Linton, and Xia (2010) (KLX hereafter). In our example, $\rho(y; \theta) = \frac{1}{2}(y - \theta)^2$ using the notation in KLX. Then $\varphi(y; \theta)$ in Assumptions A1 and A2 in KLX is $\varphi(y; \theta) = \varphi(y - \theta) = -(y - \theta)$. Then Assumption A1 is satisfied since the pdf of U_i is bounded and U_i is a bounded random vector. Assumption A2 is trivially satisfied since $\varphi(u) = -u$. Assumption A3 follows since $K(\cdot)$ has compact support and is twice continuous differentiable. Assumption A4 holds by condition (ii) since X_i and X_j are independent in our example ($i \neq j$). Assumption A5 is implied directly by Condition (i). Since we have i.i.d. data, mixing coefficients $(\gamma[k])$ using the notation of KLX) are identically zeros for any $k \geq 1$. The regression

error U_i is assumed to be bounded, so that ν_1 in KLX can be arbitrary large. Hence, to verify Assumption A6 of KLX, it suffices to check that for some $\nu_2 > 2$, $h_n \rightarrow 0$, $nh_n^d/\log n \rightarrow \infty$, $h_n^{d+2(p+1)}/\log n < \infty$, and $n^{-1}(nh_n^d/\log n)^{\nu_2/8}d_n \log n/M_n^{(2)} \rightarrow \infty$, where $d_n = (nh_n^d/\log n)^{-1/2}$ and $M_n^{(2)} = M^{1/4}(nh_n^d/\log n)^{-1/2}$ for some $M > 2$, by choosing $\lambda_2 = 1/2$ and $\lambda_1 = 3/4$ on page 1540 in KLX. By choosing a sufficiently large ν_2 (at least greater than 8), the following holds: $n^{-1}(nh_n^d)^{\nu_2/8} \rightarrow \infty$. Then condition (vi) implies Assumption A6. Finally, condition (iv) implies Assumption A7 since we have i.i.d. data. Thus, we have verified all the conditions in Kong, Linton, and Xia (2010).

Let $\delta_n = 1/\log n$. Then it follows from Corollary 1 and Lemmas 8 and 10 of Kong, Linton, and Xia (2010) that

$$\widehat{\theta}_n(z, j) - \theta_n(z, j) = \frac{1}{nh_n^d f(z)} \mathbf{e}'_1 S_p^{-1} \sum_{i=1}^n (e'_j U_i) K_h(Z_i - z) \mathbf{u}_p \left(\frac{Z_i - z}{h_n} \right) + B_n(z, j) + R_n(z, j), \quad (\text{I.1})$$

where \mathbf{e}_1 is a $|A_p| \times 1$ vector whose first element is one and all others are zeros, S_p is a $|A_p| \times |A_p|$ matrix such that $S_p = \{\int z^u (z^v)' du : u \in A_p, v \in A_p\}$, $\mathbf{u}_p(z)$ is a $|A_p| \times 1$ vector such that $\mathbf{u}_p(z) = \{z^u : u \in A_p\}$,

$$B_n(z, j) = O(h_n^{p+1}) \text{ and } R_n(z, j) = o_P \left(\frac{\delta_n}{(nh_n^d)^{1/2}} \right),$$

uniformly in $(z, j) \in \mathcal{V}$. The exact form of $B_n(z, j)$ is given in equation (12) of Kong, Linton, and Xia (2010). The result that $B_n(z, j) = O(h_n^{p+1})$ uniformly in (z, j) follows from the standard argument based on Taylor expansion given in Fan and Gijbels (1996), Kong, Linton, and Xia (2010), or Masry (1996). The condition that $nh_n^{d+2(p+1)} \rightarrow 0$ at a polynomial rate in n corresponds to the undersmoothing condition. Now the lemma follows from (I.1) immediately since $\mathbf{K}(z/h) \equiv \mathbf{e}'_1 S_p^{-1} K_h(z) \mathbf{u}_p(z/h)$ is a kernel of order $(p+1)$ (See section 3.2.2 of Fan and Gijbels (1996)). \blacksquare

APPENDIX J. LOCAL ASYMPTOTIC POWER COMPARISONS

We have shown in the main text that the test of $H_0 : \theta_{na} \leq \theta_{n0}$ of the form

$$\text{Reject } H_0 \text{ if } \theta_{na} > \widehat{\theta}_{n0}(p),$$

can reject all local alternatives θ_{na} that are more distant than $\bar{\sigma}_n \bar{a}_n$. We now provide a couple of examples of local alternatives against which our test has non-trivial power, but for which the CvM statistic of Andrews and Shi (2009), henceforth AS, does not. It is evident from the results of AS on local asymptotic power that there are also models for which their

CvM statistic will have power against some $n^{-1/2}$ alternatives, where our approach will not.²⁵ We conclude that neither approach dominates.

We consider two examples in which

$$Y_i = \theta_n(V_i) + U_i,$$

where U_i are iid with $E[U_i|V_i] = 0$ and V_i are iid random variables uniformly distributed on $[-1, 1]$. Suppose that for all $v \in [-1, 1]$ we have

$$\theta^* \leq E[Y_i|V_i = v],$$

equivalently

$$\theta^* \leq \theta_0 = \min_{v \in [-1, 1]} \theta_n(v).$$

In the examples below we consider two specifications of the bounding function $\theta_n(v)$, each with

$$\min_{v \in [-1, 1]} \theta_n(v) = 0,$$

and we analyze asymptotic power against a local alternative $\theta_{na} > \theta_0$.

Following AS, consider the CvM test statistic

$$T_n(\theta) := \int \left[n^{1/2} \frac{\bar{m}_n(g; \theta)}{\hat{\sigma}_n(g; \theta) \vee \varepsilon} \right]_-^2 dQ(g), \quad (\text{J.1})$$

for some $\varepsilon > 0$, where $[u]_- := -u1(u < 0)$ and θ is the parameter value being tested. In the present context we have

$$\bar{m}_n(g; \theta) := \frac{1}{n} \sum_{i=1}^n (Y_i - \theta) g(V_i),$$

where $g \in \mathcal{G}$ are instrument functions used to transform the conditional moment inequality $E[Y - \theta|V = v]$ a.e. $v \in \mathcal{V}$ to unconditional inequalities, and $Q(\cdot)$ is a measure on the space \mathcal{G} of instrument functions as described in AS Section 3.4. $\hat{\sigma}_n(g; \theta)$ is a uniformly consistent estimator for $\sigma_n(g; \theta)$, the standard deviation of $n^{1/2}\bar{m}_n(g; \theta)$.

We can show that $T_n(\theta) = \tilde{T}_n(\theta) + o_p(1)$, where

$$\tilde{T}_n(\theta) := \int [\beta_n(\theta, g) / (\sigma_n(g; \theta) \vee \varepsilon) + w(\theta, g)]_-^2 dQ(g),$$

²⁵For the formal results, see AS Section 7, Theorem 4. In the examples that follow their Assumption LA3' is violated, as is also the case in the example covered in their Section 13.5.

where $w(\theta, g)$ is a mean zero Gaussian process, and $\beta_n(\theta, g)$ is a deterministic function of the form

$$\beta_n(\theta, g) \equiv \sqrt{n}E\{[\theta_n(V_i) - \theta]g(V_i)\}.$$

For any θ , the testing procedure based on the CvM statistic rejects $H_0 : \theta \leq \theta_{n0}$ if

$$T_n(\theta) > c(\theta, 1 - \alpha),$$

where $c(\theta, 1 - \alpha)$ is a generalized moment selection (GMS) critical value that satisfies

$$c(\theta, 1 - \alpha) = (1 - \alpha)\text{-quantile of } \left(\int [\varphi_n(\theta, g) / (\sigma_n(g; \theta) \vee \varepsilon) + w(\theta, g)]_-^2 dQ(g) \right) + o_p(1).$$

$\varphi_n(\theta, g)$ is a GMS function that satisfies $0 \leq \varphi_n(\theta, g) \leq \beta_n(\theta, g)$ with probability approaching 1 whenever $\beta_n(\theta, g) \geq 0$, see AS Section 4.4 for further details. Relative to $\tilde{T}_n(\theta)$, in the integrand of the expression above $\varphi_n(\theta, g)$ is replaced with $\beta_n(\theta, g)$. Hence if

$$\sup_{g \in \mathcal{G}} [\beta_n(\theta_{na}, g)]_- \rightarrow 0,$$

for the sequence of local alternatives θ_{na} , then

$$\liminf_{n \rightarrow \infty} P(T_n(\theta_{na}) > c(\theta_{na}, 1 - \alpha)) \leq \alpha,$$

since asymptotically $c(\theta_{na}, 1 - \alpha)$ exceeds the $1 - \alpha$ quantile of $\tilde{T}_n(\theta)$. It follows that the CvM test has only trivial power against such a sequence of alternatives. The same conclusion holds using plug-in asymptotic critical values, since these are no smaller than GMS critical values.

In the following two examples we now verify that $\sup_{g \in \mathcal{G}} [\beta_n(\theta_{na}, g)]_- \rightarrow 0$. We assume that instrument functions are g are either boxes or cubes, defined in AS Section 3.3, and hence bounded between zero and one.

J.1. Example J.1 (Unique, well-defined optimum). Let the function $\theta(\cdot)$ be specified as

$$\theta_n(v) = |v|^a,$$

for some $a \geq 1$.

Let us now proceed to bound

$$\begin{aligned}
[\beta_n(\theta_{na}, g)]_- &= \sqrt{n} [E \{[\theta_n(V_i) - \theta_{na}] g(V_i)\}]_- \\
&\leq \sqrt{n} [E [\theta_n(V_i) - \theta_{na}]]_- \\
&\leq \sqrt{n} E \{[\theta_n(V_i) - \theta_{na}]_-\} \\
&= \sqrt{n} \int_{-1}^1 (\theta_{na} - |v|^a) \mathbf{1}\{|v|^a \leq \theta_{na}\} dv \\
&= 2\sqrt{n} \int_0^1 (\theta_{na} - v^a) \mathbf{1}\{v \leq \theta_{na}^{1/a}\} dv \\
&= \frac{2a}{a+1} \sqrt{n} \theta_{na}^{(a+1)/a} \\
&\equiv \bar{\beta}_n.
\end{aligned}$$

Note that

$$\theta_{na} = o\left(n^{-a/[2(a+1)]}\right) \Rightarrow \bar{\beta}_n \rightarrow 0.$$

Thus, in this case the asymptotic rejection probability of the CvM test for the local alternative θ_{na} is bounded above by α . On the other hand, by Theorems 1 and 2 of the main text, our test rejects all local alternatives θ_{na} that are more distant than $\bar{\sigma}_n \bar{a}_n$ with probability at least α asymptotically. It suffices to find a sequence of local alternatives θ_{na} such that $\theta_{na} = o\left(n^{-a/[2(a+1)]}\right)$ but $\theta_{na} \gg \bar{\sigma}_n \bar{a}_n$.

For instance, consider the case where $a = 2$. Then

$$\sqrt{n} \theta_{na}^{3/2} \rightarrow 0 \Rightarrow \bar{\beta}_n \rightarrow 0,$$

i.e. $\theta_{na} = o\left(n^{-1/3}\right) \Rightarrow \bar{\beta}_n \rightarrow 0$, so the CvM test has trivial asymptotic power against θ_{na} . In contrast, since this is a very smooth case, our approach can achieve $\bar{\sigma}_n \bar{a}_n = O(n^{-\delta})$ for some δ that can be close to $1/2$, for instance by using a series estimator with a slowly growing number of terms, or a higher-order kernel or local polynomial estimator. Our test would then be able to reject any θ_{na} that converges to zero faster than $n^{-1/3}$ but more slowly than $n^{-\delta}$.

J.2. Example J.2 (Deviation with Small Support). Now suppose that the form of the conditional mean function, $\theta_n(v) \equiv E[Y_i | V_i = v]$, is given by

$$\theta_n(v) := \bar{\theta}(v) - \tau_n^a (\phi(v/\tau_n) - \phi(0)),$$

where τ_n is a sequence of positive constants converging to zero and $\phi(\cdot)$ is the standard normal density function. Let $\bar{\theta}(v)$ be minimized at zero so that

$$\theta_0 = \min_{v \in [-1, 1]} \theta_n(v) = \min_{v \in [-1, 1]} \bar{\theta}(v) = 0.$$

Let the alternative by $\tilde{\theta}_{na} \equiv \tau_n^a \phi(0)$. Again, the behavior of the AS statistic is driven by $\left[\beta_n \left(\tilde{\theta}_{na}, g \right) \right]_-$, which we bound from above as follows.

$$\begin{aligned} \left[\beta_n \left(\tilde{\theta}_{na}, g \right) \right]_- &= \sqrt{n} \left[E \left\{ \left[\theta_n(V_i) - \tilde{\theta}_{na} \right] g(V_i) \right\} \right]_- \\ &\leq \sqrt{n} \left[E \left\{ \theta_n(V_i) - \tilde{\theta}_{na} \right\} \right]_- \\ &= \sqrt{n} \left[E \left\{ \bar{\theta}(V_i) - \tau_n^a \phi(V_i/\tau_n) \right\} \right]_- \\ &\leq \sqrt{n} E \left\{ \tau_n^a \phi(V_i/\tau_n) \right\} \\ &= \frac{\sqrt{n}}{2} \int_{-1}^1 \tau_n^a \phi(v/\tau_n) dv \\ &\leq \frac{\sqrt{n}}{2} \tau_n^{a+1} \equiv \bar{\beta}_n. \end{aligned}$$

Consider the case $a = 2$. If $\tau_n = o(n^{-1/6})$ then $\bar{\beta}_n \rightarrow 0$, so that again the CvM test has only trivial asymptotic power. If $\tau_n = n^{-1/6-c/2}$ for some small positive constant c , then $\tilde{\theta}_{na} \equiv n^{-1/3-c} \phi(0)$. Note that

$$f(v) := \tau_n^2 \phi(v/\tau_n) \Rightarrow f''(v) = \phi''(v/\tau_n) \leq \bar{\phi}'' < \infty,$$

for some constant $\bar{\phi}''$. Hence, if $\bar{\theta}(v)$ is twice continuously differentiable, we can use a series or kernel estimator to estimate $\theta_n(v)$ uniformly at the rate of $(\log n)^d n^{-2/5}$ for some $d > 0$, leading to non-trivial power against alternatives $\tilde{\theta}_{na}$ for sufficiently small c .

APPENDIX K. RESULTS OF ADDITIONAL MONTE CARLO EXPERIMENTS

In this section we present the results of some additional Monte Carlo experiments that illustrate the finite-sample performance of our method. We consider a Monte Carlo design that is similar to that of Manski and Pepper (2009), discussed briefly in Example B of the main text. In particular, we consider the lower bound on $\theta^* = E[Y_i(t)|V_i = v]$ under the monotone instrumental variable (MIV) assumption, where t is a treatment, $Y_i(t)$ is the corresponding potential outcome, and V_i is a monotone instrumental variable. The lower

bound on $E[Y_i(t)|V_i = v]$ can be written as

$$\max_{u \leq v} E [Y_i \cdot 1\{Z_i = t\} + y_0 \cdot 1\{Z_i \neq t\} | V_i = u], \quad (\text{K.1})$$

where Y_i is the observed outcome, Z_i is a realized treatment, and y_0 is the left end-point of the support of Y_i , see Manski and Pepper (2009). Throughout the Monte Carlo experiments, the parameter of interest is $\theta^* = E[Y_i(1)|V_i = 1.5]$.

K.1. Data-Generating Processes. We consider four cases of data-generating processes (DGPs). In the first case, which we call DGP5, $V_0 = \mathcal{V}$ and the MIV assumption has no identifying power. In other words, the bound-generating function is flat on \mathcal{V} , in which case the bias of the analog estimator is most acute, see Manski and Pepper (2009). In the second case, which we call DGP6, the MIV assumption has identifying power, and V_0 is a strict subset of \mathcal{V} . In the third and fourth cases, which we call DGP7 and DGP8, we set V_0 to be a singleton set.

Specifically, for all DGPs we generated 1000 independent samples from the following model:

$$V_i \sim \text{Unif}[-2, 2], Z_i = 1\{\varphi_0(V_i) + \varepsilon_i > 0\}, \text{ and } Y_i = \min\{\max\{-0.5, \sigma_0(V_i)U_i\}, 0.5\},$$

where $\varepsilon_i \sim N(0, 1)$, $U_i \sim N(0, 1)$, $\sigma_0(V_i) = 0.1 \times |V_i|$, and (V_i, U_i) are statistically independent ($i = 1, \dots, n$). The bounding function has the form

$$\begin{aligned} \theta(v) &:= E [Y_i \cdot 1\{Z_i = 1\} + y_0 \cdot 1\{Z_i \neq 1\} | V_i = v] \\ &= -0.5\Phi[-\varphi_0(v)], \end{aligned}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function. For DGP5, we set $\varphi_0(v) \equiv 0$. In this case, the bounding function is completely flat ($\theta_l(v) = -0.25$ for each $v \in \mathcal{V} = [-2, 1.5]$). For DGP6, an alternative specification is considered:

$$\varphi_0(v) = v1(v \leq 1) + 1(v > 1).$$

In this case, $v \mapsto \theta(v)$ is strictly increasing on $[-2, 1]$ and is flat on $[1, 2]$, and $V_0 = [1, 1.5]$ is a strict subset of $\mathcal{V} = [-2, 1.5]$. For DGP7, we consider

$$\varphi_0(v) = -2v^2.$$

In this case, $v \mapsto \theta_l(v)$ has a unique maximum at $v = 0$, and thus, $V_0 = \{0\}$ is singleton. For DGP8, we consider

$$\varphi_0(v) = -10v^2.$$

In this case, $v \mapsto \theta(v)$ has a unique maximum at $v = 0$ and is more peaked than that of DGP7.

We considered sample sizes $n = 250$, $n = 500$, and $n = 1000$, and we implemented the series estimator to estimate the bounding function $\theta(v)$ in (K.1). For basis functions we used cubic B-splines with knots equally spaced over the sample quantiles of V_i . Details of the implementation are the same as in Section 5 of the main text. Figures 1-4 show realizations of data and the bounding functions for all DGPs, including those considered in the main text.

K.2. Simulation Results. To evaluate the relative performance of our inference method, we have also implemented one of the inference methods proposed by AS, specifically their Cramér-von Mises-type (CvM) statistic with PA/Asy and GMS/Asy critical values. Turning parameters for CvM were chosen as exactly as in AS (see Section 9).²⁶

The coverage probability (CP) is evaluated at the true lower bound, say θ_0 (with the nominal level of 95%) and the false coverage probability (FCP) is evaluated at a θ value outside the identified set. For DGP5, we set $\theta = \theta_0 - 0.03$; for DGP6-DGP7, $\theta = \theta_0 - 0.05$; and for DGP8, $\theta = \theta_0 - 0.07$. These points are chosen differently across different DGPs to ensure that the FCPs have similar values. This type of FCP was reported in AS, along with a so-called “CP-correction” (similar to size correction in testing). We did not do CP-correction in our reported results. There were 1,000 replications for each experiment. Table 1 summarizes the results of Monte Carlo experiments. CLR and AS refer to our inference method and that of AS, respectively.

First, we consider Monte Carlo results for DGP5. The discrepancies between nominal and actual coverage probabilities are not large across all methods, implying that all of them perform well in finite samples. For DGP5, since the true argmax set V_0 is equal to \mathcal{V} , an estimated V_0 should be the entire set \mathcal{V} . Thus the simulation results are the same whether or not estimating V_0 since for most of simulation draws, $\widehat{V}_n = \mathcal{V}$. Similar conclusions hold for AS with CvM between PA/Asy and GMS/Asy critical values. In terms of false coverage probability, CvM with either critical value performs better than our method.

We now move to DGPs 6-8. In DGP6, the true argmax set V_0 is $[1, 1.5]$ and in DGP7 and DGP8, V_0 is a singleton set. In these cases the true argmax set V_0 is a strict subset of \mathcal{V} . Hence, we expect that it is important to estimate V_0 . On average, for DGP6, the estimated sets were $[-0.73, 1.5]$ when $n = 250$, $[-0.280, 1.5]$ when $n = 500$, and $[-0.015, 1.5]$ when

²⁶Our Monte Carlo design differs from that of AS, and alternative choices of tuning parameters could perform more or less favorably in our design. We did not examine sensitivity to the choice of tuning parameters for their method.

$n = 1,000$; for DGP7, the estimated sets were $[-0.996, 0.984]$ when $n = 250$, $[-0.837, 0.835]$ when $n = 500$, and $[-0.729, 0.728]$ when $n = 1,000$; for DGP8, the estimated sets were $[-0.71, 0.69]$ when $n = 500$, $[-0.438, 0.436]$ when $n = 500$, and $[-0.346, 0.346]$ when $n = 1,000$.

Hence, an average estimated set is larger than V_0 ; however, it is still a strict subset of \mathcal{V} and gets smaller as n gets large. For all the methods, the Monte Carlo results are consistent with asymptotic theory. Unlike in DGP5, the CLR method performs better than the AS method in terms of false coverage probability. As can be seen from the table, the CLR method performs better when V_0 is estimated in terms of making the coverage probability less conservative and also of making the false coverage probability smaller. Similar gains are obtained for the CvM with GMS/Asy critical values, relative to that with PA/Asy critical values.

The results of this section support the conclusions reached in Section 5 of the main text. In completely flat cases, the AS method outperforms our method, whereas in non-flat cases, our method outperforms the AS method. In this section we also considered one intermediate case, where the bounding function is partly-flat. In this particular case our method performed favorably, but more generally there is a wide range of intermediate cases that could be considered, and we would expect the approach of AS to perform favorably in some cases too. The main conclusions we draw from the Monte Carlo experiments are that our inference method performs well both in coverage probabilities and false coverage probabilities and that in terms of a comparison between our approach and that of AS, neither approach dominates.²⁷

²⁷As in Section 5, this conclusion will remain valid even with CP-correction as in AS, since our method performs better in DGPs 6-8 where we have over-coverage.

TABLE 1. Results for Monte Carlo Experiments

DGP	Sample Size	Critical Value	Cov. Prob.	False Cov. Prob.
CLR with Series Estimation				
		Estimating V_n ?		
5	250	No	0.924	0.720
5	250	Yes	0.924	0.720
5	500	No	0.942	0.612
5	500	Yes	0.942	0.612
5	1000	No	0.950	0.404
5	1000	Yes	0.950	0.404
6	250	No	0.967	0.689
6	250	Yes	0.956	0.636
6	500	No	0.969	0.535
6	500	Yes	0.945	0.455
6	1000	No	0.979	0.291
6	1000	Yes	0.962	0.195
7	250	No	0.982	0.892
7	250	Yes	0.974	0.851
7	500	No	0.997	0.847
7	500	Yes	0.994	0.741
7	1000	No	0.994	0.597
7	1000	Yes	0.984	0.457
8	250	No	0.994	0.923
8	250	Yes	0.988	0.832
8	500	No	0.994	0.817
8	500	Yes	0.987	0.657
8	1000	No	0.998	0.568
8	1000	Yes	0.986	0.364
AS with CvM (Cramér-von Mises-type statistic)				
5	250	PA/Asy	0.951	0.544
5	250	GMS/Asy	0.945	0.537
5	500	PA/Asy	0.949	0.306
5	500	GMS/Asy	0.945	0.305
5	1000	PA/Asy	0.962	0.068
5	1000	GMS/Asy	0.956	0.068
6	250	PA/Asy	1.000	0.941
6	250	GMS/Asy	0.990	0.802
6	500	PA/Asy	1.000	0.908
6	500	GMS/Asy	0.980	0.674
6	1000	PA/Asy	1.000	0.744
6	1000	GMS/Asy	0.980	0.341
7	250	PA/Asy	1.000	1.000
7	250	GMS/Asy	0.997	0.948
7	500	PA/Asy	1.000	0.997
7	500	GMS/Asy	0.997	0.916
7	1000	PA/Asy	1.000	0.993
7	1000	GMS/Asy	0.997	0.823
8	250	PA/Asy	1.000	1.000
8	250	GMS/Asy	1.000	0.988
8	500	PA/Asy	1.000	1.000
8	500	GMS/Asy ₃₀	0.999	0.972
8	1000	PA/Asy	1.000	1.000
8	1000	GMS/Asy	1.000	0.942

Notes: CLR and AS refer to our inference methods and those of Andrews and Shi (2009), respectively. There were 1000 replications per experiment.

FIGURE 1. Simulated Data and Bounding Functions: DGP1 and DGP2
Simulated Data and Bounding Function

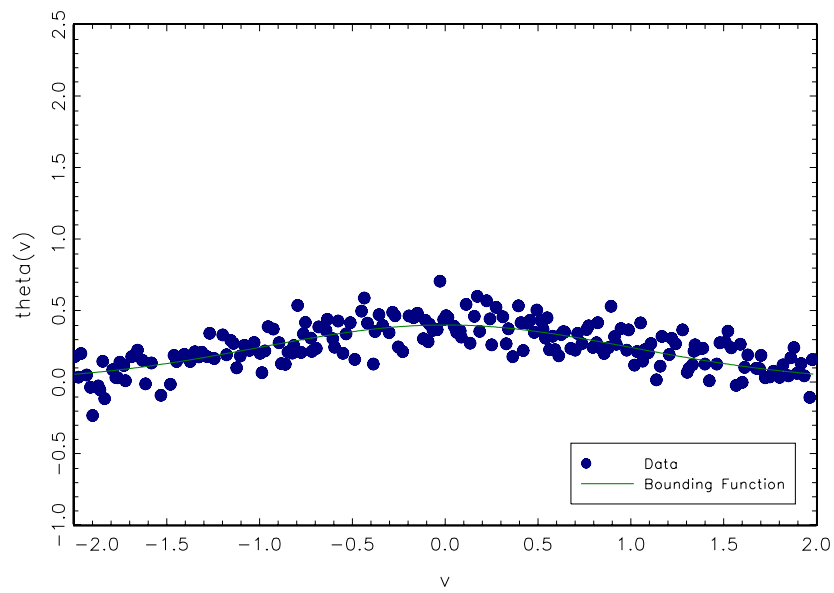
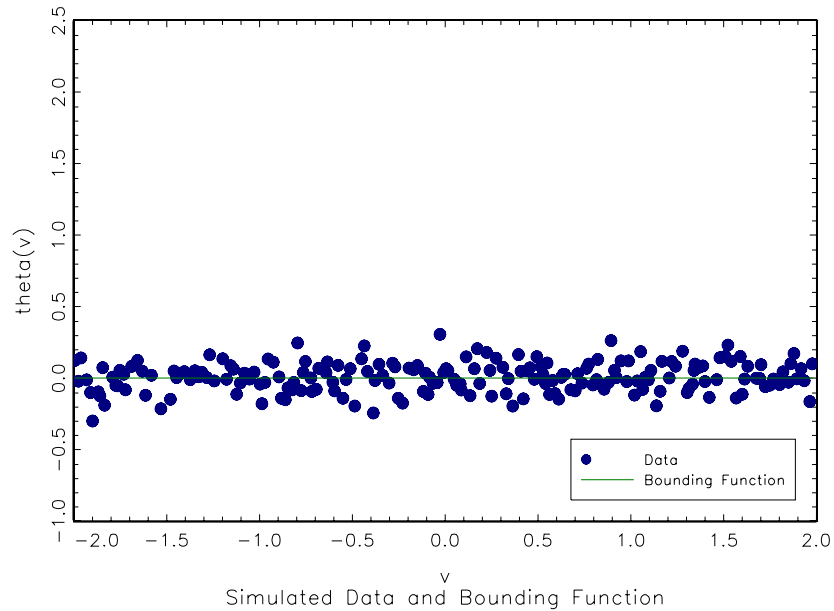


FIGURE 2. Simulated Data and Bounding Functions: DGP3 and DGP4
Simulated Data and Bounding Function

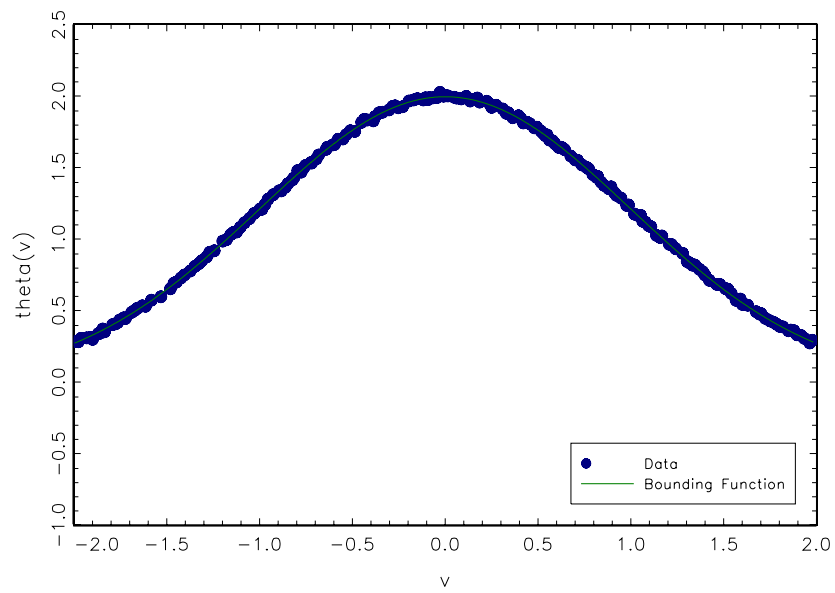
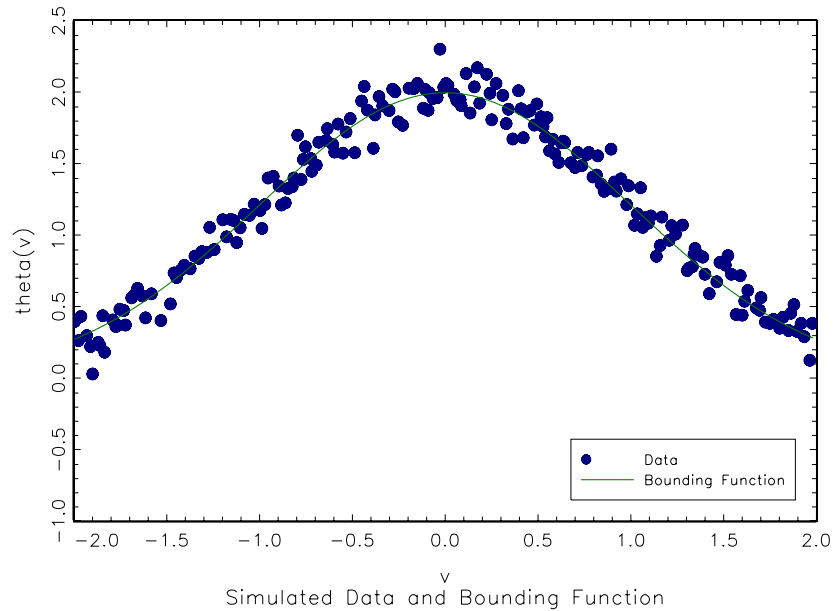


FIGURE 3. Simulated Data and Bounding Functions: DGP5 and DGP6
Simulated Data and Bounding Function

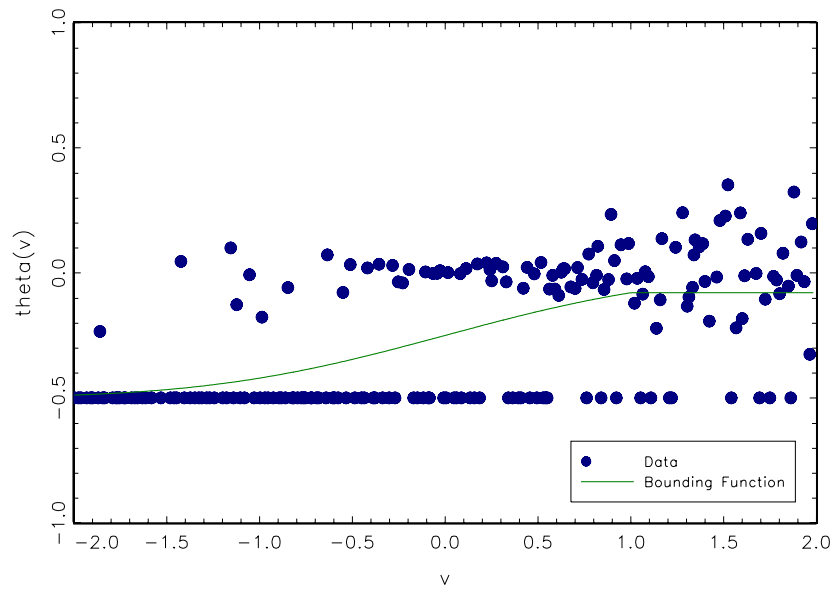
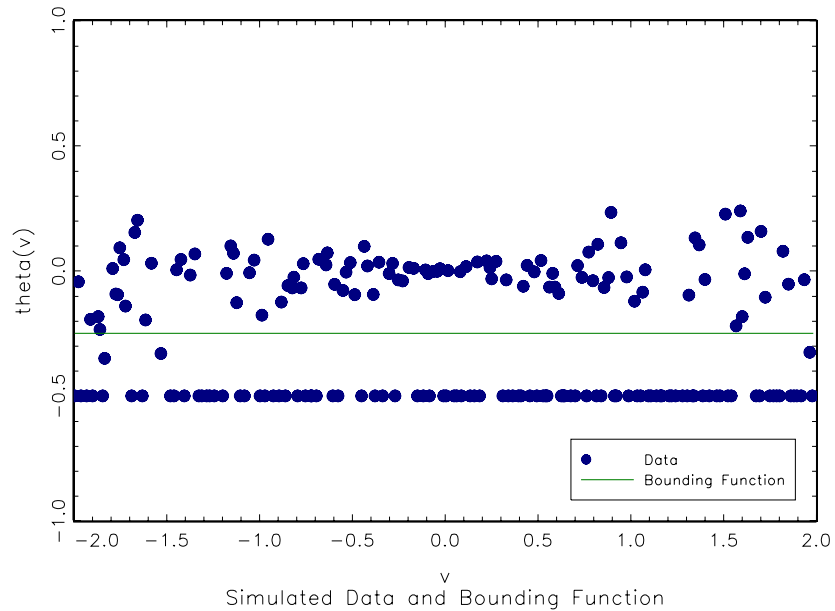
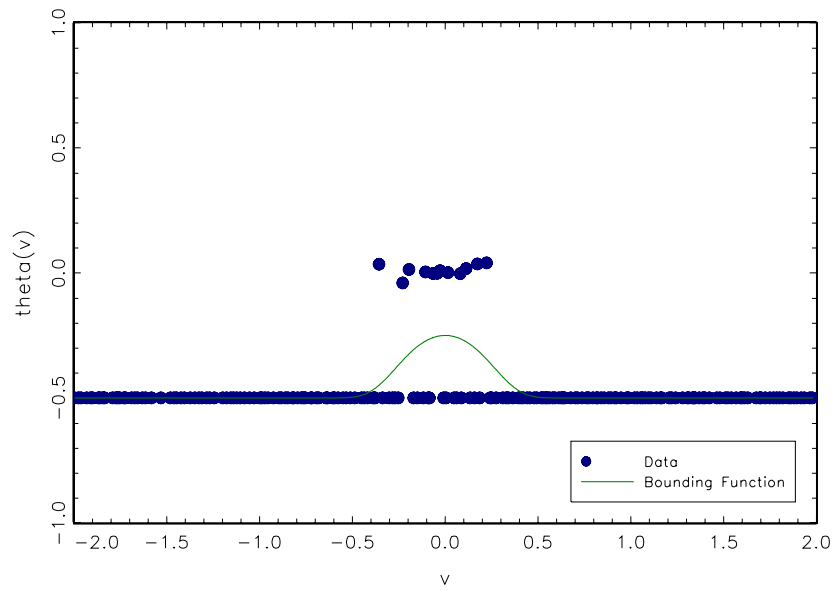
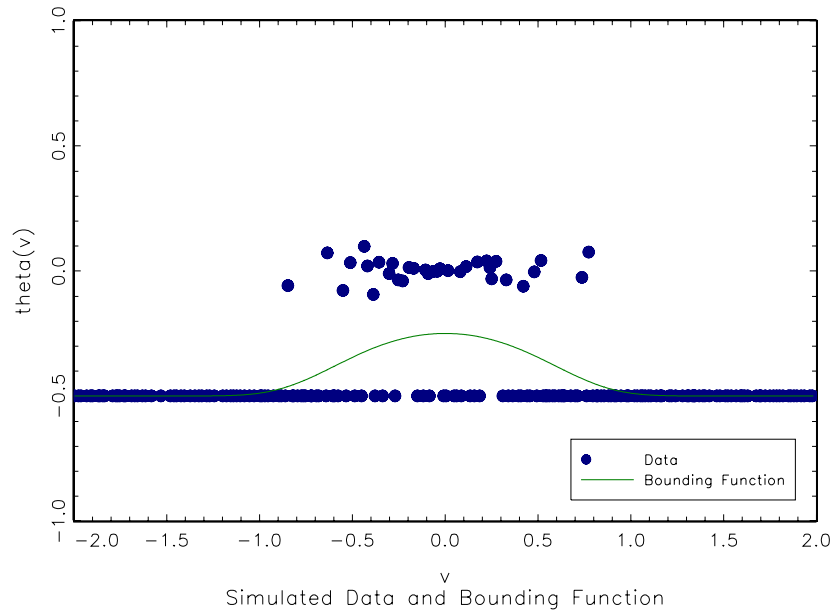


FIGURE 4. Simulated Data and Bounding Functions: DGP7 and DGP8
Simulated Data and Bounding Function



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