

Seierstad, Atle

**Working Paper**

## Open mapping theorems for directionally differentiable functions

Memorandum, No. 2004,21

**Provided in Cooperation with:**

Department of Economics, University of Oslo

*Suggested Citation:* Seierstad, Atle (2004) : Open mapping theorems for directionally differentiable functions, Memorandum, No. 2004,21, University of Oslo, Department of Economics, Oslo

This Version is available at:

<https://hdl.handle.net/10419/63084>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*

# MEMORANDUM

No 21/2004

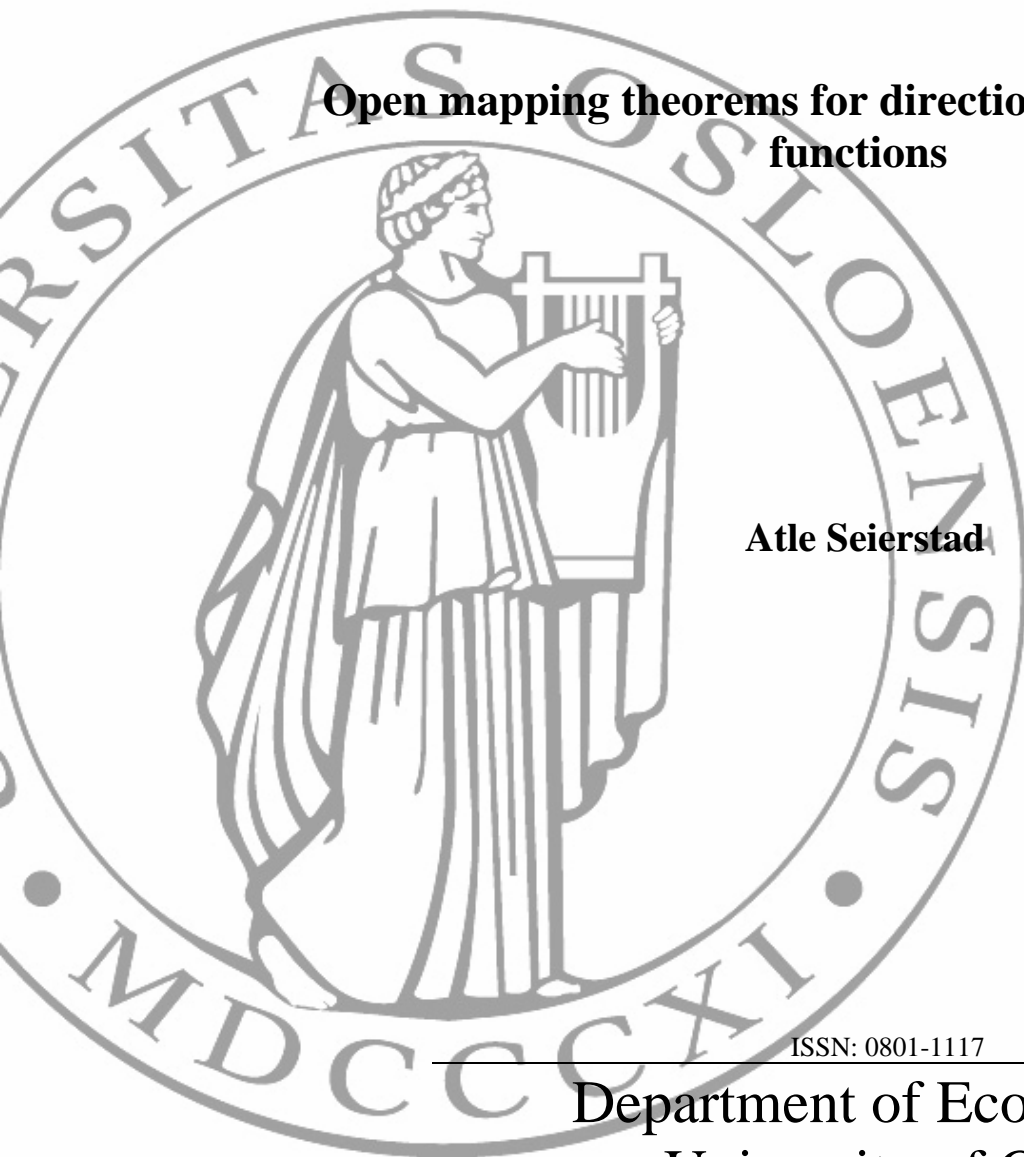
**Open mapping theorems for directionally differentiable functions**

**Atle Seierstad**

ISSN: 0801-1117

---

Department of Economics  
University of Oslo



This series is published by the  
**University of Oslo**  
**Department of Economics**

P. O.Box 1095 Blindern  
N-0317 OSLO Norway  
Telephone: + 47 22855127  
Fax: + 47 22855035  
Internet: <http://www.oekonomi.uio.no/>  
e-mail: [econdep@econ.uio.no](mailto:econdep@econ.uio.no)

In co-operation with  
**The Frisch Centre for Economic  
Research**

Gaustadalleén 21  
N-0371 OSLO Norway  
Telephone: +47 22 95 88 20  
Fax: +47 22 95 88 25  
Internet: <http://www.frisch.uio.no/>  
e-mail: [frisch@frisch.uio.no](mailto:frisch@frisch.uio.no)

List of the last 10 Memoranda:

No 20	Contract Renewal Helge Holden, Lars Holden, and Steinar Holden. 44 pp.
No 19	Jo Thori Lind Repeated surveys and the Kalman filter. 19 pp.
No 18	Kari Eika When quality today affects service needs Tomorrow. 30 pp.
No 17	Rolf Golombek and Michael Hoel Unilateral emission reductions when there are cross -country technology spillovers. 23 pp.
No 16	Kjetil Bjorvatn and Alexander W. Cappelen Globalisation, inequality and redistribution. 17 pp.
No 15	Alexander W. Cappelen and Bertil Tungodden Rewarding effort. 31 pp.
No 14	Alexander W. Cappelen and Bertil Tungodden The liberal egalitarian paradox. 15 pp.
No 13	John K. Dagsvik and Steinar Strøm Sectoral labor supply, choice restrictions and functional form. 15 pp.
No 12	Tiziano Razzolini The Norwegian market for pharmaceuticals and the non-mandatory substitution reform of 2001: the case of enalapril. 58 pp.
No 11	Rolf Golombek and Michael Hoel Climate Agreements and Technology Policy. 38 pp.

A complete list of this memo-series is available in a PDF® format at:

<http://www.oekonomi.uio.no/memo/>

# Open mapping theorems for directionally differentiable functions

Atle Seierstad, University of Oslo

**Abstract.** Open mapping theorems are proved for directionally differentiable Lipschitz continuous functions. It is indicated that generalizations to nonsmooth functions that are not directionally differentiable are possible. The results in the paper generalize the open mapping theorems for differentiable mappings, and are different from open mapping theorems for nonsmooth functions in the literature, when these are specialized to directionally differentiable functions.

**Introduction** Open mapping theorems are proved for directionally differentiable Lipschitz continuous nonlinear functions. It is indicated that generalizations to nonsmooth functions that are not directionally differentiable are possible. The results in this paper generalize the open mapping theorems for differentiable mappings, and are different from open mapping theorems for nonsmooth functions in the literature, when these are specialized to directionally differentiable functions. It is well-known that if a Lipschitz continuous function between two Euclidean spaces is differentiable at a given point, and the derivative has a maximal rank, then the function value at this point is an interior point in the range of the function. Similar open mapping theorems for set-valued functions in Banach spaces that use contingent derivatives or coderivatives can be specialized to ordinary function, and in particular to directionally differentiable Lipschitz continuous functions. (For a selection of such results, see e.g. Aubin and Ekeland (1984), Borwein and Zhu (1999), Aubin and Frankowska (1990), Ioffe (2000), Hiriart-Urruty (1979)). These specialized results, however require stronger approximation conditions than the ones applied below, somewhat similar in case of differentiability to the change from differentiability to strict differentiability in the sense of Clarke (1983). Also for applications of the Solvability Theorem of Clarke et. al. (1998), it seems that some sort of stronger conditions are needed. For nonsmooth Lipschitz continuous ordinary functions between Euclidean spaces, with a convex set of first order approximating matrices ("derivatives"), open mapping theorems of the type mentioned above hold, provided all matrices have maximal rank. Scattered examples are Clarke (1983, Ch. 7), Clarke et. al. (1998), Halkin (1976), Jeyakumar and Luc (2002), Warga (1976, 1978), and Sussmann (2001), (which also considers set-valued maps). These results can be specialized to directionally differentiable ordinary functions. However, these specialized results differ from the results below, since the results below yield open mapping properties in cases where the set of approximating matrices by necessity must include matrices with nonmaximal rank. An example is given.



## Results and proofs

For any function  $f$ , denote by  $f'(\bar{v})[x]$  the (one-sided) derivative along  $x$  at  $\bar{v}$ . If  $x$  has unit length,  $f'(\bar{v})[x]$  is called the (one-sided) directional derivative in direction  $x$ , at  $\bar{v}$ .

**Theorem 1.** Let  $v \rightarrow f(v) : R^n \rightarrow R^m$  be locally Lipschitz continuous, with (one-sided) directional derivatives at any point  $v$ . Assume that for some  $\bar{v} \in R^n$ ,  $\sigma > 0, K > 0$ ,

(A) if  $y = f'(\bar{v})[x]$ , with  $y \in \text{cl}B(0, \sigma)$  and  $x \in \text{cl}B(0, 1)$ , then for any other  $y' \in \text{cl}B(0, \sigma)$ , there exists an  $x' \in \text{cl}B(0, 1)$  such that  $y' = f'(\bar{v})[x']$  and  $|x - x'| \leq K|y - y'|$ .

Then for some  $\varepsilon > 0$ ,  $f(\bar{v}) + \text{cl}B(0, \varepsilon) \subset f(\bar{v} + \text{cl}B(0, 1))$ . □

Observe that a consequence of (A) is that  $\text{cl}B(0, \sigma) \subset f'(\bar{v})[\text{cl}B(0, 1)]$  (let  $y = 0$ ,  $x = 0$  in (A)). Note also that if  $n = m$ , and  $x \rightarrow f'(\bar{v})[x]$  has a Lipschitz continuous inverse on  $R^m$ , then (A) holds.

**Proof** We can assume  $\bar{v} = 0$  and  $f(0) = 0$ , and that  $f$  is Lipschitz continuous with rank  $F > 0$  in  $\text{cl}B(0, 1)$  (i.e.  $|f(v) - f(v')| \leq F|v - v'|$ ,  $v, v' \in \text{cl}B(0, 1)$ ). Write  $f'(0)[x] = f'[x]$ . Then  $f'[x]$  is Lipschitz continuous in  $x \in \text{cl}B(0, 1)$  with rank  $F$ . Let  $\Delta$  be a largest symmetric geometric  $n$ -dimensional simplex with barycenter in  $0$  contained in  $\text{cl}B(0, 1) \subset R^n$ , and let  $B(0, \rho)$  be the largest ball centered in  $0$  and contained in  $\Delta$ . Then  $\text{cl}B(0, \delta\rho) \subset \delta\Delta \subset \text{cl}B(0, \delta)$ , for any  $\delta > 0$ . Choose  $\alpha \in (0, \sigma/2F)$  so small that when  $x$  in  $\text{cl}B(0, \alpha)$ , then  $|f'[x] - f(x)| < \alpha\rho/4K$ . (We then also have  $|f(x)| < \sigma/2$  and in particular, these inequalities hold for  $x \in \alpha\Delta$ .) The former inequality follows from the existence of directional derivatives at  $\bar{v} = 0$ , and a uniform approximation result, shown below.

Let  $q$  be any given vector in  $B(0, \min\{\sigma/2, \alpha\rho/4K\}) \subset R^m$ . We are going to prove that  $q = f(v)$  for some  $v \in \text{cl}B(0, 1)$ .

Let  $x \in \alpha\Delta$  and let  $y := f'[x] \in \text{cl}B(0, \sigma)$ . Then  $|f(x) - q| \leq |f(x)| + |q| < \sigma$ . Since  $|f'[x] - f(x)| < \alpha\rho/4K$ , and hence  $|y - (f(x) - q)| \leq |y - f(x)| + |q| < \alpha\rho/2K$ , then by (A) (with  $y' = f(x) - q \in \text{cl}B(0, \sigma)$ ), there exists an  $x' \in \text{cl}B(0, 1)$  such that  $|x - x'| < \alpha\rho/2$  and  $x' \in f'^{-1}[f(x) - q]$ . Hence,  $x - x' \in \alpha\Delta$ .

Let  $\tilde{D}_m$  be any geometric  $n$ -dimensional simplex in the  $m$ -th barycentric subdivision of  $\alpha\Delta$ , and let  $\tilde{z}_m := \tilde{z}_{\tilde{D}_m}$  be the barycenter of  $\tilde{D}_m$ . By the argument just presented, there exists a  $\tilde{z}'_m := \tilde{z}'_{\tilde{D}_m} \in \text{cl}B(0, 1)$  such that  $|\tilde{z}_m - \tilde{z}'_m| < \alpha\rho/2$ ,  $\tilde{z}'_m \in f'^{-1}[f(\tilde{z}_m) - q]$ , hence,  $f'[\tilde{z}'_m] = f(\tilde{z}_m) - q \in \text{cl}B(0, \sigma)$ . Next, and similarly, for each  $z$  in  $\tilde{D}_m$ , there exists a  $w_z \in \text{cl}B(0, 1)$  such that  $|w_z - \tilde{z}'_m| \leq KF\tilde{D}^m$ , where  $\tilde{D}^m$  is the diameter of  $\tilde{D}_m$  and  $w_z \in f'^{-1}[f(z) - q]$ . The reason is that for  $y_m = f'[\tilde{z}'_m] \in \text{cl}B(0, \sigma)$ , for  $y' = f(z) - q \in B(0, \sigma)$  (cf. the property  $f(x) - q \in \text{cl}B(0, \sigma)$  shown above), we get that  $|y' - y_m| = |y' - f'[\tilde{z}'_m]| = |f(z) - f(\tilde{z}_m) + f(\tilde{z}_m) - f'[\tilde{z}'_m] - q| = |f(z) - f(\tilde{z}_m)| \leq F\tilde{D}^m$ , so by (A) an  $x' =: w_z \in \text{cl}B(0, 1)$  exists such that  $|w_z - \tilde{z}'_m| < KF\tilde{D}^m$  and  $w_z \in f'^{-1}[f(z) - q]$ .

Then  $|w_z - z| \leq |w_z - \tilde{z}'_m| + |\tilde{z}'_m - \tilde{z}_m| + |\tilde{z}_m - z| \leq KF\tilde{D}^m + \alpha\rho/2 + \tilde{D}^m$ . Now,  $\lim_{m \rightarrow \infty} \tilde{D}^m = 0$ . Hence, for  $m$  large enough,  $z - w_z \in B(0, \alpha\rho) \subset \alpha\Delta$ .

For any  $x \in R^n$ , let  $\beta(x)$  be the largest number in  $(0, 1]$  such that  $\beta(x)x \in \alpha\Delta$ . For any  $z \in \alpha\Delta$ , let  $I_z = \{i : (z)_i > 0\}$ , where  $(z)_i$  denotes the barycentric coordinates of  $z$  relative to the vertices  $x_i, i = 0, \dots, n$ , of  $\alpha\Delta$ . Next, let a label (or labeling rule)  $i(z)$  of the vertices  $z$  of  $\tilde{D}_m$  be the rule that  $i(z)$  is the smallest number in  $I_z$  such that  $(\beta(z - w_z)(z - w_z))_i \leq (z)_i$ . (Such a number must exist, because for any  $\tilde{z}, \tilde{z}'$  in  $\alpha\Delta$ ,  $1 = \sum_i (\tilde{z}')_i \geq \sum_{i \in I_{\tilde{z}}} (\tilde{z}')_i, \sum_{i \in I_{\tilde{z}}} (\tilde{z})_i = 1$ , so  $(\tilde{z}')_i > (\tilde{z})_i$  for all  $i \in I_{\tilde{z}}$  cannot be the case.) If  $z$  belongs to a  $k$ -dimensional face of  $\alpha\Delta$ , equal to  $\text{co}\{x_{j_0}, \dots, x_{j_k}\}$  for some subcollection  $x_{j_0}, \dots, x_{j_k}$  of the collection of vertices  $\{x_0, \dots, x_n\}$  of  $\alpha\Delta$ , then  $(z)_i = 0$ , for  $i \in \{0, \dots, n\} \setminus \{j_0, \dots, j_k\}$ , and hence  $i(z) \in \{j_0, \dots, j_k\}$ . Thus  $i(z)$  is what is called a proper label.

By Sperner's lemma, there exists a sequence of completely labeled  $n$ -dimensional simplices  $\{D_m\}_m$  with barycenters  $z_m = \tilde{z}_{D_m}$  and corresponding vectors  $z'_m = \tilde{z}'_{D_m}$ . (Completely labeled means that the vertices of  $D_m$  have labels  $\{0, \dots, n\}$ , and  $D_m$  belongs to the  $m$ -th barycentric subdivision.) Let the subsequences  $z_{m_k}$  and  $z'_{m_k}$  converge to limits  $\bar{z}$  and  $\bar{z}'$  in  $\text{cl}B(0, 1)$ , respectively. If  $\{z^j_m\}_j$  are the vertices of  $D_m$ , then, for each  $j$ ,  $\{z^j_{m_k}\}_k$  and  $\{w_{z^j_{m_k}}\}_k$  also converge to  $\bar{z}$  and  $\bar{z}'$ , respectively, (recall  $|w_z - \tilde{z}'_m| \leq KF\tilde{D}^m$ ).

As we saw above, for  $m$  large, when  $z \in \tilde{D}_m$ , then  $z - w_z \in \alpha\Delta$ , i.e.  $\beta(z - w_z) = 1$ . Hence, by the complete labeling, if  $m_k$  is large enough, for any  $i$ , for some  $j$ ,  $(z^j_{m_k} - w_{z^j_{m_k}})_i \leq (z^j_{m_k})_i$ . Letting  $k \rightarrow \infty$ , we get  $(\bar{z} - \bar{z}')_i \leq (\bar{z})_i$  for all  $i$ , i.e.  $\bar{z}' = 0$ . As  $f'[z^j_{m_k}] = f(z_{m_k}) - q$ , then, by continuity,  $0 = f'[0] = f'(\bar{z}') = f(\bar{z}) - q$ , i.e.  $f(\bar{z}) = q$ .

Theorem 1 has the following generalization, the proof of which is only a slight modification of the proof above.

**Theorem 2** Let  $v \rightarrow f(v) : R^n \rightarrow R^m$  be Lipschitz continuous in a closed bounded convex set  $A \subset R^n$ , let  $\bar{v}$  be a point in  $A$ , and assume the existence of a ball  $B(b, \xi)$  in  $R^n$  such that  $\bar{v} + \text{cl}B(b, \xi) \subset A$ . Assume that  $f$  has (one-sided) directional derivatives at  $\bar{v}$  in any direction  $v/|v|$ ,  $v$  in  $A - \bar{v}, v \neq 0$ . Assume also that positive numbers  $K$  and  $\sigma > 0$  exist such that if  $p = f'(\bar{v})[b]$ , then the following condition holds:

(B) If  $y = f'(\bar{v})[x], y \in \text{cl}B(p, \sigma), x \in \text{cl}B(b, \xi)$ , then for any other  $y' \in \text{cl}B(p, \sigma)$ , there exists an  $x' \in \text{cl}B(b, \xi)$  such that  $y' = f'(\bar{v})[x']$  and  $|x - x'| \leq K|y - y'|$ .

Then for some  $\varepsilon > 0, \gamma > 0, f(\bar{v}) + \text{cl}B(\alpha p, \alpha\varepsilon) \subset f(\bar{v} + \text{cl}B(\alpha b, \alpha\xi))$  for all  $\alpha \in (0, \gamma]$ .  $\square$

**Proof** Assume that  $f$  is Lipschitz continuous with rank  $F > 0$  on  $A$ . Then  $f'[x] := f'(\bar{v})[x]$  is Lipschitz continuous with rank  $F$  on  $\text{cone}(A - \bar{v})$ , the cone generated by  $A - \bar{v}$ . Let  $\mu = \min\{\xi, \sigma/4F\}$ . Let  $\Delta$  be a largest symmetric geometric  $n$ -dimensional

simplex with barycenter in 0, contained in  $\text{cl}B(0, \mu) \subset R^n$ , and let  $B(0, \rho)$  be the largest ball centered in 0 and contained in  $\Delta$ . Then  $\text{cl}B(0, \delta\rho) \subset \delta\Delta \subset \text{cl}B(0, \delta\mu)$  for any  $\delta > 0$ . Using the uniform approximation property shown below, choose  $\gamma$  so small that when  $x$  belongs to  $\text{cl}B(ab, \alpha\mu)$  and  $\alpha \in (0, \gamma]$ , then

$|f'[x] - (f(\bar{v} + x) - f(\bar{v}))| < \alpha \min\{\rho/4K, \sigma/4\}$ . Now, when  $x$  belongs to  $\text{cl}B(ab, \alpha\mu)$ , then, by Lipschitz continuity,  $|f'[x] - \alpha p| = |f'[x] - f'[ab]| \leq \alpha\sigma/4$  and hence,  $|f(\bar{v} + x) - f(\bar{v}) - \alpha p| < \alpha\sigma/2$ .

Fix an arbitrary number  $\alpha$  in  $(0, \gamma]$  and let  $q$  be any given vector in  $\text{cl}B(0, \min\{\alpha\sigma/2, \alpha\rho/4K\}) \subset R^m$ . We are going to prove that  $f(\bar{v}) + q + \alpha p = f(v)$  for some  $v \in \bar{v} + \text{cl}B(ab, \alpha\xi)$ .

Let  $z \in \text{cl}B(0, \alpha\mu) \subset R^n$ . Then  $y := f'[z + ab] \in \text{cl}B(\alpha p, F\alpha\mu) \subset \text{cl}B(\alpha p, \alpha\sigma)$ . Let  $x = z + ab$  and let  $y' = f(\bar{v} + ab + z) - f(\bar{v}) - q$ . Then, by the last inequality,  $|y' - \alpha p| = |f(\bar{v} + ab + z) - f(\bar{v}) - q - \alpha p| \leq |f(\bar{v} + ab + z) - f(\bar{v}) - \alpha p| + |q| < \alpha\sigma/2 + \alpha\sigma/2 = \alpha\sigma$ , so  $y' \in \text{cl}B(\alpha p, \alpha\sigma)$ . Now (B) also holds for  $\text{cl}B(b, \xi)$  and  $\text{cl}B(p, \sigma)$  replaced by  $\text{cl}B(ab, \alpha\xi)$  and  $\text{cl}B(\alpha p, \alpha\sigma)$ , respectively, denote this condition by  $(\alpha B)$ . Since  $|f'[z + ab] - (f(\bar{v} + z + ab) - f(\bar{v}))| < \alpha\rho/4K$  and hence  $|y - y'| = |y - (f(\bar{v} + ab + z) - f(\bar{v})) + q| \leq |y - (f(\bar{v} + ab + z) - f(\bar{v}))| + |q| < \alpha\rho/2K$ , then by  $(\alpha B)$  there exists an  $x' \in \text{cl}B(ab, \alpha\xi)$  such that, for  $z' := x' - ab \in \text{cl}B(0, \alpha\xi)$ ,  $|z - z'| = |x - x'| < K\alpha\rho/2K = \alpha\rho/2$  and  $x' = z' + ab \in f'^{-1}[f(\bar{v} + ab + z) - f(\bar{v}) - q]$ . Evidently,  $z - z' \in \alpha\Delta$ .

Let  $\tilde{D}_m$  be any geometric  $n$ -dimensional simplex in the  $m$ -th barycentric subdivision of  $\alpha\Delta$ , and let  $\tilde{z}_m := \tilde{z}_{\tilde{D}_m}$  be the barycenter of  $\tilde{D}_m$ . By the argument just presented, there exists an  $\tilde{z}'_m := \tilde{z}'_{\tilde{D}_m} \in \text{cl}B(0, \alpha\xi)$  such that  $|\tilde{z}_m - \tilde{z}'_m| < \alpha\rho/2$ ,  $\tilde{z}'_m + ab \in f'^{-1}[f(\bar{v} + ab + \tilde{z}_m) - f(\bar{v}) - q]$ , hence,  $f'[\tilde{z}'_m + ab] = f(\bar{v} + ab + \tilde{z}_m) - f(\bar{v}) - q \in \text{cl}B(\alpha p, \alpha\sigma)$ .

Next and similarly, for each  $z$  in  $\tilde{D}_m$  there exists a  $w_z$  such that  $|w_z - \tilde{z}'_m| \leq KF\tilde{D}_m^m$ , where  $\tilde{D}_m^m$  is the diameter of  $\tilde{D}_m$  and  $w_z + ab \in f'^{-1}[f(\bar{v} + ab + z) - f(\bar{v}) - q]$ . The reason is that for  $y_m := f'[\tilde{z}'_m + ab] \in \text{cl}B(\alpha p, \alpha\sigma)$ , then  $y' := f(\bar{v} + ab + z) - f(\bar{v}) - q \in \text{cl}B(\alpha p, \alpha\sigma)$ , (this inclusion was shown above), and we have that  $|y' - y_m| = |y' - f'[\tilde{z}'_m + ab]| = |f(\bar{v} + ab + z) - f(\bar{v}) - q - \{f(\bar{v} + ab + \tilde{z}_m) - f(\bar{v}) - q\}| = |f(\bar{v} + ab + z) - f(\bar{v}) - f(\bar{v} + ab + \tilde{z}_m) + f(\bar{v})|$  so by  $(\alpha B)$ , an  $x' := w_z + ab \in \text{cl}B(ab, \alpha\xi)$  exists such that  $|w_z - \tilde{z}'_m| \leq KF\tilde{D}_m^m$  and  $w_z + ab \in f'^{-1}[f(\bar{v} + ab + z) - f(\bar{v}) - q]$ . Then  $|w_z - z| \leq |w_z - \tilde{z}'_m| + |\tilde{z}'_m - \tilde{z}_m| + |\tilde{z}_m - z| \leq KF\tilde{D}_m^m + \alpha\rho/2 + \tilde{D}_m^m$ . Now,  $\lim_m \tilde{D}_m = 0$ . Hence, for  $m$  large enough,  $|z - w_z| \leq \alpha\rho$ , so  $z - w_z \in \alpha\Delta$ .

Again there exists a sequence of simplices  $\{D_m\}_m$  with barycenters  $z_m := \tilde{z}_{D_m}$  and corresponding vectors  $z'_m := \tilde{z}'_{D_m}$  and subsequences  $z_{m_k}$  and  $z'_{m_k}$  that converge to limits  $\bar{z}$  and  $\bar{z}'$ , respectively ( $\bar{z}$  and  $\bar{z}'$  now in  $\text{cl}B(0, \alpha\xi)$ ), such that if  $\{z^j_m\}_j$  are the vertices of  $D_m$ , then, for each  $j$ ,  $\{z^j_{m_k}\}_k$  and  $\{w^j_{z^j_{m_k}}\}_k$  also converge to  $\bar{z}$  and  $\bar{z}'$ , respectively, (recall  $|w_z - \tilde{z}'_m| \leq KF\tilde{D}_m^m$ ), and finally, such that, for any  $i$ , for some  $j$ ,  $(z^j_{m_k} - w^j_{z^j_{m_k}})_i \leq (z^j_{m_k})_i$ . Letting  $k \rightarrow \infty$ , we get  $(\bar{z} - \bar{z}')_i \leq (\bar{z})_i$  for all  $i$ , i.e.  $\bar{z}' = 0$ . As  $f'[z'_m + ab] = f(\bar{v} + ab + z_m) - f(\bar{v}) - q$ , by continuity,  $\alpha p = f'[ab] = f(\bar{v} + ab + \bar{z}) - f(\bar{v}) - q$ , i.e.  $f(v) = f(\bar{v}) + q + \alpha p$  for  $v = \bar{v} + ab + \bar{z} \in \bar{v} + \text{cl}B(ab, \alpha\xi)$ .



**Note 1.** (Comment on the uniformity of  $\varepsilon, \gamma$ .) Evidently,  $\varepsilon = \varepsilon(\sigma, F, K, \xi)$ , and  $\gamma = \gamma(\sigma, F, K, \xi)$ , i.e. these entities do not depend on  $b$ , as long as  $b$  is such that  $\bar{v} + \text{cl}B(b, \xi) \subset A$ . (By the proof of the uniform approximation property below, the above independence of  $\varepsilon$  and  $\gamma$  on  $b$  evidently holds.)

Let  $\text{rint}C$  be the relative interior of the set  $C$ .

**Corollary.** Let  $v \rightarrow f(v) : R^n \rightarrow R^m$  be Lipschitz continuous in a closed bounded convex set  $A \subset R^n$ . Assume that  $f$  has (one-sided) directional derivatives at  $\bar{v} \in A$  in any direction  $v/|v|$ ,  $v$  in  $A - \bar{v}$ . Assume also that

(C) For each  $p \in f'(\bar{v})[\text{rint}(A - \bar{v})]$  there exist a  $b_p \in \text{rint}(A - \bar{v})$  such that  $p = f'(\bar{v})[b_p]$  and positive numbers  $K_p, \xi_p, \sigma_p$ , such that  $\text{cl}B(b_p, \xi_p) \cap \text{linspan}(A - \bar{v}) \subset A - \bar{v}$ , and such that if  $y = f'(\bar{v})[x], y \in \text{cl}B(p, \sigma_p), x \in \text{cl}B(b_p, \xi_p) \cap \text{linspan}(A - \bar{v})$ , then for any other  $y' \in \text{cl}B(p, \sigma_p)$ , there exists an  $x' \in \text{cl}B(b_p, \xi_p) \cap \text{linspan}(A - \bar{v})$  such that  $y' = f'(\bar{v})[x']$  and  $|x - x'| \leq K_p|y - y'|$ .

Then, for any  $p \in f'(\bar{v})[\text{rint}(A - \bar{v})]$ , for some  $\delta^* > 0$ ,  $f(\bar{v}) + \delta p \in f(A)$  for all  $\delta \in (0, \delta^*]$ . In fact,  $\delta^* = \delta^*(K_p, \xi_p, \sigma_p)$  ( $\delta^*$  also depends on the Lipschitz rank  $F$  of  $f$ ).

**Proof.** We can assume that  $\bar{v} = 0$ . Let  $E = \text{linspan}A$ . If we restrict  $f$  to  $E$ , and  $b \in \text{rint}(A - \bar{v}) = \text{rint}(A)$ , then  $E \cap \text{cl}B(b, \xi_p) \subset A - \bar{v} = A$ , and we are back in a situation where Theorem 2 applies.

## The uniform approximation property

Let  $A'$  be a subset of the unit sphere  $\{x \in R^n : |x| = 1\}$ , let  $\bar{v} \in R^n$ , and assume that  $f : R^n \rightarrow R^m$  is Lipschitz continuous on  $A'' := B(\bar{v}, \mu) \cap [\bar{v} + \text{cone}(A')]$ ,  $\mu$  some number  $> 0$ , with directional derivatives at  $\bar{v}$  in all directions  $x$  in  $A'$ . Then  $\lim_{\lambda \downarrow 0} \lambda^{-1} \{f(\lambda x + \bar{v}) - f(\bar{v})\}$  is uniform in  $x \in A'$ .

**Proof:** Let  $\varepsilon$  be any number  $> 0$ , and let  $x_i, i = 1, \dots, i^*$ , be a finite set of unit vectors such that  $x_i \in A'$  and  $A' \subset B(\{x_1, \dots, x_{i^*}\}, \varepsilon/4F)$ ,  $F$  the Lipschitz rank of  $f$  on  $A''$ .

(Then  $F$  is also the Lipschitz rank of  $x \rightarrow f'(\bar{v})[x]$  on  $A'$ .) Choose  $\lambda^* \in (0, \mu]$  so small that  $|\lambda^{-1} \{f(\bar{v} + \lambda x_i) - f(\bar{v})\} - f'[x_i]| < \varepsilon/2$  for all  $i$ , when  $\lambda \in (0, \lambda^*]$ . Let  $x$  be any vector  $x$  in  $A'$ . Then, for some  $i, |x - x_i| \leq \varepsilon/4F$ , and

$$\begin{aligned} |\lambda^{-1} \{f(\bar{v} + \lambda x) - f(\bar{v})\} - f'[x]| &\leq |\lambda^{-1} \{f(\bar{v} + \lambda x) - f(\bar{v})\} - f'[x] - [\lambda^{-1} \{f(\bar{v} + \lambda x_i) - f(\bar{v})\} - f'[x_i]]| \\ &+ |\lambda^{-1} \{f(\bar{v} + \lambda x) - f(\bar{v} + \lambda x_i)\} + f'[x_i] - f'[x]| + |\lambda^{-1} \{f(\bar{v} + \lambda x_i) - f(\bar{v})\} - f'[x_i]| \\ &\leq \lambda^{-1} F \lambda |x - x_i| + F|x - x_i| + \varepsilon/2 \leq \varepsilon/4 + \varepsilon/4 + \varepsilon/2 = \varepsilon. \end{aligned}$$

**Example** Let  $\theta, r$  denote polar coordinates, let a point in  $R^2$  described by polar coordinates be denoted by  $[\theta; r]$  and let  $\phi$  be the function that maps points  $[\theta; 1]$  into  $[2\theta; 1]$ , for  $\theta \in [0, 3\pi/2]$ , and that maps points  $[\theta; 1]$ , for  $\theta \in [3\pi/2, 2\pi]$  into  $(-1 + 4(\theta - 3\pi/2)/\pi, 0)$ . The two definitions of  $\phi$  for  $\theta = 3\pi/2$ , and the definitions of  $\phi$  for  $\theta = 0$  and  $\theta = 2\pi$  coincide. Let  $F(x, y) : R^2 \rightarrow R^2$  be the positively linearly homogeneous extension of  $\phi$  from the unit circle  $C$  to all  $R^2$  (with  $F(0, 0) = (0, 0)$ ). On  $C$ , it is obvious that  $F$  is Lipschitz continuous, and because  $F$  is positively linearly homogeneous, it is Lipschitz continuous everywhere, (see Appendix). In this particular example, we see immediately that  $\text{cl}B(0, 1) \subset F(\text{cl}B(0, 1))$ . Let us use the Corollary to show  $B(0, \varepsilon) \subset F(\text{cl}B(0, 1))$  for some  $\varepsilon > 0$ .

For  $\theta \in [0, 3\pi/2]$ ,  $F'(0, 0)[x, y] = (\cos(2\theta), \sin(2\theta))$  if  $(x, y) = (\cos\theta, \sin\theta)$ , and for any  $p = [\theta_p; 1] \in R^2$  of unit length, we have  $p = [2\theta^p \text{ modulo } 2\pi; 1]$  for some  $\theta^p \in [\pi/8, \pi + \pi/8)$ . Note that  $F'(0, 0)[\text{cl}B(0, 1)] = \text{cl}B(0, 1)$ . To show that property (C) holds in this example it suffices to prove that it holds for any  $p = [\theta_p; 1] \in R^2, \theta_p \in [0, 2\pi)$ . Let  $b_p := [\theta^p; 1]$ ,  $\sigma_p = \pi/16, K_p = 4, \xi_p = \pi/8$ , and note that if  $a \in \text{cl}B(p, \pi/16)$ ,  $a' \in \text{cl}B(p, \pi/16)$ , then  $|a/|a| - p| \leq \pi/8, |a'/|a'| - p| \leq \pi/8$ , so  $a = [2\theta; |a|]$ ,  $a' = [2\theta'; |a'|]$  for some unique  $\theta$  and  $\theta'$  in  $[\theta^p - \pi/16, \theta^p + \pi/16] \subset [0, 3\pi/2]$ . For  $d = [\theta; |a|]$ ,  $d' = [\theta'; |a'|]$ , evidently,  $d, d' \in B(b_p, \pi/8)$  and  $|a - a'| \geq |d - d'|$ . Then the Corollary says that for each  $p$ , for some  $\delta_p^*$ ,  $(0, \delta_p^*]p \subset F(\text{cl}B(0, 1))$ , and in fact,  $\delta_p^*$  is independent of  $p$ .

Now,  $F'(0, 0)[(x, y)] = (0, 0)$  for  $\theta = 7\pi/4, (x, y) = (\cos(7\pi/4), \sin(7\pi/4))$ , which removes the possibility that any convex family of  $2 \times 2$ -matrices  $\sigma$  that yields the linear approximations of the function  $F$  contains only invertible matrices.

Frankly, this example may not be completely convincing as regards the usefulness of the above approach. I guess that refinements of the proofs in the tradition of approximating matrices are possible (or even exist) giving results that would cover examples of the type above, including more nonlinear and hence more interesting examples. (In particular, one might consider the possibility that the matrices are only approximating in certain directions, not all directions.) Of course, one could try to come up with more "substantial" examples in favor of the approach here. One difficulty then is that the sets of approximating matrices can be varied indefinitely, and all of them have to be shown to contain a matrix of nonmaximal rank. In the truly nonsmooth case, for which results are briefly sketched in the next note, an example is mentioned which in a sense seems to be more substantial in this case.

**Note 2 (Generalizations)** Generalizations to Lipschitz continuous functions not having directional derivatives and possible applications in control theory will be considered in future work. Some indications of the possibility of such generalizations can however be given. For any given vector  $\bar{v}$ , for any sequence  $\lambda^k \downarrow 0, \lambda_k > 0$ , for any vector  $v$ , a subsequence  $\bar{\lambda}_k \downarrow 0$ , exists such that  $\bar{\lambda}_k^{-1}[f(\bar{v} + \bar{\lambda}_k v) - f(\bar{v})]$  converges to some vector denoted  $f'(\bar{v})[v]$ . More generally, given a countable dense set  $V$  in  $\text{cl}B(0, 1)$ , by diagonal selection, a subsequence  $\lambda_k$  of  $\lambda^k$  can be found, such that the sequence  $\lambda_k^{-1}[f(\bar{v} + \lambda_k v) - f(\bar{v})]$  converges to some

limit denoted  $f'(\bar{v})[v]$  for all  $v \in V$ , and by the Lipschitz continuity, even for all  $v \in \text{cl}B(0, 1)$ . Of course,  $v \rightarrow f'(\bar{v})[v]$  is not positively linearly homogeneous, so the condition (A) and (B) in Theorems 1 and 2 need the following reformulations, respectively:

(D) A sequence  $\lambda_k \downarrow 0$ , and positive numbers  $K$  and  $\sigma$  exist, such that  $\lambda_k^{-1}[f(\bar{v} + \lambda_k v) - f(\bar{v})]$  converges to some limit denoted  $f'(\bar{v})[v]$  for all  $v \in \text{cl}B(0, 1)$ , and for any  $k$ , if  $y = \lambda_k f'(\bar{v})[x/\lambda_k]$ ,  $y \in \text{cl}B(0, \lambda_k \sigma)$ ,  $x \in \text{cl}B(0, \lambda_k)$ , then for any other  $y' \in \text{cl}B(0, \lambda_k \sigma)$ , there exists an  $x' \in \text{cl}B(0, \lambda_k)$  such that  $y' = \lambda_k f'(\bar{v})[x'/\lambda_k]$  and  $|x - x'| \leq K|y - y'|$ .

(E) A sequence  $\lambda_k \downarrow 0$ , and positive numbers  $K$  and  $\sigma$  exist, such that  $\lambda_k^{-1}[f(\bar{v} + \lambda_k v) - f(\bar{v})]$  converges to some limit denoted  $f'(\bar{v})[v]$  for all  $v/|v|, v \in A - \bar{v}$ ,  $v \neq 0$ , and such that for any  $k$ , if  $y = \lambda_k f'(\bar{v})[x/\lambda_k]$ ,  $x \in \text{cl}B(\lambda_k b, \lambda_k \xi)$ ,  $y \in \text{cl}B(\lambda_k f'(\bar{v})[b], \lambda_k \sigma)$ , then for any other  $y' \in \text{cl}B(\lambda_k f'(\bar{v})[b], \lambda_k \sigma)$ , there exists an  $x' \in \text{cl}B(\lambda_k b, \lambda_k \xi)$  such that  $y' = \lambda_k f'(\bar{v})[x'/\lambda_k]$  and  $|x - x'| \leq K|y - y'|$ .

The conclusion in Theorem 1 still holds, while the conclusion of Theorem 2 must be weakened slightly:

For some  $\varepsilon > 0$ ,  $\gamma > 0$ ,  $f(\bar{v}) + \text{cl}B(\alpha f'(\bar{v})[b], \alpha \varepsilon) \subset f(\bar{v} + \text{cl}B(\alpha b, \alpha \xi))$  for all  $\alpha = \lambda_k \in (0, \gamma]$ .

To obtain a proof of both results in this note, one can replace  $f'[x]$  in the proof of Theorem 2 by  $\alpha f'[x/\alpha] := \alpha f'(\bar{v})[x/\alpha]$ ,  $\alpha := \lambda_k$  chosen so small that  $|\alpha f'[x/\alpha] - (f(\bar{v} + x) - f(\bar{v}))| < \alpha \min\{\sigma/4, \rho/4K\}$ . (A uniform approximation property corresponding to the uniform approximation property used in the theorems above holds.)

For the proof see Appendix.

For a moment, one might wonder if the results in this note could work in the case of the function  $\phi(x) = x \sin(1/x)$ , ( $\phi(0) = 0$ ) to yield, for example that  $B(0, \varepsilon) \subset \phi([0, \xi])$  for some  $\varepsilon, \xi > 0$ , which of course can be seen to be true in this particular example. But  $\phi(x)$  is not Lipschitz continuous. Without being completely specific, define  $\phi(x)$ ,  $x > 0$  to be a similar, Lipschitz continuous function, that zigzags between the lines  $y = x$  and  $y = -x$  along lines shifting between having slopes 2 and  $-2$ , the oscillations being more rapid the closer  $x$  is to the origin. It may be seen that the indicated results in this note applies to this function  $\phi$ , moreover one of the approximating  $1 \times 1$  matrices of  $\phi$  is zero.

## References

Aubin, J-P. and Ekeland, I. (1984) *Applied Nonlinear Analysis*, John Wiley & Sons, New York

Aubin J-P. and Frankowska, H. (1990) *Set-Valued Analysis*, Birkhäuser, Boston

- Borwein, J. M., Zhu, Q.J. (1999) A survey of subdifferential calculus with applications, *Nonlinear Analysis* 38, 687-773 .
- Clarke, F. H. (1983) *Optimization and Nonsmooth Analysis*, John Wiley & Sons, Toronto.
- Clarke, F. H. et. al. (1998) *Nonsmooth Analysis and Control Theory*, Springer-Verlag, New York.
- Halkin, H. (1976) Mathematical Programming without differentiability, in: D. L. Russell. (ed.), *Calculus of Variations and Control Theory*, Academic Press, New York, 279-288.
- Halkin, H. (1978) Necessary conditions for optimal control problems with differentiable or nondifferentiable data. In *Mathematical Control Theory, Lecture Notes in Mathematics 680*, Springer Verlag Berlin, 77-118
- Hiriart-Urruty, J. B. (1979) Tangent cones , generalized gradients and mathematical programming in Banach spaces, *Mathematics of. Operations. Research.* 4, 79-97.
- Ioffe, A.D (2000) Metric regularity and subdifferential calculus, *Russian Mathematical Surveys*, 55, 501-558
- Jeyakumar, V., Luc, D.T.(2002) An open mapping theorem using unbounded generalized Jacobians, *Nonlinear Analysis. 50 Ser. A, Theory, Methods*, 647-663.
- Sussmann, H.J. (2001), New theories of set-valued differentials and new versions of the maximum principle of optimal control theory, in Isidori, Alberto (ed) et al. *Nonlinear Control in the Year 2000, Vol 2*, Springer-Verlag, *Lecture Notes Control, Inf. Sci.* 259, 487-526.
- Warga, J. (1976) Derivative containers, inverse functions and controllability, in: D.L. Russell. (ed.), *Calculus of Variations and Control Theory*, Academic Press, New York, 13-46.
- Warga, J. (1978) An implicit function theorem without differentiability, *Proceedings of the American Mathematical Society* 69, 65-69.

**Appendix.** A positively linearly homogeneous function  $F : R^n \rightarrow R^m$ , with  $F(0) = 0$ , is Lipschitz continuous if it is Lipschitz continuous on the unit sphere

$\{x \in R^n : |x| = 1\}$  : Let  $F^*$  be the Lipschitz rank of  $F$  on the unit sphere, let  $K = \sup_{|v|=1} |F(v)|$ , and let  $F' = 2F^* + K$ . The inequality  $|F(v) - F(w)| \leq F'|-v - w|$  is trivial if  $v = 0$  or  $w = 0$ . Assume  $v \neq 0, w \neq 0$ . Define  $a := |w| - |v| \leq |w - v|$ . Then  $|F(v) - F(w)| = ||v|F(v/|v|) + |w|F(w/|w|)| \leq ||v|F(v/|v|) + |v|F(w/|w|)| + (|w| - |v|)|F(w/|w|)| \leq F^*||v|v/|v| - |v|w/|w|| + |w - v|K = F^*||v|v/|v| - (|w| - a)w/|w|| + (|w| - |v|)K \leq F^*||v|v/|v| - |w|w/|w|| + F^*|aw/|w|| + (|w| - |v|)K \leq F^*|v - w| + F^*|v - w| + K|v - w| = F'|v - w|.$

## Proof of the variants of Theorems 1 and 2 stated in Note 2

**Proof** Assume that  $f$  is Lipschitz continuous with rank  $F > 0$  on  $A$ . Then, for any  $\alpha > 0$ ,  $\alpha f[\alpha^{-1}x] := \alpha f(\bar{v})[x/\alpha]$  is Lipschitz continuous with rank  $F$  on cone( $A - \bar{v}$ ). Let  $\mu = \min\{\xi, \sigma/4F\}$ .

Let  $\Delta$  be a largest symmetric geometric  $n$ -dimensional simplex with barycenter in  $0$ , contained in  $\text{cl}B(0, \mu) \subset R^n$ , and let  $B(0, \rho)$  be the largest ball centered in  $0$  and contained in  $\Delta$ . Then  $\text{cl}B(0, \delta\rho) \subset \delta\Delta \subset \text{cl}B(0, \delta\mu)$ , for any  $\delta \in (0, 1]$ . Now, using the uniform approximation property (which also holds in the present case), choose  $\gamma$  so small that when  $x$  belongs to  $\text{cl}B(\alpha b, \alpha\mu)$  and  $\alpha = \lambda_k \in (0, \gamma]$ , then

$|\alpha f[x/\alpha] - (f(\bar{v} + x) - f(\bar{v}))| < \alpha \min\{\rho/4K, \sigma/4\}$ . Note that for  $x$  in  $\text{cl}B(\alpha b, \alpha\mu)$ ,  $|\alpha f[\alpha^{-1}x] - \alpha f[b]| \leq \alpha F|x/\alpha - b| < \alpha\sigma/4$ , by Lipschitz continuity. The two last inequalities give  $|f(\bar{v} + x) - f(\bar{v}) - \alpha p| < \alpha\sigma/2$ .

Fix an arbitrary number  $\alpha = \lambda_k$  in  $(0, \gamma]$  and let  $q$  be any given vector in  $\text{cl}B(0, \min\{\alpha\sigma/2, \alpha\rho/4K\}) \subset R^m$ . We are going to prove that  $f(\bar{v}) + q + \alpha p = f(v)$  for some  $v \in \bar{v} + \text{cl}B(\alpha b, \alpha\xi)$ .

Let  $z \in \text{cl}B(0, \alpha\mu) \subset R^n$ . Then  $y := \alpha f[\alpha^{-1}\{z + \alpha b\}] = \alpha f[z/\alpha + b] \in \text{cl}B(\alpha p, F\alpha\mu) \subset \text{cl}B(\alpha p, \alpha\sigma)$ . Let  $x = z + \alpha b$  and let  $y' = f(\bar{v} + \alpha b + z) - f(\bar{v}) - q$ . Then, by the last inequality,

$|y' - p| = |f(\bar{v} + \alpha b + z) - f(\bar{v}) - q - \alpha p| \leq |f(\bar{v} + \alpha b + z) - f(\bar{v}) - \alpha p| + |q| < \alpha\sigma/2 + \alpha\sigma/2 = \alpha\sigma$ , so  $y' \in \text{cl}B(\alpha p, \alpha\sigma)$ . Since  $|\alpha f[\alpha^{-1}\{z + \alpha b\}] - (f(\bar{v} + z + \alpha b) - f(\bar{v}))| < \alpha\rho/4K$  and hence  $|y - y'| = |y - (f(\bar{v} + \alpha b + z) - f(\bar{v})) + q| < \alpha\rho/2K$ , then by (E), there exists an  $x' \in \text{cl}B(\alpha b, \alpha\xi)$  such that, for  $z' := x' - \alpha b$ ,  $|z - z'| = |x - x'| < K\alpha\rho/2K \leq \alpha\rho/2$  and  $x'/\alpha = (z' + \alpha b)/\alpha \in f^{-1}[\{f(\bar{v} + \alpha b + z) - f(\bar{v}) - q\}/\alpha]$ . Evidently,  $z - z' \in \alpha\Delta$ .

Let  $\tilde{D}_m$  be any geometric  $n$ -dimensional simplex in the  $m$ -th barycentric subdivision of  $\alpha\Delta$ , and let  $\tilde{z}_m$  be the barycenter of  $\tilde{D}_m$ . By the argument just presented, there exists an  $\tilde{z}'_m \in \text{cl}B(0, \alpha\xi)$  such that  $|\tilde{z}_m - \tilde{z}'_m| < \alpha\rho/2$ ,  $(\tilde{z}'_m + \alpha b)/\alpha \in f^{-1}[\{f(\bar{v} + \alpha b + \tilde{z}_m) - f(\bar{v}) - q\}/\alpha]$ , hence,  $\alpha f[\alpha^{-1}\{\tilde{z}'_m + \alpha b\}] = f(\bar{v} + \alpha b + \tilde{z}_m) - f(\bar{v}) - q$ .

Next and similarly, for each  $z$  in  $\tilde{D}_m$ , there exists a  $w_z$  such that  $|w_z - \tilde{z}'_m| \leq KF\tilde{D}^m$ , where  $\tilde{D}^m$  is the diameter of  $\tilde{D}_m$  and  $(w_z + \alpha b)/\alpha \in f^{-1}[\{f(\bar{v} + \alpha b + z) - f(\bar{v}) - q\}/\alpha]$ . The reason is that for  $y_m = \alpha f[\alpha^{-1}\{\tilde{z}'_m + \alpha b\}]$ , then for  $y' = f(\bar{v} + \alpha b + z) - f(\bar{v}) - q \in \text{cl}B(\alpha p, \alpha\sigma)$  (this inclusion was shown above), we have that  $|y' - y_m| = |y' - \alpha f[\alpha^{-1}\{\tilde{z}'_m + \alpha b\}]| = |f(\bar{v} + \alpha b + z) - f(\bar{v}) - q - \{f(\bar{v} + \alpha b + \tilde{z}_m) - f(\bar{v}) - q\}| = |f(\bar{v} + \alpha b + z) - f(\bar{v} + \alpha b + \tilde{z}_m)|$  so by (E), an  $x' := w_z + \alpha b \in \text{cl}B(\alpha b, \alpha\xi)$  exists such that  $|w_z - \tilde{z}'_m| \leq KF\tilde{D}^m$  and  $(w_z + \alpha b)/\alpha \in f^{-1}[\{f(\bar{v} + \alpha b + z) - f(\bar{v}) - q\}/\alpha]$ . Then  $|w_z - z| \leq |w_z - \tilde{z}'_m| + |z'_m - \tilde{z}_m| + |\tilde{z}_m$

$-z| \leq KF\tilde{D}^m + \alpha\rho/2 + \tilde{D}^m$ . Now,  $\lim_{m \rightarrow \infty} \tilde{D} = 0$ . Hence, for  $m$  large enough,  $|z - w_z| \leq \alpha\rho$ , so  $z - w_z \in \alpha\Delta$ .

Again there exists a sequence of simplices  $\{D_m\}_m$  with barycenters  $z_m = \tilde{z}_{D_m}$  and corresponding vectors  $z'_m = \tilde{z}'_{D_m}$  and subsequences  $z_{m_k}$  and  $z'_{m_k}$  that converge to limits  $\bar{z}$  and  $\bar{z}'$  in  $\text{cl}B(0, \alpha\xi)$ , such that if  $\{z^j_m\}_j$  are the vertices of  $D_m$ , then, for each  $j$ ,  $\{z^j_{m_k}\}_k$  and  $\{w_{z^j_{m_k}}\}_k$  also converge to  $\bar{z}$  and  $\bar{z}'$ , respectively (recall  $|w_z - \tilde{z}'_m| \leq KF\tilde{D}^m$ ), and finally, such that, for any  $i$ , for some  $j$ ,  $(z^j_{m_k} - w_{z^j_{m_k}})_i \leq (z^j_{m_k})_i$ . Letting  $k \rightarrow \infty$ , we get  $(\bar{z} - \bar{z}')_i \leq (\bar{z})_i$  for all  $i$ , i.e.  $\bar{z}' = 0$ . As  $\alpha f[\alpha^{-1}\{z'_m + ab\}] = f(\bar{v} + \alpha b + z_m) - f(\bar{v}) - q$ , by continuity,  $\alpha p = \alpha f[b] = f(\bar{v} + \alpha b + \bar{z}) - f(\bar{v}) - q$ , i.e.  $f(v) = f(\bar{v}) + q + \alpha p$  for  $v = \bar{v} + \alpha b + \bar{z} \in \bar{v} + \text{cl}B(\alpha b, \alpha\xi)$ .