

Biørn, Erik

Working Paper

The rate of capital retirement: How is it related to the form of the survival function and the investment growth path?

Memorandum, No. 2000,12

Provided in Cooperation with:

Department of Economics, University of Oslo

Suggested Citation: Biørn, Erik (2000) : The rate of capital retirement: How is it related to the form of the survival function and the investment growth path?, Memorandum, No. 2000,12, University of Oslo, Department of Economics, Oslo

This Version is available at:

<https://hdl.handle.net/10419/63068>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

MEMORANDUM

No 12/2000

The rate of capital retirement: How is it related to the form of the survival function and the investment growth path?

*By
Erik Bjørn*

ISSN: 0801-1117

Department of Economics
University of Oslo

This series is published by the
University of Oslo
Department of Economics

P. O.Box 1095 Blindern
N-0317 OSLO Norway
Telephone: + 47 22855127
Fax: + 47 22855035
Internet: <http://www.sv.uio.no/sosoek/>
e-mail: econdep@econ.uio.no

In co-operation with
**The Frisch Centre for Economic
Research**

Gaustadalleén 21
N-0371 OSLO Norway
Telephone: +47 22 95 88 20
Fax: +47 22 95 88 25
Internet: <http://www.frisch.uio.no/>
e-mail: frisch@frisch.uio.no

List of the last 10 Memoranda:

No 11	By Geir B. Asheim and Wolfgang Buchholz: The Hartwick rule: Myths and facts. 33 p.
No 10	By Tore Nilssen: Risk Externalities in Payments Oligopoly. 31p.
No 09	By Nils-Henrik M. von der Fehr and Lone Semmingsen: Influencing Bureaucratic Decisions. 31 p.
No 08	By Geir B. Aasheim and Martin Dufwenberg: Deductive Reasoning in Extensive Games. 22 p.
No 07	By Geir B. Asheim and Martin Dufwenberg: Admissibility and Common Belief. 29 p.
No 06	By Nico Keilman: Demographic Translation: From Period to Cohort Perspective and Back. 17 p.
No 05	By Pål Longva and Oddbjørn Raaum: Earnings Assimilation of Immigrants in Norway – A Reappraisal. 28 p.
No 04	By Jia Zhiyang: Family Labor Supply when the Husband is Eligible for Early Retirement: Some Empirical Evidences. 54 p.
No 03	By Mads Greaker: Strategic Environmental Policy; Eco-dumping or a green strategy? 33 p
No 02	By Ole J. Røgeberg: Married Men and Early Retirement Under the AFP Scheme. 78 p.

A complete list of this memo-series is available in a PDF® format at:
<http://www..sv.uio.no/sosoek/memo/>

**THE RATE OF CAPITAL RETIREMENT:
HOW IS IT RELATED TO
THE FORM OF THE SURVIVAL FUNCTION
AND THE INVESTMENT GROWTH PATH?*)**

by

ERIK BIØRN

ABSTRACT

We discuss the relationship between the retirement process of the capital, as formalized by its survival function, and the average retirement rate, and how this relationship is affected by changes in the investment path. The effect of the survival function on the age distribution of the capital goods, both those existing and those retired in each period, is also considered. These issues are illustrated by means of parametric (convex and concave) functions and numerical examples. We find that the retirement rate is a declining function of the growth rate of investment (except in the exponential decay case) and quite sensitive to the value of this parameter over a reasonable interval. Approximating the retirement rate by the inverse of the capital's maximal life-time or twice this value ('double declining balance') will in many cases produce very inaccurate results. The response of the capital/investment ratio and the retirement/investment ratio to changes in the investment growth rate and in the curvature of the survival function is also investigated.

Keywords: Capital. Retirement. Survival function. Age distribution.

Exponential decay. Mortality rate

JEL classification: C51, D24, E22, O47

* I thank Nico Keilman, Kjersti-Gro Lindquist, and Terje Skjerpen for valuable comments.

1 Introduction

The retirement of capital goods is a variable of considerable interest in economics and econometrics, and retirement (mortality) rates are important for several research purposes. In most countries, replacement investment accounts for a substantial part of gross investment and GNP, according to national accounts calculations. The process describing how capital retires is a basic determinant of the age distribution of the capital stock, as is also the time path of the investment in new assets. Using demographic analogues, we may associate (gross) investment with the birth of capital and retirement with death. From basic demography we know that the pattern of mortality rates of a population of persons and the time profile of the births jointly determine the age distribution of the population (when adjustment is made for migrations). Similarly, changes in the retirement process and in the time path of new investments jointly determine the average age of the capital and other characteristics of its age distribution (when purchases and sales of used capital are adjusted for).

In this paper, we discuss the relationship between the retirement process of the capital, formalized by its survival function, and the average retirement rate, and how this relationship is affected by changes in the investment growth path. The average retirement rate of capital goods corresponds to the average mortality rate of a population of persons and the ratio between the gross investment and the capital stock corresponds to the birth rate. We will also discuss how changes in the survival function affects the age distribution of the capital goods, partly in general terms, partly by means of parametric (convex and concave) functions, and partly through numerical examples. An integration of our analysis into a model of the firm's investment demand is, however, beyond our scope. Neither do we present econometric illustrations of the relationships under discussion.

It is well known that the average rate of capital retirement will be a constant independent of the time path of gross investment in the particular case where the survival function is exponentially declining in the age of the capital (exponential decay, declining balance). Exponential decay is the *only* case in which changes in the investment profile do not affect the average retirement rate; see Feldstein and Rothschild (1974, section 2). Often exponential decay is considered a benchmark case of capital retirement, and it has been argued that it is an acceptable simplifying approximation for long-term analysis, for instance in modelling economic growth, cf. Jorgenson (1963, p. 251, 1974). Yet, average retirement rates are applied by researchers even if exponential decay is not warranted, for instance in the calculation of capital service prices. This practise may be criticized,

see Biørn (1989, section 11.7).¹

An examination of the relationship between the form of the retirement process and the average retirement rate of capital under non-exponential decay is interesting for several purposes. We may, for example, wish to interpret time series for the retirement rates calculated from investment data by the perpetual inventory method with a given retirement process on a ‘scale’ comparable to the retirement rate under exponential decay. Which is the appropriate ‘translation’ of the retirement rate, δ , if the retirement pattern of the capital corresponds to, say, simultaneous retirement at a fixed age N , or to linear retirement up to a fixed life-time, N ? Often $\delta = 1/N$ is used as a ‘rule of translation’ for the former and $\delta = 2/N$ for the latter.² Under which conditions are these translation rules valid, or acceptable, approximations? If direct registrations of the capital stock are available, when will constancy of the implied retirement rate hold as a satisfactory approximation? A further, more general question is: How will the changes in the curvature of the survival function affect the average retirement rate for different growth profiles of gross investment?

The following sections are disposed as follows. In Section 2, we describe the model and define the basic concepts. Section 3 considers the age distribution of the capital and the related age distribution of retirement. We next derive the relationship between the overall retirement rates, the hazard rate of the mortality distribution of the capital, and its age distribution. The hazard rates can be interpreted as age specific retirement rates. In Section 4, we consider the special case with constant gross investment. We show that in this case, the overall retirement rate coincides with the inverse of the total service flow from one capital unit during its life-time.

Three parametric survival functions are defined in Section 5, characterized by, respectively, exponentially declining survival function with infinite maximal life-time (exponential decay), convex survival function with finite maximal life-time, and concave survival function with finite maximal life-time. We discuss the way in which the curvature of the survival function interferes with the growth rate of investment in determining the age distribution of the capital and its average retirement rate. Three special cases – linear, simultaneous, and immediate retirement – are considered in Section 6. The translation formulae $\delta = 1/N$ for the first and $\delta = 2/N$ for the second hold under stationarity of gross investment only and may be quite inaccurate otherwise. Numerical illustrations of

¹Retirement, as the term is being used here, is distinctly different from *depreciation*, which is related to the decline in the *value* of the capital, rather than to its deterioration as an input in production. A recent survey of empirical studies of depreciation is Jorgenson (1996).

²The latter is often denoted as the ‘double declining balance’ pattern; cf., *e.g.*, Fraumeni and Jorgenson (1980, p. 24).

the sensitivity of the retirement rate and properties of the age distribution to changes in the curvature of the survival function and the growth pattern of investment are given in Section 7. A main conclusion is that the retirement rate is a declining function of the growth rate of investment and quite sensitive to the value of this growth rate over a reasonable interval. If, for example, the (continuous) growth rate of investment increases from 0 to 5 per cent and all capital retires simultaneously at age 20, the retirement rate decreases from 5 per cent to 2.9 per cent. With linear retirement over 20 years, the corresponding retirement rate decreases from 10 per cent to 8.6 per cent.

2 Model and basic concepts

Let $J(t)$ denote gross investment at time t , with t continuous. The investment brings an increase in the productive capacity of the capital stock, which disappears gradually as the age of the capital increases. This is characterized by the *survival function, or survival curve*, $B(s)$, indicating the proportion of the capacity of a capital stock which survives at age s (≥ 0).³ It is assumed to be time invariant and satisfies

$$(1) \quad B(s) \in [0, 1], \quad B'(s) \leq 0, \quad B(0) = 1, \quad B(\infty) = 0.$$

The capital volume of age s at time t , *i.e.*, belonging to vintage $t-s$ at time t , is

$$(2) \quad K(t, s) = B(s) J(t-s).$$

We make the usual assumption that *one unit of capital produces one unit of capital services per unit of time*, so that $K(t, s)$ both has a stock interpretation and represents the instantaneous flow of capital services produced at time t by the capital which is of age s . Neo-classical theory of production implies that capital is malleable, *i.e.*, capital units belonging to different vintages are perfect substitutes at any point in time, so that the capital input in a production function at time t can be represented by the aggregate

$$(3) \quad K(t) = \int_0^\infty K(t, s) ds = \int_0^\infty B(s) J(t-s) ds.$$

This variable, the *gross capital stock*, has the joint interpretation as the total number of capital units at time t and the instantaneous flow of services ‘produced’ by these units

³This function represents both the *loss of efficiency* of existing capital units and their *physical disappearance*. We may therefore, in principle, consider $B(s)$ as the product of two factors, one indicating the relative number of capital units surviving at age s (the survival function), the other indicating the relative decline in efficiency of each remaining unit (the efficiency function). We will not focus on such a decomposition here. However, for the purpose of measuring capital stocks from market data, the distinction between the survival function and the efficiency function is important; see Biørn (1998).

at time t .⁴

The flow of capital goods retired at time t is the difference between the gross investment, $J(t)$, and the increase in the gross capital stock, $\dot{K}(t)$, the “dot” indicating the time derivative. Using (1) and (3), the *retirement* at time t can be expressed as

$$(4) \quad D(t) = J(t) - \dot{K}(t) = \int_0^\infty b(s)J(t-s) ds$$

where

$$(5) \quad b(s) = -B'(s), \quad s \geq 0.$$

The function $b(s)$, denoted as the *(relative) retirement function, or retirement (mortality) curve*, indicates the structure of the retirement process: $b(s)ds$ is the share of an *initial investment* of one unit which disappears from s to $s + ds$ years after installation. From (1) and (5) it follows that $b(s)$ is non-negative with $\int_0^\infty b(s)ds = 1$.

Let

$$(6) \quad \beta(s) = \frac{b(s)}{B(s)} = -\frac{d \ln B(s)}{ds},$$

so that $\beta(s)ds$ is the share of the *remaining capital* at age s which disappears from s to $s + ds$ years after installation. The difference between $b(s)$ and $\beta(s)$ follows from their different normalization: $b(s)$ relates to normalization with respect to the *initial* investment, $\beta(s)$ relates to normalization against the capital *remaining*. The latter corresponds to age specific mortality rates in demography.⁵

The volume of retirement at time t can be decomposed by vintages by

$$(7) \quad D(t, s) = b(s)J(t-s) = \beta(s)K(t, s)$$

which denotes the retirement of capital of age s at time t , *i.e.*, belonging to vintage $t-s$ at time t , so that (4) can be written as

$$(8) \quad D(t) = \int_0^\infty b(s)J(t-s) ds = \int_0^\infty D(t, s)ds.$$

⁴This contrasts with the putty-clay approach in which (i) each capital vintage is assumed to have its specific input coefficients *ex post*, determined at the time of investment [cf. Johansen (1959, 1972)], (ii) capital is not malleable, and (iii) (planned and realized) retirement of capital is determined from profitability conditions, *e.g.*, formalized by means of a quasi-rent criterion [cf. Biørn and Frenger (1992)].

⁵In a probabilistic setting, $B(s)$ can be interpreted as the survival probability, $1 - B(s)$ as the non-survival probability, $b(s)$ as the density function and $\beta(s)$ as the failure rate or hazard rate function; see Barlow and Proschan (1975, section 3.1) and Kalbfleisch and Prentice (1980, section 1.2).

3 The age distribution of capital and retirement

We next consider the age distributions of the capital and the retirement. Using (2) and (3), the *age distribution of the capital stock* at time t can be characterized by the share of the capital which at time t is of age s ,

$$(9) \quad k(t, s) = \frac{K(t, s)}{K(t)} = \frac{B(s)J(t-s)}{\int_0^\infty B(s)J(t-s)ds}, \quad s \geq 0.$$

From (7) and (8) it follows that the *age distribution of the retirement of capital* at time t can be characterized by the share of the retirement which at time t relates to capital of age s as

$$(10) \quad d(t, s) = \frac{D(t, s)}{D(t)} = \frac{\beta(s)K(t, s)}{\int_0^\infty \beta(s)K(t, s)ds} = \frac{b(s)J(t-s)}{\int_0^\infty b(s)J(t-s)ds}, \quad s \geq 0.$$

Obviously, $k(t, s)$ and $d(t, s)$ are non-negative with $\int_0^\infty k(t, s)ds = \int_0^\infty d(t, s)ds = 1$.

From (7) we find that the *age specific retirement rate* at time t and age s is

$$(11) \quad \delta(t, s) = \frac{D(t, s)}{K(t, s)} = \beta(s),$$

and from (3), (8), and (9) it follows that the *overall retirement rate* can be expressed as

$$(12) \quad \delta(t) = \frac{D(t)}{K(t)} = \frac{\int_0^\infty D(t, s)ds}{\int_0^\infty K(t, s)ds} = \frac{\int_0^\infty b(s)J(t-s)ds}{\int_0^\infty B(s)J(t-s)ds} = \int_0^\infty \beta(s)k(t, s)ds.$$

This rate is an average of the age specific retirement rates, weighted by the age distribution of gross capital at time t . It will therefore, in general, be *time dependent*. If, however, $\beta(s)$ is constant or the age distribution $k(t, s)$ is time invariant, $\delta(t)$ will be constant. These are the only two cases in which $\delta(t)$ will be time independent.

From (9) and (10) it follows that the relationship between the age distributions of retirement and capital can be expressed as

$$(13) \quad d(t, s) = \frac{\beta(s)k(t, s)}{\int_0^\infty \beta(s)k(t, s)ds}.$$

Using (12), this relationship can be rewritten as

$$(14) \quad \frac{d(t, s)}{k(t, s)} = \frac{\beta(s)}{\delta(t)},$$

i.e., the ratio between the age specific retirement and capital distributions at time t and age s equals the ratio between the age specific retirement rate at age s and the overall retirement rate at time t .

Let now $\gamma(t, s) = J(t-s)/J(t)$, $s \geq 0$, denote the *growth path of gross investment* up to time t . The rates $k(t, s)$, $d(t, s)$, and $\delta(t)$ then can be expressed in a simpler form. The age distribution of the capital can be expressed in terms of the survival function $B(s)$ and the growth path of investment $\gamma(t, s)$ as

$$(15) \quad k(t, s) = \frac{B(s)\gamma(t, s)}{\int_0^\infty B(s)\gamma(t, s)ds},$$

the age distribution of the retirement can be expressed in terms of the retirement function $b(s)$ and the path $\gamma(t, s)$ as

$$(16) \quad d(t, s) = \frac{b(s)\gamma(t, s)}{\int_0^\infty b(s)\gamma(t, s)ds},$$

and the overall retirement rate can be expressed in terms of the survival function $B(s)$, the retirement function $b(s)$, and the growth path $\gamma(t, s)$ as

$$(17) \quad \delta(t) = \frac{\int_0^\infty \beta(s)B(s)\gamma(t, s)ds}{\int_0^\infty B(s)\gamma(t, s)ds} = \frac{\int_0^\infty b(s)\gamma(t, s)ds}{\int_0^\infty B(s)\gamma(t, s)ds}.$$

It follows from (15) – (17) that $k(t, s)$, $d(t, s)$, and $\delta(t)$ are all time invariant for any $B(s)$ if and only if $\gamma(t, s)$ is time invariant. The corresponds to a *stable population* in demography [cf. Coale (1972) and Keyfitz (1977)]. For $\gamma(t, s) = J(t-s)/J(t)$ to be independent of t for all t and s , it must be an *exponential function* of the form $\gamma(t, s) = e^{-\alpha s}$.⁶ The latter assumption will be specifically considered in Sections 5 – 7. If $\beta(s)$ is constant, which implies that $B(s)$ is an exponential function [cf. (6)], $\delta(t)$ will be time invariant and equal to this constant for any $\gamma(t, s)$. *Hence, constancy of $\delta(t)$ implies that at least one of the functions $B(s)$ or $\gamma(t, s)$ must be exponential functions in s .*

⁶Nickell comments on this result by saying that if $\delta(t)$ “is to settle down to a constant level in the long run in the absence of exponential decay, investment must settle to an exponential growth path in the long run. This is, in fact the result of renewal theory which has been quoted as a justification for assuming that the potential replacement-capital ratio is constant. Now, if investment was indeed moving on a fixed exponential growth path, a proliferation of investment equations would hardly be required to explain this movement and consequently any serious theoretical justification of the assumption of a fixed potential replacement-capital ratio for empirical work must fall back on exponential decay” [Nickell (1978, p. 119)].

4 The constant investment case

If gross investment is constant, then $\gamma(t, s) = 1$ for all t and s , and as a consequence, the age distributions of capital and retirement and the overall retirement rates will be constant and depend on the form of the survival function only. Let us take a look at this case.

Let Φ denote the total flow of services produced by one capital unit during its entire life-time, *i.e.*,

$$(18) \quad \Phi = \int_0^{\infty} B(s)ds.$$

If gross investment is constant and equal to J , it follows from (3) and (4) that

$$(19) \quad K(t) = K = \Phi J,$$

$$(20) \quad D(t) = D = J.$$

Then the (constant) capital stock equals the (constant) gross investment times the per unit of capital service flow during the unit's life-time; net investment, \dot{K} , is zero. This corresponds to a *stationary population* in demography. Hence,

$$(21) \quad \delta(t) = \delta = \frac{D}{K} = \frac{1}{\Phi},$$

i.e., under constant gross investment, the overall retirement rate is constant and equal to the inverse of the total service flow per capital unit during its life-time.⁷

From (9) and (10) we find that under constant gross investment, the age distribution of capital and retirement are time invariant and equal to

$$(22) \quad k(t, s) = k(s) = \frac{B(s)}{\Phi},$$

$$(23) \quad d(t, s) = d(s) = b(s),$$

for all t . Then we have

$$(24) \quad \frac{d(t, s)}{k(t, s)} = \frac{d(s)}{k(s)} = \beta(s)\Phi,$$

so that the two age distributions will coincide if and only if the hazard rate $\beta(s)$ is constant and equal to the inverse of the total service flow per capital unit during its life-time. This particular case is 'exponential decay', which will be discussed in Section 5.

⁷If the survival function declines so rapidly that $\Phi < 1$ – which is rather unlikely to occur in practice – the overall retirement rate, with our continuous time formulation of the retirement process will exceed 1.

5 Three parametric survival functions

In this section, we describe three examples with parametric survival functions. The examples are characterized by, respectively, (i) exponentially declining survival function with infinite maximal life-time, (ii) convex survival function with finite maximal life-time, and (iii) concave survival function with finite maximal life-time. Example (i) is one-parametric, Examples (ii) and (iii) are two-parametric, with one parameter representing the maximal life-time and the other describing the curvature of the function.

In characterizing the convex and the concave survival functions and their properties we need the following function, defined by the integral

$$(25) \quad F(i, z) = \int_0^1 e^{z\theta} (1 - \theta)^i d\theta = \int_0^1 e^{z(1-\lambda)} \lambda^i d\lambda = e^z \int_0^1 e^{-z\lambda} \lambda^i d\lambda, \quad i = 0, 1, 2, \dots,$$

where the second equality is obtained by substituting $\lambda = 1 - \theta$. Some properties of this function to be used below are proved in the Appendix.

Exponential decay

This, probably the most famous example of a survival function of capital in the literature, is the case where the function declines exponentially at the rate δ , which is its only parameter, and we have

$$(26) \quad B(s) = e^{-\delta s}, \quad \delta \geq 0, \quad s \geq 0,$$

$$(27) \quad b(s) = \delta e^{-\delta s},$$

$$(28) \quad \beta(s) = \delta,$$

$$(29) \quad \Phi = \frac{1}{\delta}.$$

This distribution thus has a *constant hazard rate, equal to δ* . We also find

$$(30) \quad k(t, s) = d(t, s) = \frac{e^{-\delta s} \gamma(t, s)}{\int_0^\infty e^{-\delta s} \gamma(t, s) ds},$$

$$(31) \quad \delta(t) = \delta,$$

so that this survival function is characterized by identical age distributions of capital and retirement and a constant overall retirement rate.

Assume now *constant rate of investment growth* at the rate α , $J(t-s) = J(t)e^{-\alpha s}$, *i.e.*,

$$(32) \quad \gamma(t, s) = e^{-\alpha s}, \quad \text{for all } s, t.$$

Then the ratio between capital and gross investment, which may be interpreted as the *inverse birth rate* of capital, can, for an arbitrary survival function, be written as

$$(33) \quad \frac{K(t)}{J(t)} = \int_0^\infty B(s)e^{-\alpha s} ds.$$

Similarly, the ratio between retirement and gross investment, which can be interpreted as the *death to birth ratio* of capital, can be written as

$$(34) \quad \frac{D(t)}{J(t)} = \int_0^\infty b(s)e^{-\alpha s} ds.$$

Under exponential decay and exponential investment growth, we get the following particular expressions for the capital stock and retirement:

$$(35) \quad K(t) = \frac{J(t)}{\delta + \alpha},$$

$$(36) \quad D(t) = \frac{J(t)\delta}{\delta + \alpha}.$$

It is easy to show that net investment then equals

$$\dot{K}(t) = \frac{J(t)\alpha}{\delta + \alpha},$$

which implies that the net investment/gross investment ratio is larger the larger is the growth rate of the investment. There is proportionality between $K(t)$, $D(t)$, $\dot{K}(t)$, and $J(t)$. If δ and α are positive, the net investment/retirement ratio equals α/δ . The capital/gross investment ratio and the retirement/gross investment ratio are smaller the larger is the past growth rate of the investment. If investment is declining, with $\delta > -\alpha$, retirement will exceed gross investment, *i.e.*, net investment will be negative. The two age distributions coincide in this case,

$$(37) \quad k(t, s) = d(t, s) = (\delta + \alpha)e^{-(\delta + \alpha)s},$$

i.e., they follow an exponential decay retirement function with the parameter δ replaced by $(\delta + \alpha)$.

Under *constant investment* ($\alpha = 0$), we have in particular

$$K(t) = \frac{J(t)}{\delta} = \frac{D(t)}{\delta}$$

$$k(t, s) = d(t, s) = b(s) = \delta e^{-\delta s}.$$

Then the age distribution of capital and retirement both coincide with the retirement function.

Convex survival function

In this example, the survival function is two-parametric and convex. One parameter, N , is the maximal life-time, the other, τ , characterizes the curvature of the survival function. It is given by⁸

$$(38) \quad B(s) = \left(1 - \frac{s}{N}\right)^\tau, \quad N > 0, \tau \geq 1, 0 \leq s \leq N,$$

$$(39) \quad b(s) = \frac{\tau}{N} \left(1 - \frac{s}{N}\right)^{\tau-1},$$

$$(40) \quad \beta(s) = \frac{\tau}{N-s},$$

$$(41) \quad \Phi = \frac{N}{\tau+1}.$$

The retirement function and the survival function belong to the same class of functions, as was also the case for exponential decay. The hazard rate is a monotonically increasing function of the capital's age, increasing in τ , and goes to infinity at the maximal age N . The total per unit service flow is proportional to the maximal life-time and monotonically *decreasing* in τ . The survival function is strictly convex for $\tau > 1$ and linear for $\tau = 1$.

Inserting (38) and (39) into (15) – (17), we find the following expressions for the age distributions of capital and retirement and the (time dependent) overall retirement rate for an arbitrary growth path of investment, $\gamma(t, s)$:

$$(42) \quad k(t, s) = \frac{\left(1 - \frac{s}{N}\right)^\tau \gamma(t, s)}{\int_0^N \left(1 - \frac{s}{N}\right)^\tau \gamma(t, s) ds},$$

$$(43) \quad d(t, s) = \frac{\left(1 - \frac{s}{N}\right)^{\tau-1} \gamma(t, s)}{\int_0^N \left(1 - \frac{s}{N}\right)^{\tau-1} \gamma(t, s) ds},$$

$$(44) \quad \delta(t) = \frac{\tau \int_0^N \left(1 - \frac{s}{N}\right)^{\tau-1} \gamma(t, s) ds}{\int_0^N \left(1 - \frac{s}{N}\right)^\tau \gamma(t, s) ds}.$$

We again assume exponentially growing investment, (32). Since, from (A.1) and (A.5), by substituting $\theta = s/N$ and $i = \tau$,

$$\int_0^N \left(1 - \frac{s}{N}\right)^\tau e^{-\alpha s} ds = NF(\tau, -\alpha N) = \frac{N}{\tau+1} [1 - \alpha NF(\tau+1, -\alpha N)],$$

⁸This survival function is also considered in Biørn (1989, section 11.2.2).

we can write the relationship between capital, gross investment, and retirement at time t as

$$(45) \quad K(t) = J(t)NF(\tau, -\alpha N) = J(t)\frac{N}{\tau+1}[1 - \alpha NF(\tau+1, -\alpha N)],$$

$$(46) \quad D(t) = J(t)\tau F(\tau-1, -\alpha N) = J(t)[1 - \alpha NF(\tau, -\alpha N)].$$

From (32), (42), and (43) it then follows that

$$(47) \quad k(t, s) = \frac{e^{-\alpha s}}{NF(\tau, -\alpha N)} \left(1 - \frac{s}{N}\right)^\tau,$$

$$(48) \quad d(t, s) = \frac{e^{-\alpha s}}{NF(\tau-1, -\alpha N)} \left(1 - \frac{s}{N}\right)^{\tau-1}.$$

From (45) and (46) we find that *the overall retirement rate is time invariant* and can be written as the following function of N , τ , and α :

$$(49) \quad \delta(t) = \delta = \frac{\tau}{N} \frac{F(\tau-1, -\alpha N)}{F(\tau, -\alpha N)} = \frac{\tau+1}{N} \frac{1 - \alpha NF(\tau, -\alpha N)}{1 - \alpha NF(\tau+1, -\alpha N)},$$

where we in the second equality have used (A.5). If we approximate $F(\tau, -\alpha N)$ and $F(\tau+1, -\alpha N)$ by the *first* term in their Taylor expansions, which are $1/(\tau+1)$ and $1/(\tau+2)$, respectively [cf. (A.6)], we find the following approximation formula for the retirement rate, provided that αN is not too large:

$$(50) \quad \delta \approx \frac{\tau+1}{N} \frac{1 - \frac{\alpha N}{\tau+1}}{1 - \frac{\alpha N}{\tau+2}} \approx \frac{\tau+1}{N} \left[1 - \left(\frac{1}{\tau+1} - \frac{1}{\tau+2}\right) \alpha N\right] = \frac{\tau+1}{N} - \frac{\alpha}{\tau+2}.$$

Under *constant investment* ($\alpha = 0$), this survival function implies

$$\begin{aligned} K(t) &= J(t)\frac{N}{\tau+1}, \\ k(t, s) &= \frac{\tau+1}{N} \left(1 - \frac{s}{N}\right)^\tau, \\ d(t, s) &= \frac{\tau}{N} \left(1 - \frac{s}{N}\right)^{\tau-1}, \\ \delta &= \frac{\tau+1}{N}. \end{aligned}$$

Comparing the last expression with (49) and (50), we see that we can interpret the retirement rate when gross investment grows at the rate α as obtained by multiplying its value under constant investment by a correction factor which is one for $\alpha = 0$ and in general depends on αN . Approximately, when α is not too large, this correction simply implies that we should deduct a share $1/(\tau+2)$ of the growth rate of investment. This share declines from $1/3$ to zero as τ increases from 1 to infinity. Numerical illustrations are given in Section 7.

Concave survival function

In this example, the survival function is two-parametric and concave. Again, one parameter, N , is the maximal life-time, the other, σ , characterizes the curvature of the survival function. It is given by⁹

$$(51) \quad B(s) = 1 - \left(\frac{s}{N}\right)^\sigma, \quad N > 0, \sigma \geq 1, 0 \leq s \leq N,$$

$$(52) \quad b(s) = \frac{\sigma}{N} \left(\frac{s}{N}\right)^{\sigma-1},$$

$$(53) \quad \beta(s) = \frac{\frac{\sigma}{N} \left(\frac{s}{N}\right)^{\sigma-1}}{1 - \left(\frac{s}{N}\right)^\sigma} = \frac{\sigma s^{\sigma-1}}{N^\sigma - s^\sigma},$$

$$(54) \quad \Phi = \frac{N\sigma}{\sigma + 1}.$$

In this case, the retirement function and the survival function do *not* belong to the same class of functions. The hazard rate is a monotonically increasing function of the capital's age and goes to infinity at the maximal age N . The total per unit service flow is proportional to the maximal life-time and monotonically *increasing* in σ . The survival function is strictly concave for $\sigma > 1$ and linear for $\sigma = 1$.

Inserting (51) and (52) into (15) – (17), we find the following expressions for the age distributions of capital and retirement and the (time dependent) overall retirement rate for an arbitrary growth path of investment, $\gamma(t, s)$:

$$(55) \quad k(t, s) = \frac{\left[1 - \left(\frac{s}{N}\right)^\sigma\right] \gamma(t, s)}{\int_0^N \left[1 - \left(\frac{s}{N}\right)^\sigma\right] \gamma(t, s) ds},$$

$$(56) \quad d(t, s) = \frac{\left(\frac{s}{N}\right)^{\sigma-1} \gamma(t, s)}{\int_0^N \left(\frac{s}{N}\right)^{\sigma-1} \gamma(t, s) ds},$$

$$(57) \quad \delta(t) = \frac{\sigma \int_0^N \left(\frac{s}{N}\right)^{\sigma-1} \gamma(t, s) ds}{N \int_0^N \left[1 - \left(\frac{s}{N}\right)^\sigma\right] \gamma(t, s) ds}.$$

We again assume exponentially growing investment, (32). Since, from (A.1) and (A.5), by substituting $\lambda = s/N$ and $i = \sigma$,

$$\int_0^N \left(\frac{s}{N}\right)^\sigma e^{-\alpha s} ds = N e^{-\alpha N} F(\sigma, \alpha N) = \frac{N}{\sigma+1} e^{-\alpha N} [1 + \alpha N F(\sigma + 1, \alpha N)],$$

⁹This survival function is also considered in Biørn (1989, section 11.2.1).

so that

$$\int_0^N \left[1 - \left(\frac{s}{N}\right)^\sigma\right] e^{-\alpha s} ds = N e^{-\alpha N} [F(0, \alpha N) - F(\sigma, \alpha N)],$$

we can write the relationship between capital, gross investment, and retirement at time t as

$$(58) \quad K(t) = J(t) N e^{-\alpha N} [F(0, \alpha N) - F(\sigma, \alpha N)],$$

$$(59) \quad D(t) = J(t) \sigma e^{-\alpha N} F(\sigma - 1, \alpha N) = J(t) e^{-\alpha N} [1 + \alpha N F(\sigma, \alpha N)].$$

From (32), (55), and (56) it follows that

$$(60) \quad k(t, s) = \frac{e^{-\alpha s}}{N e^{-\alpha N} [F(0, \alpha N) - F(\sigma, \alpha N)]} \left[1 - \left(\frac{s}{N}\right)^\sigma\right],$$

$$(61) \quad d(t, s) = \frac{e^{-\alpha s}}{N e^{-\alpha N} F(\sigma - 1, \alpha N)} \left(\frac{s}{N}\right)^{\sigma-1}.$$

From (58) and (59) we find that *the overall retirement rate is time invariant* and can be written as the following function of N , σ , and α :

$$(62) \quad \delta(t) = \delta = \frac{\sigma}{N} \frac{F(\sigma - 1, \alpha N)}{F(0, \alpha N) - F(\sigma, \alpha N)}.$$

If we approximate $F(\sigma - 1, \alpha N)$, $F(0, \alpha N)$ and $F(\sigma, \alpha N)$ by the *first and second* term in their Taylor expansions, which are $1/\sigma + \alpha N/(\sigma(\sigma+1))$, $1 + \alpha N/2$ and $1/(\sigma+1) + \alpha N/((\sigma+1)(\sigma+2))$, respectively, [cf. (A.6)] we find the following approximation formula for the retirement rate provided that αN is not too large:

$$(63) \quad \begin{aligned} \delta &\approx \frac{\sigma}{N} \frac{\frac{1}{\sigma} + \frac{\alpha N}{\sigma(\sigma+1)}}{\left(1 + \frac{\alpha N}{2}\right) - \left(\frac{1}{\sigma+1} + \frac{\alpha N}{(\sigma+1)(\sigma+2)}\right)} \\ &= \frac{1}{N} \frac{1 + \frac{\alpha N}{\sigma+1}}{\frac{\sigma}{\sigma+1} + \left(\frac{1}{2} - \frac{1}{(\sigma+1)(\sigma+2)}\right) \alpha N} = \frac{\sigma+1}{\sigma N} \frac{1 + \frac{\alpha N}{\sigma+1}}{1 + \frac{\sigma+3}{2(\sigma+2)} \alpha N} \\ &\approx \frac{\sigma+1}{\sigma N} \left[1 - \left(\frac{\sigma+3}{2(\sigma+2)} - \frac{1}{\sigma+1}\right) \alpha N\right] = \frac{\sigma+1}{\sigma N} - \left(\frac{(\sigma+1)(\sigma+3)}{2\sigma(\sigma+2)} - \frac{1}{\sigma}\right) \alpha. \end{aligned}$$

Under *constant investment* ($\alpha = 0$), when we use (A.2), this survival function implies

$$\begin{aligned} K(t) &= J(t) \frac{N\sigma}{\sigma+1}, \\ k(t, s) &= \frac{\sigma+1}{N\sigma} \left[1 - \left(\frac{s}{N}\right)^\sigma\right], \\ d(t, s) &= \frac{\sigma}{N} \left(\frac{s}{N}\right)^{\sigma-1}, \\ \delta &= \frac{\sigma+1}{N\sigma}. \end{aligned}$$

Comparing the last expression with (62) and using (A.2), we see that we can interpret the retirement rate when gross investment grows at the rate α as obtained by multiplying its value under constant investment by a correction factor which is one for $\alpha = 0$ and in general depends on αN . Approximately, when α is not too large, this correction implies that we should deduct a share of the growth rate of investment which increases from $1/3$ to $1/2$ as σ increases from 1 to infinity. Numerical illustrations are given in Section 7.

6 Linear, simultaneous, and immediate retirement

Let us briefly consider retirement processes characterized by linear retirement, simultaneous retirement at a finite positive age, and immediate retirement after the first service period. They are special cases of the second and third examples in Section 5. We still assume a constant rate of investment growth.

Linear retirement

Linear retirement, in which a constant proportion, $1/N$, of the initial investment vanishes in each period is the special case of both the convex survival function (38) with $\tau = 1$ and of the concave survival function (51) with $\sigma = 1$, and we have

$$B(s) = 1 - \frac{s}{N},$$

$$b(s) = \frac{1}{N}.$$

It follows from (45) and (46) with $\tau = 1$ that capital and retirement become

$$K(t) = J(t)NF(1, -\alpha N) = J(t)\frac{1}{\alpha} \left[1 - \frac{1}{\alpha N}(1 - e^{-\alpha N}) \right],$$

$$D(t) = J(t)F(0, -\alpha N) = J(t)\frac{1}{\alpha N}(1 - e^{-\alpha N}).$$

The overall retirement rate, obtained from (49), is

$$\delta = \frac{1}{N} \frac{F(0, -\alpha N)}{F(1, -\alpha N)} = \frac{1}{N} \frac{1 - e^{-\alpha N}}{1 - \frac{1}{\alpha N}(1 - e^{-\alpha N})}.$$

Simultaneous retirement

Simultaneous retirement (often called “sudden death” or “one-horse shay”), in which all capital invested vanishes completely at age N , can be considered the special case of the concave survival function (51) with $\sigma \rightarrow \infty$,

$$B(s) = 1 - \lim_{\sigma \rightarrow \infty} \left(\frac{s}{N} \right)^\sigma = \begin{cases} 1, & s < N, \\ 0, & s = N. \end{cases}$$

It follows from (58) and (59), by inserting $\sigma \rightarrow \infty$ and using (A.3) and (A.4), that capital and retirement become

$$\begin{aligned} K(t) &= J(t)N e^{-\alpha N} F(0, \alpha N) = J(t)N F(0, -\alpha N), \\ D(t) &= J(t-N) = J(t)e^{-\alpha N}. \end{aligned}$$

The overall retirement rate, obtained from (62), is

$$\delta = \frac{1}{N F(0, \alpha N)} = \frac{e^{-\alpha N}}{N F(0, -\alpha N)}.$$

Immediate retirement

This case, in which all capital goods vanish immediately after investment, may be considered the special case of the convex survival function (38) with $\tau \rightarrow \infty$, so that

$$B(s) = \lim_{\tau \rightarrow \infty} \left(1 - \frac{s}{N}\right)^\tau = \begin{cases} 1, & s = 0, \\ 0, & s > 0. \end{cases}$$

Since $K(t)$ is a stock concept and $J(t)$ and $D(t)$ are flow concepts, this specification, however, is not convenient. Instead, we consider immediate retirement as the special case of simultaneous retirement where the life-time is $N = 1$. We then get

$$\begin{aligned} K(t) &= J(t)e^{-\alpha} F(0, \alpha) = J(t)F(0, -\alpha), \\ D(t) &= J(t)e^{-\alpha}, \end{aligned}$$

and the overall retirement rate becomes

$$\delta = \frac{1}{F(0, \alpha)} = \frac{e^{-\alpha}}{F(0, -\alpha)}.$$

If we let α go to zero – which has the interpretation that investment is constant over the age interval $(0, 1)$ – we simply get $K(t) = D(t) = J(t)$ and $\delta = 1$, which agree with our intuitive understanding of immediate retirement when the life-time is arbitrarily short.

7 Numerical illustrations

In this section, we present numerical illustrations, relating to the two-parametric convex and concave survival functions described in Section 5, under exponential investment growth. Using a demographic analogue, what we describe here are characteristics of a stable population of capital assets for different parameter values. Six values of the maximal life-time N are considered, 6, 10, 20, 30, 50, and 100 years, and six values of the

curvature parameters τ and σ are specified for each class of functions. The six panels of the tables below, a – f, represent the six values of the curvature parameters. Seven values of the (continuous) growth rate of investment, α , are considered, ranging from -5 to +15 per cent. Tables 1 – 3 illustrate the convex class, Tables 4 – 6 illustrate the concave class of survival functions.¹⁰

Tables 1 and 4 contain the retirement rates (in per cent). Tables 2, 3, 5, and 6 describe properties of the implied age distributions. Tables 2 and 5 give the capital/investment ratio (K/J) – which is analogous to the inverse birth rate of capital – and Tables 3 and 6 give the retirement/investment ratio (D/J) – which is analogous to the death to birth ratio.

Retirement rates

From the columns of Tables 1 and 4 we see that the retirement rate under exponential investment growth shows considerable *sensitivity to the growth rate*. The relative sensitivity is larger the higher is the maximal life-time. In the concave class with the curvature parameter $\sigma = 2$ and the maximal life time $N = 6$, for example, the retirement rate declines from $\delta = 27.3$ per cent for $\alpha = -0.05$ via $\delta = 25$ per cent for $\alpha = 0$ to $\delta = 19.2$ per cent for $\alpha = 0.15$. With $\sigma = 2, N = 50$, the retirement rate declines from $\delta = 5.97$ per cent for $\alpha = -0.05$ via $\delta = 3$ per cent for $\alpha = 0$ to $\delta = 0.55$ per cent for $\alpha = 0.15$. For each value of the maximal life time N , the sensitivity of δ is smaller the higher is the curvature parameter τ for the convex class of functions, and larger the higher is the curvature parameter σ for the concave class.

The rows for $\alpha = 0$ in Tables 1 and 4, which represent the *constant investment case*, give the value $\delta = 1/\Phi = (\tau + 1)/N$ for the convex class of survival functions [cf. (41), (49), and (50)] and the value $\delta = (\sigma + 1)/(\sigma N)$ for the concave class [cf. (54), (62), and (63)], as can be easily checked. The ‘double declining balance’ hypothesis, corresponding to $\delta = 2/N$, is represented by the $\alpha = 0$ rows for $\tau = 1$ in Table 1 and for $\sigma = 1$ in Table 4 (linear retirement). The ‘inverse life-time’ hypothesis, corresponding to $\delta = 1/N$, is represented by the $\alpha = 0$ rows for $\tau = 0$ ¹¹ in Table 1 and (approximately) for $\sigma = 1000$ in Table 4 (simultaneous retirement)

The formulae (50) and (63) give very good approximations to the exact δ value for $\alpha \in (-0.01, +0.01)$, $N \leq 50$ and fairly good approximations for $\alpha \in (-0.05, -0.01)$,

¹⁰The function $F(i, z)$ is evaluated by means of (A.6) (by truncating the expansion after a suitable number of terms), and the calculations are performed by means of routines in the Gauss software code constructed by the author.

¹¹Strictly, (38) is defined for $\tau \geq 1$ only. The case $\tau = 0$ gives, however, simultaneous retirement. even if (39) and (40) and expressions derived from them are undefined.

$N \leq 20$ and $\alpha \in (0.01, 0.05)$, $N \leq 20$. For $\alpha = 0.01, \tau = 1, N = 30$ in the *convex* class, the exact formula gives for example $\delta = 6.35$ per cent and the approximate formula $\delta = 6.33$ per cent. For $\alpha = 0.05, \tau = 1, N = 20$ the exact formula gives $\delta = 8.59$ per cent and the approximate formula $\delta = 8.33$ per cent. For $\alpha = -0.01, \sigma = 2, N = 30$ in the *concave* class, the exact formula gives $\delta = 5.46$ per cent and the approximate formula $\delta = 5.44$ per cent. For $\alpha = -0.05, \sigma = 2, N = 20$, the exact formula gives $\delta = 10.00$ per cent and the approximate formula $\delta = 9.68$ per cent.

Capital/investment ratios

By examining the columns of Tables 2 and 5, we see how changes in the investment growth rate, for each given maximal life-time and each value of the curvature parameter, affects an important property of the age distribution of the capital, the capital/investment ratio, or the inverse birth ratio of capital. As a benchmark case, we may consider the constant investment case, where $K/J = N/(\tau + 1)$ for the convex class and $K/J = N\sigma/(\sigma + 1)$ for the concave class. These constants also represent the total number of services per capital unit during its life time and the inverse retirement rates under constant investment. The latter can be confirmed by comparing the $\alpha = 0$ rows in Tables 2 and 5 with the corresponding rows in Tables 1 and 4.

Increasing α implies a growing and gradually younger stock of capital (decreasing K/J). A few examples illustrate this. In the convex class with $\tau = 2, N = 6$, the capital/investment ratio declines from $K/J = 2.16$ for $\alpha = -0.05$ via $K/J = 2.00$ for $\alpha = 0$ to $K/J = 1.62$ for $\alpha = 0.15$. With $\tau = 2, N = 50$, the ratio declines from $K/J = 35.6$ for $\alpha = -0.05$ via $K/J = 16.7$ for $\alpha = 0$ to $K/J = 5.1$ for $\alpha = 0.15$. In the concave class with $\sigma = 2, N = 6$, the capital/investment ratio declines from $K/J = 4.49$ for $\alpha = -0.05$ via $K/J = 4.00$ for $\alpha = 0$ to $K/J = 2.92$ for $\alpha = 0.15$. With $\sigma = 2, N = 50$, the ratio declines from $K/J = 103.4$ for $\alpha = -0.05$ via $K/J = 33.3$ for $\alpha = 0$ to $K/J = 6.4$ for $\alpha = 0.15$. For each N and α , increasing τ leads to a younger stock of capital for the convex class (decreasing K/J), and increasing σ leads to an older stock of capital for the concave class (increasing K/J).

Retirement/investment ratios

By examining the columns of Tables 3 and 6 we see how changes in the investment growth rate, for each given maximal life-time and each value of the curvature parameter, affects another basic property of the age distribution of the capital, the retirement/investment ratio, or the death to birth ratio of the capital. As a benchmark case, we may consider the constant investment case, in which $D/J = 1$ for any survival functions.

Increasing α leads to a decline in the death to birth ratio. This is also an indication

of a gradually younger capital stock. A few examples illustrate this. In the convex class with $\tau = 2, N = 6$, the retirement/investment ratio declines from $D/J = 1.11$ for $\alpha = -0.05$ to $D/J = 0.75$ for $\alpha = 0.15$, whereas with $\tau = 2, N = 50$, the ratio declines from $D/J = 2.78$ for $\alpha = -0.05$ to $D/J = 0.23$ for $\alpha = 0.15$. In the concave class with $\sigma = 2, N = 6$, the retirement/investment ratio declines from $D/J = 1.22$ for $\alpha = -0.05$ to $D/J = 0.56$ for $\alpha = 0.15$, whereas with $\sigma = 2, N = 50$, the ratio declines from $D/J = 6.17$ for $\alpha = -0.05$ to $D/J = 0.04$ for $\alpha = 0.15$. For each N and α , increasing τ leads to a decreasing/increasing D/J ratio for the convex class if α is negative/positive, and increasing σ leads to increasing/decreasing D/J ratio for the concave class if α is negative/positive.

8 Concluding remarks

In this paper, we have investigated the relationship between the retirement process of the capital, formalized by its survival function, and the average retirement rate and how this relationship is affected by changes in the investment path. As is well known, the average retirement rate will be a constant independent of the investment path and equal for all capital vintages only in the particular case where the survival function is exponentially declining (exponential decay). This is often regarded as a benchmark case in the literature. Otherwise, the average retirement rate will depend on both the growth profile of gross investment and the parameters describing the survival function. An interesting question for practical purposes is: How sensitive is it? Can constant average retirement rate be defended as a useful practical simplification?

Our numerical illustrations of how the average retirement rate responds to changes in the curvature of survival functions and to the growth pattern of investment under non-exponential survival functions show considerable sensitivity to both. The survival functions we have considered are two-parametric and characterized by the maximal life-time and a curvature parameter indicating the degree of convexity or concavity. We find that the retirement rate is a declining function of the growth rate of investment and quite sensitive to its value over a reasonable interval. Furthermore, we have illustrated how two basic properties of the age distribution of the capital, the capital/investment ratio and the retirement/investment ratio, are affected by changes in the growth rate of investment and in the form of the survival function.

Simultaneous retirement (sudden death) at a fixed age N , and linear retirement up to a fixed life-time, N are often applied in practical research or in national accounting. Which is the appropriate ‘translation’ of the retirement rate, δ , in these cases? We

have shown, as particular cases of our parametric functions, that the ‘translation’ rules $\delta = 1/N$ for the former and $\delta = 2/N$ for the latter hold under constant investment, but may be very inaccurate otherwise.

Some limitations of the paper should be mentioned: First, an integration of our analysis into a model of the firm growth process is beyond our scope. Second, although our formulae in the first part of the paper can be used to represent cases with cyclical variation in the retirement rates and in the age distribution of the capital following from cyclical variations in investment, we have not illustrated such situations numerically in the last part. Third, due to lack of reliable empirical information on survival functions, we have not presented econometric evidence on the parameters and relationships under discussion.

Table 1:

TABLE 1. RETIREMENT RATE, PER CENT.

CONVEX SURVIVAL FUNCTIONS.

Curvature Parameter = τ .Maximal life time = N .Rate of investment growth, per cent = α *a.* $\tau = 0$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	19.29148	12.70747	7.90988	6.43608	5.44713	5.03392
-1	17.17167	10.50833	5.51666	3.85830	2.54149	1.58198
0	16.66667	10.00000	5.00000	3.33333	2.00000	1.00000
1	16.17167	9.50833	4.51666	2.85830	1.54149	0.58198
5	14.29148	7.70747	2.90988	1.43608	0.44713	0.03392
10	12.16369	5.81977	1.56518	0.52396	0.06784	0.00045
15	10.27677	4.30825	0.78594	0.16851	0.00830	0.00000

b. $\tau = 1$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	35.08496	21.80997	11.96106	8.78465	6.43968	5.17555
-1	33.67001	20.33893	10.34459	7.01699	4.36199	2.39221
0	33.33333	20.00000	10.00000	6.66667	4.00000	2.00000
1	33.00332	19.67218	9.67763	6.34966	3.69348	1.71828
5	31.74829	18.46742	8.59141	5.37158	2.90097	1.23949
10	30.31943	17.18282	7.61594	4.63567	2.47898	1.11106
15	29.03567	16.11473	6.95350	4.22470	2.30622	1.07143

c. $\tau = 2$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	51.30766	31.34766	16.45308	11.56598	7.81152	5.48109
-1	50.25226	30.25378	15.25763	10.26153	6.26953	3.29062
0	50.00000	30.00000	15.00000	10.00000	6.00000	3.00000
1	49.75224	29.75372	14.75738	9.76097	5.76797	2.78442
5	48.80485	28.83987	13.92211	8.99707	5.12694	2.35877
10	47.71383	27.84422	13.13035	8.36575	4.71755	2.19514
15	46.71877	26.99121	12.54863	7.96303	4.50917	2.13198

Table 1:

TABLE 1 (CONT.). RETIREMENT RATE, PER CENT.

CONVEX SURVIVAL FUNCTIONS.

Curvature Parameter = τ .

Maximal life time = N .

Rate of investment growth, per cent = α

d. $\tau = 3$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	67.70771	41.06957	21.14513	14.56018	9.40887	5.95496
-1	66.86827	40.20269	20.20543	13.54154	8.21391	4.22903
0	66.66667	40.00000	20.00000	13.33333	8.00000	4.00000
1	66.46826	39.80264	19.80524	13.14113	7.81277	3.82447
5	65.70565	39.06386	19.12235	12.50917	7.26932	3.44133
10	64.81863	38.24470	18.44938	11.95350	6.88267	3.26259
15	63.99986	37.52750	17.93025	11.57137	6.66214	3.18427

e. $\tau = 4$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	84.19718	50.88505	25.94119	17.66868	11.13855	6.56805
-1	83.50120	50.16867	25.17070	16.83944	10.17701	5.18824
0	83.33333	50.00000	25.00000	16.66667	10.00000	5.00000
1	83.16785	49.83530	24.83724	16.50581	9.84285	4.85160
5	82.52903	49.21425	24.25800	15.96488	9.36918	4.50315
10	81.77993	48.51601	23.67037	15.46761	9.00630	4.31844
15	81.08197	47.89465	23.20141	15.10799	8.78340	4.23004

f. $\tau = 5$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	100.73780	60.75411	30.79721	20.84382	12.94851	7.28450
-1	100.14378	60.14440	30.14597	20.14756	12.15082	6.15944
0	100.00000	60.00000	30.00000	20.00000	12.00000	6.00000
1	99.85806	59.85866	29.86016	19.86163	11.86450	5.87130
5	99.30814	59.32250	29.35648	19.38789	11.44380	5.55167
10	98.65903	58.71297	28.83385	18.93737	11.10334	5.36570
15	98.04970	58.16366	28.40606	18.59937	10.88266	5.27054

Table 2:

TABLE 2. CAPITAL/INVESTMENT RATIO.

CONVEX SURVIVAL FUNCTIONS.

Curvature Parameter = τ .Maximal life time = N .Rate of investment growth, per cent = α *a.* $\tau = 0$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	6.99718	12.97443	34.36564	69.63378	223.64988	2948.26318
-1	6.18365	10.51709	22.14028	34.98588	64.87213	171.82818
0	6.00000	10.00000	20.00000	30.00000	50.00000	100.00000
1	5.82355	9.51626	18.12692	25.91818	39.34693	63.21206
5	5.18364	7.86939	12.64241	15.53740	18.35830	19.86524
10	4.51188	6.32121	8.64665	9.50213	9.93262	9.99955
15	3.95620	5.17913	6.33475	6.59261	6.66298	6.66666

b. $\tau = 1$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	3.32392	5.94885	14.36564	26.42252	69.45995	569.65264
-1	3.06091	5.17092	10.70138	16.61960	29.74425	71.82818
0	3.00000	5.00000	10.00000	15.00000	25.00000	50.00000
1	2.94089	4.83742	9.36538	13.60607	21.30613	36.78794
5	2.72121	4.26123	7.35759	9.64174	12.65668	16.02695
10	2.48019	3.67879	5.67668	6.83262	8.01348	9.00005
15	2.27089	3.21391	4.55508	5.20164	5.77827	6.22222

c. $\tau = 2$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	2.15947	3.79540	8.73127	15.23003	35.56796	207.86105
-1	2.03036	3.41836	7.01379	10.79735	18.97702	43.65637
0	2.00000	3.33333	6.66667	10.00000	16.66667	33.33333
1	1.97036	3.25164	6.34623	9.29284	14.77547	26.42411
5	1.85857	2.95509	5.28482	7.14435	9.87466	13.58922
10	1.73269	2.64241	4.32332	5.44492	6.79461	8.19999
15	1.62025	2.38145	3.62995	4.35483	5.12579	5.83704

Table 2:

TABLE 2 (CONT.). CAPITAL/INVESTMENT RATIO.
 CONVEX SURVIVAL FUNCTIONS.
 Curvature Parameter = τ .
 Maximal life time = N .
 Rate of investment growth, per cent = α

d. $\tau = 3$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	1.59470	2.77242	6.19382	10.46006	22.68155	104.71663
-1	1.51818	2.55085	5.20686	7.97350	13.86210	30.96910
0	1.50000	2.50000	5.00000	7.50000	12.50000	25.00000
1	1.48218	2.45082	4.80648	7.07157	11.34717	20.72766
5	1.41431	2.26943	4.14553	5.71129	8.15041	11.84647
10	1.33657	2.07277	3.51501	4.55508	5.92323	7.54000
15	1.26582	1.90376	3.03672	3.76345	4.61635	5.49926

e. $\tau = 4$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	1.26267	2.17936	4.77528	7.89348	6.29049	63.77331
-1	1.21210	2.03382	4.13724	6.31336	10.89679	23.87639
0	1.20000	2.00000	4.00000	6.00000	10.00000	20.00000
1	1.18810	1.96714	3.87038	5.71239	9.22267	17.08934
5	1.14248	1.84453	3.41787	4.76988	6.95934	10.52283
10	1.08956	1.70893	2.96997	3.92656	5.26141	6.98400
15	1.04078	1.58996	2.61770	3.32138	4.20461	5.20020

f. $\tau = 5$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	1.04452	1.79359	3.87639	6.31161	12.58097	43.77331
-1	1.00864	1.69078	3.43101	5.22260	8.96795	19.38194
0	1.00000	1.66667	3.33333	5.00000	8.33333	16.66667
1	0.99149	1.64315	3.24042	4.79349	7.77333	14.55329
5	0.95870	1.55467	2.91066	4.10040	6.08132	9.47717
10	0.92031	1.45533	2.57507	3.45574	4.73859	6.50800
15	0.88457	1.36680	2.30383	2.97625	3.86359	4.93327

Table 3:

TABLE 3. RETIREMENT/INVESTMENT RATIO.

CONVEX SURVIVAL FUNCTIONS.

Curvature Parameter = τ .Maximal life time = N .Rate of investment growth, per cent = α *a.* $\tau = 0$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	1.34986	1.64872	2.71828	4.48169	12.18249	148.41316
-1	1.06184	1.10517	1.22140	1.34986	1.64872	2.71828
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1	0.94176	0.90484	0.81873	0.74082	0.60653	0.36788
5	0.74082	0.60653	0.36788	0.22313	0.08208	0.00674
10	0.54881	0.36788	0.13534	0.04979	0.50337	0.00005
15	0.40657	0.22313	0.04979	0.01111	0.00055	0.00000

b. $\tau = 1$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	1.16620	1.29744	1.71828	2.32113	4.47300	29.48263
-1	1.03061	1.05171	1.10701	1.16620	1.29744	1.71828
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1	0.97059	0.95163	0.90635	0.86394	0.78694	0.63212
5	0.86394	0.78694	0.63212	0.51791	0.36717	0.19865
10	0.75198	0.63212	0.43233	0.31674	0.59933	0.10000
15	0.65937	0.51791	0.31674	0.21975	0.13326	0.06667

c. $\tau = 2$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	1.10797	1.18977	1.43656	1.76150	2.77840	11.39305
-1	1.02030	1.03418	1.07014	1.10797	1.18977	1.43656
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1	0.98030	0.96748	0.93654	0.90707	0.85225	0.73576
5	0.90707	0.85225	0.73576	0.64278	0.50627	0.32054
10	0.82673	0.73576	0.56767	0.45551	0.66027	0.18000
15	0.75696	0.64278	0.45551	0.34678	0.23113	0.12444

Table 3:

TABLE 3 (CONT.). RETIREMENT/INVESTMENT RATIO.
 CONVEX SURVIVAL FUNCTIONS.
 Curvature Parameter = τ .
 Maximal life time = N .
 Rate of investment growth, per cent = α

d. $\tau = 3$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	1.07974	1.13862	1.30969	1.52300	2.13408	6.23583
-1	1.01518	1.02551	1.05207	1.07974	1.13862	1.30969
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1	0.98518	0.97549	0.95194	0.92928	0.88653	0.79272
5	0.92928	0.88653	0.79272	0.71444	0.59248	0.40768
10	0.86634	0.79272	0.64850	0.54449	0.70384	0.24600
15	0.81013	0.71444	0.54449	0.43548	0.30755	0.17511

e. $\tau = 4$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	1.06313	1.10897	1.23876	1.39467	1.81452	4.18867
-1	1.01212	1.02034	1.04137	1.06313	1.10897	1.23876
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1	0.98812	0.98033	0.96130	0.94288	0.90777	0.82911
5	0.94288	0.90777	0.82911	0.76151	0.65203	0.47386
10	0.89104	0.82911	0.70300	0.60734	0.73693	0.30160
15	0.84388	0.76151	0.60734	0.50179	0.36931	0.21997

f. $\tau = 5$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	1.05223	1.08968	1.19382	1.31558	1.62905	3.18867
-1	1.01009	1.01691	1.03431	1.05223	1.08968	1.19382
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1	0.99009	0.98357	0.96760	0.95207	0.92227	0.85447
5	0.95207	0.92227	0.85447	0.79498	0.69593	0.52614
10	0.90797	0.85447	0.74249	0.65443	0.76307	0.34920
15	0.86731	0.79498	0.65443	0.55356	0.42046	0.26001

Table 4:

TABLE 4. RETIREMENT RATE, PER CENT.

CONCAVE SURVIVAL FUNCTIONS.

Curvature Parameter = σ .Maximal life time = N .Rate of investment growth, per cent = α *a.* $\sigma = 1$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	35.08496	21.80997	11.96106	8.78465	6.43968	5.17555
-1	33.67001	20.33893	10.34459	7.01699	4.36199	2.39221
0	33.33333	20.00000	10.00000	6.66667	4.00000	2.00000
1	33.00332	19.67218	9.67763	6.34966	3.69348	1.71828
5	31.74829	18.46742	8.59141	5.37158	2.90097	1.23949
10	30.31943	17.18282	7.61594	4.63567	2.47898	1.11106
15	29.03567	16.11473	6.95350	4.22470	2.30622	1.07143

b. $\sigma = 2$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	27.27980	17.34218	10.00000	7.65851	5.96757	5.10736
-1	25.44116	15.44361	7.94977	5.45596	3.46844	2.00000
0	25.00000	15.00000	7.50000	5.00000	3.00000	1.50000
1	24.56615	14.56857	7.07460	4.58057	2.59237	1.12081
5	22.90287	12.96184	5.60405	3.23783	1.47723	0.41574
10	20.98180	11.20811	4.22469	2.16496	0.83149	0.20398
15	19.22876	9.71349	3.24744	1.53313	0.55030	0.13453

c. $\sigma = 3$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	24.65389	15.83099	9.32958	7.27079	5.80413	5.08403
-1	22.69281	13.80653	7.14642	4.93078	3.16620	1.86592
0	22.22222	13.33333	6.66667	4.44444	2.66667	1.33333
1	21.75946	12.87317	6.21297	3.99719	2.23215	0.92979
5	19.98593	11.16076	4.64894	2.57378	1.06190	0.21930
10	17.94009	9.29788	3.20119	1.47026	0.43860	0.06019
15	16.07784	7.72135	2.20539	0.86319	0.21197	0.02671

Table 4:

TABLE 4 (CONT.). RETIREMENT RATE, PER CENT.

CONCAVE SURVIVAL FUNCTIONS.

Curvature Parameter = σ .Maximal life time = N .Rate of investment growth, per cent = α *d.* $\sigma = 4$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	23.33183	15.06714	8.98805	7.07225	5.72011	5.07212
-1	21.31660	12.98600	6.74284	4.66637	3.01343	1.79761
0	20.83333	12.50000	6.25000	4.16667	2.50000	1.25000
1	20.35826	12.02764	5.78443	3.70785	2.05458	0.83728
5	18.53924	10.27291	4.18638	2.25894	0.87506	0.14521
10	16.44555	8.37275	2.72774	1.16724	0.29043	0.02381
15	14.54601	6.77681	1.75087	0.60052	0.10782	0.00711

e. $\sigma = 5$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	22.53442	14.60503	8.78020	6.95095	5.66861	5.06486
-1	20.48994	12.49276	6.49982	4.50688	2.92101	1.75604
0	20.00000	12.00000	6.00000	4.00000	2.40000	1.20000
1	19.51851	11.52132	5.52833	3.53532	1.94919	0.78311
5	17.67658	9.74596	3.91554	2.07760	0.77181	0.10978
10	15.56072	7.83108	2.46026	1.00386	0.21957	0.01166
15	13.64646	6.23279	1.50579	0.47074	0.06612	0.00237

f. $\sigma = 1000$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	19.30802	12.71726	7.91448	6.43886	5.44834	5.03409
-1	17.18833	10.51832	5.52164	3.86160	2.54345	1.58290
0	16.68333	10.01000	5.00500	3.33667	2.00200	1.00100
1	16.18833	9.51832	4.52164	2.86160	1.54345	0.58290
5	14.30802	7.71727	2.91449	1.43886	0.44835	0.03409
10	12.17987	5.82898	1.56880	0.52561	0.06818	0.00046
15	10.29236	4.31658	0.78842	0.16928	0.00836	0.00000

Table 5:

TABLE 5. CAPITAL/INVESTMENT RATIO.

CONCAVE SURVIVAL FUNCTIONS.

Curvature Parameter = σ .Maximal life time = N .Rate of investment growth, per cent = α *a.* $\sigma = 1$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	3.32392	5.94885	14.36564	26.42252	69.45995	569.65264
-1	3.06091	5.17092	10.70138	16.61960	29.74425	71.82818
0	3.00000	5.00000	10.00000	15.00000	25.00000	50.00000
1	2.94089	4.83742	9.36538	13.60607	21.30613	36.78794
5	2.72121	4.26123	7.35759	9.64174	12.65668	16.02695
10	2.48019	3.67879	5.67668	6.83262	8.01348	9.00005
15	2.27089	3.21391	4.55508	5.20164	5.77827	6.22222

b. $\sigma = 2$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	4.48837	8.10230	20.00000	37.61501	103.35194	931.44422
-1	4.09146	6.92347	14.38897	22.44185	40.51149	100.00000
0	4.00000	6.66667	13.33333	20.00000	33.33333	66.66667
1	3.91142	6.42320	12.38452	17.91930	27.83679	47.15178
5	3.58386	5.56736	9.43036	12.13912	15.43870	18.46468
10	3.22770	4.71518	7.03003	8.22033	9.23234	9.80010
15	2.92152	4.04637	5.48022	6.04846	6.43074	6.60741

c. $\sigma = 3$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	5.08805	9.23276	23.09691	44.03753	124.35752	1190.09138
-1	4.60982	7.80852	16.26963	25.44026	46.16381	115.48455
0	4.50000	7.50000	15.00000	22.50000	37.50000	75.00000
1	4.39378	7.20816	13.86391	20.01126	30.93915	51.81916
5	4.00225	6.18783	10.36383	13.20344	16.49648	19.15967
10	3.57909	5.18192	7.57507	8.71820	9.57983	9.94017
15	3.21773	4.40115	5.81213	6.30390	6.57377	6.65482

Table 5:

TABLE 5 (CONT.). CAPITAL/INVESTMENT RATIO.

CONCAVE SURVIVAL FUNCTIONS.

Curvature Parameter = σ .Maximal life time = N .Rate of investment growth, per cent = α *d.* $\sigma = 4$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	5.45499	9.93330	25.07491	48.25666	138.86777	1386.53744
-1	4.92208	8.34307	17.41298	27.27496	49.66652	125.37454
0	4.80000	8.00000	16.00000	24.00000	40.00000	80.00000
1	4.68203	7.67599	14.73964	21.24113	32.73769	54.42842
5	4.24823	6.54754	10.88568	13.77612	17.02110	19.43554
10	3.78135	5.44284	7.85685	8.95476	9.71777	9.97625
15	3.38455	4.59204	5.96984	6.41004	6.61909	6.66351

e. $\sigma = 5$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	5.70307	10.41121	26.45365	51.25708	149.56422	1541.72574
-1	5.13085	8.70113	18.18241	28.51535	52.05607	132.26823
0	5.00000	8.33333	16.66667	25.00000	41.66667	83.33333
1	4.87365	7.98638	15.31785	22.04918	33.90760	56.08184
5	4.40984	6.78152	11.21637	14.12909	17.32559	19.57030
10	3.91225	5.60818	8.02551	9.08772	9.78515	9.98835
15	3.49083	4.70970	6.05848	6.46382	6.63741	6.66561

f. $\sigma = 1000$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	6.98909	12.95796	34.31138	69.49967	223.04288	2933.51031
-1	6.17729	10.50605	22.11588	34.94544	64.78981	171.55690
0	5.99401	9.99001	19.98002	29.97003	49.95005	99.90010
1	5.81790	9.50722	18.11056	25.89597	39.31662	63.17527
5	5.17919	7.86332	12.63505	15.53070	18.35419	19.86456
10	4.50859	6.31753	8.64394	9.50063	9.93228	9.99954
15	3.95376	5.17690	6.33376	6.59227	6.66295	6.66666

Table 6:

TABLE 6. RETIREMENT/INVESTMENT RATIO.

CONCAVE SURVIVAL FUNCTIONS.

Curvature Parameter = σ .Maximal life time = N .Rate of investment growth, per cent = α *a.* $\sigma = 1$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	1.16620	1.29744	1.71828	2.32113	4.47300	29.48263
-1	1.03061	1.05171	1.10701	1.16620	1.29744	1.71828
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1	0.97059	0.95163	0.90635	0.86394	0.78694	0.63212
5	0.86394	0.78694	0.63212	0.51791	0.36717	0.19865
10	0.75198	0.63212	0.43233	0.31674	0.19865	0.10000
15	0.65937	0.51791	0.31674	0.21975	0.13326	0.06667

b. $\sigma = 2$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	1.22442	1.40511	2.00000	2.88075	6.16760	47.57221
-1	1.04091	1.06923	1.14389	1.22442	1.40511	2.00000
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1	0.96089	0.93577	0.87615	0.82081	0.72163	0.52848
5	0.82081	0.72163	0.52848	0.39304	0.22806	0.07677
10	0.67723	0.52848	0.29700	0.17797	0.07677	0.01999
15	0.56177	0.39304	0.17797	0.09273	0.03539	0.00889

c. $\sigma = 3$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	1.25440	1.46164	2.15485	3.20188	7.21788	60.50457
-1	1.04610	1.07809	1.16270	1.25440	1.46164	2.15485
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1	0.95606	0.92792	0.86136	0.79989	0.69061	0.48181
5	0.79989	0.69061	0.48181	0.33983	0.17518	0.04202
10	0.64209	0.48181	0.24249	0.12818	0.04202	0.00598
15	0.51734	0.33983	0.12818	0.05441	0.01393	0.00178

Table 6:

TABLE 6 (CONT.). RETIREMENT/INVESTMENT RATIO.
CONCAVE SURVIVAL FUNCTIONS.

Curvature Parameter = σ .

Maximal life time = N .

Rate of investment growth, per cent = α

d. $\sigma = 4$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	1.27275	1.49667	2.25375	3.41283	7.94339	70.32687
-1	1.04922	1.08343	1.17413	1.27275	1.49667	2.25375
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1	0.95318	0.92324	0.85260	0.78759	0.67262	0.45572
5	0.78759	0.67262	0.45572	0.31119	0.14895	0.02822
10	0.62186	0.45572	0.21431	0.10452	0.02822	0.00238
15	0.49232	0.31119	0.10452	0.03849	0.00714	0.00047

e. $\sigma = 5$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	1.28515	1.52056	2.32268	3.56285	8.47821	78.08629
-1	1.05131	1.08701	1.18182	1.28515	1.52056	2.32268
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1	0.95126	0.92014	0.84682	0.77951	0.66092	0.43918
5	0.77951	0.66092	0.43918	0.29355	0.13372	0.02149
10	0.60877	0.43918	0.19745	0.09123	0.02149	0.00116
15	0.47638	0.29355	0.09123	0.03043	0.00439	0.00016

f. $\sigma = 1000$

α	$N = 6$	$N = 10$	$N = 20$	$N = 30$	$N = 50$	$N = 100$
-5	1.34945	1.64790	2.71557	4.47498	12.15214	147.67552
-1	1.06177	1.10506	1.22116	1.34945	1.64790	2.71557
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1	0.94182	0.90493	0.81889	0.74104	0.60683	0.36825
5	0.74104	0.60683	0.36825	0.22347	0.08229	0.00677
10	0.54914	0.36825	0.13561	0.04994	0.00677	0.00005
15	0.40694	0.22347	0.04994	0.01116	0.00056	0.00000

Appendix: Properties of the function (25)

In this appendix, we prove some useful properties of the auxiliary function (25), defined by the integral

$$(A.1) \quad F(i, z) = \int_0^1 e^{z\theta} (1-\theta)^i d\theta = \int_0^1 e^{z(1-\lambda)} \lambda^i d\lambda = e^z \int_0^1 e^{-z\lambda} \lambda^i d\lambda, \quad i = 0, 1, 2, \dots$$

It is straightforward to show that

$$(A.2) \quad F(i, 0) = \frac{1}{i+1},$$

$$(A.3) \quad \lim_{i \rightarrow \infty} F(i, z) = 0 \quad \text{for all } z,$$

$$(A.4) \quad \lim_{i \rightarrow \infty} (i+1)F(i, z) = 1 \quad \text{for all } z.$$

If $z \neq 0$, the function satisfies the recursion

$$(A.5) \quad \begin{cases} F(0, z) = \frac{1}{z}(e^z - 1), \\ F(i, z) = \frac{1}{z}[iF(i-1, z) - 1], \quad i = 1, 2, \dots, \end{cases}$$

i.e.,

$$\begin{aligned} F(0, -z) &= e^{-z}F(0, z) = \frac{1}{z}(1 - e^{-z}), \\ F(i, -z) &= \frac{1}{z}[1 - iF(i-1, -z)], \quad i = 1, 2, \dots, \end{aligned}$$

which can be shown by integration by parts. For $i = 1$ and $i = 2$, we have in particular (when $z \neq 0$)

$$\begin{aligned} F(1, z) &= \frac{1}{z} \left[\frac{1}{z}(e^z - 1) - 1 \right], \\ F(2, z) &= \frac{1}{z} \left[\frac{2}{z} \left(\frac{1}{z}(e^z - 1) - 1 \right) - 1 \right], \end{aligned}$$

i.e.,

$$\begin{aligned} F(1, -z) &= \frac{1}{z} \left[1 - \frac{1}{z}(1 - e^{-z}) \right], \\ F(2, -z) &= \frac{1}{z} \left[1 - \frac{2}{z} \left(1 - \frac{1}{z}(1 - e^{-z}) \right) \right]. \end{aligned}$$

If z is small and i is large, the above recursion may give inaccurate results in numerical calculations. A better way of keeping the accuracy under control is to represent $F(i, z)$ by its Taylor expansion. Expanding e^z by Taylor's formula, we get, for $i = 0$,

$$F(0, z) = \frac{1}{z} \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots - 1 \right) = 1 + \frac{z}{1 \cdot 2} + \frac{z^2}{1 \cdot 2 \cdot 3} + \frac{z^3}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

Substituting this expression in (A.5), it follows that

$$\begin{aligned} F(1, z) &= \frac{1}{z}[F(0, z) - 1] = \frac{1}{2} + \frac{z}{2 \cdot 3} + \frac{z^2}{2 \cdot 3 \cdot 4} + \frac{z^3}{2 \cdot 3 \cdot 4 \cdot 5} + \dots, \\ F(2, z) &= \frac{1}{z}[2F(0, z) - 1] = \frac{1}{3} + \frac{z}{3 \cdot 4} + \frac{z^2}{3 \cdot 4 \cdot 5} + \frac{z^3}{3 \cdot 4 \cdot 5 \cdot 6} + \dots, \end{aligned}$$

and in general,

$$(A.6) \quad F(i, z) = \frac{1}{i+1} + \frac{z}{(i+1)(i+2)} + \frac{z^2}{(i+1)(i+2)(i+3)} + \cdots = \sum_{j=1}^{\infty} \frac{i! z^{j-1}}{(i+j)!},$$

$i = 0, 1, 2, \dots$,

which can be verified by induction.

References

- Barlow, R.E., and Proschan, F. (1975): *Statistical Theory of Reliability and Life Testing*. New York: Holt, Rinehart and Winston, 1975.
- Biørn, E. (1989): *Taxation, Technology and the User Cost of Capital*. North-Holland, Amsterdam, 1989.
- Biørn, E. (1998): Survival and Efficiency Curves for Capital and the Time-Age-Profile of Vintage Prices. *Empirical Economics*, 23 (1998), 611 – 633.
- Biørn, E., and Frenger, P. (1992): Expectations, Substitution, and Scrapping in a Putty-Clay Model. *Journal of Economics*, 56 (1992), 157 – 184.
- Coale, A. (1972): *The Growth and Structure of Human Populations*. Princeton: Princeton University Press, 1972.
- Feldstein, M.S., and Rothschild, M. (1974): Towards an Economic Theory of Replacement Investment. *Econometrica*, 42 (1974), 393 – 423.
- Fraumeni, B.M., and Jorgenson, D.W. (1980): The Role of Capital in U.S. Economic Growth, 1948 – 1976. Chapter 2 in: G.M. von Furstenberg (ed.): *Capital, Efficiency and Growth*. Cambridge, Mass.: Ballinger, 1980.
- Johansen, L. (1959): Substitution versus Fixed Production Coefficients in the Theory of Economic Growth: A Synthesis. *Econometrica*, 27 (1959), 157 – 176.
- Johansen, L. (1972): *Production Functions*. Amsterdam: North-Holland, 1972.
- Jorgenson, D.W. (1963): Capital Theory and Investment Behavior. *American Economic Review*, 53 (1963), 247 – 259.
- Jorgenson, D.W. (1974): The Economic Theory of Replacement and Depreciation. In: Sellekaerts, W. (ed.): *Econometrics and Economic Theory. Essays in Honour of Jan Tinbergen*. London: Macmillan, 1974, pp. 189 – 221.
- Jorgenson, D.W. (1996): Empirical Studies of Depreciation. *Economic Inquiry*, 34 (1996), 24 – 42.
- Kalbfleisch, J.D., and Prentice, R.L. (1980): *The Statistical Analysis of Failure Time Data*. New York: Wiley, 1980.
- Keyfitz, N. (1977): *Introduction to the Mathematics of Population*. Reading, Mass.: Addison-Wesley, 1977.
- Nickell, S.J. (1978): *The Investment Decisions of Firms*. Cambridge: Cambridge University Press, 1978.