

# Department of Economics

## Balanced Growth with a Network of Ideas

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Working Paper No. 546

September 2005

ISSN 1473-0278



# Balanced growth with a network of ideas <sup>1</sup>

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**Abstract:** *We propose a model of economic growth in which technological progress is modelled as an expanding random network of ideas. New ideas are created by combining successful old ideas. Old ideas are chosen according to their visibility as ideas, success as generators of innovations and age but the process is stochastic. The productivity of an innovation on the other hand depends on the number, importance and success of the neighbors to the parent idea. Within this framework, we isolate the conditions on the law governing the growth of the network compatible with balanced growth. The paper can be viewed as an attempt to provide microfoundations to the set of production functions compatible with the stylized facts of economic growth.*

**Keywords:** *Economic growth, Technological progress, Innovations, Random growing network, Ideas, Scale-free distributions.*

*Journal of Economic Literature* Classification Numbers: D30, D50, D90, O41.

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<sup>1</sup>We thank R. Becker, C. Le Van, M. Boldrin, T. Hens, P. Peretto, K. Shell, J. Stachurski and J. Zweimuller for useful comments and suggestions.

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# 1 Introduction

A striking fact of economic growth is the apparent constancy of the growth rate of US GDP over the last 150 years. This is remarkable as technology, defined in a broad sense, has gone through deep changes during this period of time. The reason for the existence of a linear long-run trend despite the radical changes has to be found outside the realm of standard “technologies”. As the human brain has not evolved one may expect some regularity in the production of ideas. Having this in mind, our aim is to provide micro-foundations to a production function exhibiting technological progress and compatible with the stylized facts. In fact we impose that when imbedded in an optimal growth model the technology produces a constant growth rate of output. Even though the model we propose does not completely escape the linearity critique<sup>3</sup> we believe this is introduced in a natural way.

We assume that new ideas are produced from existing ideas. As ideas are produced by the human brain we assume that there is a natural constraints on the process of creation: only a bounded number of existing ideas can be processed to produce an idea. Of course, the ideas themselves can become more and more complex and their productivity may grow without bounds. Importantly, successful ideas are more likely to be chosen again in the creative process than forgotten ones but the choice is stochastic. Consequently, clusters of visible ideas are expected to occur. The production of ideas needs ideas. The stock of ideas plays a role because the larger is the pool of ideas the lower is the “cost” of producing them. However, when ideas are abundant this constraint becomes less compelling. In this case the number of ideas produced is expected to depend on the amount of labor used in the research sector and to a lesser extent capital. Finally, ideas are realized into innovations according to some stochastic process. The probability that an idea becomes an innovation depends on the resources dedicated to this process but may also depend on the visibility of the idea itself.

To model the properties outlined in the previous paragraph, ideas are assumed to be the nodes of a random growing network. New nodes are linked to a given number of older nodes. The choice depends on their weighted connectivity, or *strength*, where the weights capture the importance and age of the nodes. Under some plausible assumptions the strength of the nodes follows asymptotically a generalized power law distribution. Innovations are

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<sup>3</sup>This critique asserts that the constant growth in output produced by a model is a direct consequence of a linearity implicitly contained in the assumptions (see Jones (2004)).

extracted either by a blind draw or by one that takes into account the success of the idea. Given the way new nodes are preferentially attached, it is natural to define productivity as the strength of the parent idea. Furthermore, in this way productivity also reflects the existence externalities across ideas. We then show how to modify the Houthakker's approach, as formalized by Jones (2005), in order to show that average output is produced by a Cobb-Douglas production technology. Finally, imbedded in a neoclassical model this allows to obtain a balanced growth path. In this way the model is able to reproduce the observed regularity in the long run growth rate of output.

The model implicitly assumes that the interconnection between knowledge accumulation and the production of innovations is weak. This is in line with the Shumpeterian view of technological progress. A consequence is that variations in the growth rate of ideas have little immediate impact on the growth rate of innovations and reciprocally. The properties of the business cycle are dictated by this feature. The model predicts that in most cases temporary and permanent shocks have no permanent effect on the growth rate of output. On the other hand, the duration of the effects of a shock on the level of output depends on the properties of the shock itself and on the properties of innovations. In particular, temporary shocks have a temporary effect which persist for some time, at least as long as the "cyclical" innovations are in use. This fact agrees with the data as it implies that the economy recovers its original trend even after large fluctuations as in the 30's.

A detailed discussion of the assumptions is pursued in the next sections of the paper. We believe these are not unrealistic although in some cases other equally plausible assumptions could be proposed. There are mainly two areas in which the model can be criticized. A first difficulty is that data about the production of "ideas" are impossible to obtain. The most closely related processes are the production of *patents* and the production of *scientific articles*. The production of research has been investigated in a large number of studies since Lotka (1926). For example, a recent analysis of the distribution of citations of patents can be found in Leiva Bertran (2003). The large majority of the studies are compatible with power law distributions. The other weakness concern the innovative sector. The fact that there is no strategic behavior from the part of the innovator is clearly a weakness.

## 2 Production of ideas

Ideas are produced one at a time but the number of ideas produced between  $t$  and  $t + 1$  grows. We first focus on the constraints on the production of each idea, letting the determination of the number of ideas produced for later. New ideas are produced using existing ideas. The machine producing ideas is the human brain. The process of creation is bounded in the sense that only a bounded number of existing ideas are processed at once. This bound is mainly due to the physiological capacity of the human brain. We may assume that the bound is tight or simply that the average number of links established by the new idea is a constant. Let this number be  $m$ . An important issue is what guides the scientist in his choice of old ideas. Ideas that have been often used in the past in the combinations (for example the Dixit-Stiglitz model) are more visible and have a higher chance of being used again. Clearly, being successful in this sense is not the whole story. Feedbacks between the “world of ideas” and the “physical world” of innovations probably exists. Ideas that were realized as successful innovations are in the spotlight and have a higher chance of being used again in combinations. Furthermore, it may be expected that clusters of good ideas develop as the brightness of an idea shines on its neighbors, i.e. on the combinations that use this successful idea. Finally, ideas lose their appeal as time goes by.

To integrate these properties, the set of ideas is modelled as a graph. Ideas are the vertices (or nodes) of a random growing network. New ideas are added sequentially. The edges (or links) of the network represent the fact that two ideas are linked. Intuitively a new idea embodies completely a linked older one. Producing orange juice with an electric device includes the older idea of squeezing the oranges manually. However, some investigation shows that this is not as obvious as it may seem. We simply assume that the existence of a link between two ideas represents the fact that these have something in common, without any time orientation. We also include weights so that both the strength of the link and the age of the idea can be taken into account.

Two remarks. The construction implies that the set of ideas doesn't possess any spatial characteristic. In the adopted topology the distance between two ideas is represented by the minimal number of edges separating the two associated nodes. Finally, at this level it is not possible to give a “value” to an idea, as it is impossible to define the “productivity” of an idea before this is realized as an innovation.

The next issue is to define the factors determining the level of idea production, i.e. how many ideas are produced in a unit interval. Producing ideas needs ideas. Clearly, the larger is the pool of ideas the lower is the “cost” of producing them. Furthermore, as we separate ideas from innovations we may disregard the role of physical inputs at this level. Let  $\Delta N_t$  be the number of ideas produced at time  $t$  and  $N_t$  be the stock of ideas. We assume that

$$\Delta N_t = \sigma_N N_{t-1}$$

where  $\sigma_N$  is a strictly positive constant. We then obtain

$$N_t = (1 + \sigma_N)^t N_0.$$

We may criticize this formulation by pointing out that when ideas are abundant it is not obvious that for very large stocks of ideas a further increase would significantly reduce the cost of producing a new idea. In other words, the cost of producing ideas may be bounded from below when  $N \rightarrow \infty$ <sup>4</sup>. In this case physical inputs would matter, and more appropriate specification would be to assume that the production of ideas is an homogeneous function of degree one in capital and labor as

$$\Delta N_t = A_N K_t^\xi L_t^{1-\xi} N_{t-1}^\gamma$$

with  $0 \leq \xi \leq 1$ ,  $0 \leq \gamma \leq 1$  and  $A_N$  strictly positive constant. Pushing further this line of thought we set  $\gamma = 0$  and assume that in the production of fundamental ideas physical capital is not a binding constraint. Finally we obtain

$$\Delta N_t = A_N L_t^N$$

where  $L_t^N$  designates the labor force in the research sector.

Historical data shows that even though the population may be assumed to grow at a constant rate  $\mu$  the labor force employed in the production of ideas grows at a higher rate, so  $\sigma_N > \mu$ . This is compatible with the observation that the employment in the research departments rose by an average 5% since the 50's in the US. Finally,  $N_t$  grows at a constant rate  $\sigma_N$  because in this case  $\Delta N_t = N_t - N_{t-1} = (1 + \sigma_N)^{t-1} N_0 \sigma_N$  (together with the condition  $\sigma_N N_0 = A^N L_0$ ). Note that if ideas were produced with a “creation-by-doing” process then the entire population would participate to the creation of ideas and  $\sigma_N = \mu$ . Clearly, assuming that  $\sigma_N > \mu$  is not sustainable in the very long run.

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<sup>4</sup>a similar assumption is made by Weizmann (1997)

To conclude this section let us point out that in many cases it is useless to obtain the exact value of the growth rate of  $N_t$  as long as it is large enough. Indeed, as we will see in Section 4 very often the constraints come from the resources needed to transform ideas into innovations because ideas are sufficiently abundant. The assumption  $N_t = (1 + \sigma_N)^t N_0$  then does not appear too restrictive.

### 3 The set of ideas as a growing network

Ideas are modelled as the nodes of a random growing network. Little can be said about the topology in the early stages of the network growth. However, when the stock of ideas becomes large, i.e.  $N_t \rightarrow \infty$ , stationary patterns may appear, as for example in the distribution of the connectivity of the nodes. The main issue is to define the law used in the production of new combinations. In our framework, new nodes are linked to existing ones according to a non-uniform law. The early random graph theory developed by Erdos and Renyi (1959) assumed a purely random assignment but recent results have been obtained for networks with the so-called “preferential attachment”.

The simplest way to formalize “preferential attachment” is to assume that the probability that a given node is used depends on the connectivity of the node. In this case the probability function  $\pi$  that a new node is connected to a node  $i$  with  $k_i$  links is, at least asymptotically in the number of nodes, of the form

$$\pi_i \sim A + Bk_i^\eta$$

with  $\eta$  a positive parameter,  $A$  a positive constant representing the “initial attractiveness” of an isolated node and  $B$  a positive constant. The value of  $\eta$  plays a major role in the type of behavior of the economy. When  $\pi$  is an affine function, the distribution of connectivity  $k$  evolves toward a scale-free power law.

The previous formalism has several limitations. In particular, only the number of connected ideas matters, not the “quality” of the linked ideas or the strength of the links. In this way, second order effects produced by the rest of the network as the effect of the neighbors’s neighbors are excluded. Furthermore, no new information as the productivity of a realized idea can be taken into account.

To cure these problems, a weighted network is constructed in which each edge is given a weight (see for. e.g. Yook et al. (2001)). Let  $w_{ij}$  be the weight of edge  $i \leftrightarrow j$  connecting nodes  $i$  and  $j$ . A node  $i$  is then also characterized by its strength, defined by  $s_i = \sum_{i'} w_{ii'}$  where  $i'$  designates all nodes directly connected to  $i$ . Finally, preferential attachment can now be stated in terms of the strength of the nodes, i.e. the sum of the weights of the connected edges.

**Assumption 1** *The probability  $\Pi_i$  that a new node is connected to a node  $i$  of strength  $s_i = \sum_{i'} w_{ii'}$  is of the form*

$$\Pi_i = \frac{s_i}{\sum_{i'} s_{i'}}$$

where  $i'$  designate the direct neighbors of node  $i$ .

The crucial part is how the weights are determined. In a static weighted model the weights are determined once the edge is added to the network and do not evolve afterwards. We adopt a dynamic weighted networks (see for ex. Barrat, Barthélemy and Vespignani (2004)) in which the weights may change after the edge has been introduced in the network. In particular, the weight associated to a link  $w_{ij}$  is updated whenever a new edge is attached either to node  $i$  or  $j$ . In this way the neighbor's of the nodes chosen by the new idea also benefit from its success as their strengths are increased. The effect of the new links on these nodes may however depend on the property of the nodes. A node  $i$  is assumed to be affected with a coupling  $\delta_i$  by the addition of the new edge. Clearly, how the  $\delta_i$  is then distributed across the edges matters. It seems natural to assume that the effect of the shock  $\delta_i$  is distributed across the edges connected to  $i$  according to the present values of the edges's weights  $w_{ij}$ . The weights are updated according to  $w_{ij} \rightarrow w_{ij} + \delta_i \frac{w_{ij}}{s_i}$ . Finally all the relevant nodes strength's are updated.

The topology of the network depends on the underlying distribution from which  $\delta_i$  is drawn. There are three obvious candidates. The  $\delta_i$ 's may be independent of the chosen node  $i$ , they may be extracted from a uniform distribution or depend on the strength of the node. More precisely.

**Assumption 2** *Let  $w_0 > 0$  be the common value of the edges weights when they are added to the network. Let the weights be updated according to*



$w_{ij} \rightarrow w_{ij} + \delta_i \frac{w_{ij}}{s_i}$  where  $\delta_i$  is the coupling or shock. The following cases are considered:

1. The coupling is independent of the node,  $\delta_i = \delta$  for all  $i$ .
2. The coupling depends on the strength of the node according to the expression

$$\delta_i = \delta \left[ s_0 \tanh \left( \frac{s_i}{s_0} \right) \right]^a$$

where  $\delta, s_0$  and  $a$  are parameters.

3. The coupling  $\delta_i$  is drawn from a uniform distribution on some closed interval  $[a, b]$

Finally, following Dorogovtsev and Mendes (2000) we introduce aging.

**Assumption 3** Aging follows a power law of the type  $\tau^{-\nu}$  where  $\tau_i = t - t_i$  is the age of node  $i$ ,  $t_i$  the date of birth of node  $i$  and  $\nu$  a tunable parameter such that  $\nu < 1$ .

**Remark.** If aging is exponential, i.e. of the type  $e^{-\tau\nu}$ , then the power law distribution is lost (see Zhu, Wang and Zhu (2003) and Xu, Wu and Wang (2005))

We can now state the main result of this section.

**Lemma 1** Suppose that Assumption 1 and Assumption 3 hold.

- If Assumption 2.1 or Assumption 2.2 hold then the resulting distribution of strength follows asymptotically a power law of the form

$$P(s) \propto s^{-\xi}$$

- If Assumption 2.3 holds then the resulting distribution of strength follows asymptotically a corrected power law of the form

$$arabas(s) \propto \frac{s^{-\xi}}{\log(s)}$$

where  $\xi$  depends also on the support of the distribution of  $\delta_i$ .

**Sketch of Proof:** A dynamic weighted network is analyzed in Barrat, Barthelemy and Vespignani (2004). Their results need to be extended in two directions. Aging as formalized in Assumption 3 is analyzed in Dorogovtsev and Mendes (2000) for a model without weights. They obtain a power law with an exponent depending on  $\nu$ . Inspection of the proofs allows to conclude that the result still holds here. Second, in the present model the time interval between two successive births is shrinking because of the strictly positive growth rate of new ideas. Indeed, the number of vertices born in a unit interval of time depends on the stock of vertices so that the length of the time interval  $l_t$  between the birth of two vertices is inversely proportional to the number of vertices, i.e.  $l_t \sim 1/N_t$ . However, this does not affect the proofs as the only relevant assumption is that the growth is linear in the sense that both the number of nodes and the number of edges grow in parallel (see Dorogovtsev and Mendes (2002)). Q.E.D.

*Remark:* An extension to affine attachment functions  $\pi_i \sim A + Bs_i$  is possible. Indeed, Dorogovtsev, Mendes and Samukhin (2000) and Albert and Barabasi (2001) extended Barabasi and Albert (1999) to arbitrary  $A$  and  $B$ .

*Remark:* The number of new links  $m$  affects  $P(k)$  in a separable multiplicative way while it does not affect the exponent  $\xi$ . Furthermore, it has been shown that  $A$  affects continuously the power exponent  $\xi$ .

Finally when the attachment function is not an affine function of  $k$  the network behaves completely differently. In this case the topology depends on whether  $\eta$  is larger or smaller than one. If  $\eta < 1$  the distribution is a stretched exponential while if  $\eta > 1$  the number of nodes with a given (or larger) number of edges is finite even when the network grows without bounds. The result is made precise in the following lemma.

**Lemma 2** *Let the attachment function be non-linear in the number of edges  $\pi(k) \sim A + Bk^\eta$ , with  $\eta \neq 1$ . If  $\eta < 1$  then the proportion  $P(k)$  of nodes with  $k$  edges follows a stretched exponential, i.e.  $P(k) = \frac{C}{k^\alpha} \prod_{j=1}^k (1 + \frac{C}{j^\alpha})^{-1}$  with  $0 < C \leq 2$ . If  $\eta > 1$  a gelation-like pattern in which a single node links to nearly all other nodes is obtained. If  $\eta > l/(l-1)$  a winner-takes-all node appear in which all remaining ideas have only one edge while for  $(l+1)/l < \eta < l/(l-1)$  the number of nodes with more than  $l$  edges is finite and the number of nodes with  $k \leq l$  edges follows  $N_k \sim t^{k-(k-1)\eta}$ .*

*Proof* : See Krapivisky, Redner and Leyvraz (2000). Q.E.D.

We will see in Section 5 that, within our framework, a strength distribution within the basin of attraction of the Frechet distribution is a necessary condition to obtain an aggregate Cobb-Douglas production function. The necessary and sufficient conditions for this to happen are not met by the stretched exponential obtained with a sublinear attachment function, i.e.  $\eta < 1$ . On the other hand, the gelation pattern obtained for  $\eta > 1$  implies that the economy stops to grow as the probability of finding ideas with higher and higher productivity decreases with  $t$ . This is a situation in which new ideas increase the productivity of a takes-all old idea, like the wheel used over and over again.

## 4 From ideas to innovations

In the present model the stock of available ideas increases at least exponentially and faster than the population growth rate. However, not all ideas become innovations. By innovation we mean the realization of an idea. There are two issues, one concerning the quantity of innovations and the other the quality. The first issue is to individuate the determinants of the level of production of innovations and the second issue is to characterize the subset of ideas that are transformed into innovations. Clearly one expects that both the size of the pool of new ideas and the available resources matter in determining the number of innovations produced. For example it can be assumed that total resources increase the chances an idea has to be successfully transformed into an innovation, similarly to the assumption in Weitzman (1998) that the rate of transformation is a function of the per-idea resource. On the other hand, the role of the stock of ideas is expected to be small when the number of ideas is very large and the number of innovations may rather be determined by resources alone, i.e. the number of innovations is directly related to aggregated output and not to the stock or the flow of ideas.

To be more specific, let  $\Delta n_t$  be the number of innovations produced at time  $t$ . We assume a constant returns production function of the form

$$\Delta n_t = H(K_t^n, N_t) = \sigma K_t^{n\theta} N_t^{1-\theta} = \sigma N_t \left( \frac{K_t^n}{N_t} \right)^\theta$$

where  $K_t^n$  is the amount of capital used in the R & D sector,  $N_t$  the number of ideas available in period  $t$ ,  $\sigma$  is positive real constant and  $0 \leq \theta \leq 1$ . In particular we will focus on the two polar cases  $\theta = 0$  and  $\theta = 1$ , i.e.  $\Delta n_t = \sigma$  and  $\Delta n_t = \sigma k_t$  with  $k_t$  being the per-idea resources. In the former configuration,  $\sigma$  can be interpreted as the fact that the probability that an idea decays into an innovation is constant. In a more general framework  $H$  may change through time. However, these changes are of second order in the determination of the growth rate of output. As noted before we could also assume that only new ideas can be realized as an innovation. Finally, we could assume that the production of innovations does not require any specific investment and in which case  $K_t^n$  would represent the overall capital. These alternative specifications would not change the results while making the model less appealing.

Regarding the second issue, one simple approach is to assume that the process is blind and that the intrinsic properties of the ideas do not affect their probability of being transformed into innovations. The assumption that the transformation of ideas into innovations depends on the aggregate performance of the economy and not on some intrinsic property of the ideas themselves is not unusual. For example, Jones (2005) assumes that ideas are drawn from a Pareto distribution and that the productivity is related to the value of the parameter that is realized by the drawn. The economic rationale being that the innovator does not know the quality of the idea before this is realized into an innovation.

In a more sophisticated model, the probability of decay may depend on the connectivity or the strength of the idea. This would allow to take into account that successful and young ideas have a higher success to be used by the innovator<sup>5</sup>. Provided the distribution of the strengths of the nodes is stationary the expression  $\Delta n_t = H(K_t, N_t)$  would remain unaffected. The strength distributions of ideas and of innovations would still be generalized power laws but would differ. If the probability follows a power function then the distribution of innovations also follows a power law.

The productivity of an innovation is assumed to be given by the strength of the parent idea. This quantity is obtained by summing the weighted edges between the idea and its immediate neighbors. Considering how the weights are constructed (see the previous section) the strength of the idea primarily

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<sup>5</sup>Another way to model aging would be to assume that only new ideas can become innovations.

reflects the success of the parent idea as a generator of new ideas and these links are weighted by their importance. However, the strength also reflects the realizations of the shock  $\delta_i$  and therefore reflects the productivity of the innovation associated to the idea or to some of its neighbors. Finally it also captures the existence of externalities across ideas: if two ideas are related they both benefit from this it. Note that the connectivity of a *given* idea grows with time as the number of links increases and this even when the network is in scale-free configuration. However, aging affects the strength of the nodes a reduces the appeal of an old node with a high connectivity. This mimics the fact that innovations are most likely to be obtained from slightly older ideas and these are also the most productive. In some sense, there is a time lag between a smart idea is produced and its most productive realization.

A production processes, or recipe, is a multidimensional object. A production process uses  $q$  physical inputs. We assume that productivity enters a standard production function as an input augmenting progress. Indeed, this type of progress leads to a balanced growth path (see for ex. Jones (2005)). In general a recipe is characterized by  $q$  innovations and its productivity is given by a vector of  $q$  scalars. Without too much loss of generality we assume that there are only two physical inputs, so  $q = 2$ .

## 5 From innovations to the production possibility frontier

We adopt the strategy of Jones (2005) to find the PPF. The global frontier is obtained by computing the maximal output obtainable from the given set of inputs. Before performing this maximum, it is then necessary to define the punctual production function associated to a given recipe  $i$ . We assume that this production function is Leontief but any other function with a sufficiently low degree of substitution would work. We assume the following.

**Assumption 4** *The output using a recipe  $i$  is produced by a Leontief production function  $Y_i = \min[b_i K, a_i L]$ . The parameters  $a_i$  and  $b_i$  are given by the strengths of the two-dimensional idea  $i$  evaluated along the two independent dimensions “capital” and “labor”.*

According to Lemma 1, Assumption 4 implies that the parameters  $a_i$  and  $b_i$  are blindly extracted from two generalized power laws. As noted before we could as well assume that they are extracted according to a draw depending on the strength or visibility of the idea. Formally,

$$P[a_i \leq a] = 1 - \frac{1}{\log(\frac{a}{\gamma_a})} (\frac{a}{\gamma_a})^{-\alpha}, a \geq \gamma_a > 0$$

$$P[b_i \leq b] = 1 - \frac{1}{\log(\frac{b}{\gamma_b})} (\frac{b}{\gamma_b})^{-\beta}, b \geq \gamma_b > 0$$

where  $\alpha > 1$  and  $\beta > 1$ . The value of the power exponent  $\alpha$  and  $\beta$  depend on the form of the attachment function and the other parameters of the model (note however that the attachment function need to be linear or affine in order to obtain a scale-free distribution). The value of  $\gamma_a$  and  $\gamma_b$  are related to the number of links a new idea establishes (or uses). The fact that the two laws are not identical is motivated by empirical evidence across sectors. As we adopt Leontief production functions, the distribution over output is obtained from the joint distribution over  $a$  and  $b$

$$P[a_i > a, b_i > b] = (\frac{a}{\gamma_a})^{-\alpha} (\frac{b}{\gamma_b})^{-\beta} \frac{1}{\log(\frac{a}{\gamma_a})} \frac{1}{\log(\frac{b}{\gamma_b})}$$

In other words,

$$\begin{aligned} P(Y_i > y) &= P[a_i L > y, b_i K > y] \\ &= (\frac{y}{L})^{-\alpha} (\frac{y}{K})^{-\beta} \frac{1}{\log(\frac{y}{L})} \frac{1}{\log(\frac{y}{K})} (\gamma_a \gamma_b)^{\alpha+\beta} \frac{1}{\log(\gamma_a)} \frac{1}{\log(\gamma_b)} \end{aligned}$$

giving

$$P(Y_i > y) = \gamma K^\beta L^\alpha y^{-(\alpha+\beta)} \frac{1}{\log(y) - \log(L)} \frac{1}{\log(y) - \log(K)}$$

with  $\gamma = \gamma_a \gamma_b$ .

Let  $n_t$  be the stock of innovations available in period  $t$ . At this stage we may consider that innovations are durable or that they last for one period only. Indeed, since the network is assumed to be in a stationary configuration extracting all the  $n_t$  innovations in time  $t$  and supposing they live only one period or extracting only a part of the  $n_t$  in period  $t$  but assume that innovations are durable makes no difference.

The aggregated production function is obtained from the  $n_t$  innovations:

$$F(K_t, L_t, n_t) = \max_{i=1, \dots, n_t} F(b_i K_t, a_i L_t)$$

As the draws are independent, provided the distribution of ideas is stationary, i.e. the network has already reached its asymptotic configuration, the distribution of the global production function satisfies

$$\Pr(Y \leq y) = \prod_{i=1, \dots, n_t} P(Y_i \leq y) = (1 - P(Y_i > y))^{n_t}$$

$$\Pr(Y \leq y) = (1 - \gamma K^\beta L^\alpha y^{-(\alpha+\beta)} \frac{1}{\log(y) - \log(L)} \frac{1}{\log(y) - \log(K)})^{n_t}$$

We are interested in the limit as  $n_t \rightarrow \infty$ . This is obtained from the theory of extremal value. In fact the distribution  $F(y) = P(Y_i \leq y)$  is in the domain of attraction of the Frechet distribution. The necessary and sufficient conditions for this to be true are well known (see Galambos (1987)) In particular, there should be a  $\phi$  such that for all  $c$

$$\lim_{y \rightarrow \infty} \frac{1 - F(y)}{1 - F(cy)} = c^\phi$$

In the present model the distribution is

$$1 - F(y) = \gamma K^\beta L^\alpha y^{-\gamma} \frac{1}{\log(y) - \log(L)} \frac{1}{\log(y) - \log(K)}$$

so that for given  $(K, L)$  we have

$$\begin{aligned} \lim_{y \rightarrow \infty} \frac{1 - F(y)}{1 - F(cy)} &= \lim_{y \rightarrow \infty} \frac{y^{-\gamma} \frac{1}{\log(y) - \log(L)} \frac{1}{\log(y) - \log(K)}}{(cy)^{-\gamma} \frac{1}{\log(cy) - \log(L)} \frac{1}{\log(cy) - \log(K)}} \\ &= c^\gamma \lim_{y \rightarrow \infty} \frac{\log(cy) \log(cy)}{\log(y) \log(y)} \\ &= c^\gamma \lim_{y \rightarrow \infty} \left( \frac{\log(c) + \log(y)}{\log(y)} \right)^2 = c^\gamma \end{aligned}$$

As  $F(y) = P(Y_i \leq y)$  is in the domain of attraction of the Frechet distribution we obtain the following result.

**Lemma 3** *As the number of innovations becomes large,  $n_t \rightarrow \infty$ ,*

$$Y_t \approx (\gamma K_t^\beta L_t^\alpha n_t)^{\frac{1}{\alpha+\beta}} \varepsilon_t$$

*where  $\varepsilon_t$  follows a Frechet distribution of parameter  $\alpha + \beta$  and unit mean.*

**Remark:** Note that the number of draws  $n_i$  is strictly speaking a random variable because of its dependence on output. However, the results can be extended (see Proposition 3.2 in Kortum (1997)).

According to Lemma 3 the technology is on average Cobb-Douglas with a capital share of  $\frac{\beta}{\alpha+\beta}$ . The parameter  $\alpha$  and  $\beta$  depend explicitly on the exponents of the distributions of ideas related to the use of capital and labor. As noticed by Jones (2005) the easiest is to find ideas the smallest is the exponent associated to capital in the aggregate production function.

## 6 The balanced growth path

We consider the standard one sector growth model with constant saving rate  $s$  and depreciation  $\delta$ . The saving rate  $s$  and the associated investment rates includes several exogenously determined components, particularly the investment in the R & D sector. The capital accumulation equation reads

$$K_{t+1} = (1 - \delta)K_t + sY_t$$

with  $\delta, s \in (0, 1)$ . According to the previous section,  $Y_t \approx (\gamma K_t^\beta L_t^\alpha n_t)^{\frac{1}{\alpha+\beta}} \varepsilon_t$  where the number of available innovations  $n_t$  is the result of present and past innovative effort. Formally,  $n_t = n_0 + \sum_{s=1}^t \Delta n_s$  or alternatively  $n_t = n_{t-1} + \Delta n_t$ . This formulation implies that in general the stock of (durable) innovations depend on the whole history. Such property is important when considering the transition dynamics and the stability properties of the b.g.p. However, it can be ignored when the focus is on the b.g.p. itself. Indeed, the capital accumulation equation stated above shows that along a balanced growth path, the growth rate of total output and of capital are equal and set to be a constant  $\sigma_Y$ .

As the number of ideas  $N_t$  grows exponentially at a rate  $\sigma_N$  the adopted formulation

$$\Delta n_t = \sigma K_t^\theta N_t^{1-\theta}$$



implies that  $\Delta n_t$  grows at a constant rate given by

$$\frac{\Delta n_{t+1}}{\Delta n_t} = \frac{K_{t+1}^\theta}{K_t^\theta} \frac{N_{t+1}^{1-\theta}}{N_t^{1-\theta}} = (1 + \sigma_Y)^\theta (1 + \sigma_N)^{1-\theta}$$

In the two polar cases  $\theta = 0$  and  $\theta = 1$ ,  $\Delta n_t$  grows either at the rate of  $N_t$  or at the rate of  $K_t$ , i.e.  $\sigma_N$  or  $\sigma_Y$ . Independently of  $\theta$ , the growth rate of the stock of innovations  $n_t$  is constant and noted  $\sigma_n$ . If  $n_t = (1 + \sigma_n)^t n_0$  then  $\Delta n_t = n_t - n_{t-1} = (1 + \sigma_n)^t n_0 - (1 + \sigma_n)^{t-1} n_0 = (1 + \sigma_n)^{t-1} n_0 \sigma_n$  so that  $\Delta n_t$  grows at the constant growth rate  $\sigma_n$ .

The next result states that a balanced growth path exists and that this is independent of the value of  $\beta$ .

**Proposition 1** *Assume that Assumption 1 to Assumption 4 hold. Then a non-trivial balanced growth path exists and  $Exp[\frac{Y_{t+1}}{Y_t}] = 1 + \sigma_Y = (1 + \mu)^{\frac{\alpha}{\alpha-\theta}} (1 + \sigma_N)^{\frac{1-\theta}{\alpha-\theta}}$ . In particular for  $\theta = 0$  we obtain  $Exp[\frac{Y_{t+1}}{Y_t}] = (1 + \mu)(1 + \sigma_N)^{\frac{1}{\alpha}}$  while for  $\theta = 1$ ,  $Exp[\frac{Y_{t+1}}{Y_t}] = (1 + \mu)^{\frac{\alpha}{\alpha-1}}$ . Furthermore along the b.g.p. the average growth rate of innovations is constant and given by  $1 + \sigma_n = (1 + \mu)^{\frac{\alpha\theta}{\alpha-\theta}} (1 + \sigma_N)^{\alpha \frac{1-\theta}{\alpha-\theta}}$*

**Proof:** See Appendix.

Q.E.D.

**Remark:** Note that the labor force that participate to the production of output is assumed to be the entire population. Of course, this implies that fluctuations in employment are disregarded and that the decomposition of the labor force in the production of the physical good and in research is not modelled. However, considering the size of the labor force which is historically devoted to research and the fact that a significant part of creation is a side product of the process of production of the physical good this simplification is not too restrictive.

As noted before, Lemma 2 shows that the network is in a scale-free configuration only if attachment is an affine function of the number of edges. A corollary is that typically in this case a b.g.p. does not exist if the attachment function has a higher term. How the economy behaves when the attachment function is not an affine function is the scope of the next paragraphs. To be done.

## 7 Stability of the balanced growth path

In the previous section we have seen that a b.g.p. exists provided  $\frac{n_{t+1}}{n_t}$  is constant along the b.g.p. According to the ‘‘Samuelson principle’’, the balanced growth path characterizes the long-run behavior of the economy provided it is dynamically stable. In the present framework stability has to be defined carefully. Indeed, in a given period output depends on capital, labor and the stock of available innovations. However, as these are durable the stock of innovations depends on the entire history of present and past innovative effort. In such a contest, local stability of the b.g.p. is naturally defined by considering the effect of deviations from the b.g.p. only in present innovative effort. This means that the sequence  $(n_t)_{t=0}^{t-1}$  is given and only deviations in

$$\Delta n_t = \sigma K_t^\theta N_t^{1-\theta}$$

are considered. We will then only consider the direct and indirect effects of deviations in capital assuming that  $N_t$  is unaffected. First, assume that the rate of transformation of ideas into innovations is independent of capital, i.e.  $\theta = 0$ . The growth rate of innovations is then  $\sigma_N$  even during the shock on capital. The path of innovations is not affected by the status of the economy. Then the sequence of innovations is given and the model has exactly the usual properties of the Solow model with exogenous technological progress and Cobb-Douglas production function. The b.g.p. is then asymptotically stable.

In the general case  $\theta \neq 0$  the analysis is more difficult. The equation governing capital accumulation is

$$\begin{aligned} K_{t+1} &= (1 - \delta)K_t + sY_t \\ &= (1 - \delta)K_t + s(\gamma K_t^\beta L_t^\alpha n_t)^{\frac{1}{\alpha+\beta}} \\ &= (1 - \delta)K_t + s\{\gamma K_t^\beta L_t^\alpha ((1 + \sigma_n)^{t-1} n_0 + \sigma K_t^\theta N_t^{1-\theta})\}^{\frac{1}{\alpha+\beta}} \end{aligned}$$

where  $L_t = (1 + \mu)^t L_0$  and  $N_t = (1 + \sigma_N)^t N_0$ . In order to analyze the dynamical system in a neighborhood of the b.g.p. capital  $K_t$  needs to be normalized by  $[(1 + \sigma_n)^t n_0]^{\frac{1}{\alpha}} L_t$ . The following result holds.

**Proposition 2** *In the case of Proposition 1, the b.g.p. is unique and dynamically stable*

**Proof:** See Appendix. Q.E.D.

## 8 Comparison with the data

The data on the research effort (as for example summarized by Kortum (1997)) is characterized by a strong expansion of the research sector over the past forty years. On the other hand, the number of patents is relatively constant over this period. It seems then increasingly harder to come up with relevant innovations while these are increasingly more productive. It can be expected that patenting of an innovation occurs when there is some indication that it brings a sufficiently large increase in productivity. In our model patenting would occur after ideas are transformed into innovations. The innovations that do not improve over past innovations would not be patented. Consequently, the adopted framework does not contradict the evidence.

The model predicts that the distribution of “productivity parameters” follows a stationary generalized power law when the networks has reached its asymptotic configuration. Direct evidence is not available but several studies suggests that productivity behaves roughly that way. For example, patent citations follows a power law (Leiva Bertran (2003)). Similarly to Jones (2005) the choice of the parameters value is however somewhat problematic. Indeed,  $\frac{\beta}{\alpha+\beta}$  need to match the empirical values of the income shares so that  $\frac{\beta}{\alpha+\beta} \in [0.3, 0.7]$ . On the other hand, most of the cited indirect evidence on power laws indicates exponents in the range  $\xi \in [0.5, 1.5]$ . Does it means that  $\alpha, \beta$  need to be taken in the range implied by the previous two intervals? Jones (2005) takes  $\alpha$  and  $\beta$  in the interval  $[2.5, 5]$  because lower values would produce excessive macroeconomic jumps when new technologies are adopted. The issue of compatibility in the values of the exponents in the power laws could probably be resolved if more sectors would be considered.

## 9 Relation with the literature

The philosophy of the present paper on how technological progress occurs is related to Weitzman (1998), Auerswald, Kauffman, Lobo and Shell (2000) and Olsson (2000) and Olsson (2005)<sup>6</sup>. In Weitzman’s world, progress is due to hybridizations of past innovations. In his terminology an innovation is

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<sup>6</sup>See also Peretto and Smulders (2002)

a “cultivar”. New cultivars arise from  $m$ -at-a-time combinations of unused cultivars. However, not all combinations succeed in becoming a cultivar, the rate of success  $\pi$  being typically a function of the amount of resources per combination devoted to this activity. The stock of cultivars (innovations) constitutes the available knowledge. Productivity is given by the number of cultivars. Finally the production function for physical output is constant returns in knowledge and physical capital.

The approach followed in this paper differs in a number of aspects. First, in the model new ideas arise from combinations of past ideas even unused, not of innovations, or cultivars, as in Weitzman. The evolution of knowledge depends on how parent ideas are selected and not on how these are transformed into innovations. A second difference is that in the present model the productivity of an innovation is endogenous and evolves with the success of the parent idea and the externalities across them. In Weitzman’s model the productivity of a cultivar is a fixed quantity (that can be normalized to unity) so that technological knowledge is only given by the number of cultivars or innovations. Finally, in our model there are two standard factors of production, capital and labor. Nothing is assumed on the production function at the aggregated level except that this function is obtained as the result of a large number of individual production processes, each corresponding to a given innovation. These innovations enter the individual production functions as labor and capital enhancing factors. In Weitzman, the factors are capital and knowledge and the function is assumed to be constant returns to scale in these two factors. Finally, in Weitzman’s model a balanced growth path is obtained provided the real cost of producing an innovation is strictly positive, even in the limit when the pool of existing ideas is infinite. In our model, the existence of a b.g.p. is, at least within the assumptions, endogenously produced by the network.

Our model is also related to Auerswald, Kauffman, Lobo and Shell (2000). Their model is not intended to explain economic growth as their main application is a microeconomic model for shop-floor learning curves. In their paper a production plan is described by an input-output specification together with the recipe employed. The recipe is described by vector of production operations. The model is given more structure by describing how one recipe is related to another one. It is assumed that the distance between two recipes is given by the number of operations that has to be changed. Given this topology, the model tells which of the technologies are likely to be uncovered by shop-floor operations. New recipes are discovered

by trials. Finally there is a single input, the production function being with constant returns. The efficiency of a recipe typically depends additively on the efficiency of each operation and on the properties of the neighbors, reflecting the existence of externalities. On the other hand, in absence of prior knowledge, the efficiency of an operation is a random variable with uniform distribution. The model is able to explain the shape of the empirical learning curves.

The similarities between Auerswald, Kauffman, Lobo and Shell (2000) and the present paper are insightful. The most significant is the existence of externalities across recipes. Furthermore, trials in Auerswald, Kauffman, Lobo and Shell (2000) can be interpreted as draws on an existing distribution of uncovered recipes. In the present paper, the draws transform ideas into innovations in a way which is somewhat similar to the uncovering by trial occurring in Auerswald et al. (2000). However, here the network is actually expanding so the underlying space is also expanding. Another important distinction is that in Auerswald, Kauffman, Lobo and Shell (2000) the distance between two recipes matters in the sense that the closest ones have more chances of being uncovered. Distances between ideas could also be defined in the present framework by taking into account the minimum number of edges separating two ideas. This is not what we do and such a distance would not affect the “cost” of discovery. Instead, the “easiest” ideas to find are those issued from successful parents.

The present paper is also related to Olsson (2000) and Olsson (2005). In Olsson’s view, ideas are elements of the positive orthant of a Euclidean space. The available knowledge is then a subset of the positive orthant which defines a technological frontier. Incremental progress, or normal science, consists in convexifying the set of available knowledge. Reformulated in our framework, normal science would consist in adding new nodes in a way to relate any two existing nodes. In our framework, the technological frontier is stochastic and it is related to the strengths of the ideas in each fundamental direction. The two models are then fairly different.

## 10 Conclusion

One of the most mysterious fact of economic development is the apparent constancy of the growth rate of US GDP over a long period of time. We propose a model of economic growth in which technological progress is modelled as an expanding random network of ideas. New ideas are created by

combining successful old ideas. Old ideas are chosen according to their visibility as ideas, success as generators of innovations and age but the process is stochastic. Within this framework, we isolate the conditions on the law governing the growth of the network compatible with balanced growth.

The paper helps identifying the conditions on the law governing the growth of the network compatible with balanced growth. Two features of the model are critical to obtain the results. First, independently of the level of complexity of the ideas, these are produced on average by a combinations of  $m$  ideas, with  $m$  a constant. This assumption captures the physiological limitations of the brain. Second, successful ideas are more likely to be used in the combinations to produce new ideas and the relation is linear. These assumptions reflect the psychological attitude to focus attention on more visible objects. An idea can be visible both as a prolific generator of new ideas or a generator of high productivity innovations. Although no direct evidence is available, the assumptions seem not unreasonable at least as a first approximation. These conditions shed some light on how economic knowledge is acquired and on the constraints the human brain faces in the production of new ideas.

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## 12 Appendix

### 12.1 Proof of Proposition 1

In order to explicitly find the balanced growth path it is useful to normalize the variables at time  $t$  by  $n_t^{\frac{1}{\alpha}} L_t$ . Then

$$\begin{aligned} y_t &= \frac{Y_t}{L_t n_t^{\frac{1}{\alpha}}} = (\gamma K_t^\beta L_t^\alpha n_t \frac{1}{L_t^{\alpha+\beta} n_t^{\frac{\alpha+\beta}{\alpha}}})^{\frac{1}{\alpha+\beta}} \varepsilon_t = (\lambda K_t^\beta L_t^\alpha \frac{1}{L_t^{\alpha+\beta} n_t^{\frac{\beta}{\alpha}}})^{\frac{1}{\alpha+\beta}} \varepsilon_t \\ &= (\gamma \frac{K_t^\beta}{L_t^\beta n_t^{\frac{\beta}{\alpha}}})^{\frac{1}{\alpha+\beta}} \varepsilon_t = (\gamma k_t^\beta)^{\frac{1}{\alpha+\beta}} \varepsilon_t = f(k_t) \varepsilon_t \end{aligned}$$

where

$$f(k) = (\gamma k^\beta)^{\frac{1}{\alpha+\beta}}$$

Taking averages, the capital accumulation equation in normalized variables becomes

$$(1 + \mu) \left( \frac{n_{t+1}}{n_t} \right)^{\frac{1}{\alpha}} k_{t+1} = s (\gamma k_t^\beta)^{\frac{1}{\alpha+\beta}} + (1 - \delta) k_t$$

Note that here  $n_t$  is variable and implicitly depends on the whole past history. First, note that taking averages

$$\frac{Y_{t+1}}{Y_t} = \frac{(\gamma K_{t+1}^\beta L_{t+1}^\alpha n_{t+1})^{\frac{1}{\alpha+\beta}}}{(\gamma K_t^\beta L_t^\alpha n_t)^{\frac{1}{\alpha+\beta}}} = \left( \left( \frac{K_{t+1}}{K_t} \right)^\beta \left( \frac{L_{t+1}}{L_t} \right)^\alpha \frac{n_{t+1}}{n_t} \right)^{\frac{1}{\alpha+\beta}}$$

On the other hand from  $K_{t+1} = (1 - \delta)K_t + sY_t$  it follows that along the b.g.p.  $1 + \sigma_K = 1 + \sigma_Y$ . Consequently,

$$1 + \sigma_Y = [(1 + \sigma_K)^\beta (1 + \mu)^\alpha (1 + \sigma_n)]^{\frac{1}{\alpha+\beta}}$$

giving

$$1 + \sigma_Y = (1 + \mu)(1 + \sigma_n)^{\frac{1}{\alpha}}$$

Furthermore, we have seen that

$$\Delta n_t = \sigma K_t^\theta N_t^{1-\theta}$$

implies that

$$\frac{\Delta n_{t+1}}{\Delta n_t} = \frac{K_{t+1}^\theta N_{t+1}^{1-\theta}}{K_t^\theta N_t^{1-\theta}} = (1 + \sigma_Y)^\theta (1 + \sigma_N)^{1-\theta}$$

As  $\Delta n_t = n_t - n_{t-1} = (1 + \sigma_n)^t n_0 - (1 + \sigma_n)^{t-1} n_0 = (1 + \sigma_n)^{t-1} n_0 \sigma_n$  we obtain that



$$1 + \sigma_n = (1 + \sigma_Y)^\theta (1 + \sigma_N)^{1-\theta}$$

Then

$$1 + \sigma_Y = (1 + \mu) [(1 + \sigma_Y)^\theta (1 + \sigma_N)^{1-\theta}]^{\frac{1}{\alpha}}$$

Collecting the terms with  $\sigma_Y$  we get the stated result. The value of the growth rate of innovations is obtained from

$$(1 + \mu)^{\frac{\alpha}{\alpha-\theta}} (1 + \sigma_N)^{\frac{1-\theta}{\alpha-\theta}} = 1 + \sigma_Y = (1 + \mu)(1 + \sigma_n)^{\frac{1}{\alpha}}$$

which gives

$$(1 + \mu)^{\frac{\theta}{\alpha-\theta}} (1 + \sigma_N)^{\frac{1-\theta}{\alpha-\theta}} = (1 + \sigma_n)^{\frac{1}{\alpha}}$$

A b.g.p. exists. Indeed, the normalized capital accumulation equation admits as a solution

$$k = \left[ \frac{(1 + \mu)(1 + \sigma_n)^{\frac{1}{\alpha}} - (1 - \delta)}{s\gamma^{\frac{1}{\alpha+\beta}}} \right]^{-\frac{\alpha+\beta}{\alpha}}$$

or

$$k = \left[ \frac{(1 + \mu)^{\frac{\alpha}{\alpha-\theta}} (1 + \sigma_N)^{\frac{1-\theta}{\alpha-\theta}} - (1 - \delta)}{s\gamma^{\frac{1}{\alpha+\beta}}} \right]^{-\frac{\alpha+\beta}{\alpha}}$$

## 12.2 Proof of Proposition 2

The equation governing capital accumulation can be written as

$$\begin{aligned} & K_{t+1} \frac{1}{[(1 + \sigma_n)^{t+1} n_0]^{\frac{1}{\alpha}} L_{t+1}} \frac{[(1 + \sigma_n)^{t+1} n_0]^{\frac{1}{\alpha}} L_{t+1}}{(1 + \sigma_n)^t n_0]^{\frac{1}{\alpha}} L_t} \\ &= \frac{1}{[(1 + \sigma_n)^t n_0]^{\frac{1}{\alpha}} L_t} [(1 - \delta)K_t + s\{\gamma K_t^\beta L_t^\alpha ((1 + \sigma_n)^{t-1} n_0 + \sigma K_t^\theta N_t^{1-\theta})\}^{\frac{1}{\alpha+\beta}}] \\ &= \frac{(1 - \delta)K_t}{[(1 + \sigma_n)^t n_0]^{\frac{1}{\alpha}} L_t} + s \left[ \frac{\gamma K_t^\beta L_t^\alpha ((1 + \sigma_n)^{t-1} n_0 + \sigma K_t^\theta N_t^{1-\theta})}{[(1 + \sigma_n)^t n_0]^{1+\frac{\beta}{\alpha}} L_t^{\alpha+\beta}} \right]^{\frac{1}{\alpha+\beta}} \end{aligned}$$

In normalized variables we get

$$(1 + \mu)(1 + \sigma_n)^{\frac{1}{\alpha}} k_{t+1} = (1 - \delta)k_t + s(\gamma \frac{(1 + \sigma_n)^{t-1} n_0 + \sigma K_t^\theta N_t^{1-\theta}}{(1 + \sigma_n)^t n_0} k_t^\beta)^{\frac{1}{\alpha+\beta}}$$

The second term in r.h.s. can be written in terms of  $k_t$  only. Indeed,

$$\begin{aligned}
& s\left(\gamma\left[\frac{1}{(1+\sigma_n)n_0} + \sigma\frac{K_t^\theta N_t^{1-\theta}}{((1+\sigma_n)^t n_0)^{\frac{1}{\alpha}}((1+\sigma_n)^t n_0)^{1-\frac{1}{\alpha}}}\right]k_t^\beta\right)^{\frac{1}{\alpha+\beta}} \\
&= s\left(\gamma\left[\frac{1}{(1+\sigma_n)n_0} + \sigma\left(\frac{K_t}{((1+\sigma_n)^t n_0)^{\frac{1}{\alpha}}}\right)^\theta\left(\frac{N_t}{((1+\sigma_n)^t n_0)^{\frac{1}{\alpha} + \frac{\alpha-1}{\alpha(1-\theta)}}}\right)^{1-\theta}\right]k_t^\beta\right)^{\frac{1}{\alpha+\beta}} \\
&= s\left(\gamma\left[\frac{1}{(1+\sigma_n)n_0} + \sigma k_t^\theta L_t^\theta\left(\frac{(1+\sigma_N)^t N_0}{(1+\sigma_n)^t n_0}\right)^{1-\theta}\right]k_t^\beta\right)^{\frac{1}{\alpha+\beta}} \\
&= s\left(\gamma\left[\frac{1}{(1+\sigma_n)n_0} + \sigma k_t^\theta (L_0(1+\mu)^t)^\theta\left(\left\{\frac{1+\sigma_N}{(1+\sigma_n)^{\frac{1}{\alpha} + \frac{\alpha-1}{\alpha(1-\theta)}}}\right\}^t \frac{N_0}{n_0^{\frac{1}{\alpha} + \frac{\alpha-1}{\alpha(1-\theta)}}}\right)^{1-\theta}\right]k_t^\beta\right)^{\frac{1}{\alpha+\beta}}
\end{aligned}$$

However,

$$\frac{(1+\sigma_N)(1+\mu)}{(1+\sigma_n)^{\frac{1}{\alpha} + \frac{\alpha-1}{\alpha(1-\theta)}}} = \frac{(1+\sigma_N)(1+\mu)}{(1+\sigma_n)^{\frac{1-\theta+\alpha-1}{\alpha(1-\theta)}}} = \frac{(1+\sigma_N)(1+\mu)}{(1+\sigma_n)^{\frac{\alpha-\theta}{\alpha(1-\theta)}}} = 1$$

Indeed, using the expression

$$((1+\mu)(1+\sigma_n)^{\frac{1}{\alpha}})^\theta (1+\sigma_N)^{1-\theta} \equiv 1 + \sigma_n$$

we get

$$(1+\mu)^\theta (1+\sigma_N)^{1-\theta} = (1+\sigma_n)^{1-\frac{\theta}{\alpha}}$$

giving

$$1 + \sigma_N = [(1+\sigma_n)^{1-\frac{\theta}{\alpha}} (1+\mu)^{-\theta}]^{\frac{1}{1-\theta}} = (1+\sigma_n)^{\frac{\alpha-\theta}{\alpha(1-\theta)}} (1+\mu)^{\frac{-\theta}{1-\theta}}$$

Finally we get

$$(1+\mu)(1+\sigma_n)^{\frac{1}{\alpha}} k_{t+1} = (1-\delta)k_t + s\left(\gamma\left[\frac{1}{(1+\sigma_n)n_0} + \sigma k_t^\theta \frac{N_0^{1-\theta} L_0^\theta}{n_0^{1-\frac{\theta}{\alpha}}}\right]k_t^\beta\right)^{\frac{1}{\alpha+\beta}}$$

Along the b.g.p. this equation becomes

$$(1+\mu)(1+\sigma_n)^{\frac{1}{\alpha}} k = s(\gamma k^\beta)^{\frac{1}{\alpha+\beta}} + (1-\delta)k$$

with

$$1 + \sigma_n = (1+\mu)^{\frac{\alpha-\theta}{\alpha(1-\theta)}} (1+\sigma_N)^{\alpha\frac{1-\theta}{\alpha-\theta}}$$

The solution is unique. Stability is then obtained because the slope of the “dynamical curve” is larger than one at the origin. Indeed, computing the derivative in respect to  $k_t$  evaluated along the b.g.p. we get

$$(1 - \delta) + s \frac{1}{\alpha + \beta} \gamma^{\frac{1}{\alpha + \beta}} \left( \left[ \frac{1}{(1 + \sigma_n)n_0} + \sigma k^\theta \frac{N_0^{1-\theta} L_0^\theta}{n_0^{1-\frac{\theta}{\alpha}}} \right] k^\beta \right)^{\frac{1}{\alpha + \beta} - 1} \\ \times \left( \frac{\beta}{(1 + \sigma_n)n_0} k^{\beta-1} + \sigma(\beta + \theta) \frac{N_0^{1-\theta} L_0^\theta}{n_0^{1-\frac{\theta}{\alpha}}} k^{\beta+\theta-1} \right)$$

Letting  $k \rightarrow 0$  this expression tend to  $+\infty$  showing that the slope is  $+\infty$ . The b.g.p. is then asymptotically stable.

**This working paper has been produced by  
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