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Abstract

The paradigm of a factor model is very appealing and has been used extensively in economic analyses. Underlying the factor model is the idea that a large number of economic variables can be adequately modelled by a small number of indicator variables. Throughout this extensive research activity on large dimensional factor models a major preoccupation has been the development of tools for determining the number of factors needed for modelling. This paper provides an alternative method to information criteria as tools for estimating the number of factors in large dimensional factor models. The theoretical properties of the method are explored and an extensive Monte Carlo study is undertaken. Results are favourable for the new method and suggest that it is a reasonable alternative to existing methods.

Keywords: Factor Models, Large Sample Covariance Matrix, Maximum Eigenvalue
JEL Codes: C12, C15, C23

1 Introduction

The paradigm of a factor model is very appealing and has been used extensively in economic analyses. Underlying the factor model is the idea that a large number of economic variables can be adequately modelled by a small number of indicator variables. Factor analysis has been used fruitfully to model, among other cases, asset returns, macroeconomic aggregates and Engel curves (see, e.g., Stock and Watson (1989), Lewbel (1991) and others).

Most analyses have traditionally been focused on small datasets meaning that the number of variables, N , to be modelled via a factor model is finite. Recently, Stock and Watson (2002) have put forward the case for analysing large datasets via factor analysis, where N is allowed to tend to infinity. Stock and Watson (2002) suggest the use of principal components for estimating factors in this context. Further work has been carried out by, e.g.,

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Forni, Hallin, Lippi, and Reichlin (2000) and Forni, Hallin, Lippi, and Reichlin (2004) in which use of dynamic principal components has been made.

Throughout this extensive research activity on large dimensional factor models a major preoccupation has been the development of tools for determining the number of factors needed for modelling. The only tool for estimating the number of factors for large dimensional datasets is the use of information criteria developed by Bai and Ng (2002). The criteria developed are modifications of standard information criteria such Akaike's information criterion where the penalty terms needed for consistent estimation of the number of factors depend both on the number of observations T as well as N , unlike the traditional criteria where the penalty terms depend only on T .

This paper aims to provide an alternative to information criteria as tools for estimating the number of factors in large dimensional factor models. The reasons for proposing a new method are two fold. Firstly, the new method seems to be a genuinely novel approach to the determination of the number of factors and is therefore interesting on theoretical grounds. Secondly, and perhaps more importantly, Monte Carlo evidence suggests that it is a much more robust method than information criteria in determining the number of factors.

The basis of the method is a remarkable fact that is well known in the statistical literature but little known in econometrics relating to the behaviour of the eigenvalues of large sample covariance matrices. In particular, under certain regularity conditions, it can be shown that the largest eigenvalue of the sample covariance matrix of a dataset converges almost surely to a fixed constant as both N and T tend to infinity, if the dataset does not have a factor structure. If on the other hand the dataset can be modelled using r factors then it is well known that the r largest eigenvalues of the covariance matrix tend to infinity (see, e.g., Chamberlain and Rothschild (1991)). Using these two facts and ideas from Altissimo and Corradi (2004), we formalise a sequential procedure for determining the number of factors.

The behaviour of the largest eigenvalue of large dimensional covariance matrices has been fully explored for i.i.d. and cross-sectionally independent data. This implies that the method is valid for a strict factor model. As the popularity of approximate factor models is increasing in the literature due to the flexibility of this setup, we further extend the analysis to moderate forms of serial and cross sectional dependence of the idiosyncratic components of the dataset. Monte Carlo evidence suggest that the new method is more robust than the information criteria methods and is therefore worthy of exploration as an alternative tool

for estimating the number of factors.

The paper is organised as follows: Section 2 surveys the available results on the behaviour of the eigenvalues of large sample covariance matrices. Section 3 discusses the new method and provides some theoretical results for strict factor models. Section 4 provides some extensions to approximate factor models. Results from a Monte Carlo study are presented in Section 5. Finally, Section 5.3 concludes.

2 Preliminaries

The factor model we consider for a given dataset for cross sectional unit i at time t , is given by

$$y_{i,t} = f_t' \lambda_i + \epsilon_{i,t} \quad (1)$$

where f_t is an r -dimensional vector of factors at time t , λ_i is an r -dimensional vector of factor loadings for cross sectional unit i and $\epsilon_{i,t}$ is the idiosyncratic part of $y_{i,t}$. For the time being we will assume that $\epsilon_{i,t}$ are i.i.d. across both i and t . This assumption will be relaxed at a later section.

Rewriting the above model in matrix notation gives

$$Y = F\Lambda + \epsilon \quad (2)$$

where $Y = (Y_1, \dots, Y_N)$, $F = (F_1, \dots, F_r)$, $\Lambda = (\lambda_1, \dots, \lambda_N)$, $\epsilon = (\epsilon_1, \dots, \epsilon_N)$, $Y_i = (y_{i,1}, \dots, y_{i,T})'$, $F_i = (f_{i,1}, \dots, f_{i,T})'$ and $\epsilon_i = (\epsilon_{i,1}, \dots, \epsilon_{i,N})'$. Following Chamberlain and Rothschild (1991) and assuming uncorrelatedness between the factors and the idiosyncratic components $\epsilon_{i,t}$, it is easy to see that the variance covariance matrix of the dataset is given by

$$\Sigma_Y = \Sigma_f + \Sigma_\epsilon \quad (3)$$

where Σ_f is a matrix with finite rank r and Σ_ϵ is the covariance matrix of the idiosyncratic component which is assumed to have bounded eigenvalues for all N . It is obvious that, by the rank requirement on Σ_f , the largest r eigenvalues of Σ_f will tend to infinity at rate N whereas the rest will be equal to zero.

Before outlining in intuitive terms the new methodology we quote some results on large dimensional covariance matrices. Let $\varepsilon = [\varepsilon_{i,t}]$ denote a $T \times N$ matrix of i.i.d. mean zero and unit variance random variables. Let $\hat{\Sigma}_\varepsilon$ denote the sample covariance matrix given by $\frac{1}{T} \varepsilon' \varepsilon$. Then, the largest eigenvalue of $\hat{\Sigma}_\varepsilon$ denoted μ_{max} converges almost surely to $(1 + \sqrt{c})^2$ where

$c = \lim_{N,T \rightarrow \infty} \frac{N}{T}$. The result is remarkable in its simplicity. For example, for $N = T$ the largest eigenvalue converges almost surely to 4. This result has been proven repeatedly under successively weaker conditions culminating in the work of Yin, Bai, and Krishnaiah (1988) who proved the result showing that a necessary and sufficient condition is that $E(\varepsilon_{i,t}^4) < \infty$. In this context it has also been shown that the minimum eigenvalue of $\hat{\Sigma}_\varepsilon$ converges almost surely to $(1 - \sqrt{c})^2$ as long as $N < T$ and, obviously, zero otherwise. We note that the condition $E(\varepsilon_{i,t}^4) < \infty$ is crucial. If this condition does not hold the maximum eigenvalue tends to infinity.

The result has been extended to more complicated setups. To appreciate the following result we note that in the case of large dimensional matrices, where the dimension of the matrix tends to infinity, focus has been placed on the limit of the empirical distribution of the eigenvalues of the matrix. Thus, it has been shown, among other things, by Bai and Silverstein (1998), for a $N \times N$ nonnegative definite symmetric matrix Q_N , that the limit as $N, T \rightarrow \infty$ of the empirical distribution of the eigenvalues of $\frac{1}{T} Q_N^{1/2} \varepsilon' \varepsilon Q_N^{1/2}$ has a support which is almost surely contained in the support of the limit of the empirical distribution of the eigenvalues of Q_N . The latter support, of course, depends on c . This result may form the basis for extensions of the method that will be suggested below.

We now outline the suggested estimation method for the number of factors. The method starts by checking whether a factor structure is supported by the data. Thus, the maximum eigenvalue of the sample covariance matrix of the normalised data is compared to $(1 + \sqrt{c})^2$. By the above discussion, if the number of factors is zero, i.e. no factor structure exists and assuming for the time being a strict factor model setup, the maximum eigenvalue of $\hat{\Sigma}_y$ should not exceed $(1 + \sqrt{c})^2$ almost surely. If a factor structure exists then the maximum eigenvalue will tend to infinity. Of course, in finite samples, the maximum eigenvalue will sometimes exceed $(1 + \sqrt{c})^2$. Following, Altissimo and Corradi (2004) who uses similar ideas we suggest that the bound be $b = (1 + \sqrt{c})^2 + d$ where $d > 0$ is chosen a priori. In fact, setting d to the average eigenvalue of the sample covariance matrix of the normalised data has performed very well in all our work. As the data are normalised this is simply equal to 1. If the maximum eigenvalue is seen to exceed the almost sure bound, b , then it is concluded that a factor structure exists.

The next step involves extracting the largest principal component from the data to give

$$y_{i,t}^{(1)} = y_{i,t} - \hat{\lambda}_{1,i} \hat{f}_{1,t}$$

This is the residual from a regression of the data on the first principal component. Note that the first principal component is a consistent estimate of a linear combination of the true factors, if more than one factor actually exist. The procedure is repeated on $y_{i,t}^{(1)}$, i.e. $y_{i,t}^{(1)}$ is normalised, the sample covariance matrix is constructed and its maximum eigenvalue is compared to b . If the maximum eigenvalue does not exceed b , it is concluded that the data are adequately described by a one-factor structure. Otherwise, the dataset is regressed on the second principal component. The procedure is repeated until, for some dataset $y_{i,t}^{(j)}$, $j = 2, \dots, r^{max}$, the bound, b , is not exceeded by the maximum eigenvalue of the sample covariance matrix of the normalised data. Note that this step-by-step sequence of regressions on the principal components is equivalent to a single regression on the whole set of principal components considered, since every principal component is uncorrelated, by construction, with all the other principal components.

The idea behind our method is simple. We check for the presence of a factor structure and if we find such evidence we keep adding factors until the residuals from the factor analysis do not display evidence of neglected factor structure. This is similar to the use of information criteria. The new elements relates to the method used to check for the presence of factor structure and uses the properties of the maximum eigenvalue of the sample covariance matrix of the data.

Note that this idea may easily be extended to the dynamic factor analysis suggested in a number of papers by Forni, Lippi Hallin and Reichlin (see, e.g., Forni and Reichlin (1998), Forni, Hallin, Lippi, and Reichlin (2000), and others). Their model is given by

$$y_{i,t} = b_{1,i}(L)u_{1,t} + \dots + b_{r,i}(L)u_{r,t} + \epsilon_{i,t} \quad (4)$$

where $b_{j,i}$ are lag polynomials and $u_{j,t}$ are white noise processes. In this case the common component (as the term $b_{1,i}(L)u_{1,t} + \dots + b_{r,i}(L)u_{r,t}$ is referred to) can be estimated for different factor numbers. In this method, there is one more parameter of influence for factor extraction which is the bandwidth used for the estimation of the spectral density needed for the final estimation of the common component. A sequential search of two loops (an outer loop of factor numbers and an inner loop for the bandwidth) can then be specified and again a search for factor structure via the examination of the largest eigenvalue of the sample covariance matrix can be implemented. We do not consider the theoretical properties of this setup further but leave work on this issue for future research. However, in the Monte Carlo section we consider an experiment which should be analysed using dynamic factors.

3 Theory

In this section we discuss the theoretical properties of the new method. For that we provide the following set of assumptions.

Assumption 1 $E\|f_t\|^4 \leq M < \infty$, $T^{-1} \sum_{t=1}^T f_t f_t' \xrightarrow{p} \Sigma$ for some $k \times k$ positive definite matrix Σ .

Assumption 2 $\|\lambda_i\| \leq \bar{\lambda} < \infty$ and $\|\Lambda' \Lambda / N - D\| \rightarrow 0$ for some positive definite matrix D .

Assumption 3 $E(\epsilon_{i,t}) = 0$, $E(\epsilon_{i,t}^2) = \sigma_i^2$, $E|\epsilon_{i,t}|^8 \leq M$

Assumption 4 $\epsilon_{i,t}$ are i.i.d. across i and t .

Assumption 5 f_t and $\epsilon_{i,t}$ are independent

Assumption 6 $\lim_{N,T \rightarrow \infty} N/T \rightarrow c$, where $0 \leq c < \infty$

These assumptions are similar to Bai and Ng (2002) apart from assumption 4 which specifies a strict factor model and is clearly restrictive and assumption 6 which governs the relative rates for N and T . Aspects of these assumptions will be relaxed in Section 4. Now, we provide a formal definition of the new estimator through the following algorithm

Algorithm 1 *Estimation of number of factors*

Step 1 Demean the data $y_{i,t}$. Set $y_{i,t}^f = y_{i,t}$. Normalise $y_{i,t}^f$ by dividing every observation of each series with the estimated standard deviation of that series. Estimate the first r^{\max} principal components of $y_{i,t}^f$. Denote the estimates by $\hat{f}_{s,t}$, $s = 1, \dots, r^{\max}$. Set $b = (1 + \sqrt{N/T})^2 + 1$, $j = 0$ and $y_{i,t}^{(j)} = y_{i,t}$.

Step 2 Normalise $y_{i,t}^{(j)}$ by dividing every observation of each series with the estimated standard deviation of that series. Denote the resulting dataset by $\tilde{y}_{i,t}^{(j)}$.

Step 3 Calculate the maximum eigenvalue of the estimated covariance matrix of $\tilde{y}_{i,t}^{(j)}$, denoted $\mu_{\max}^{(j)}$. If $\mu_{\max}^{(j)} < b$, then set $\hat{r} = j$ and stop the algorithm, otherwise go to Step 4.

Step 4 Regress $y_{i,t}^{(j)}$ for all i on $\hat{f}_{j+1,t}$. Denote the estimated regression coefficients $\gamma_i^{(j)}$. Let $y_{i,t}^{(j+1)} = y_{i,t}^{(j)} - \gamma_i^{(j)} \hat{f}_{j+1,t}$. Set $j = j + 1$. If $j > r^{\max}$ stop else go to step 2.

This algorithm will be referred to as ME (Maximum Eigenvalue). We denote the true number of factors by r^0 . Note that the algorithm is equivalent to one where, in Step 4, $y_{i,t}^{(j)}$ is regressed on $\hat{f}_{1,t}, \dots, \hat{f}_{j+1,t}$ instead.

Then, we have the following theorem.

Theorem 1 *Under assumptions 1-5, and as $N, T \rightarrow \infty$, \hat{r} converges in probability to r^0 .*

The proof for this theorem is given in the Appendix.

4 Extensions

The result presented in the previous section is relevant for strict factor models where the idiosyncratic component is i.i.d. across both N and T . As discussed in the introduction, this setup is quite restrictive and given the results of Bai and Ng (2002) not very satisfactory. This section provides some extensions by relaxing some of the assumptions made in the previous section. In particular, we relax assumption 4 regarding temporal and cross sectional dependence in the idiosyncratic component and instead assume that

Assumption 7 $\epsilon_t = (\epsilon_{i,1}, \dots, \epsilon_{i,T})' = T_N^{1/2} \nu_t$ where ν_t is an $N \times 1$ vector of cross sectionally independently distributed and martingale difference random variables, $E(\nu_{i,t}) = 0$, $E(\nu_{i,t}^2) = 1$, $E(\nu_{i,t}^8) < \infty$ and $T_N = [\sigma_{i,j}]$. $\sigma_{i,i} = 1$ and $\sigma_{i,j}$, $i - j < M$, $M < \infty$, are i.i.d. random variables with $E(\sigma_{i,j}) = 0$ such that the elements of the square root factorisation of T_N are bounded between $-\frac{1}{M^\beta}$ and $\frac{1}{M^\beta}$, where $\beta > 1 - \frac{1}{\ln M}$ and zero otherwise.

This assumption allows for non-diagonal covariances matrices for the idiosyncratic component as long as the covariances are on average zero. In fact, M can tend to infinity with N as long as it allows $E(\epsilon_t^8) < \infty$. The assumption may look complicated as far as the bounds on the elements of the square root factorisation of T_N are concerned but due to the fact that both $\nu_{i,t}$ and $\epsilon_{i,t}$ have unit variance (the $\epsilon_{i,t}$ do so by construction) these elements would be of order $\frac{1}{M}$ anyway. The particular specification of the bounds is needed for the particular version of the proof of the theorem we choose to adapt from the literature. Then, we have

Theorem 2 *Under assumptions 1-3 and 5-7, and as $N, T \rightarrow \infty$, \hat{r} converges in probability to r^0 .*

A variety of other specifications for $\epsilon_{i,t}$ may be constructed such that results analogous to Theorem 2 may be obtained. Clearly, assumption 7 is more restrictive than assumption C of Bai and Ng (2002), in the sense that it only allows for significantly more restrictive forms

of temporal dependence. On the other hand, assumption *refass7* and assumption C of Bai and Ng (2002) are comparable in terms of cross-sectional dependence. Note that assumption C of Bai and Ng (2002) is quite restrictive in allowing only for sparse covariance matrices for the idiosyncratic component. The assumption of zero average cross-sectional covariance is not far-fetched as any global cross sectional covariance characteristics of the dataset are likely to be captured by the factor structure.

Another extension which seems possible but will not be pursued rigorously in this paper, is the case $c \rightarrow \infty$. The results of Yin, Bai, and Krishnaiah (1988) will go through in this case as well and it is easily seen that μ_{max} will tend to infinity at rate c . This rate is clearly lower than N which is the rate at which μ_{max} tends to infinity when a factor structure is present. So the method should in principle work in this context as well. Note further that the only case where one cannot distinguish the factor structure from the no factor structure is if T is finite, where in any case none of the results go through.

Finally, we note that the temporal dependence structure allowed by Assumption 7 should in principal be relaxable to a very large extent. We know that the asymptotic properties of covariance matrix estimates are not different in the case of weakly dependent data compared to independent data. This should imply that the results reported here could be extended to general forms of weak temporal dependence. The Monte Carlo study reported in the next section supports this. However, it is also clear that the method of proof used in the appendix cannot be of use. Recent work by Bai and Silverstein (1998) suggests alternative avenues for proofs based on Stieljes transforms that might be of use in this respect. This is clearly an interesting topic for future research, not only aimed at improving factor analysis but at augmenting the large literature on the theoretical properties of large dimensional sample covariance matrices.

5 Monte Carlo Study

5.1 Monte Carlo Setup

In this section we provide a detailed Monte Carlo study of the new number of factors estimator compared with the information criteria suggested by Bai and Ng (2002). We consider two sets of experiments. The set of experiments A considers strict factor models, whereas the set of experiments B considers approximate factor models.

The general model we consider has many similarities with Bai and Ng (2002) and is given

by

$$y_{i,t} = \sum_{j=1}^{r^0} \lambda_{j,i} f_{j,t} + \epsilon_{i,t}, \quad i = 1, \dots, N, \quad t = 1, \dots, T$$

$$\epsilon_t = \Sigma^{1/2} \nu_t$$

$$\nu_{i,t} = \rho_i \nu_{i,t} + \xi_{i,t}$$

We set $f_{j,t} \sim N(0, 1)$, $\xi_{i,t} \sim N(0, \theta r^0)$, $\lambda_{j,i} \sim N(0, 1)$, $N = 50, 100, 200$, $T = 50, 100, 200, 500, 1000$. The choice of N and T reflects the characteristics of the datasets used in empirical analysis in conjunction with factor models. For example, the best known dataset used for factor analysis is that in Stock and Watson (2002) where $N = 147$ and $T = 478$. Financial datasets as well are covered here. For example, see the work of Cipollini and Kapetanios (2004) where the volatility characteristics of the S&P 100 with about 2000 observations are analysed via a factor model.

One of the most important determinants of the performance of the number of factor estimators is the proportion of variance explained by the factors. This is controlled by θ . So for $\theta = 1$, R^2 is 0.5 whereas for $\theta = 9$, R^2 is 0.10. Evidence seems to suggest that in many datasets this R^2 is quite low. Hence, it is crucial that any method works well in these circumstances. We consider $\theta = 1, 9, 19$ leading to R^2 of 0.5, 0.1 and 0.05. The latter value may seem extreme but it will provide an envelope for the performance of the methods for most circumstances. Also we consider $r^0 = 1, 5$ and $r^{max} = 8$. For Experiments A, $\Sigma = I$, $\rho_i = 0$. For experiments B, $\Sigma = [\sigma_{i,j}]$, $\sigma_{i,i} = 1$, $\sigma_{i,j} = \sigma_{j,i} \sim U(-0.15, 0.15)$ for $|i-j| \leq 5$ and $\rho_i = 0.2$.

So the approximate factor models allow for considerable cross-sectional dependence and moderate temporal dependence even though the asymptotic analysis does not cover the serial correlation case we consider. We also consider another set of experiments referred to as experiments C where the setup is as before but now the model is given by

$$y_{i,t} = \sum_{j=1}^2 \lambda_{j,i} f_{j,t} + \sum_{j=1}^{r^0} \lambda_{2,j,i} f_{j,t-1} + \epsilon_{i,t}, \quad i = 1, \dots, N, \quad t = 1, \dots, T$$

$\lambda_{2,j,i} \sim N(0, 1)$. This is a model akin to that suggested by Forni, Hallin, Lippi, and Reichlin (2000), where lags of the factor enter the model. Clearly, the number of static factors for this model is $2 \times r^0$. Experiment C(A) is a strict factor model whereas experiment C(B) is an approximate factor model.

We compare the new method with the information criteria suggested by Bai and Ng (2002). These criteria, which are minimised over r , are given below

$$PC_1(r) = V_r + r\hat{\sigma}^2 \left(\frac{N+T}{NT} \right) \ln \left(\frac{NT}{N+T} \right)$$

$$PC_2(r) = V_r + r\hat{\sigma}^2 \left(\frac{N+T}{NT} \right) \ln C_{NT}^2$$

$$PC_3(r) = V_r + r\hat{\sigma}^2 \left(\frac{\ln C_{NT}^2}{C_{NT}^2} \right)$$

$$IC_1(r) = V_r + r \left(\frac{N+T}{NT} \right) \ln \left(\frac{NT}{N+T} \right)$$

$$IC_2(r) = V_r + r \left(\frac{N+T}{NT} \right) \ln C_{NT}^2$$

$$IC_3(r) = V_r + r \left(\frac{\ln C_{NT}^2}{C_{NT}^2} \right)$$

where

$$V_r = (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \left(y_{i-t} - \sum_{j=1}^r \hat{\lambda}_{j,i} \hat{f}_{j,t} \right)^2$$

and $\hat{\sigma}^2 = V_{r_{max}}$. Note that we choose to start the search at $r = 0$ for the ME algorithm but at $r = 1$ for the information criteria. The choice for the information criteria is motivated by comparability reasons with the results of Bai and Ng (2002). However, such a search does not address the very interesting problem of whether a given dataset supports a factor structure at all. Assuming the presence of at least one factor does not really seem as innocuous as usually presumed in the literature. Hence, the task for the ME algorithm is more difficult.

5.2 Monte Carlo Results

Tables 1-6 report the average selected number of factors over 1000 replications. The MSE of the number of factors estimator over 1000 replications, is reported in parentheses. Results make interesting reading. We start with results in Table 1. The setup here is one where $r^0 = 5$. For $\theta = 1$ all methods do well. In particular, the ME algorithm does particularly well for all experiments with $T \geq 100$ with the estimated number of factor, practically always, being chosen to be equal to 5. The information criteria do quite well too with some problems being encountered at $N = 50$ for all values of T .

Moving on to the case of $\theta = 9$ we see that the performance worsens for all methods. Looking at ME first we note that it has serious problems for $N = 50$ but otherwise works relatively well with impressive performance at $N = 200$ where the correct number of factors is chosen practically always. The information criteria do less well. All criteria apart from two

choose almost always one factor. Performance does not seem to improve for larger N or T . The only criteria which do marginally better are PC_3 and IC_3 . Even they underestimate the true number of factors considerably choosing in the best case about 3.5 factors on average. The performance for $\theta = 19$ worsens further. The ME algorithm still chooses about 3 factors for $N = 200$ and $T = 1000$. All information criteria choose invariably one factor.

Moving on to Table 2 we see that both ME and the information criteria work well. There is no underestimation for the criteria but there could not be any anyway as the true number of factors is equal to the minimum one being considered. The ME chooses sometimes fewer factors (< 1) but that is to be expected given that it allows for the possibility that no factor structure exists. Moving on to the experiments B relating to approximate factor models we get similar results. The performance of the ME does not seem to be affected. If anything it becomes slightly better. Again for $\theta = 5, 10$ ME outperforms the criteria overwhelmingly for the case of $r^0 = 5$. Finally, we consider experiments C. Again ME does better and the performance is not affected by cross sectional and temporal dependence. Surprisingly, the information criteria do better although still they perform worse than ME.

To conclude, ME seems to outperform the information criteria across a variety of Monte Carlo experiments. It seems insensitive to moderate cross sectional and temporal dependence. Importantly it seems less sensitive to low R^2 for the factor equations compared to the information criteria. Given that factors are likely to explain a relatively small average proportion of the variance of empirical datasets due to the extreme parsimony of the factor model such a property is highly prized. The performance of ME coupled with the ease of its implementation makes the method a reasonable alternative to information criteria.

5.3 Conclusions

Factor models for large datasets have gained much prominence in empirical and theoretical econometric work recently. Following on from the path breaking work of Stock and Watson (2002) a series of papers by Bai and Ng (Bai and Ng (2002), Bai (2003), Bai (2004)) have provided the theoretical foundations of static factor models for large datasets. Work in Forni, Hallin, Lippi, and Reichlin (2000) and other papers by these authors have provided an alternative explicitly dynamic approach to factor analysis. An important issue in this work is choosing the number of factors to be included in the factor model. The only rigorous method for doing this has been developed in an influential paper by Bai and Ng (2002) and uses information criteria.

This paper suggests a new method for this problem. The method is based on a remarkable fact concerning the behaviour of the eigenvalues of a large sample covariance matrix when no factor structure exists. In particular there exists a large literature on the fact that the maximum eigenvalue of such a covariance matrix tends to a constant asymptotically. Since the behaviour of the eigenvalues of the covariance matrix tend to infinity when a factor structure exists a method for distinguishing these two cases suggests itself. The paper develops rigorously this idea for a variety of settings.

Monte Carlo analysis indicates that the method works very well. In a majority of instances of empirical interest it outperforms information criteria methods. Thus, it provides a useful alternative to existing methods. Future research can concentrate on extending the applicability of the method to allow for stronger forms of temporal and especially cross-sectional dependence. Also, applications to empirical settings will provide a better test of its potential.

Appendix

Proof of Theorem 1

To prove the theorem, we need to show that Algorithm 1 will stop at step 3 when and only when $j = r^0$ in probability. Before continuing we note that Lemma 2.1 of Yin, Bai, and Krishnaiah (1988) also holds as $c \rightarrow 0$. Hence, the proof of Theorem 1 of Yin, Bai, and Krishnaiah (1988) although discussed for $0 < c < \infty$, is also valid for $c \rightarrow 0$. By Theorem 1 of Bai (2003) we know that \hat{f}_t is a $\min\{\sqrt{N}, T\}$ -consistent estimator for Pf_t where P is some nonsingular matrix. Without loss of generality we assume that $P = I$. By Theorem 2 of Bai (2003) we have that $\hat{\lambda}_i - \lambda_i$ is $O_p(\min\{N, \sqrt{T}\})$. Thus, we conclude that $\hat{\epsilon}_{i,t} - \epsilon_{i,t} = O_p(\min\{\sqrt{N}, \sqrt{T}\})$. These results hold for $r \geq r^0$. Hence, it follows that

$$1/T \sum_{t=1}^T \epsilon_{i,t} \epsilon_{j,t} - 1/T \sum_{t=1}^T \hat{\epsilon}_{i,t} \hat{\epsilon}_{j,t} = o_p(1) \quad (5)$$

uniformly over N . Since the maximum eigenvalue is a continuous function of the elements of the sample covariance matrix, it follows from (5), that $\mu_{max}^{(r)}$ converges in probability to the maximum eigenvalue of $1/T \epsilon' \epsilon$. Thus, by theorem 3.1 of Yin, Bai, and Krishnaiah (1988) we have that $\hat{\mu}_{max}^{(j)} < b$ for $j = r^0$ in probability as $N, T \rightarrow \infty$. This guarantees that $\hat{r} \leq r^0$ in probability.

We next need to show $\hat{r} \geq r^0 - 1$ in probability. Let $\lambda_i = (\lambda_{1,i}, \dots, \lambda_{r^0,i})'$. For all $1 \leq j < r^0$ we have that asymptotically

$$y_{i,t}^{(j)} = \lambda_{j+1,i} f_{j+1,t} + \dots + \lambda_{r^0,i} f_{r^0,t} + \epsilon_{i,t} \quad (6)$$

Note that this is the case since $\hat{\lambda}_{j,i}$, $j < r^0$ enjoys the asymptotic properties given in the previous paragraph even if estimated from a regression with fewer than r^0 factors, since every estimated factor is orthogonal with all the other estimated factors. Thus, the population covariance matrix of $y_{i,t}^{(j)}$ is given by

$$\Lambda^{(j+1)'} \Sigma^{(j+1)} \Lambda^{(j+1)} + \Sigma_\epsilon \quad (7)$$

where $\Lambda^{(j+1)} = (\lambda_1^{(j+1)}, \dots, \lambda_N^{(j+1)})$, $\lambda_i^{(j+1)} = (\lambda_{j+1,i}, \lambda_{r^0,i})'$ and $\Sigma^{(j+1)}$ is the covariance matrix of $f_{j+1,t}, \dots, f_{r^0,t}$. The maximum eigenvalue of the covariance matrix in (7) tends to infinity at rate N as $N \rightarrow \infty$. Hence, by Corollary 1 of Bai and Silverstein (1998), the maximum eigenvalue of the sample covariance matrix of $y_{i,t}^{(j)}$ will exceed b almost surely and thus in probability. Hence, the result is proven.

Proof of Theorem 2

We have to prove that Theorem 3.1 of Yin, Bai, and Krishnaiah (1988) holds under assumption 7. To do this it suffices to show that

$$\sum_{N=1}^{\infty} E \left(\frac{\mu_{max}(N)}{z} \right)^{k_N} < \infty$$

where $k_N = k$ is some sequence tending to infinity as $N \rightarrow \infty$ for all $z > (1 + \sqrt{c})^2$. Note that

$$\begin{aligned} E(\mu_{max}(N))^k &\leq Etr \left(\frac{1}{N} \epsilon' \epsilon \right)^k = N^{-k} Etr(\epsilon' \epsilon) = \\ &N^{-k} \sum E \epsilon_{i_1, j_1} \epsilon_{i_2, j_1} \epsilon_{i_2, j_2} \dots \epsilon_{i_k, j_k} \epsilon_{i_1, j_k} \end{aligned}$$

This summation is taken in such a way that i_1, \dots, i_k run over all integers in $\{1, \dots, N\}$ and j_1, \dots, j_k run over all integers in $\{1, \dots, T\}$. To show that Theorem 3.1 of Yin, Bai, and Krishnaiah (1988) holds it suffices to show firstly that no term of the above summation which is zero under assumption 4 is nonzero under assumption 7 and secondly that all terms which are nonzero are bounded by the same bound in absolute value under assumption 7 and under assumption 4. The first part follows from assumption 7 which restricts all autocovariances to be zero and all cross sectional covariances to be, on average, zero. To examine the second part we note that the bound used in Yin, Bai, and Krishnaiah (1988) is

$$E|\epsilon_{i_1, j_1} \epsilon_{i_2, j_1} \epsilon_{i_2, j_2} \dots \epsilon_{i_k, j_k}| < k^t (\delta \sqrt{T})^{2k-2l-t}$$

for some $t \geq 1$ and $l \geq 1$ such that $2k - 2l - t > 0$. We further note that, under assumption 7, for all i, j , $\epsilon_{i,t}$ is of the form

$$\epsilon_{i,t} = \sum_{s=0}^M \tilde{\sigma}_s \nu_{i+s,t}$$

where $\tilde{\sigma}_s$ are random variables independent of $\nu_{i,t}$ and bounded between -1 and 1. This bound arises since the data are normalised to have unit variance. Thus $E|\epsilon_{i_1, j_1} \epsilon_{i_2, j_1} \epsilon_{i_2, j_2} \dots \epsilon_{i_k, j_k}|$ is bounded by

$$M^k \left(\frac{1}{M^\beta} \right)^k E|\nu_{i_1, j_1} \nu_{i_2, j_1} \nu_{i_2, j_2} \dots \nu_{i_k, j_k}|$$

It is permissible following Yin, Bai, and Krishnaiah (1988) to have $k = \log T^{1+\alpha}$, for some $\alpha > 0$. Hence, $M^k \left(\frac{1}{M^\beta} \right)^k \sim T^\zeta$, $\zeta < 1$. Since $E(\nu_{i,t}^8) < \infty$, by simply modifying assumptions (4) and (5) of Yin, Bai, and Krishnaiah (1988) from

$$E(\nu_{i,t})^l \leq (\delta \sqrt{T})^{l-1}, l \geq 2, \quad E(\nu_{i,t})^l \leq c(\delta \sqrt{T})^{l-3}, l \geq 3,$$

to

$$E(\nu_{i,t})^l \leq (\delta \sqrt{T})^{l-3}, l \geq 8, \quad E(\nu_{i,t})^l \leq c(\delta \sqrt{T})^{l-5}, l \geq 8,$$

gives the required result.

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Table 1. Experiment A ($r^0 = 5$)								
N	T	ME	PC_1	PC_2	PC_3	IC_1	IC_2	IC_3
$\theta = 1$								
50	50	4.89 _(0.10)	5.03 _(0.04)	4.87 _(0.11)	8.00 _(0.00)	4.54 _(0.40)	3.13 _(1.48)	7.99 _(0.02)
100	50	5.00 _(0.00)	5.00 _(0.00)	4.99 _(0.01)	5.26 _(0.19)	4.93 _(0.07)	4.74 _(0.24)	5.00 _(0.00)
200	50	5.03 _(0.04)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	4.99 _(0.01)	4.97 _(0.02)	5.00 _(0.00)
50	100	5.00 _(0.00)	5.00 _(0.00)	4.99 _(0.01)	5.50 _(0.31)	4.92 _(0.08)	4.73 _(0.26)	5.00 _(0.00)
100	100	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	6.23 _(0.40)	5.00 _(0.00)	4.99 _(0.01)	5.13 _(0.14)
200	100	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
50	200	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	4.99 _(0.01)	4.97 _(0.03)	5.00 _(0.00)
100	200	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
200	200	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
50	500	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
100	500	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
200	500	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
50	1000	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
100	1000	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
200	1000	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
$\theta = 9$								
50	50	0.01 _(0.00)	1.84 _(0.42)	1.02 _(0.02)	7.92 _(0.08)	1.00 _(0.00)	1.00 _(0.00)	4.72 _(8.73)
100	50	0.23 _(0.20)	1.02 _(0.02)	1.00 _(0.00)	3.18 _(0.47)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	50	2.09 _(0.98)	1.00 _(0.00)	1.00 _(0.00)	1.15 _(0.13)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	100	0.01 _(0.01)	1.02 _(0.02)	1.00 _(0.00)	3.08 _(0.41)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	100	0.61 _(0.43)	1.00 _(0.00)	1.00 _(0.00)	4.67 _(0.36)	1.00 _(0.00)	1.00 _(0.00)	1.98 _(0.67)
200	100	3.50 _(0.67)	1.00 _(0.00)	1.00 _(0.00)	2.17 _(0.41)	1.00 _(0.00)	1.00 _(0.00)	1.01 _(0.01)
50	200	0.02 _(0.02)	1.00 _(0.00)	1.00 _(0.00)	1.11 _(0.10)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	200	1.48 _(0.60)	1.00 _(0.00)	1.00 _(0.00)	2.15 _(0.45)	1.00 _(0.00)	1.00 _(0.00)	1.01 _(0.01)
200	200	4.74 _(0.20)	1.00 _(0.00)	1.00 _(0.00)	4.88 _(0.10)	1.00 _(0.00)	1.00 _(0.00)	4.45 _(0.36)
50	500	0.10 _(0.09)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	500	3.08 _(0.68)	1.00 _(0.00)	1.00 _(0.00)	1.07 _(0.07)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	500	5.00 _(0.00)	1.27 _(0.21)	1.03 _(0.03)	3.94 _(0.34)	1.00 _(0.00)	1.00 _(0.00)	3.08 _(0.65)
50	1000	0.29 _(0.23)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	1000	4.09 _(0.45)	1.00 _(0.00)	1.00 _(0.00)	1.01 _(0.01)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	1000	5.00 _(0.00)	1.82 _(0.41)	1.48 _(0.32)	3.47 _(0.39)	1.03 _(0.03)	1.00 _(0.00)	2.44 _(0.66)
$\theta = 19$								
50	50	0.00 _(0.00)	1.29 _(0.23)	1.00 _(0.00)	7.86 _(0.13)	1.00 _(0.00)	1.00 _(0.00)	2.38 _(5.91)
100	50	0.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	2.16 _(0.46)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	50	0.08 _(0.08)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	100	0.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	2.02 _(0.41)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	100	0.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	3.41 _(0.50)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	100	0.16 _(0.14)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	200	0.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	200	0.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	200	0.51 _(0.37)	1.00 _(0.00)	1.00 _(0.00)	1.70 _(0.37)	1.00 _(0.00)	1.00 _(0.00)	1.01 _(0.01)
50	500	0.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	500	0.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	500	1.87 _(0.58)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	1000	0.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	1000	0.02 _(0.02)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	1000	3.38 _(0.53)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)

Table 3. Experiment B ($r^0 = 5$)

N	T	ME	PC_1	PC_2	PC_3	IC_1	IC_2	IC_3
$\theta = 1$								
50	50	4.94 _(0.13)	5.32 _(0.23)	4.91 _(0.10)	8.00 _(0.00)	4.54 _(0.43)	3.00 _(1.41)	8.00 _(0.00)
100	50	5.33 _(0.39)	5.00 _(0.00)	4.99 _(0.01)	6.23 _(0.38)	4.88 _(0.12)	4.63 _(0.36)	5.06 _(0.06)
200	50	7.61 _(0.65)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	4.98 _(0.02)	4.95 _(0.05)	5.00 _(0.00)
50	100	5.00 _(0.01)	5.00 _(0.00)	4.99 _(0.01)	6.39 _(0.38)	4.90 _(0.10)	4.66 _(0.32)	5.12 _(0.12)
100	100	5.01 _(0.01)	5.00 _(0.00)	5.00 _(0.00)	7.66 _(0.25)	5.00 _(0.00)	4.99 _(0.01)	6.89 _(1.13)
200	100	5.20 _(0.24)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
50	200	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	4.98 _(0.02)	4.96 _(0.04)	5.00 _(0.00)
100	200	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
200	200	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.11 _(0.09)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
50	500	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
100	500	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
200	500	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
50	1000	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
100	1000	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
200	1000	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)	5.00 _(0.00)
$\theta = 9$								
50	50	0.02 _(0.02)	2.76 _(0.50)	1.18 _(0.16)	8.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	7.76 _(1.12)
100	50	0.59 _(0.51)	1.19 _(0.16)	1.01 _(0.01)	4.43 _(0.47)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	50	4.43 _(3.88)	1.01 _(0.01)	1.00 _(0.00)	1.50 _(0.31)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	100	0.02 _(0.02)	1.14 _(0.13)	1.00 _(0.00)	4.29 _(0.46)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	100	0.84 _(0.55)	1.00 _(0.00)	1.00 _(0.00)	6.19 _(0.46)	1.00 _(0.00)	1.00 _(0.00)	2.67 _(1.27)
200	100	4.13 _(0.72)	1.00 _(0.00)	1.00 _(0.00)	2.44 _(0.41)	1.00 _(0.00)	1.00 _(0.00)	1.01 _(0.01)
50	200	0.04 _(0.04)	1.00 _(0.00)	1.00 _(0.00)	1.36 _(0.26)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	200	1.56 _(0.64)	1.00 _(0.00)	1.00 _(0.00)	2.33 _(0.42)	1.00 _(0.00)	1.00 _(0.00)	1.01 _(0.01)
200	200	4.77 _(0.20)	1.00 _(0.00)	1.00 _(0.00)	4.92 _(0.08)	1.00 _(0.00)	1.00 _(0.00)	4.46 _(0.34)
50	500	0.11 _(0.10)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	500	3.06 _(0.68)	1.00 _(0.00)	1.00 _(0.00)	1.06 _(0.05)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	500	5.00 _(0.00)	1.18 _(0.15)	1.02 _(0.02)	3.84 _(0.36)	1.00 _(0.00)	1.00 _(0.00)	2.81 _(0.65)
50	1000	0.30 _(0.26)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	1000	4.09 _(0.50)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	1000	5.00 _(0.00)	1.62 _(0.33)	1.29 _(0.21)	3.24 _(0.38)	1.01 _(0.01)	1.00 _(0.00)	2.07 _(0.63)
$\theta = 19$								
50	50	0.00 _(0.00)	2.46 _(0.49)	1.05 _(0.05)	7.99 _(0.01)	1.00 _(0.00)	1.00 _(0.00)	7.51 _(2.33)
100	50	0.08 _(0.08)	1.00 _(0.00)	1.00 _(0.00)	3.93 _(0.50)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	50	1.23 _(1.41)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	100	0.00 _(0.00)	1.01 _(0.01)	1.00 _(0.00)	3.64 _(0.52)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	100	0.01 _(0.01)	1.00 _(0.00)	1.00 _(0.00)	5.42 _(0.53)	1.00 _(0.00)	1.00 _(0.00)	1.06 _(0.09)
200	100	0.59 _(0.46)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	200	0.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	200	0.01 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	200	0.79 _(0.47)	1.00 _(0.00)	1.00 _(0.00)	2.35 _(0.38)	1.00 _(0.00)	1.00 _(0.00)	1.01 _(0.01)
50	500	0.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	500	0.01 _(0.01)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	500	2.00 _(0.61)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	1000	0.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	1000	0.03 _(0.03)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	1000	3.42 _(0.56)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)

Table 4. Experiment B ($r^0 = 1$)

N	T	ME	PC_1	PC_2	PC_3	IC_1	IC_2	IC_3
$\theta = 1$								
50	50	1.00 _(0.00)	3.99 _(0.53)	2.11 _(0.42)	8.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	8.00 _(0.00)
100	50	1.08 _(0.08)	1.57 _(0.33)	1.06 _(0.05)	5.22 _(0.52)	1.00 _(0.00)	1.00 _(0.00)	1.02 _(0.02)
200	50	2.34 _(2.42)	1.00 _(0.00)	1.00 _(0.00)	1.28 _(0.21)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	100	1.00 _(0.00)	1.97 _(0.45)	1.20 _(0.18)	5.84 _(0.66)	1.00 _(0.00)	1.00 _(0.00)	1.07 _(0.09)
100	100	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	7.23 _(0.49)	1.00 _(0.00)	1.00 _(0.00)	3.33 _(4.13)
200	100	1.04 _(0.05)	1.00 _(0.00)	1.00 _(0.00)	1.07 _(0.07)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	200	1.00 _(0.00)	1.05 _(0.05)	1.01 _(0.01)	2.14 _(0.60)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	200	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.32 _(0.24)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	200	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	2.83 _(0.54)	1.00 _(0.00)	1.00 _(0.00)	1.01 _(0.01)
50	500	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	500	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	500	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	1000	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	1000	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	1000	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
$\theta = 9$								
50	50	0.89 _(0.10)	2.81 _(0.49)	1.28 _(0.22)	8.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	7.72 _(1.29)
100	50	1.07 _(0.07)	1.10 _(0.09)	1.00 _(0.00)	4.22 _(0.47)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	50	2.10 _(1.62)	1.00 _(0.00)	1.00 _(0.00)	1.04 _(0.04)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	100	0.98 _(0.01)	1.05 _(0.05)	1.00 _(0.00)	4.04 _(0.50)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	100	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	5.49 _(0.44)	1.00 _(0.00)	1.00 _(0.00)	1.18 _(0.18)
200	100	1.04 _(0.04)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	200	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.01 _(0.01)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	200	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	200	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.15 _(0.13)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	500	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	500	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	500	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	1000	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	1000	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	1000	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
$\theta = 19$								
50	50	0.22 _(0.18)	2.67 _(0.49)	1.16 _(0.14)	8.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	7.74 _(1.14)
100	50	0.90 _(0.20)	1.07 _(0.06)	1.00 _(0.00)	4.06 _(0.48)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	50	2.03 _(1.33)	1.00 _(0.00)	1.00 _(0.00)	1.02 _(0.02)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	100	0.39 _(0.24)	1.02 _(0.02)	1.00 _(0.00)	3.86 _(0.48)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	100	1.00 _(0.01)	1.00 _(0.00)	1.00 _(0.00)	5.39 _(0.51)	1.00 _(0.00)	1.00 _(0.00)	1.14 _(0.16)
200	100	1.04 _(0.04)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	200	0.69 _(0.21)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	200	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	200	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.12 _(0.11)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	500	0.92 _(0.07)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	500	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	500	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	1000	0.98 _(0.02)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	1000	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	1000	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)

Table 5. Experiment C(A) ($r^0 = 2$)

N	T	ME	PC_1	PC_2	PC_3	IC_1	IC_2	IC_3
$\theta = 1$								
50	50	4.00 _(0.00)	4.64 _(0.35)	4.02 _(0.02)	8.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	8.00 _(0.00)
100	50	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	5.27 _(0.41)	4.00 _(0.00)	4.00 _(0.00)	4.01 _(0.01)
200	50	4.01 _(0.01)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
50	100	4.00 _(0.00)	4.04 _(0.04)	4.00 _(0.00)	6.13 _(0.56)	4.00 _(0.00)	4.00 _(0.00)	4.16 _(0.21)
100	100	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	7.00 _(0.48)	4.00 _(0.00)	4.00 _(0.00)	5.10 _(1.32)
200	100	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
50	200	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.22 _(0.19)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
100	200	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.01 _(0.01)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
200	200	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.08 _(0.08)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
50	500	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
100	500	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
200	500	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
50	1000	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
100	1000	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
200	1000	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
$\theta = 5$								
50	50	0.76 _(0.61)	3.10 _(0.40)	1.89 _(0.41)	7.97 _(0.03)	1.00 _(0.00)	1.00 _(0.00)	7.02 _(2.81)
100	50	2.96 _(0.75)	2.62 _(0.48)	1.98 _(0.45)	4.08 _(0.18)	1.01 _(0.01)	1.00 _(0.00)	2.21 _(0.87)
200	50	3.97 _(0.05)	2.48 _(0.48)	2.12 _(0.47)	3.40 _(0.39)	1.04 _(0.04)	1.01 _(0.01)	1.51 _(0.49)
50	100	1.55 _(0.79)	2.55 _(0.41)	1.92 _(0.41)	4.05 _(0.15)	1.00 _(0.00)	1.00 _(0.00)	2.15 _(0.76)
100	100	3.82 _(0.17)	2.79 _(0.46)	1.81 _(0.39)	4.71 _(0.35)	1.14 _(0.14)	1.00 _(0.00)	3.98 _(0.02)
200	100	4.00 _(0.00)	3.30 _(0.41)	2.78 _(0.47)	3.99 _(0.01)	1.90 _(0.72)	1.28 _(0.29)	3.90 _(0.09)
50	200	2.53 _(0.73)	2.46 _(0.42)	2.08 _(0.42)	3.35 _(0.33)	1.01 _(0.01)	1.00 _(0.00)	1.46 _(0.43)
100	200	4.00 _(0.00)	3.25 _(0.37)	2.72 _(0.40)	4.00 _(0.00)	1.84 _(0.64)	1.23 _(0.23)	3.90 _(0.09)
200	200	4.00 _(0.00)	3.96 _(0.04)	3.61 _(0.27)	4.00 _(0.00)	3.75 _(0.22)	2.76 _(0.74)	4.00 _(0.00)
50	500	3.45 _(0.40)	2.42 _(0.38)	2.29 _(0.39)	2.82 _(0.38)	1.04 _(0.04)	1.02 _(0.02)	1.19 _(0.18)
100	500	4.00 _(0.00)	3.71 _(0.21)	3.54 _(0.29)	3.97 _(0.03)	2.99 _(0.65)	2.61 _(0.72)	3.82 _(0.16)
200	500	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
50	1000	3.77 _(0.20)	2.39 _(0.43)	2.32 _(0.43)	2.60 _(0.43)	1.05 _(0.06)	1.04 _(0.04)	1.12 _(0.13)
100	1000	4.00 _(0.00)	3.87 _(0.12)	3.82 _(0.15)	3.96 _(0.04)	3.41 _(0.43)	3.24 _(0.52)	3.81 _(0.17)
200	1000	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
$\theta = 10$								
50	50	0.01 _(0.01)	1.92 _(0.43)	1.03 _(0.03)	7.93 _(0.06)	1.00 _(0.00)	1.00 _(0.00)	4.75 _(8.46)
100	50	0.43 _(0.38)	1.06 _(0.06)	1.00 _(0.00)	3.19 _(0.42)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	50	2.41 _(0.94)	1.01 _(0.01)	1.00 _(0.00)	1.33 _(0.26)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	100	0.03 _(0.03)	1.05 _(0.05)	1.00 _(0.00)	3.07 _(0.42)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	100	0.99 _(0.63)	1.01 _(0.01)	1.00 _(0.00)	4.41 _(0.34)	1.00 _(0.00)	1.00 _(0.00)	2.36 _(0.80)
200	100	3.51 _(0.41)	1.01 _(0.01)	1.00 _(0.00)	2.48 _(0.47)	1.00 _(0.00)	1.00 _(0.00)	1.13 _(0.13)
50	200	0.08 _(0.07)	1.00 _(0.00)	1.00 _(0.00)	1.26 _(0.20)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	200	2.11 _(0.71)	1.00 _(0.00)	1.00 _(0.00)	2.48 _(0.42)	1.00 _(0.00)	1.00 _(0.00)	1.12 _(0.11)
200	200	3.98 _(0.02)	1.12 _(0.12)	1.00 _(0.00)	4.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	3.95 _(0.05)
50	500	0.30 _(0.24)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	500	3.40 _(0.39)	1.01 _(0.01)	1.00 _(0.00)	1.37 _(0.26)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	500	4.00 _(0.00)	1.95 _(0.39)	1.43 _(0.30)	3.90 _(0.09)	1.11 _(0.10)	1.01 _(0.01)	3.60 _(0.32)
50	1000	0.65 _(0.43)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	1000	3.87 _(0.12)	1.01 _(0.01)	1.00 _(0.00)	1.11 _(0.10)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	1000	4.00 _(0.00)	2.70 _(0.41)	2.36 _(0.42)	3.76 _(0.19)	1.72 _(0.52)	1.38 _(0.34)	3.40 _(0.38)

Table 6. Experiment C(B) ($r^0 = 2$)

N	T	ME	PC_1	PC_2	PC_3	IC_1	IC_2	IC_3
$\theta = 1$								
50	50	4.03 _(0.04)	5.37 _(0.39)	4.24 _(0.19)	8.00 _(0.00)	4.01 _(0.01)	3.99 _(0.01)	8.00 _(0.00)
100	50	4.34 _(0.46)	4.08 _(0.07)	4.00 _(0.00)	6.42 _(0.42)	4.00 _(0.00)	4.00 _(0.00)	4.23 _(0.27)
200	50	7.11 _(1.84)	4.00 _(0.00)	4.00 _(0.00)	4.02 _(0.02)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
50	100	4.00 _(0.00)	4.29 _(0.22)	4.03 _(0.03)	6.98 _(0.46)	4.00 _(0.00)	4.00 _(0.00)	4.79 _(1.06)
100	100	4.01 _(0.01)	4.00 _(0.00)	4.00 _(0.00)	7.93 _(0.07)	4.00 _(0.00)	4.00 _(0.00)	7.62 _(0.68)
200	100	4.16 _(0.20)	4.00 _(0.00)	4.00 _(0.00)	4.01 _(0.01)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
50	200	4.00 _(0.00)	4.03 _(0.03)	4.00 _(0.00)	4.71 _(0.40)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
100	200	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.21 _(0.17)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
200	200	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	5.39 _(0.43)	4.00 _(0.00)	4.00 _(0.00)	4.17 _(0.18)
50	500	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.03 _(0.03)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
100	500	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
200	500	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
50	1000	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
100	1000	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
200	1000	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
$\theta = 5$								
50	50	0.88 _(0.70)	3.68 _(0.45)	2.20 _(0.44)	8.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	7.98 _(0.07)
100	50	3.29 _(0.93)	2.82 _(0.43)	2.12 _(0.47)	5.22 _(0.40)	1.01 _(0.01)	1.00 _(0.00)	2.23 _(0.91)
200	50	6.41 _(2.29)	2.62 _(0.46)	2.26 _(0.46)	3.53 _(0.33)	1.03 _(0.03)	1.01 _(0.01)	1.49 _(0.49)
50	100	1.62 _(0.78)	2.75 _(0.41)	2.09 _(0.39)	5.02 _(0.39)	1.00 _(0.00)	1.00 _(0.00)	2.21 _(0.80)
100	100	3.83 _(0.17)	2.84 _(0.42)	1.87 _(0.42)	6.56 _(0.43)	1.14 _(0.14)	1.00 _(0.00)	4.44 _(0.53)
200	100	4.13 _(0.13)	3.24 _(0.40)	2.68 _(0.44)	3.99 _(0.01)	1.71 _(0.63)	1.19 _(0.21)	3.87 _(0.12)
50	200	2.52 _(0.70)	2.54 _(0.40)	2.19 _(0.39)	3.48 _(0.29)	1.01 _(0.01)	1.00 _(0.00)	1.42 _(0.39)
100	200	4.00 _(0.00)	3.28 _(0.36)	2.75 _(0.40)	4.00 _(0.00)	1.67 _(0.57)	1.17 _(0.18)	3.90 _(0.09)
200	200	4.00 _(0.00)	3.96 _(0.04)	3.52 _(0.31)	4.00 _(0.00)	3.64 _(0.31)	2.50 _(0.82)	4.00 _(0.00)
50	500	3.46 _(0.39)	2.49 _(0.42)	2.34 _(0.40)	2.92 _(0.39)	1.03 _(0.03)	1.01 _(0.02)	1.13 _(0.14)
100	500	4.00 _(0.00)	3.68 _(0.23)	3.51 _(0.29)	3.97 _(0.03)	2.81 _(0.68)	2.38 _(0.73)	3.75 _(0.21)
200	500	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
50	1000	3.79 _(0.18)	2.48 _(0.42)	2.40 _(0.43)	2.67 _(0.42)	1.02 _(0.02)	1.02 _(0.02)	1.08 _(0.08)
100	1000	4.00 _(0.00)	3.80 _(0.17)	3.72 _(0.21)	3.94 _(0.05)	3.23 _(0.51)	3.05 _(0.60)	3.67 _(0.28)
200	1000	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)	4.00 _(0.00)
$\theta = 10$								
50	50	0.06 _(0.06)	2.83 _(0.47)	1.22 _(0.17)	8.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	7.83 _(0.69)
100	50	0.78 _(0.65)	1.27 _(0.22)	1.02 _(0.02)	4.50 _(0.49)	1.00 _(0.00)	1.00 _(0.00)	1.01 _(0.01)
200	50	4.28 _(3.75)	1.02 _(0.02)	1.00 _(0.00)	1.61 _(0.36)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
50	100	0.06 _(0.06)	1.21 _(0.17)	1.01 _(0.01)	4.33 _(0.50)	1.00 _(0.00)	1.00 _(0.00)	1.01 _(0.01)
100	100	1.21 _(0.71)	1.02 _(0.02)	1.00 _(0.00)	6.13 _(0.50)	1.00 _(0.00)	1.00 _(0.00)	2.89 _(1.12)
200	100	3.69 _(0.42)	1.01 _(0.01)	1.00 _(0.00)	2.63 _(0.43)	1.00 _(0.00)	1.00 _(0.00)	1.13 _(0.13)
50	200	0.10 _(0.09)	1.01 _(0.01)	1.00 _(0.00)	1.50 _(0.33)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	200	2.15 _(0.73)	1.01 _(0.01)	1.00 _(0.00)	2.63 _(0.41)	1.00 _(0.00)	1.00 _(0.00)	1.11 _(0.10)
200	200	3.99 _(0.01)	1.11 _(0.10)	1.00 _(0.00)	4.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	3.94 _(0.06)
50	500	0.34 _(0.27)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	500	3.38 _(0.41)	1.01 _(0.01)	1.00 _(0.00)	1.35 _(0.26)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	500	4.00 _(0.00)	1.85 _(0.42)	1.32 _(0.25)	3.86 _(0.12)	1.06 _(0.06)	1.01 _(0.01)	3.47 _(0.37)
50	1000	0.74 _(0.44)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
100	1000	3.86 _(0.13)	1.00 _(0.00)	1.00 _(0.00)	1.10 _(0.09)	1.00 _(0.00)	1.00 _(0.00)	1.00 _(0.00)
200	1000	4.00 _(0.00)	2.54 _(0.39)	2.17 _(0.41)	3.69 _(0.22)	1.46 _(0.36)	1.20 _(0.18)	3.20 _(0.47)

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