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Modelling the Yield Curve: A Two Components Approach *

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Abstract

Using parametric return autocorrelation tests and non parametric variance ratio statistics show that the UK and US short-term interest rates are unit root processes with significant mean reverting components. Congruent with this empirical evidence, we develop a new continuous time term structure model which assumes that the dynamics of the instantaneous interest rate are given by the joint effect of a (stationary) mean reverting component and a (nonstationary) martingale component. We provide a closed-form solution for the equilibrium yield curve when the temporary component is modelled as an Ornstein-Uhlenbeck process and the permanent component is modelled as an Arithmetic Brownian motion process.

Keywords: Term structure, mean reversion, random walk, brownian motion, variance ratio, linear regression.

JEL Classifications: C20, E43, G12

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1 Introduction

The issue of nonstationarity of the "short-term" interest rate is still puzzling the literature on term structure modelling. Whilst empirical evidence based on nearly all kinds of unit root tests overwhelmingly shows that an entire range of "short-term" interest rates appears nonstationary, in practice one of the most distinguished patterns of interest rates is that high (low) values of rates, in historical terms, tend to be followed by a decrease (increase) in rates more frequently than by an increase (decrease).

The important empirical question that we first address in this paper is how significant are the permanent and transitory components in an observed "short-term" interest rate series. Focusing on the UK and US markets, we apply both parametric return autocorrelation and non parametric variance ratio tests to ascertain statistically the size of the random walk/mean reverting components in our dataset.

It is widely accepted that the choice of a specific interest rate process is crucial for yield curve modelling, valuation of interest rate sensitive securities and general risk management. However, despite the plethora of equilibrium market single and multi-factor yield curve models, there has been, to the best of our knowledge, no previous attempt in the literature to solve for the equilibrium term structure which is compatible with joint martingale and mean reverting dynamics of the spot interest rate.[1]

In order to fill this void in the literature, in the theoretical part of the paper we provide a closed-form solution for the equilibrium yield curve generated by a two components model in which the state variable is the instantaneous interest rate and its dynamics are generated by the joint effect of stationary and nonstationary components. The stationary component, modelled using Vasicek's (1977) Gaussian term structure framework, induces temporary effects and hence mean reversion in the interest rate. The nonstationary component is identified by using Merton's (1973) random walk model, and induces permanent effects which account for the martingale behavior of the riskless rate.

Our term structure framework belongs in the affine class of yield curve models (Duffie and Kan (1996), Dai and Singleton (2000)), thus allowing the transformation of the unobserved state variable, i.e. the instantaneous interest rate, into a set of spot rates and its potential calibration to spot rates of chosen maturities.

The remaining of this paper is organized as follows. Section 2 tests the significance of the permanent and temporary components in the short-term interest rates of UK and US.

[1] Term structure models with nonlinear drift (for example, Ait-Sahalia (1996), Pfann et al. (1996), Stanton (1997)) are capable of producing mixed random walk and mean reverting patterns in the interest rate process. However, they imply nonlinear cross-sectional relations between the short rate and long yields. This makes it extremely difficult to derive closed form solutions for such models except for very special and empirically uninteresting cases.

Section 3 presents our continuous time term structure model. Section 4 concludes.

2 Testing for Random Walk and Stationary Components

In this section we investigate the significance of the permanent and the temporary components in the short term interest rate using both parametric and non parametric tests. As it is standard practice in related empirical work, an observed interest rate series will be used as a proxy of the latent variable, i.e. the instantaneous interest rate, that drives the entire term structure.

2.1 Description of the Data

We use monthly time series data for the UK and US 3-month end period Tbill rates.[2] (Source: Datastream.) The UK 435 monthly observations cover the period 1/1968-3/2004, while the US sample includes 387 observations from 1/1972 to 3/2004. Figures 1a and 1b present the plots of the UK and US 3-month Tbill rates.

Figure 1a
UK, 3 month T-bill rate, 1/1968 - 3/2004

Figure 1b
US, 3 month T-bill rate, 1/1972 - 3/2004

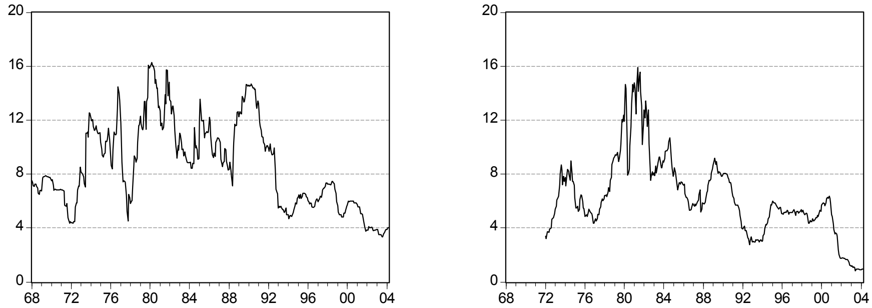

Table 1 below gives the Dickey-Fuller (DF), Phillips-Perron (PP), and Kwiatkowski-Phillips-Schmidt-Shin (KPSS) unit root test statistics for the two time series. The null hypothesis in the DF and PP tests is that the nominal interest rate is generated by an integrated of order one, I(1), process. The KPSS statistic tests the hypothesis that the nominal interest rate follows an I(0), i.e., a stationary stochastic process. The subscript in the reported statistics denotes the degree of augmentation of the parametric DF tests and

[2] See Duffee (1996) for an interesting discussion of alternative interest rate series.

the lag truncation parameter of the non parametric PP and KPSS tests.[3] The superscript c implies the use of a constant in the auxiliary regressions.[4] The p-values of the DF and PP statistics clearly show that the null of a unit root cannot be rejected at conventional significance levels.[5] In addition, the KPSS statistic rejects the null of stationarity at all conventional sizes of the test. Thus, application of the DF, PP, and KPSS unit root tests indicates that the UK and US 3-month Tbill rates can be treated as I(1), i.e. first difference stationary processes.

Table 1: Unit root tests

		UK 3-month Tbill rate		
$DF^c = -1.86$ (p-value=0.35)	$DF = -0.89$ (p-value=0.33)	$PP_6^c = -2.15$ (p-value=0.22)	$PP_5 = -0.95$ (p-value=0.31)	$KPSS_{16}^c = 0.77$ (1% critical value=0.74)
		US 3-month Tbill rate		
$DF_1^c = -2.02$ (p-value=0.28)	$DF_1 = -0.99$ (p-value=0.29)	$PP_6^c = -1.73$ (p-value=0.41)	$PP_8 = -0.86$ (p-value=0.34)	$KPSS_{15}^c = 1.01$ (1% critical value=0.74)

2.2 Variance Ratio Statistic

Cochrane (1988) uses the Beveridge and Nelson (1981) decomposition to express a first-difference stationary process (r_t) as the sum of (covariance) stationary (z_t) and random walk (q_t) components: $r_t = z_t + q_t$. So speaking, a measurement of the size of the random walk component can be a better guide to the proper statistical characterization of the series than a simple unit root test.

He proposes a non-parametric method for determining the magnitudes of the random walk and stationary components of a time series,[6] and argues that the Variance Ratio $\left(VR = \sigma^2_{\Delta q}/\sigma^2_{\Delta r}\right)$, i.e. the ratio of the variance of a change in the permanent component of the interest rate to the variance of the actual change, can be thought of as a measure of the

[3] Kwiatkowski, Phillips, Schmidt, and Shin (1992) developed their test for the null hypothesis that the observable series is stationary around a deterministic trend and only report critical values (c.v.) for the case of (i) a constant in the auxilliary regression: 1% c.v.=0.74, 2.5% c.v.=0.57, 5% c.v.=0.46, 10% c.v.=0.35, and (ii) both a constant and a trend: 1% c.v.=0.22, 2.5% c.v.=0.18, 5% c.v.=0.15, 10% c.v.=0.12.

[4] The argument for not using a constant in the tests is that if the nominal interest rate is I(1) with positive drift it will converge to infinity; this is rather unrealistic. On the other hand, if it is I(1) without positive drift it can take negative values with positive probability. This is also implausible, since the nominal interest rate is a positive variable. (See Bierens (1997).)

[5] These results are not sensitive to the order of augmentation (trucation lag) of the DF (PP) tests, or to the inclusion of a constant (and trend) in the auxilliary regressions.

[6] Cochrane's approach is based on the following argument. If the interest rate is adequately captured by a random walk model, i.e. $r_t = \mu + r_{t-1} + \varepsilon_t$, where $\varepsilon_t \sim iid\left(0, \sigma^2_\varepsilon\right)$, then $Var\left(r_t - r_{t-k}\right) = k\sigma^2_\varepsilon$. In other words, $\frac{1}{k}$ times the variance of the k-differences of r_t remains constant at σ^2_ε as k increases. If, on the other hand, the interest rate is (trend) stationary $\frac{1}{k}$ times the variance of the k-differences of r_t approaches zero as k increases (in this case it is easy to show that $Var\left(r_t - r_{t-k}\right) \to 2\sigma^2_r$, where σ^2_r is the unconditional variance of r_t).

relative significance of the random walk component in the series.[7] Cochrane (1988) proves that

$$\left(\frac{1}{k}\right) Var\left(r_t - r_{t-k}\right) = \left[1 + 2\sum_{j=1}^{k-1}\left(1 - \frac{j}{k}\right)\rho_j\right]\sigma^2_{\Delta r}, \tag{1}$$

where Δ is the first-difference operator, ρ_j is the jth autocorrelation coefficient of Δr_t, and the limit of the above is given by

$$\lim_{k\to\infty}\left[\frac{Var\left(r_t - r_{t-k}\right)}{k}\right] = \left(1 + 2\sum_{j=1}^{\infty}\rho_j\right)\sigma^2_{\Delta r} = \sigma^2_{\Delta q}. \tag{2}$$

Therefore, one way to estimate the Variance Ratio is by replacing the population values ρ_j in eq. (1) with the sample autocorrelations $\widehat{\rho}_j$ (see Huizinga (1987), and Campbell and Mankiw (1988)):

$$\widehat{VR}_k = 1 + 2\sum_{j=1}^{k-1}\left(1 - \frac{j}{k}\right)\widehat{\rho}_j. \tag{3}$$

Estimates of the Variance Ratio, $\widehat{VR}_k$, close to zero (one) indicate that the underlying stochastic process is stationary (a random walk). Values between zero and one indicate that the series contains both random walk and stationary components. Stated differently, there is evidence for mean reversion when $\widehat{VR}_k$ stabilizes below unity as the lag truncation parameter k increases. This implies that an increase in the level of the current interest rate will be reversed by decreases in the future.[8]

The Variance Ratio statistic in eq. (3) can be interpreted as the normalized Bartlett estimator of the spectral density at frequency zero, and thus its asymptotic standard error is given by

$$s.e.\left(\widehat{VR}_k\right) = \frac{\widehat{VR}_k}{\sqrt{\frac{3}{4}\left(\frac{T}{k}\right)}}, \tag{4}$$

where T is the number of observations.[9]

Figure 2 plots the Variance Ratio statistic for different values of the lag truncation parameter (order of the variance ratio) k.

[7] For example, suppose that the interest rate r_t follows an ARIMA(0,1,1) stochastic process with a moving average parameter equal to -0.6. In this case, it is easy to show that the permanent component accounts for only 12% of the actual change in r_t, in the long-run.

[8] $\widehat{VR}_k$ is a consistent estimate of VR when $\frac{k}{T} \to 0$ as $T \to \infty$. The choice of the lag truncation parameter k is usually made arbitrarily.

[9] See Priestley (1989), p.463.

Figure 2: Variance ratio statistic

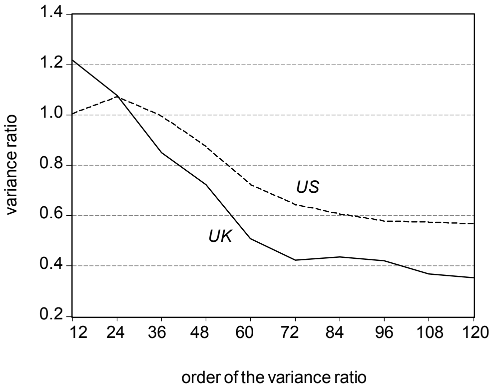

Table 2 presents the values of the Variance Ratio statistics and their standard errors in parentheses. It is evident that both the US and UK short term interest rate series have significant mean reverting components since the Variance Ratio Statistic stabilizes below unity. In particular, the variance of the change in the permanent component of the US nominal interest rate is roughly 60% of the variance of the monthly actual change in the nominal interest rate. For the UK interest rate series the magnitude of the Variance Ratio statistic falls to approximately 40%.

Table 2: Variance ratio statistics

UK 3-month TBill rate										
k	12	24	36	48	60	72	84	96	108	120
$\widehat{VR}_k$	1.22	1.08	0.85	0.72	0.51	0.42	0.44	0.42	0.37	0.35
	(0.23)	(0.29)	(0.28)	(0.28)	(0.22)	(0.20)	(0.22)	(0.23)	(0.21)	(0.21)
US 3-month TBill rate										
k	12	24	36	48	60	72	84	96	108	120
$\widehat{VR}_k$	1.00	1.07	0.99	0.87	0.72	0.64	0.61	0.58	0.57	0.57
	(0.20)	(0.31)	(0.35)	(0.36)	(0.33)	(0.32)	(0.33)	(0.33)	(0.35)	(0.36)

2.3 Slope Coefficients (Regression Analysis)

The size of the stationary component of an I(1) series can also be measured by a regression procedure (see Fama and French (1988), and Huizinga (1987)) which involves estimation of the serial correlation of interest rate changes over various horizons. In particular, we estimate the k-th autocorrelation of the change in the interest rate over k periods as follows:

$$\beta_k = \frac{Cov\left(r_{t+k} - r_t, r_t - r_{t-k}\right)}{Var\left(r_t - r_{t-k}\right)}. \tag{5}$$

Observe that β_k is the slope coefficient of the following regression:

$$r_{t+k} - r_t = \alpha_k + \beta_k (r_t - r_{t-k}) + \varepsilon_{k,t}. \tag{6}$$

The above equation (6) is estimated for various values of k. When β_k is zero then the behavior of the interest rate is consistent with that predicted by a random walk model. Negative (positive) values of β_k provide evidence for (against) mean reversion.

Consider the case where the random walk and stationary components of the nominal interest rate are given by

$$q_t = \mu + q_{t-1} + u_t \quad \text{and} \quad z_t = \phi z_{t-1} + \eta_t, \tag{7}$$

respectively, and where $|\phi| < 1$ and the error terms are white noise processes uncorrelated with one another. It is not difficult to show that

$$\beta_k = -\frac{\left(1-\phi^k\right)^2 Var(z_t)}{2\left(1-\phi^k\right) Var(z_t) + kVar(u_t)}. \tag{8}$$

By inspecting eq. (8) we can distinguish three possibilities: (i) in the absence of the temporary component the slope coefficients are zero ($\beta_k = 0$) for all k; (ii) in the absence of a permanent component $Var(u_t) = 0$, and so β_k is negative and approaches $-\frac{1}{2}$ as k increases; and (iii) in the presence of both permanent and stationary components β_k is negative but close to zero for small k, it moves towards -0.5 as k increases, and finally it gradually returns to zero for large k.

From the above discussion it is apparent that $\frac{Cov(z_{t+k}-z_t, z_t-z_{t-k})}{Var(z_t-z_{t-k})}$ approaches $-\frac{1}{2}$ as k goes to infinity. Using this limiting argument, eq. (5) can be expressed as

$$\frac{Var(z_{t+k} - z_t)}{Var(r_t - r_{t-k})} = -2\beta_k, \text{ for large } k. \tag{9}$$

As Fama and French (1988) note, since we do not observe the stationary component (z_t) ,we can infer its existence and properties from the behavior of the slope coefficient (β_k) in eq. (6). When nominal interest rates have both random walk and slowly decaying stationary components, the plot of β_k as a function of k might be U-shaped. The slope coefficient β_k is close to zero at short horizons (small k) as the slowly decaying stationary component does not allow mean reversion to manifest itself. As k increases, the temporary component begins to operate and pushes β_k to more negative values. The random walk component dominates in the long-run, and thus the slopes return to zero at long horizons ($\beta_k \to 0$ as $k \to 0$).

Figure 3 shows that when the slope coefficients of eq. (6) are plotted against k they form a U-shaped pattern.

Figure 3: Estimated slope coefficients

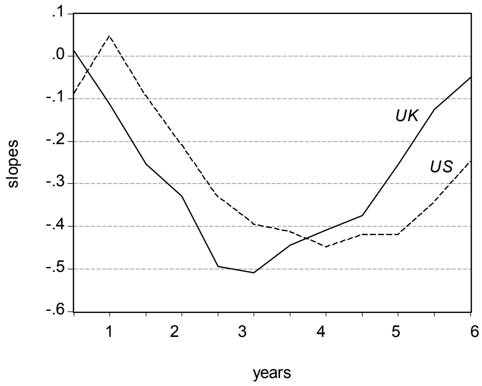

Table 3 reports the estimated slopes $\left(\widehat{\beta_k}\right)$ and their standard errors in parentheses.[10] The results indicate the existence of both temporary and random walk components. For $k = 60$ (i.e., over a period of 5 years) the regression slopes are -0.26 for the UK and -0.42 for the US. Therefore, following the limiting argument in eq. (9), we can infer that roughly 50% (80%) of the variance of a 5-year change in the UK (US) nominal short term interest rate is due to the stationary component of the series.

Table 3: Estimated slope coefficients

					UK 3-month TBill rate					
k	6	12	18	24	30	36	42	48	54	60
$\widehat{\beta}_k$	0.01 (0.10)	-0.11 (0.13)	-0.25 (0.11)	-0.33 (0.10)	-0.50 (0.10)	-0.51 (0.10)	-0.44 (0.10)	-0.41 (0.11)	-0.37 (0.09)	-0.26 (0.10)
					US 3-month TBill rate					
k	6	12	18	24	30	36	42	48	54	60
$\widehat{\beta}_k$	-0.09 (0.14)	0.05 (0.11)	-0.09 (0.11)	-0.21 (0.13)	-0.33 (0.13)	-0.40 (0.10)	-0.41 (0.09)	-0.45 (0.07)	-0.42 (0.07)	-0.42 (0.08)

Although in this paper we do not investigate the implications for the business cycle of monetary policies in the UK and US, it is worth pointing out that in our sample period there is empirical evidence that in the long-run (as documented by the variance ratio statistic) the US interest rate is more "random-walk" compared to the UK, whereas over a 5-year medium-term period (as highlighted by the returns autocorrelation tests) the US interest rate appears more stationary vis-a-vis the UK one. A bird's eye view points to different

[10] The standard error of β_k has been computed using the Newey and West (1987) covariance estimator that is consistent in the presence of autocorrelation and/or heteroskedasticity.

interest rate management by the Federal reserve Board and the Chancellor/Bank of England which could be valuable for the analysis of the real effects of interest rate policies in the UK and US during the last three decades or so. Meanwhile, for the specific purposes of this paper, our statistical investigation calls for the development of a two-components theoretical term structure model which addresses the empirical caveats of interest rate series.

3 A Simple Term Structure Model

Congruent with the empirical results of the previous section, the maintained hypothesis in our paper is that the state variable, i.e., the instantaneous interest rate is a difference stationary process in the spirit of Nelson and Plosser (1982). In a seminal paper, Beveridge and Nelson (1981) introduced a general procedure of decomposition, in discrete time, of non-stationary time series into permanent and transitory components. They showed that the permanent component is a random walk with drift and the transitory component is a stationary process with mean zero. This approach was used by Fama and French (1988) to model (log) stock prices as the sum of an autoregressive of order one (AR(1)) process and a random walk with drift. Our continuous time term structure model is similar in spirit to Fama and French's discrete-time framework.

Let $P(t,\tau)$ be the price as of calendar time t of a discount bond maturing at time $\tau = t + T, T \geq 0$, with unit maturity value, i.e., $P(\tau,\tau) = 1$. The yield-to-maturity, $R(t,\tau)$, at time t for a bond maturing at time τ can be defined, given $P(t,\tau)$, as the steady state at which the price should increase if the bond is to worth one currency unit at time τ. It then follows that:

$$R(t,\tau) = -\frac{1}{T}\log P(t,\tau). \tag{10}$$

The spot interest rate is defined as:

$$r_t = R(t,t). \tag{11}$$

As in Vasicek (1977), we proceed with the following assumptions.

Assumption A1: The spot interest rate r_t is modelled as the sum of a permanent (non-stationary) component q_t, and a temporary (stationary) component z_t which follow unobserved continuous Markov processes:

$$r_t = q_t + z_t, \tag{12}$$

where

$$dq_t = f^{(q)}(r,t)\,dt + \rho^{(q)}(r,t)\,dW^{(q)}, \tag{13}$$

$$dz_t = f^{(z)}(r,t)\,dt + \rho^{(z)}(r,t)\,dW^{(z)}. \tag{14}$$

Both the drift, $f^{(q)}(r,t)$, $f^{(z)}(r,t)$, and diffusion functions, $\rho^{(q)}(r,t)$, $\rho^{(z)}(r,t)$, are sufficiently well behaved for an application of Ito's Lemma (see Arnold (1974)). In general, the Wiener processes $W^{(q)}$ and $W^{(z)}$ will be uncorrelated. We denote the instantaneous covariance matrix by

$$\sum = \begin{bmatrix} \rho^{(q)^2} & 0 \\ 0 & \rho^{(z)^2} \end{bmatrix}. \tag{15}$$

Applying the two-dimensional Ito's Lemma, and using expressions (12)-(15), we obtain the stochastic law of motion of the spot rate in terms of its unobserved components:

$$dr_t = dq_t + dz_t. \tag{16}$$

Assumption A2: The price $P(t,\tau)$ of a discount bond is determined by the assessment at time t, of the segment $\{r_s, t \le s \le \tau\}$ of the spot rate process over the term of the bond.

Assumption A3: The market is efficient, there are no transaction costs, information is available to all investors simultaneously and every investor acts rationally.

Assumptions A1, A2, and A3 imply that the magnitude of the spot rate is the only determinant of the whole term structure and expectations formed with the knowledge of all past developments (including the present) are equivalent to expectations conditional only on the present value of the spot rate. Furthermore, the current value of the spot rate is given by the interaction of two unobserved independent stochastic components, one causing permanent effects and the other causing transitory effects.

Proposition 1. *Under assumptions A1, A2, and A3, the price $P(r,t,\tau)$ of a discount bond at time t of maturity $\tau = t + T$, given the state variable r_t will be given by*

$$P(r,t,\tau) = E_t\{\exp[-\int_t^\tau (q_s + z_s)\,ds - \frac{1}{2}\int_t^\tau \left(\phi^{(q)^2} + \phi^{(q)^2}\right) ds + \int_t^\tau \phi^{(q)} dW^{(q)} + \int_t^\tau \phi^{(z)} dW^{(z)}]\}, \tag{17}$$

were E_t denotes expectations formed in period t. (Note that in the expression above we have suppressed functional dependencies for notational brevity.)

Proof.

Assumptions A1, A2, and A3 imply that the following system (suppressing functional dependencies) is in order:

$$
\begin{aligned}
P(t,\tau) &= P(t,s,r),\\
dr_t &= dq_t + dz_t,\\
dq_t &= f^{(q)}dt + \rho^{(q)}dW^{(q)},\\
dz_t &= f^{(z)}dt + \rho^{(z)}dW^{(z)}.
\end{aligned}
$$

Applying Ito's differential rule, the discount bond's instantaneous rate of return satisfies the following stochastic differential equation:

$$\frac{dP}{P} = \lambda dt + \sigma^{(q)}dW^{(q)} + \sigma^{(z)}dW^{(z)}, \tag{18}$$

where

$$\sigma^{(q)} = \frac{P_q \rho^{(q)}}{P}, \tag{19}$$

$$\sigma^{(z)} = \frac{P_z \rho^{(z)}}{P}, \tag{20}$$

$$\lambda = \frac{(1/2)\,\rho^{(q)^2} P_{qq} + (1/2)\,\rho^{(z)^2} P_{zz} + f^{(q)} P_q + f^{(z)} P_z + P_t}{P}. \tag{21}$$

(Note that $P_z, P_{zz}, P_t, P_q, P_{qq}$ denote partial derivatives.)

By forming a riskless Black and Scholes (1973) portfolio, using Merton's (1973) accumulation equation and employing an arbitrage argument (given in Richard (1978)), we can show that the following equation holds for any arbitrary maturity, say τ, bond:

$$\lambda(\tau) - r = -\phi^{(q)}\sigma^{(q)}(\tau) - \phi^{(z)}\sigma^{(z)}(\tau), \tag{22}$$

where $\phi^{(q)}$ can be interpreted as the market price of the "permanent risk" and $\phi^{(z)}$ as the market price of the "temporary risk". Note that both $\phi^{(q)}$ and $\phi^{(z)}$ are independent of maturity τ.

Expression (22) is the standard no-arbitrage condition in partial equilibrium models of the term structure: the market prices of risk (i.e., permanent and temporary) multiplied by the unit percentage volatilities will specify the excess return over the riskless rate required by investors in order to compensate them "correctly" for the extra risk (price volatility) borne by holding a discount bond.

Substituting (19)-(21) into expression (22) and rearranging terms, we obtain:

$$\frac{1}{2}\left(\rho^{(q)^2}P_{qq}+\rho^{(z)^2}P_{zz}\right)+$$
$$\left(f^{(q)}P_q+f^{(z)}P_z\right)+P_t-rP+P_q\rho^{(q)}\phi^{(q)}+P_z\rho^{(z)}\phi^{(z)}\;=0. \tag{23}$$

The above is the fundamental partial differential equation for pricing discount bonds in a market characterized by assumptions A1, A2, A3. Bond prices are obtained by solving eq. (23) subject to the boundary condition: $P(r,\tau,\tau)=1$. The term structure, $R(t,\tau)$ of interest rates is then readily evaluated from the definitional equation (10).

Following similar steps as in Richard (1978) we can show that the probabilistic solution of eq. (23) is

$$P(r,t,\tau)=E_t\left\{\exp\left[-\int_t^\tau r_s ds-\int_t^\tau \frac{1}{2}\left(X'\Sigma^{-1}X\right)ds+\int_t^\tau X'\Sigma^{-1}\rho dW\right]\right\}, \tag{24}$$

where

$$X=\begin{bmatrix} X^{(q)} \\ X^{(z)} \end{bmatrix}=\begin{bmatrix} \rho^{(q)}\phi^{(q)} \\ \rho^{(z)}\phi^{(z)} \end{bmatrix}, \tag{25}$$

$$\rho dW=\begin{bmatrix} \rho^{(q)}dW^{(q)} \\ \rho^{(z)}dW^{(z)} \end{bmatrix}. \tag{26}$$

Substitution of eq. (25), (26), and (15) into (24) gives (17). This completes the proof of Proposition 1.

To illustrate the general model, the term structure of interest rates will now be obtained explicitly in the situation characterized by the following assumptions.

Assumption A4: The temporary component of the spot interest rate follows the Ornstein-Uhlenbeck process:

$$dz_t=\alpha\left(\gamma-z_t\right)dt+\rho^{(z)}dW^{(z)}, \tag{27}$$

where α is the speed-of-adjustment coefficient ($\alpha>0$), γ is the long run mean of the process, and $\rho^{(z)}$ is the diffusion coefficient which allows the process to fluctuate around its long run mean in a continuous but erratic way. The interest rate diffusion in expression (27) is also known as an elastic random walk. It is both Gaussian and Markovian and it was used by Vasicek (1977) in his celebrated term structure model. The permanent component of the spot interest rate follows the Arithmetic Brownian Motion process

$$dq_t=\mu dt+\rho^{(q)}dW^{(q)}, \tag{28}$$

where μ and $\rho^{(q)}$ are constants. This parametrization of the spot rate dynamics was used by Merton (1973). Finally, we assume that $dW^{(z)}$ and $dW^{(q)}$ are independent (standard) Wiener processes.

Assumption A5: The two market prices of risk, $\phi^{(q)}$ and $\phi^{(z)}$, are constant, i.e., independent of the calendar time and the level of the temporary and permanent components.

The following Theorem is the key result in our paper.

Theorem 1. *Under assumptions A1-A5, the solution of the term structure equation (17) is*

$$\ln\left[P\left(r,t,\tau\right)\right]=\left[R\left(\infty\right)-z_t\right]\left(\frac{1}{\alpha}\right)\left(1-e^{-\alpha T}\right)-R\left(\infty\right)T$$
$$-\left(\frac{\rho^{(z)^2}}{4\alpha^3}\right)\left(1-e^{\alpha T}\right)^2-q_tT-\frac{1}{2}\left(\mu+\rho^{(q)}\phi^{(q)}\right)T^2+\frac{1}{6}\rho^{(q)^2}T^3, \tag{29}$$

where

$$R\left(\infty\right)=\gamma+\frac{\rho^{(z)}\phi^{(z)}}{\alpha}-\frac{1}{2}\left(\frac{\rho^{(z)^2}}{\alpha^2}\right). \tag{30}$$

As in Vasicek (1977), $R\left(\infty\right)$ is the yield-to-maturity of a consol, where the interest rate follows the Ornstein-Uhlenbeck process in (27) alone (that is, we do not have a permanent component).

Proof.

The solution of the Arithmetic Brownian Motion process in (28) is given by

$$q_\tau=q_t+\mu T+\rho^{(q)}\int_t^\tau dW^{(q)}\left(s\right), \tag{31}$$

for $T=\tau-t$. (See Arnold (1974).)

From the above equation it follows that:

$$E_t\left[\int_t^\tau qds\right]^2=q_t^2T^2+q_t\mu T^3+\left(\frac{\rho^{(q)^2}}{3}\right)T^3+\left(\frac{\mu^2}{4}\right)T^4, \tag{32}$$
$$E_t\left[\int_t^\tau qds\right]=q_tT+\left(\frac{\mu}{2}\right)T^2. \tag{33}$$

Eq. (32) and (33) imply that

$$Var_t\left[\int_t^\tau qds\right]=\left(\frac{\rho^{(q)^2}}{3}\right)T^3, \tag{34}$$

where Var_t is the conditional variance operator. Furthermore, it is straightforward to show that

$$Cov_t\left[-\int_t^\tau qds\ ,\ \phi^{(q)}\int_t^\tau dW^{(q)}\right]=-\phi^{(q)}\rho^{(q)}\left(\frac{T^2}{2}\right), \tag{35}$$

where Cov_t denotes the conditional covariance.

The scalar stochastic differential equation in (27) is narrow-sense linear and autonomous, and its solution is given by

$$z_\tau=\gamma+e^{-\alpha T}\left(z_t-\gamma\right)+\rho^{(z)}e^{-\alpha T}\int_t^\tau e^{\alpha(s-t)}dW^{(z)}\left(s\right), \tag{36}$$

for $T=\tau-t$. (See Arnold (1974).) From eq. (36) we have:

$$E_t\left[\int_t^\tau zds\right] = \gamma T+\left(\frac{1}{\alpha}\right)\left(z_t-\gamma\right)\left(1-e^{-\alpha T}\right), \tag{37}$$

$$Var_t\left[\int_t^\tau zds\right] = \left(\frac{\rho^{(z)^2}}{\alpha^2}\right)T-\left(\frac{2\rho^{(z)^2}}{\alpha^3}\right)\left(1-e^{-\alpha T}\right)+\left(\frac{\rho^{(z)^2}}{2\alpha^3}\right)\left(1-e^{-2\alpha T}\right), \tag{38}$$

$$Cov_t\left[\int_t^\tau zds\ ,\ \int_t^\tau \phi^{(z)}dW^{(z)}\right] = -\left(\frac{\rho^{(z)}\phi^{(z)}}{\alpha^2}\right)\left(1-e^{-\alpha T}\right)+\left(\frac{2\rho^{(z)}\phi^{(z)}}{\alpha}\right)T. \tag{39}$$

Recall that both the Arithmetic Brownian Motion and the Ornstein-Uhlenbeck processes imply normally distributed increments. Since the exponent (i.e., the term in square brackets) in expression (17) is the sum of two independent normal distributions, it is also normally distributed. Thus, for constant market prices of risk, expression (17) is equivalent to

$$\begin{aligned}\ln\left[P\left(r,t,\tau\right)\right]=&-E_t\int_t^\tau\left(q+z\right)ds+\frac{1}{2}Var_t\left[\int_t^\tau\left(q+z\right)ds\right]\\ &-Cov_t\left[\int_t^\tau qds\ ,\ \int_t^\tau\phi^{(q)}dW^{(q)}\right]\\ &-Cov_t\left[\int_t^\tau zds\ ,\ \int_t^\tau\phi^{(z)}dW^{(z)}\right].\end{aligned} \tag{40}$$

Substituting expressions (33)-(35) and (37)-(39) into (40), and rearranging, we obtain (29). This completes the proof of Theorem 1.[11]

Since our term structure framework belongs in the affine class of interest rate models, it will be an interesting question for future research to calibrate our model using spot rates of different maturities and assess its performance for pricing short- and long-term debt.

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[11] For an alternative proof of Theorem 1, based on a guess solution, see Hatgioannides, Karanasos and Karanassou (2004).

4 Conclusion

In this paper we shed some light on the important jigshaw of the nonstationarity of interest rates and how it could be fruitfully accommodated in a theoretical term structure model.

Application of the variance ratio statistic and regression analysis showed that the UK and US short-term nominal interest rates are unit root process with significant mean reverting components.

In turn, we modelled the dynamics of the instantaneous interest rate as the combined effect of a stationary component that induces mean reversion and a nonstationary component which accounts for the martingale behavior of the riskless rate. The principal result of the paper is a closed-form solution of the yield curve when the interest rate is given by a mix of autoregressive and random walk processes. What remains to be seen in future research work is, given the obvious advantages of a closed-form solution, how well our model fits observed yield curves.

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