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## Estimating Deterministically Time-Varying Variances in Regression Models

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# Estimating Deterministically Time-Varying Variances in Regression Models

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## Abstract

The problem of structural change justifiably attracts considerable attention in econometrics. A number of different paradigms have been adopted ranging from structural breaks which are sudden and rare to time varying coefficient models which exhibit structural change more frequently and continuously. This paper is concerned with parametric econometric models whose coefficients change deterministically and smoothly over time. In particular we provide a new estimator for unconditional time varying variances in regression models. A small Monte Carlo study indicates that the method works reasonably well for moderately large sample sizes.

*Keywords:* *Structural Change, Non-Stationarity, Deterministic Time-Variation*  
*JEL code:* *C10, C14*

## 1 Introduction

The investigation of structural change in econometric models has been assuming increasing importance in the literature over the past couple of decades. This focus is not surprising. Assuming wrongly that the structure of a model remains fixed over time, has very significant and adverse implications. The first obvious implication is inconsistency of the parameter estimates. A distinct, yet related, implication is the fact that structural change chance is likely to be responsible for most major forecast failures of time series models.

As a result a huge literature on modelling structural change has emerged. Most of the work assumes that structural changes in parametric models occur rarely and are abrupt. Another more recent strand of the literature takes a different approach. In this approach the coefficients of parametric models are assumed to evolve over time. To achieve this the

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parameters are assumed to be stochastic processes leading to stochastic time varying coefficient (STVC) models. Such models bear resemblance to simple nonlinear econometric models such as bilinear models (see Tong (1990)). STVC models have been used recently in applied macroeconometric work by, e.g., Cogley and Sargent (2002), to model the evolution of macroeconomic variables such as US inflation in the post WWII era. In this case coefficients have been assumed to evolve as random walks over time.

Yet another strand of the literature assumes that coefficients change but in a smooth deterministic way. Such modelling attempts have a long pedigree in statistics starting with the work of Priestley (1965). This paper suggested that processes may have time varying spectral densities which change slowly over time. The context of this work is nonparametric. This work has more recently been followed up by Dahlhaus (1996). A parametric alternative to this approach has been pursued by Robinson (1989) for linear regression models and Robinson (1991) for nonlinear parametric models. Recently, Orbe, Ferreira, and Rodriguez-Poo (2005) extended these results to include time varying seasonal effects. We will refer to such parametric models as deterministic time varying coefficient (DTVC) models. A disadvantage of such an approach is that the coefficient change cannot be modelled or, for that matter, forecast. Both of these are theoretically possible with STVC. However, an important assumption underlying DTVC is that coefficients change slowly. As a result forecasting may be carried out by assuming that the coefficients remain at their end-of-observed-sample value.

In the existing literature on DTVC models, the focus has been primarily on estimating regression coefficients. However, the question of whether the variance of the error term changes over time is perhaps equally relevant. As mentioned above, questions such as the evolution of inflation and its variance are of direct interest to macroeconomists and policymakers. This paper addresses this estimation problem. The note is structured as follows: Section 2 discusses the theoretical framework and proposes a new estimator. Section 3 presents a Monte Carlo study. Finally, Section 4 concludes.

## 2 Theory

Let the model of interest be given by

$$y_t = \beta(t)'x_t + u_t \tag{1}$$

where  $y_t$  and  $x_t$  are the scalar dependent and  $k$ -dimensional explanatory variables respectively.  $u_t$  is given by  $\sigma(t)v_t$ . The following assumptions provide information on the detailed

specification of the above model:

**Assumption 1**  $\beta(t) = \beta_{t/T}$  where each element of  $\beta_\tau$ ,  $\beta_{i,\tau}$ ,  $i = 1, \dots, k$ ,  $\tau \in (0, 1)$ , is continuous and twice differentiable on  $(0, 1)$ .  $\sigma(t)^2 = \sigma_{t/T}^2$  where  $\sigma_\tau^2$ ,  $\tau \in (0, 1)$ , is continuous and twice differentiable on  $(0, 1)$ .

**Assumption 2**  $x_t$  is an  $\alpha$ -mixing sequence with size  $-4/3$  and finite 8-th moments.  $E(x_{is}x_{jt}) = m_{ij,s,t} = m_{ij}(s/T, t/T) + O(T^{-1})$  where  $m_{ij}(\cdot, \cdot)$  is a twice differentiable function of both its arguments.

**Assumption 3**  $v_t$  is a stationary  $(0, 1)$  martingale difference sequence with finite 4-th moments which is independent of  $x_t$  at all leads and lags.

**Assumption 4** The function  $K^h(\cdot)$  is a second order kernel with compact support  $[-1, 1]$  and absolutely integrable Fourier transform.

Assumption 1 is a crucial assumption. It specifies both  $\beta(t)$  and  $\sigma^2(t)$  to be smooth deterministic functions of time. It is interesting to note that they depend not only on the point in time  $t$  but also on the sample size  $T$ . This is necessary since in order to estimate consistently a particular parameter one needs the sample size that relates to that parameter to tend to infinity. This is achieved in this context by allowing an increasing number of neighboring observations to be informative about  $\beta$  and  $\sigma^2$  at time  $t$ . In other words we have to assume that as the sample size grows the functions  $\beta_\tau$  and  $\sigma_\tau$  stretch to cover the whole period of the sample. This setup has precedents in the statistical literature. For example, the concept of slowly varying processes of Priestley (1965) forms an early instance of similar ideas. Assumptions 2 and 3 are standard mixing and moment conditions for the explanatory variables and the remainder of the error term. It is important to note that  $x_t$  is also allowed to be nonstationary. Finally, assumption 4 relates to the kernel function that will be used for estimation.

This specification for the variance of the error term, is clearly related, yet distinct, from ARCH type models. Obviously, ARCH models specify the conditional variance of  $u_t$  whereas our setup focuses on the unconditional variance. One implication is that our specification has nothing to say about the generating mechanism of  $\sigma_\tau$ . This has advantages and disadvantages. One advantage over parametric conditional variance specifications is that these specifications may be wrong whereas our approach which is nonparametric is less likely to be so, as mentioned by Robinson (1991) for regression models. Another implication is that

parametric conditional specifications may possibly imply stationarity for  $y_t$  whereas the unconditional specification we suggest is nonstationary unless  $\sigma^2(t) = \sigma^2$ .

Following Robinson (1989) and Orbe, Ferreira, and Rodriguez-Poo (2005), we propose the following estimator for  $\beta_\tau$ .

$$\hat{\beta}_\tau = \left( \sum_{t=1}^T K_{t,\tau}^h x_t x_t' \right)^{-1} \left( \sum_{t=1}^T K_{t,\tau}^h x_t y_t \right) \quad (2)$$

where  $K_{t,\tau}^h = (Th)^{-1}K((\tau - t)/Th)$ . This estimator bears close resemblance to the standard OLS estimator and it is easy to see that it is the closed form solution of the following optimisation

$$\min_{\beta} \sum_{t=1}^T K_{t,\tau}^h (y_t - \beta(t)' x_t)^2 \quad (3)$$

Following estimation of  $\beta_\tau$  we propose the following simple estimator for  $\sigma_\tau^2$ .

$$\hat{\sigma}_\tau^2 = \frac{\sum_{t=1}^T K_{t,\tau}^h \hat{u}_t^2}{\sum_{t=1}^T K_{t,\tau}^h} = \sum_{t=1}^T \tilde{K}_{t,\tau}^h \hat{u}_t^2 \quad (4)$$

where

$$\hat{u}_t = y_t - \hat{\beta}(t)' x_t \quad (5)$$

and  $\hat{\beta}(t) = \hat{\beta}_{t/T}$ . Here, we have assumed that the  $h$  used in (2) is the same as that used in (4). However, they clearly do not need to be the same. Let us denote the parameter  $h$  used in (2) by  $h_\beta$  whereas the parameter  $h$  used in (4) is denoted by  $h_\sigma$ . In fact for the theorem that follows we require  $h_\beta$  and  $h_\sigma$  to be different. Now, denoting the Euclidean matrix norm by  $\|\cdot\|$ , we can show the following theorem

**Theorem 1** *Under assumptions 1-4 and if*

$$h_\sigma = o(1), h_\beta = o(1) \quad (6)$$

*then*

$$\hat{\sigma}_\tau^2 - \sigma_\tau^2 = o_p(1) \quad (7)$$

*If further the following two conditions hold*

$$h_\beta = O(T^{-(1-a_\beta)}), h_\sigma = O(T^{-(1-a_\sigma)}) \quad \text{where } 4/5 > a_\sigma > 2/3, 1 > a_\beta > 4/5 \text{ and } a_\sigma < 2a_\beta - 1 \quad (8)$$

$$\inf_\tau \left\| \sum_{t=1}^T K_{t,\tau}^{h_\beta} x_t x_t' \right\| > 0, \quad \forall T \quad (9)$$

then, it follows that

$$V_\tau^{-1/2} (\hat{\sigma}_\tau^2 - E(\hat{\sigma}_\tau^2)) \xrightarrow{d} N(0, 1) \quad (10)$$

where  $V_\tau = \lim_{T \rightarrow \infty} \sum_{t=1}^T \left( \tilde{K}_{t,\tau}^{h_\sigma} \right)^2 \sigma^2(t) = O(T^{-1} h_\sigma^{-1}) = O(T^{-a_\sigma})$

**Proof.** We will prove (10). In the course of the proof it will become obvious that (7) is obtained without using (8)-(9). As a first step we need to explore the properties of  $\hat{\beta}$ . We show that

$$\hat{\beta}_\tau - \beta_\tau = O_p(V_\tau^{1/2}) \quad (11)$$

where  $V_\tau^0 = \lim_{T \rightarrow \infty} \sum_{t=1}^T \left( \tilde{K}_{t,\tau}^{h_\beta} \right)^2 = O(T^{-a_\beta})$ . Orbe, Ferreira, and Rodriguez-Poo (2005) discuss the asymptotic properties of  $\hat{\beta}_\tau$  but only under stationarity of  $u_t$  and so (11) is also of independent interest. For this result we retrace the proof of Theorem 1 of Orbe, Ferreira, and Rodriguez-Poo (2005). It is the case that the only point in the proof where the use of assumptions 1 and 3 relating to  $u_t$  make a difference is in expression (A.7) of Orbe, Ferreira, and Rodriguez-Poo (2005). This expression is made up of terms of the form  $E(x_{it} x_{jt} u_t^2)$ . In the case of Orbe, Ferreira, and Rodriguez-Poo (2005), and under an additional martingale difference assumption for  $v_t$ , these terms take the form  $m_{ij\tau} \sigma^2$  whereas in our case they take the form  $m_{ij\tau} \sigma_\tau^2$ . Using this, the result of Theorem 1 of Orbe, Ferreira, and Rodriguez-Poo (2005) goes through establishing (11). Next we examine

$$\sum_{t=1}^T \tilde{K}_{t,\tau}^{h_\sigma} u_t^2 - \sum_{t=1}^T \tilde{K}_{t,\tau}^{h_\sigma} \hat{u}_t^2 = \sum_{t=1}^T \tilde{K}_{t,\tau}^{h_\sigma} \hat{u}_t (\hat{u}_t - u_t) - \sum_{t=1}^T \tilde{K}_{t,\tau}^{h_\sigma} \hat{u}_t (u_t - \hat{u}_t) \leq \quad (12)$$

$$2 \max_t (\hat{\beta}_t - \beta_t)' \sum_{t=1}^T \tilde{K}_{t,\tau}^{h_\sigma} x_t u_t + \left( \max_t (\hat{\beta}_t - \beta_t)' \right) \left( \max_t (\hat{\beta}_t - \beta_t) \right) \sum_{t=1}^T \tilde{K}_{t,\tau}^{h_\sigma} x_t' x_t \quad (13)$$

We consider the framework of fixed design regression of Fan (1990). Let  $\kappa_t = x_t u_t$  and  $\delta_t = x_t' x_t$ . Then,  $\sum_{t=1}^T \tilde{K}_{t,\tau}^{h_\sigma} x_t u_t$  and  $\sum_{t=1}^T \tilde{K}_{t,\tau}^{h_\sigma} x_t' x_t$  are simply the estimators of  $E(\kappa_t | t)$  and  $E(\delta_t | t)$  in the fixed regressor regressions given by  $\kappa_t = \mu_t + \nu_t$  and  $\delta_t = \eta_t + \theta_t$  where  $\mu_t = E(\kappa_t | t) = 0$ ,  $\nu_t = x_t u_t - E(\kappa_t | t)$ ,  $\eta_t = E(\delta_t | t) = m_{t,t}$ ,  $\theta_t = x_t' x_t - E(\delta_t | t)$  and  $m_{t,t} = \sum_{j=1}^k m_{jj,t}$ . We first examine  $\max_t (\hat{\beta}_t - \beta_t)$ .

$$\max_t (\hat{\beta}_t - \beta_t) \leq C \max_\tau \sum_{t=1}^T \tilde{K}_{t,\tau}^{h_\beta} x_t u_t \leq C \left( T^{a_\beta} \max_\tau \tilde{K}_{t,\tau}^{h_\beta} \right) \left( T^{-a_\beta} \sum_{t=1}^T x_t u_t \right) = \quad (14)$$

$$O_p(T^{-a_\beta+1/2}) = o_p(T^{-1/2} h_\sigma^{-1/2})$$

for some constant  $C > 0$ , by (8), (9), example 3.1 of Fan (1990) and Theorem 2.6 of Fan (1990). Further, by an application of Theorem 2.3 of Fan (1990) on the regression of  $\delta_t$  on  $t$ , we get that (13) is  $o_p(T^{-1/2} h_\sigma^{-1/2})$ . We, then, simply need to show that

$$V_\tau^{-1/2} (\bar{\sigma}_\tau^2 - E(\bar{\sigma}_\tau^2)) \xrightarrow{d} N(0, 1) \quad (15)$$

where  $\bar{\sigma}_\tau^2 = \sum_{t=1}^T \tilde{K}_{t,\tau}^{h_\sigma} u_t^2$ . But, defining  $\xi_t = u_t^2$ ,  $\bar{\sigma}_\tau^2$  is again an estimator of  $E(\xi_t|t)$  in the fixed regressor regression given by  $\xi_t = \psi_t + \zeta_t$  where  $\psi_t = E(\xi_t|t) = \sigma(t)^2$  and  $\zeta_t = u_t^2 - \sigma(t)^2$ . Then, by Theorem 3.1 of Fan (1990) and under assumptions 1, 3, 4 and (8), (15) follows. ■

**Remark 1** *Condition (9) is a positive definiteness regularity condition resembling standard regularity conditions in regression analysis.*

**Remark 2** *The conditions of Theorem 1 are sufficient for normality but we have not investigated whether they are necessary. Results from the standard nonparametric literature such as those obtained by, e.g., Ziegelmann (2002) suggest that a sharper normality result may be obtainable in our case which allows for using the same bandwidth for  $\beta_\tau$  and  $\sigma_\tau^2$ . Our result is based on deriving a uniform rate of convergence as in (14). It is possible that an alternative line of proof can strengthen the result of Theorem 1.*

## 2.1 Choice of $h$

An important question relating to the estimation of  $\sigma_\tau^2$  concerns the choice of  $h$ . For  $h_\beta$  we suggest using a leave-one-out penalised residual sum of squares objective function. In particular  $h_\beta$  can be determined by minimising numerically

$$\sum_{t=1}^T \left( y_t - \tilde{\beta}(t)' x_t \right)^2 p_\beta(h_\beta) \quad (16)$$

where

$$p_\beta(h_\beta) = \left( 1 - \frac{1}{T\sqrt{2\pi}} \sum_{t=1}^T x_t' \left( \sum_{i=1}^T K_{t,t/T}^{h_\beta} x_i x_i' \right) \right)^{-1} \quad (17)$$

and

$$\tilde{\beta}(t) = \tilde{\beta}_{t/T} = \left( \sum_{i=1, i \neq t}^T K_{i,i/T}^{h_\beta} x_i x_i' \right)^{-1} \left( \sum_{i=1, i \neq t}^T K_{i,i/T}^{h_\beta} x_i y_i \right) \quad (18)$$

For more details, see also Orbe, Ferreira, and Rodriguez-Poo (2005). Similarly,  $h_\sigma$  can be determined by minimising numerically

$$\sum_{t=1}^T \left( \hat{u}_t^2 - \tilde{\sigma}_{t/T}^2 \right)^2 p_\sigma(h_\sigma) \quad (19)$$

where, using the Rice criterion,

$$p_\sigma(h_\sigma) = \left( 1 - \frac{2}{Th_\sigma\sqrt{2\pi}} \right)^{-1} \quad (20)$$

and

$$\tilde{\sigma}_{t/T}^2 = \frac{\sum_{i=1, i \neq t}^T K_{i,i/T}^{h_\sigma} \hat{u}_i^2}{\sum_{i=1, i \neq t}^T K_{i,i/T}^{h_\sigma}} \quad (21)$$

Table 1: Average MSE for  $\hat{\beta}_\tau$  and  $\hat{\sigma}_\tau^2$

$n_b/T$	100	200	400
	$\hat{\beta}_\tau$		
4	0.174	0.130	0.060
8	0.193	0.175	0.156
$n_s$	$\hat{\sigma}_\tau^2$		
3	0.647	0.355	0.194
6	0.714	0.581	0.388

### 3 Monte Carlo Study

#### 3.1 Monte Carlo Setup

In this section we present a Monte Carlo study on the small sample properties of the new estimator. We consider the following model.

$$y_t = \beta_t x_t + u_t \quad (22)$$

where  $x_t \sim N(0, 1)$

$$\beta_t = \sin\left(\frac{n_b \pi}{T}\right) \left(1 - \frac{t}{T}\right) \quad (23)$$

and  $u_t \sim N(0, \sigma_t^2)$  where

$$\sigma_t^2 = \sin\left(\frac{n_s \pi}{T}\right) + 2 \quad (24)$$

$T = 100, 200, 400$ . The parameters  $n_b$  and  $n_s$  control the rate at which the functions change over time. We set  $n_b = 4, 8$  and  $n_s = 3, 6$ . We set  $h_\beta$  and  $h_\sigma$  using (16)-(21). The truncated standard normal kernel is used throughout.

#### 3.2 Monte Carlo Results

Results are reported in Table 1 and Figures 1-2. Table 1 reports the average MSE for  $\hat{\beta}_\tau$  and  $\hat{\sigma}_\tau^2$  across all replications for the six experiments we consider. Figure 1 presents the average  $\hat{\sigma}_\tau^2$  across replications over time together with the true  $\sigma_\tau^2$ . The first column of panels for Figure 1 present results for  $n_b = 4$  and  $n_s = 3$ . The second column of panels present results for  $n_b = 8$  and  $n_s = 6$ . Results clearly indicate that performance of the estimator improves with the number of observations and deteriorates with the speed at which the functions change over time.



## 4 Conclusion

Structural change is justifiably a major concern in econometric modelling. A number of different paradigms have been adopted ranging from structural breaks which are sudden and rare to time varying coefficient models which exhibit structural change more frequently and continuously. This paper is concerned with parametric econometric models whose coefficients and error variance change deterministically and smoothly over time.

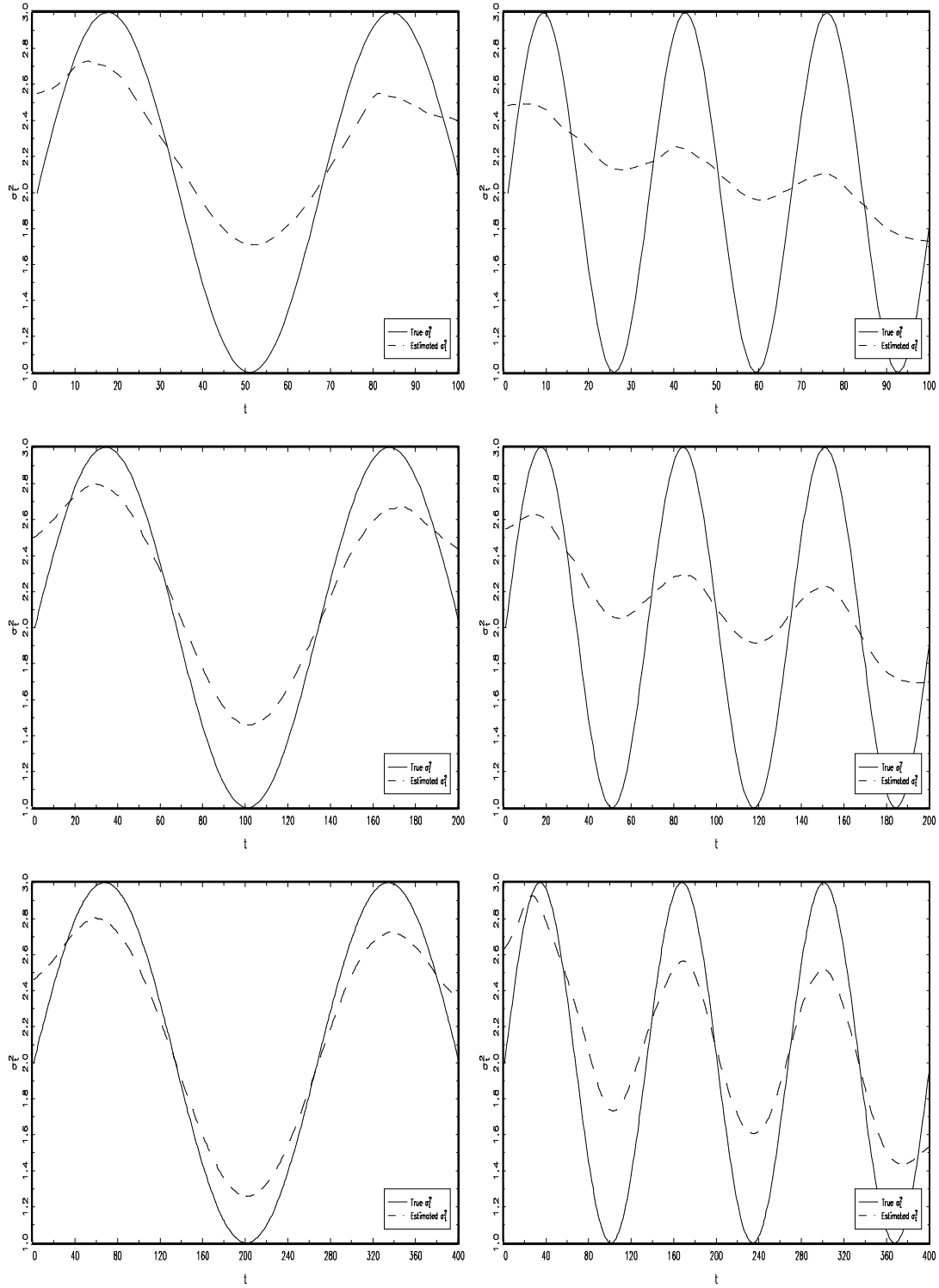
In particular we provide and discuss the theoretical properties of an estimator for unconditional time varying variances in regression models. A small Monte Carlo study indicates that the method works reasonably well for moderately large sample sizes.

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Figure 1: The first column of panels for Figure 1 present results for  $n_b = 4$  and  $n_s = 3$ . The second column of panels present results for  $n_b = 8$  and  $n_s = 6$ . For the three rows of panels,  $T = 100, 200, 400$  respectively.



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