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Working Paper

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SFB 373 Discussion Paper, No. 2001,14

Provided in Cooperation with:
Collaborative Research Center 373: Quantification and Simulation of Economic Processes, Humboldt University Berlin

Suggested Citation: Küchler, Uwe; Vasiliev, Vjatscheslav A. (2001) : On guaranteed parameter estimation of stochastic differential equations with time delay by noisy observations, SFB 373 Discussion Paper, No. 2001,14, Humboldt University of Berlin, Interdisciplinary Research Project 373: Quantification and Simulation of Economic Processes, Berlin, https://nbn-resolving.de/urn:nbn:de:kobv:11-10048746

This Version is available at:
https://hdl.handle.net/10419/62749

On guaranteed parameter estimation of stochastic differential equations with time delay by noisy observations*

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February 23, 2001

Abstract

Let $(X(t), t \geq -1)$ and $(Y(t), t \geq 0)$ be stochastic processes satisfying

$$dX(t) = aX(t)dt + bX(t-1)dt + dW(t),$$

$$dY(t) = X(t)dt + dV(t),$$

where $(W(t), t \geq 0)$ and $(V(t), t \geq 0)$ are standard Wiener processes and $\vartheta = (a, b)'$ is assumed to be an unknown parameter from some subset Θ of $\mathcal{R}^2$.
The ϑ is to be estimated based on continuous observation of $(Y(t), t \geq 0)$.
A strong consistent and sequential estimators for ϑ with preassigned least square accuracy are constructed. The limit behaviour of the duration of the estimation procedure with given accuracy is obtained.

Keywords: stochastic differential equations; time delay; noisy observations; sequential analysis; least square accuracy.

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* The research on this paper was carried out within the Sonderforschungbereich 373 at Humboldt University Berlin

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1 Introduction

Let us given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}(t), t \geq 0), P)$ and two realvalued standard Wiener processes $W = (W(t),\ t \geq 0)$ and $V = (V(t),\ t \geq 0)$ with respect to $(\mathcal{F}(t))$. Moreover, assume that $X_0 = (X_0(t),\ t \in [-1, 0])$ and Y_0 are a realvalued cadlag process and a realvalued random variable on $(\Omega, \mathcal{F}, (\mathcal{F}(t), t \geq 0), P)$ respectively with

$$E \int_{-1}^{0} X_0^2(s)ds < \infty \quad \text{and} \quad EY_0^2 < \infty.$$

Futhermore, let Y_0 and $X_0(s)$ be $\mathcal{F}_0-$measurable for every s from $[-1, 0]$, $W,\ V,\ X_0$ and Y_0 are assumed to be mutually independent.
Consider a two–dimensional random process $(X, Y) = (X(t), Y(t))$ described by the system of stochastic differential equations

$$dX(t) = aX(t)dt + bX(t-1)dt + dW(t),\ t \geq 0, \tag{1}$$

$$dY(t) = X(t)dt + dV(t),\ t \geq 0. \tag{2}$$

and the initial conditions $X(t) = X_0(t),\ t \in [-1, 0]$ and $Y(0) = Y_0$.
The parameter $\vartheta = (a, b)'$ with $a, b \in \mathcal{R}^1$ is supposed to be unknown and shall be estimated on the base of the observation of Y.
The equation (1) is a very special case of stochastic differential equations with time delay, which occur in modelising phenomenas e.g. in economics, technics, biology and medicine, which are influenced by time delayed effects, see [Kol/My] and [Mo] for examples.
The question of estimation of ϑ based on the observations of Y arises in a modified form e.g. in [Vas/Ko].
Here we shall construct sequential plans $(T_\varepsilon, \vartheta_\varepsilon^*)$ for estimation of the parameter ϑ which has a preassigned least square accuracy and shall derive asymptotic properties of the duration T_ε of these plans.
The method used below is to reduce the equations (1) and (2) to a single equation (see (4) below) for the process $(Y(t), t \geq 0)$, which can be treated similar to [Vas/Ko]. To do this we have to restrict the set of possible parameter Θ. This is because we use the matrix-valued Fisher information process related to (1), the asymptotic properties of which heavily depend on $\vartheta \in \mathcal{R}^2$. It is shown, that in two cases for the choice of Θ the used construction works.
The organisation of this paper is as follows. In Section 2 some known properties of the equation (1) are summarized which are necessary in the sequel. The two mentioned cases for Θ namely Θ_1 and Θ_2 are established and the equations (1), (2) are transformed in a new one for the one-dimensional process $(Y(t), t \geq 0)$ (see (4)) carrying all information of the original ones. In Section 3 the results are formulated, and Section 4 includes the proof. Two examples close the considerations.

2 Preliminaries

The equation (1) was studied in detail e.g. in [Gu/Ku]. Together with the described initial condition the equation (1) has a uniquely determined solution X which can be represented as follows:

$$X(t) = x_0(t)X_0(t) + b\int_{-1}^{0} x_0(t-s-1)X_0(s)ds + \int_0^t x_0(t-s)dW(s),\ t \geq 0,$$

and wich has the property $E\int_0^T X^2(s)ds < \infty$ for every $T > 0$. Here $x_0 = (x_0(t), t \geq -1)$ denotes the so-called fundamental solution of the deterministic equation

$$\dot{x}(t) = ax(t) + bx(t-1),\ t > 0 \ \text{ almost everywhere},$$

i.e. $x_0(\cdot)$ is absolutely continuous on $[0,\infty)$ and

$$x(t) = 0,\ t \in [-1,0);\ x(0) = 1.$$

The limit behavior of $X(t)$ for $t \to \infty$ is very connected with the properties of the set $\Lambda = \{\lambda \in K | \lambda = a + be^{-\lambda}\}$ (K denotes the set of complex numbers). It is well known, that Λ is nonvoid, denumberable infinite, has the finite accumulation points and has the property

$$v_0(\vartheta) := \sup\{Re\lambda | \lambda \in \Lambda\} < \infty.$$

Define

$$v_1(\vartheta) := \sup\{Re\lambda | \lambda \in \Lambda,\ Re\lambda < v_0(\vartheta)\}.$$

We assume that ϑ belongs to some fixed $\Theta \subseteq \mathcal{R}^2$ (which will be specified later) and we shall construct a sequential estimator for $\vartheta = (a,b)'$ having a preassigned accuracy in the mean square sense. Fix a positive real $\overline{\vartheta} > 0$. Our construction works in two cases for Θ.

Case I.

Assume L is an arbitrary line in the $\vartheta-$plane:

$$L = \{\tilde{\vartheta} = (\tilde{a},\ \tilde{b})' | \alpha\tilde{a} + \beta\tilde{b} = c\}.$$

Let $\tilde{\Theta}$ be the segment $L \cap \{||\tilde{\vartheta}|| \leq \overline{\vartheta}\}$ ($\overline{\vartheta}$ is choosen in such a way, that $\tilde{\Theta}$ is nonvoid). Now we introduce the finite set S by

$$S = \{\vartheta \in \tilde{\Theta} |\ v_0(\vartheta) \cdot v_1(\vartheta) = 0 \vee (a > 1, b = -e^{(a-1)})\}$$

and put $\Theta_1 = \tilde{\Theta} \setminus S$.

Remark 1. The exclusion of the elements of S from $\tilde{\Theta}$ is necessary because of the unappropriate behaviour of $X(t)$ for $t \to \infty$ if $\vartheta \in S$, see [Gu/Ku] for details.

Case II.

Define

$$\Theta_2 = \{\vartheta \in \mathcal{R}^2 |\ ||\vartheta|| \leq \overline{\vartheta},\ v_0(\vartheta) < 0 \text{ or } (v_0(\vartheta) > 0 \text{ and } v_0(\vartheta) \notin \Lambda)\}.$$

In this case the information matrix $G_X(T)$ given by

$$G_X(T) = \begin{pmatrix} \int\limits_0^T X^2(t)dt & \int\limits_0^T X(t)X(t-1)dt \\ \int\limits_0^T X(t)X(t-1)dt & \int\limits_0^T X^2(t-1)dt \end{pmatrix}$$

has the asymptotic property [Gu/Ku], [Ku/Vas]

$$\lim_{T\to\infty} |\varphi^{-1}(T)G_X(T) - I_\infty(T)| = 0 \quad P_\vartheta - \text{a.s.}, \tag{3}$$

where

$$\varphi(T) = \begin{cases} T, & if\ v_0 < 0, \\ e^{2v_0T}, & if\ v_0 > 0,\ v_0 \notin \Lambda. \end{cases}$$

If $v_0 < 0$ then (1) admits a stationary solution and $I_\infty(T) \equiv I_\infty$ is a constant positive definite $2\times 2-$matrix (further – stationary case); if $v_0 > 0$ and $v_0(\vartheta) \notin \Lambda$, then $I_\infty(T)$ is periodic with the period $\Delta = \pi/Im\lambda_0$, where λ_0 is the unique element of Λ with $Re\lambda_0 = v_0(\vartheta)$ and $Im\lambda_0 > 0$ [Gu/Ku] (further – periodic case).
The problem of sequential estimation ϑ by observation without noises by the condition (3) was considered in [Ku/Vas].
Now we shall reduce the system (1), (2) to the form

$$dY(t) = \vartheta' A(t)dt + \xi(t)dt + dV(t), \tag{4}$$

where the observable process $(A(t),\ t \geq 0)$ and the noise $\xi = (\xi(t),\ t \geq 0)$ are some $(\mathcal{F}(t))$ - measured processes. The problem of guaranteed estimation of ϑ in the model of type (4) was considered in [Vas/Ko].
Putting $X(t)dt$ from (2) into the first one in system (1)–(2) we obtain

$$dX(t) = adY(t) + bdY(t-1) - adV(t) - bdV(t-1) + dW(t),\ t \geq 1.$$

Using the integrated form of this equation and putting $Y(t)$ in the equation (2) we get the equation for the process under observation Y

$$dY(t) = [aY(t)+bY(t-1)]dt+[X(1)-aY(1)-bY(0)+aV(1)-W(1)-aV(t)-bV(t-1)+$$
$$+W(t)]dt + dV(t),\ t \geq 1.$$

From here it follows the equation of form (4) for process Y with

$$A(t) = (Y(t), Y(t-1))',$$

$$\xi(t) = X(1) - aY(1) - bY(0) + aV(1) - aV(t) - W(1) - bV(t-1) + W(t).$$

Let us introduce for any continuous function f the increments operator $\tilde{\Delta}f(t) = f(t) - f(t-1)$. The functions $A(t)$ and $\xi(t)$ are $\mathcal{F}(t)$-measurable for every $t \geq 1$ and all the conditions of type (3) in [Vas/Ko]

$$E\int_1^T (||A(t)||_1 + |\xi(t)|)dt < \infty,\ T > 1,$$

$$E[\tilde{\Delta}\xi(t)|\mathcal{F}(t-2)] = 0,\ E[(\tilde{\Delta}\xi(t))^2|\mathcal{F}(t-2)] \le \overline{s}^2,\ t \ge 2, \tag{5}$$

$$\overline{s}^2 = 1 + \overline{\vartheta}^2,\ ||A||_1 = \sum_i |A_i|$$

hold.
Taking into account the statistical properties (5) of the noise ξ we obtain the equation for increments of the process Y

$$d\tilde{\Delta}Y(t) = a\tilde{\Delta}Y(t)dt + b\tilde{\Delta}Y(t-1)dt + \tilde{\Delta}\xi(t)dt + dV(t) - dV(t-1),\ t \ge 2. \tag{6}$$

We have reduced the system (1)–(2) to a single differential equation (6) for the observed process $(\tilde{\Delta}Y(t), t \ge 2)$ including the unknown parameters a and b. The term $\tilde{\Delta}\xi(t)$ also contains a and b, but is controllable in some sence by using (5).
Nevertheless, a and b cannot be estimated from (6) by asymptotic or sequential maximum likelihood method as in [Gu/Ku] or [Ku/Vas] because of the appearance of $\tilde{\Delta}\xi(t)dt$ and $dV(t-1)$. Another way is necessary. We will follow to the idea of the method of [Vas/Ko].

3 Results

3.1 Sequential estimation procedure I

Consider the estimation problem of a linear combination $\theta = l'\vartheta,\ \vartheta \in \Theta_1$, where $l = (l_1, l_2)'$ is a some known constant vector such that $\sigma = l_1\beta - l_2\alpha \ne 0$.
Let us introduce the processes $Z_1,\ Z_2$ and Ψ by formulae

$$dZ_1(t) = \sigma^{-1}(\beta d\tilde{\Delta}Y(t) - c\tilde{\Delta}Y(t-1)dt),$$

$$dZ_2(t) = -\sigma^{-1}(\alpha d\tilde{\Delta}Y(t) - c\tilde{\Delta}Y(t)dt),$$

$$\Psi(t) = \begin{cases} \sigma^{-1}(\beta\tilde{\Delta}Y(t) - \alpha\tilde{\Delta}Y(t-1)), & t \ge 2, \\ 0, & t < 2. \end{cases}$$

From (6) and by the definition of Θ_1 we can get for $t \ge 2$ the system of equations

$$dZ_1(t) = a\Psi(t)dt + \beta\sigma^{-1}(\tilde{\Delta}\xi(t)dt + d\tilde{\Delta}V(t)),$$

$$dZ_2(t) = b\Psi(t)dt - \alpha\sigma^{-1}(\tilde{\Delta}\xi(t)dt + d\tilde{\Delta}V(t)),\ t \ge 2.$$

From here we obtain the equation for the scalar process $Z(t) = l_1Z_1(t) + l_2Z_2(t)$:

$$dZ(t) = \theta\Psi(t)dt + \tilde{\Delta}\xi(t)dt + d\tilde{\Delta}V(t),\ t \ge 2 \tag{7}$$

with unknown parameter θ. Put $Z(t) = 0,\ t < 2$. Similar to [Vas/Ko] we can define a sequential plan for estimation of θ with mean square deviation less than a given

positive ε.
The sequential estimation plans of θ have been constructed in [Vas/Ko] on the bases of estimators of the form

$$\theta^*(T) = G^{-1}(T,u)\Phi(T,u), \tag{8}$$

$$G(T,u) = \int_0^T \Psi(t-u)\Psi(t)dt, \quad \Phi(T,u) = \int_0^T \Psi(t-u)dZ(t), \quad u \geq 2.$$

By the condition $u \geq 2$ the function $\Psi(t-u)$ and the noises in the equation (7) are uncorrelated. Note that these estimators may be obtained from general criteria LSE [Vas/Ko].
From (7) and (8) we find the deviation of the estimator $\theta^*(T)$:

$$\theta^*(T) - \theta = G^{-1}(T,u)\zeta(T,u), \tag{9}$$

$$\zeta(T,u) = \zeta(T,u,1) + \zeta(T,u,2) + \zeta(T,u,3), \quad \zeta(T,u,1) = \int_0^T \Psi(t-u)\tilde{\Delta}\xi(t)dt,$$

$$\zeta(T,u,2) = \int_0^T \Psi(t-u)dV(t), \quad \zeta(T,u,3) = -\int_0^T \Psi(t-u)dV(t-1).$$

As we can see from the proof of Theorem 1 (Section 4), there are some increasing functions $\varphi(T)$ corresponding to the various regions for the parameter ϑ such that asymptotically on T the function $g(T,u) = \varphi^{-1}(T)G(T,u)$ is
a) nonrandom with $mes\{u \in (2,3] : \ g(u) = \lim\limits_{T\to\infty} g(T,u) = 0\} = 0$ and $g(0) > 0$ ($mes\{B\}$ is the Lebesque measure of the set B),
b) random with $P_\vartheta\{g(u) = \lim\limits_{T\to\infty} g(T,u) = 0\} = 0, \ u \geq 0$,
c) random periodic on T with the period $\Delta > 1$:
for some random periodic function $\tilde{g}(T,u)$ the next relationship

$$P_\vartheta\{\lim_{T\to\infty} |g(T,u) - \tilde{g}(T,u)| = 0\} = 1, \ u \geq 0,$$

holds.
Besides in the periodic case c) the limit function $\tilde{g}(T,u)$ have for every $u \geq 0$ two roots as a maximum on every interval of the lenght Δ. Then the normalized matrix $G(T,u)$ and as follows the deviation $\theta^*(T) - \theta$ may be unbounded.
To exclude this effect we need in a discretization of the observation time. Note that in the case of observations without noises we also need in similar discretization (by using Δ) for the investigation of an asymptotic properties of maximum likelihood estimators [Gu/Ku]. Our procedure is nonasymptotic and we can not use the unknown value Δ in construction of estimators.
Put for some $h \in (0, 1/3]$

$$r_n = \arg\max_{k=\overline{1,3}} |G(nh - kh, 2 + 3h)|.$$

Such choise of the value of h implies that for every $n \geq 1$ and $T > 0$ there is one or more values $nh - kh,\ k = \overline{1,3}$, where $\tilde{g}(nh - kh, T) \neq 0$. In such a way (see proof of Theorem 1) the normalized sequence $\{G(nh - r_n h, 2 + 3h),\ n \geq 1\}$ is nondegenerate in the case c) for any $h \in (0, 1/3]$ in asymptotic.
To construct the estimators with preassigned accuracy we change firstly the value n in the argument of G on the markovian stopping times. As we can see later (the inequalities (11)) this substitution gives us the possibility to control the second moments of the noise ζ.
Let $(c_n,\ n \geq 1)$ is some unboundedly increasing sequence of positive numbers.
We shall define the stopping times $(\tau_\varepsilon(n), n \geq 1)$ from the discrete sequence $\{kh,\ k \geq 1\}$ with the known step h by formula

$$\tau_\varepsilon(n) = h \inf\{k \geq 1 :\ \int_0^{kh} \Psi^2(t - 2 - 3h)dt \geq \varepsilon^{-1} c_n\},\ n \geq 1 \tag{10}$$

and put for $k = \overline{1,3},\ n \geq 1$

$$G_\varepsilon(n,k) = G(\tau_\varepsilon(n) - kh, 2 + 3h), \quad \Phi_\varepsilon(n,k) = \Phi_\varepsilon(\tau_\varepsilon(n) - kh, 2 + 3h),$$

$$\zeta_\varepsilon(n,k) = \zeta(\tau_\varepsilon(n) - kh, 2 + 3h);$$

$$k_n = \arg\max_{k=\overline{1,3}}\{|G_\varepsilon(n,k)|\},\ n \geq 1.$$

Now we introduce the sequence of estimators

$$\theta_\varepsilon(n) = G_\varepsilon^{-1}(n)\Phi_\varepsilon(n),\ n \geq 1,$$

$$G_\varepsilon(n) = G_\varepsilon(n, k_n), \quad \Phi_\varepsilon(n) = \Phi_\varepsilon(n, k_n),$$

which have the deviation

$$\theta_\varepsilon(n) - \theta = G_\varepsilon^{-1}(n)\zeta_\varepsilon(n), \quad \zeta_\varepsilon(n) = \zeta_\varepsilon(n, k_n),\ n \geq 1.$$

Further for some $h_0 \in (0, 1/3)$ we put a random $\mathcal{F}(0)$-measurable value h from some continue distribution on the interval $[h_0, 1/3]$. We need in such randomization in the case a) for the almost surely nondegeneration of the limit $g(2 + 3h) = \lim\limits_{n\to\infty} \varphi^{-1}(\tau_\varepsilon(n) - k_n h)G_\varepsilon(n, k_n)$.
Let us show that the second moments of the noise ζ calculated at times $\tau_\varepsilon(n) - k_n h,\ n \geq 1$ have known upper bounds. Note that the processes $(\zeta(T,i), \mathcal{F}(T)),\ i = \overline{1,3}$ are square integrable martingales and times $\tau_\varepsilon(n) - kh,\ n \geq 1,\ k = \overline{1,3}$ are markovian with respect to the system $(\mathcal{F}(T-2))$. By the properties of martingales [Li/Shi] and by the definition of $\tau_\varepsilon(n)$ we can obtain for all $\vartheta \in \mathcal{R}^2,\ k = \overline{1,3}$ and $n \geq 1$ the inequalities

$$E_\vartheta \zeta^2(\tau_\varepsilon(n) - kh, 2 + 3h, 1) \leq \overline{s}^2 E_\vartheta \int_0^{\tau_\varepsilon(n)-kh} \Psi^2(t - 2 - 3h)dt < \overline{s}^2 \varepsilon^{-1} c_n,$$

$$E_\vartheta \zeta^2(\tau_\varepsilon(n) - kh, 2+3h, i) \le \varepsilon^{-1} c_n,\ i = 2, 3.$$

Then for all $\varepsilon > 0$ and $n \ge 1$ the sequence $(\zeta_\varepsilon(n),\ n \ge 1)$ satisfies the inequalities

$$E_\vartheta \zeta_\varepsilon^2(n) \le \sum_{k=1}^{3} E_\vartheta \zeta^2(\tau_\varepsilon(n) - kh, 2+3h) \le$$

$$\le 3 \sum_{k=1}^{3} \sum_{i=1}^{3} E_\vartheta \zeta^2(\tau_\varepsilon(n) - kh, 2+3h, i) \le 9(2+\overline{s}^2)\varepsilon^{-1} c_n. \tag{11}$$

The asymptotic properties of the sequence $(G_\varepsilon(n), n \ge 1)$ and the inequalities (11) imply that the estimation of the parameter θ should be perfomed at times $\tau_\varepsilon(n) - k_n h,\ n \ge 1$. Note that the estimators $\theta_\varepsilon(n)$ are strong consistent (see Theorem 1).
In order to obtain the estimators with fixed least square deviation, taking into account the representation for the deviation of estimators $\theta_\varepsilon(n)$, one should control the behaviour of the sequence of random values $G_\varepsilon(n),\ n \ge 1$. It can be achieved by conducting the observation up to the time $\tau_\varepsilon(n) - k_n h$ with a specially chosen number n.
Let $(\kappa_n,\ n \ge 1)$ is some unboundedly increasing sequence of positive numbers. Put the stopping time

$$\nu_\varepsilon = \inf\{n \ge 1 : |G_\varepsilon(n)| \ge \rho^{1/2} \varepsilon^{-1} \kappa_n\},$$

where

$$\rho = 9(2+\overline{s}^2) \sum_{n \ge 1} c_n / \kappa_n^2.$$

We define the sequential plan $(T(\varepsilon), \theta_\varepsilon^*)$ of estimation θ by formulae

$$T(\varepsilon) = \tau_\varepsilon(\nu_\varepsilon),\ \theta_\varepsilon^* = \theta_\varepsilon(\nu_\varepsilon) = G_\varepsilon^{-1}(\nu_\varepsilon)\Phi_\varepsilon(\nu_\varepsilon). \tag{12}$$

It should be pointed out that the estimator (12) coincides with the sequential estimator which is obtained from general least square criteria [Vas/Ko].
The following theorem presents the conditions under which $T(\varepsilon)$ and θ_ε^* are well-defined and have the wanted property of preassigned mean square accuracy.
First we divide the parameter set Θ_1 into nine subsets, according to definitions of Section I.
Define the functions $u(a),\ a < 1$ and $w(a), a \in \mathcal{R}^1$ similar to [Gu/Ku]: consider a parametric curve $(a(\xi), b(\xi)),\ \xi > 0,\ \xi \ne \pi, 2\pi, \ldots,$ in $\mathcal{R}^2$ by

$$a(\xi) = \xi \cot \xi,\ b(\xi) = -\xi / \sin \xi,$$

then $b = u(a)$ and $b = w(a)$ are the branches of this curve corresponding to $\xi \in (0, \pi)$ and $(\pi, 2\pi)$ respectively. Put also $v(a) = -e^{a-1}$ and introduce the indices

$$i = \begin{cases} 0, & \alpha \ne \beta e^{v_0}, \\ 1, & \alpha = \beta e^{v_0}, \end{cases}$$

$$j = \begin{cases} 1, & \vartheta_0 < 1,\ u(\vartheta_0) < \vartheta_1 < -\vartheta_0, \\ 2, & -\vartheta_0 < \vartheta_1 < w(\vartheta_0), \\ 3, & \vartheta_0 > 1,\ v(\vartheta_0) < \vartheta_1 < -\vartheta_0, \\ 4, & \vartheta_0 > 1,\ \vartheta_1 = v(\vartheta_0), \\ 5, & \vartheta_1 > w(\vartheta_0), \\ 6, & \vartheta_0 < 1,\ \vartheta_1 < u(\vartheta_0) \text{ or } \vartheta_0 \geq 1,\ \vartheta_1 < v(\vartheta_0), \\ 7, & \vartheta_0 < 1,\ \vartheta_1 = -\vartheta_0,\ \vartheta_0 \neq 0, \\ 8, & \vartheta_0 > 1,\ \vartheta_1 = -\vartheta_0, \\ 9, & \vartheta_1 = w(\vartheta_0). \end{cases}$$

Note that sets, corresponds to various values of j are disjoint and the union of all the cases corresponds to $j = \overline{1,9}$ is the whole plane $\mathcal{R}^2$ minus some one-dimensional smooth curve. Besides we know, that $v_0 < 0$ if $j = 1$; $v_0 = 0$ if $j = 7$ and $v_0 > 0$ in all other cases; $v_1 < 0$ if $j = 1, 2, 7$; $v_1 = 0$ if $j = 8, 9$ and $v_1 > 0$ if $j = 3, 5$ [Gu/Ku]. Introduce the sets

$$I_1 = \{(0,1),(1,1),(1,2),(1,7)\},$$

$$I_2 = \{(0,2),(0,3),(0,5),(0,8),(0,9),(1,4)\},$$

$$I_3 = \{(1,3)\}, \quad I_4 = \{(0,4)\}, \quad I_5 = \{(0,6),(1,5),(1,6)\},$$

$$I_6 = I_2 \cup I_5 \setminus \{(1,5)\}, \quad I_7 = I_3 \cup \{(1,5)\}.$$

Theorem 1. *Let the conditions*

$$\sum_{n \geq 1} \frac{c_n}{\kappa_n^2} < \infty \tag{13}$$

and

$$\lim_{n \to \infty} \kappa_n / c_n = 0 \tag{14}$$

on the sequences (c_n) *and* (κ_n) *hold . Then*

I. *For any* $\varepsilon > 0$ *and every* $\theta \in \Theta_1$ *the sequential plan* $(T(\varepsilon), \theta_\varepsilon^*)$ *defined by* (12) *is closed* $(T(\varepsilon) < \infty\ P_\vartheta - a.s.)$ *and has the following properties*:

1°. *for any* $\varepsilon > 0$

$$\sup_{\Theta_1} E_\vartheta (\theta_\varepsilon^* - \theta)^2 \leq \varepsilon;$$

2°. *for every* $\theta \in \Theta_1$ *the following relationships with* $v_0 > v_1 > 0$ *hold:*

- *for* $(i,j) \in I_1$

$$0 < \underline{\lim_{\varepsilon \to 0}}\ \varepsilon \cdot T(\varepsilon)\ \leq\ \overline{\lim_{\varepsilon \to 0}}\ \varepsilon \cdot T(\varepsilon)\ <\ \infty\ \ P_\vartheta - a.s.,$$

- *for* $(i,j) \in I_2 \cup I_3 \cup I_5$

$$0 < \varliminf_{\varepsilon\to 0} [T(\varepsilon) - \frac{1}{2v_i}\ln\varepsilon^{-1}] \ \le \ \varlimsup_{\varepsilon\to 0} [T(\varepsilon) - \frac{1}{2v_i}\ln\varepsilon^{-1}] \ < \ \infty \ \ P_\vartheta - a.s.,$$

- *for* $(i,j) \in I_4$

$$0 < \varliminf_{\varepsilon\to 0} \ \varepsilon T^2(\varepsilon) e^{2v_0 T(\varepsilon)} \le \varlimsup_{\varepsilon\to 0} \ \varepsilon T^2(\varepsilon) e^{2v_0 T(\varepsilon)} < \infty \ \ P_\vartheta - a.s.$$

II. *For any* $\varepsilon > 0$ *and every* $\theta \in \Theta_1$ *the estimator* $\theta_\varepsilon(n)$ *is strong consistent:*

$$\lim_{n\to\infty} \theta_\varepsilon(n) = \theta \ \ P_\vartheta - a.s.$$

The proofs of this and subsequent theorem are given in Section 4.

3.2 Sequential estimation procedure II

Consider the estimation problem of $\vartheta \in \Theta_2$. Now we define on the basis of equation (6) the estimation procedure by the way of Section 1.
Let the value $\tilde{h}_0 \in (0, 1/5)$ and put $\tilde{h} \in [\tilde{h}_0, 1/5]$ as a known $\mathcal{F}(0)$-measured positive random value with continuous distribution.
Put the functions

$$\tilde{\Psi}_s(t) = \begin{cases} (\tilde{\Delta}Y(t), \tilde{\Delta}Y(t-s))', \ t \ge 1+s, \\ (0,0)', \ t < 1+s; \end{cases}$$

the sequence of stopping times

$$\tilde{\tau}_\varepsilon(n) = \tilde{h}\inf\{k \ge 1 : \ \int_0^{k\tilde{h}} ||\tilde{\Psi}_{\tilde{h}}(t-2-5\tilde{h})||^2 dt \ge \varepsilon^{-1} c_n\}, \ n \ge 1;$$

the matrices

$$\tilde{G}_\varepsilon(n,k) = \int_0^{\tilde{\tau}_\varepsilon(n)-k\tilde{h}} \tilde{\Psi}_{\tilde{h}}(t-2-5\tilde{h})\tilde{\Psi}_1'(t)dt, \ \ \tilde{\Phi}_\varepsilon(n,k) = \int_0^{\tilde{\tau}_\varepsilon(n)-k\tilde{h}} \tilde{\Psi}_{\tilde{h}}(t-2-5\tilde{h})d\tilde{\Delta}Y(t);$$

the times

$$\tilde{k}_n = \arg\min_{k=\overline{1,5}} ||\tilde{G}_\varepsilon^{-1}(n,k)||, \ n \ge 1;$$

the estimators

$$\tilde{\vartheta}_\varepsilon(n) = \tilde{G}_\varepsilon^{-1}(n)\tilde{\Phi}_\varepsilon(n), \ n \ge 1,$$
$$\tilde{G}_\varepsilon(n) = \tilde{G}_\varepsilon(n, \tilde{k}_n), \ \tilde{\Phi}_\varepsilon(n) = \tilde{\Phi}_\varepsilon(n, \tilde{k}_n);$$

and the stopping time

$$\tilde{\nu}_\varepsilon = \inf\{n \ge 1 : ||\tilde{G}_\varepsilon^{-1}(n)|| \le \varepsilon(\tilde{\rho}^{1/2}\kappa_n)^{-1}\}, \quad \tilde{\rho} = 15(2+\overline{s}^2)\sum_{n\ge 1} c_n/\kappa_n^2.$$

We take the discretisation step $\tilde{h}$ in such a way, because the determinant of the matrix $\tilde{G}$ has on every interval of the lenght $\Delta > 1$ asymptotically four roots as a maximum in the periodic case (see proof of Theorem 2).
Define the sequential estimation plan of ϑ by formulae

$$\tilde{T}(\varepsilon) = \tilde{\tau}_\varepsilon(\tilde{\nu}_\varepsilon), \quad \tilde{\vartheta}(\varepsilon) = \tilde{\vartheta}_\varepsilon(\tilde{\nu}_\varepsilon) = \tilde{G}_\varepsilon^{-1}(\tilde{\nu}_\varepsilon)\tilde{\Phi}_\varepsilon(\tilde{\nu}_\varepsilon). \tag{15}$$

We will state the results on the estimation of the parameter $\vartheta \in \Theta_2$ in the following theorem.
Theorem 2. *Let the conditions* (13) *and* (14) *on the sequences* (c_n) *and* (κ_n) *hold and let the parameter* $\vartheta = (a, b)'$ *in* (1) *is such that we have the stationary or periodic case. Then*

I. *For any* $\varepsilon > 0$ *and every* $\vartheta \in \Theta_2$ *the sequential plan* $(\tilde{T}(\varepsilon), \tilde{\vartheta}(\varepsilon))$ *defined by* (15) *is closed and possesses the following properties:*

1^o. *for any* $\varepsilon > 0$

$$\sup_{\Theta_2} E_\vartheta ||\tilde{\vartheta}(\varepsilon) - \vartheta||^2 \le \varepsilon;$$

2°. *for every* $\theta \in \Theta_2$ *the limiting inequalities*
- *in stationary case* $(v_0 < 0)$

$$0 < \underline{\lim_{\varepsilon\to 0}}\ \varepsilon \cdot \tilde{T}(\varepsilon) \ \le\ \overline{\lim_{\varepsilon\to 0}}\ \varepsilon \cdot \tilde{T}(\varepsilon) \ <\ \infty \quad P_\vartheta - a.s.,$$

- *in periodic case* $(v_0 > 0,\ v_0 \notin \Lambda)$

$$0 < \underline{\lim_{\varepsilon\to 0}}\ [\tilde{T}(\varepsilon) - \frac{1}{2v_0}\ln\varepsilon^{-1}] \ \le\ \overline{\lim_{\varepsilon\to 0}}\ [\tilde{T}(\varepsilon) - \frac{1}{2v_0}\ln\varepsilon^{-1}] \ <\ \infty \quad P_\vartheta - a.s.$$

are fulfilled.

II. *For any* $\varepsilon > 0$ *and every* $\vartheta \in \Theta_2$ *the estimator* $\vartheta_\varepsilon(n)$ *is strong consistent:*

$$\lim_{n\to\infty} \tilde{\vartheta}_\varepsilon(n) = \vartheta \quad P_\vartheta - a.s.$$

4 Proofs

Proof of Theorem 1. Firstly we establish the finiteness of stopping times $T(\varepsilon)$.
Put

$$\varphi_{ij}(T) = \begin{cases} T, & (i,j) \in I_1, \\ e^{2v_i T}, & (i,j) \in I_2 \cup I_3 \cup I_5, \\ T^2 e^{2v_0 T}, & (i,j) \in I_4. \end{cases}$$

Now we establish the auxiliary equalities for $u = 0$ and $u \ge 2$:
– for $(i,j) \in I_1 \cup I_2 \cup I_3 \cup I_4$

$$\lim_{T\to\infty} \frac{1}{\varphi_{ij}(T)} \int_0^T \Psi(t-u)\Psi(t)dt = f_{iju} \quad P_\vartheta - \text{a.s.}, \tag{16}$$

where f_{iju} are some constants or random values;
– for $(i,j) \in I_5$

$$\lim_{T\to\infty} |\frac{1}{\varphi_{ij}(T)} \int_0^T \Psi(t-u)\Psi(t)dt - f_{iju}(T)| = 0 \quad P_\vartheta - \text{a.s.}, \tag{17}$$

where $f_{iju}(T)$ are periodic random functions with the period $\Delta = 2\pi/\xi_0,\ \xi_0 \in (0,\pi)$ if $(i,j) = \{(0,6),(1,6)\}$ and $\Delta = 2\pi/\xi_1,\ \xi_1 \in (\pi, 2\pi)$ if $(i,j) = (1,5)$.
According to [Gu/Ku] (1997) by $\vartheta \in \Theta_1$ the solution $X(t)$ of (1) has the representation

$$X(t) = x_0(t)X_0(0) + b\int_{-1}^{0} x_0(t-s-1)X_0(s)ds + \int_0^t x_0(t-s)dW(s),\ t \geq 0, \tag{18}$$

where $x_0(t) = 0,\ t \in [-1,0),\ x_0(0) = 1$ and by $t \to \infty$

$$x_0(t) = \begin{cases} o(e^{\gamma t}), & \gamma < 0, \quad j = 1, \\ \frac{1}{v_0-a+1}e^{v_0 t} + o(e^{\gamma t}), & \gamma < 0, \quad j = 2, \\ \frac{1}{v_0-a+1}e^{v_0 t} + \frac{1}{a-v_1-1}e^{v_1 t} + o(e^{\gamma_1 t}), & \gamma_1 < v_1,\ j = 3, \\ (2t+\frac{2}{3})e^{v_0 t} + o(e^{\gamma_0 t}), & \gamma_0 < v_0,\ j = 4, \\ \frac{1}{v_0-a+1}e^{v_0 t} + \phi_1(t)e^{v_1 t} + o(e^{\gamma_1 t}), & \gamma_1 < v_1,\ j = 5, \\ \phi_0(t)e^{v_0 t} + o(e^{\gamma_0 t}), & \gamma_0 < v_0,\ j = 6, \\ \frac{1}{1-a} + o(e^{\gamma t}), & \gamma < 0, \quad j = 7, \\ \frac{1}{v_0-a+1}e^{v_0 t} - \frac{1}{a-1} + o(e^{\gamma t}), & \gamma < 0, \quad j = 8, \\ \frac{1}{v_0-a+1}e^{v_0 t} + \phi_1(t) + o(e^{\gamma t}), & \gamma < 0, \quad j = 9, \end{cases}$$

$$\phi_i(t) = A_i \cos \xi_i t + B_i \sin \xi_i t,$$

$$A_i = \frac{2(v_i - a + 1)}{(v_i - a + 1)^2 + \xi_i^2},\ B_i = \frac{2\xi_i}{(v_i - a + 1)^2 + \xi_i^2},\ i = 0, 1.$$

By the definition of Ψ we have

$$\Psi(t) = \tilde{\Psi}(t) + \tilde{V}(t),\ t \geq -1, \tag{19}$$

$$\tilde{\Psi}(t) = \begin{cases} \sigma^{-1}(\beta\tilde{X}(t) - \alpha\tilde{X}(t-1)), & t \geq 2, \\ 0, & t \in [-1,2], \end{cases}$$

$$\tilde{X}(t) = \int_{t-1}^t X(s)ds,$$

$$\tilde{V}(t) = \begin{cases} \sigma^{-1}(\beta\tilde{\Delta}V(t) - \alpha\tilde{\Delta}V(t-1)), & t \geq 2, \\ 0, & t \in [-1,2]. \end{cases}$$

It is easily to show that the process $(\tilde{X}(t))$ may be represented in the form

$$\tilde{X}(t) = \sigma^{-1}(\tilde{x}_0(t)X_0(0) + b\int_{-1}^{0} \tilde{x}_0(t-s-1)X_0(s)ds + \int_0^t \tilde{x}_0(t-s)dW(s))$$

for $t \geq 1,\ \tilde{X}(t) = \int_{t-1}^{0} X_0(s)ds + \int_0^t X(s)ds$ for $t \in [0,1)$ and $\tilde{X}(t) = 0$ for $t \in [-1,0)$. Here function $\tilde{x}_0(t) = \int_{t-1}^{t} x_0(s)ds$ may be defined as $\tilde{x}_0(t) = 0,\ t \in [-1,0]$ and by $t \to \infty$

$$\tilde{x}_0(t) = \begin{cases} o(e^{\gamma t}), & \gamma < 0, \quad j = 1, \\ \frac{1-e^{-v_0}}{v_0(v_0-a+1)}e^{v_0 t} + o(e^{\gamma t}), & \gamma < 0, \quad j = 2, \\ \frac{1-e^{-v_0}}{v_0(v_0-a+1)}e^{v_0 t} + \frac{1-e^{-v_1}}{v_1(a-v_1-1)}e^{v_1 t} + o(e^{\gamma_1 t}), & \gamma_1 < v_1,\ j = 3, \\ \frac{2}{v_0}[(1-e^{-v_0})t + e^{-v_0} - \frac{1-e^{-v_0}}{v_0}]e^{v_0 t} + o(e^{\gamma_0 t}), & \gamma_0 < v_0,\ j = 4, \\ \frac{1-e^{-v_0}}{v_0(v_0-a+1)}e^{v_0 t} + \tilde{\phi}_1(t)e^{v_1 t} + o(e^{\gamma_1 t}), & \gamma_1 < v_1,\ j = 5, \\ \tilde{\phi}_0(t)e^{v_0 t} + o(e^{\gamma_0 t}), & \gamma_0 < v_0,\ j = 6, \\ \frac{1}{1-a} + o(e^{\gamma t}), & \gamma < 0, \quad j = 7, \\ \frac{1-e^{-v_0}}{v_0(v_0-a+1)}e^{v_0 t} - \frac{1}{a-1} + o(e^{\gamma t}), & \gamma < 0, \quad j = 8, \\ \frac{1-e^{-v_0}}{v_0(v_0-a+1)}e^{v_0 t} + \tilde{\phi}_1(t) + o(e^{\gamma t}), & \gamma < 0, \quad j = 9, \end{cases}$$

where

$$\tilde{\phi}_i(t) = \tilde{A}_i \cos \xi_i t + \tilde{B}_i \sin \xi_i t,$$

$$\tilde{A}_i = \frac{1}{v_i^2 + \xi_i^2}[\xi_i e^{-v_i} \sin \xi_i - v_i e^{-v_i} \cos \xi_i + v_i]A_i + \frac{1}{v_i^2 - \xi_i^2}[v_i e^{-v_i} \sin \xi_i + v_i e^{-v_i} \cos \xi_i - \xi_i]B_i,$$

$$\tilde{B}_i = \frac{1}{v_i^2 + \xi_i^2}[\xi_i - v_i e^{-v_i} \sin \xi_i - \xi_i e^{-v_i} \cos \xi_i]A_i + \frac{1}{v_i^2 - \xi_i^2}[\xi_i e^{-v_i} \sin \xi_i - v_i e^{-v_i} \cos \xi_i + v_i]B_i.$$

Analogiously we can get the representation for the process $\tilde{\Psi}(t)$ with $x_\Psi(t) = \beta\tilde{x}_0(t) - \alpha\tilde{x}_0(t-1)$:

$$\tilde{\Psi}(t) = \sigma^{-1}(x_\Psi(t)X_0(0) + b\int_{-1}^{0} x_\Psi(t-s-1)X_0(s)ds + \int_0^t x_\Psi(t-s)dW(s)) \quad (20)$$

for $t \geq 2;\ x_\Psi(t) = 0$ for $t \in [-1,0]$; by $t \to \infty$

$$x_\Psi(t) = \begin{cases} o(e^{\gamma t}), & \gamma < 0, \quad j = 1, \\ \frac{(1-e^{-v_0})(\beta-\alpha e^{-v_0})}{v_0(v_0-a+1)}e^{v_0 t} + o(e^{\gamma t}), & \gamma < 0, \quad j = 2, \\ \frac{(1-e^{-v_0})(\beta-\alpha e^{-v_0})}{v_0(v_0-a+1)}e^{v_0 t} + \frac{(1-e^{-v_1})(\beta-\alpha e^{-v_1})}{v_1(a-v_1-1)}e^{v_1 t} + o(e^{\gamma_1 t}), & \gamma_1 < v_1,\ j = 3, \\ \frac{2}{v_0}\{[(1-e^{-v_0})t + e^{-v_0} - \frac{1-e^{-v_0}}{v_0}](\beta - \alpha e^{-v_0}) & \\ +\alpha e^{-v_0}(1-e^{-v_0})\}e^{v_0 t} + o(e^{\gamma_0 t}), & \gamma_0 < v_0,\ j = 4, \\ \frac{(1-e^{-v_0})(\beta-\alpha e^{-v_0})}{v_0(v_0-a+1)}e^{v_0 t} + \phi_1^*(t)e^{v_1 t} + o(e^{\gamma_1 t}), & \gamma_1 < v_1,\ j = 5, \\ \phi_0^*(t)e^{v_0 t} + o(e^{\gamma_0 t}), & \gamma_0 < v_0,\ j = 6. \\ \frac{\beta-\alpha}{1-a} + o(e^{\gamma t}), & \gamma < 0, \quad j = 7, \\ \frac{(1-e^{-v_0})(\beta-\alpha e^{-v_0})}{v_0(v_0-a+1)}e^{v_0 t} - \frac{\beta-\alpha}{a-1} + o(e^{\gamma t}), & \gamma < 0, \quad j = 8, \\ \frac{(1-e^{-v_0})(\beta-\alpha e^{-v_0})}{v_0(v_0-a+1)}e^{v_0 t} + \phi_1^*(t) + o(e^{\gamma t}), & \gamma < 0, \quad j = 9. \end{cases}$$

Here

$$\phi_i^*(t) = A_i^* \cos \xi_i t + B_i^* \sin \xi_i t,$$

$$A_i^* = \beta\tilde{A}_i - \alpha\tilde{A}_i e^{-v_i}\cos\xi_i - \alpha\tilde{B}_i e^{-v_i}\sin\xi_i,$$
$$B_i^* = \beta\tilde{B}_i - \alpha\tilde{A}_i e^{-v_i}\sin\xi_i - \alpha\tilde{B}_i e^{-v_i}\cos\xi_i,\ i = 0, 1.$$

Processes $\tilde{\Psi}(t)$ and $\tilde{V}(t)$ are mutually independent and process $\tilde{\Psi}(t)$ is the process of type (18). Then we can get the following limiting relationships by the way of [Gu/Ku] and [Ku/Vas]:

– for $(i,j) \in I_1$

$$f_{iju} = \begin{cases} \sigma^{-2}(\int_0^\infty x_\Psi^2(t)dt + 1), & u = 0, \\ \sigma^{-2}\int_0^\infty x_\Psi(t+u)x_\Psi(t)dt, & u \geq 2; \end{cases}$$

– for $(i,j) \in I_2 \cup I_3$

$$\lim_{t\to\infty} e^{-v_i t}\tilde{\Psi}(t) = \tilde{c}_{ij}U_i \quad P_\vartheta - \text{a.s.},$$

$$U_i = X_0(0) + b\int_{-1}^0 e^{-v_i(s+1)}X_0(s)ds + \int_0^\infty e^{-v_i s}dW(s),$$

$$\tilde{c}_{0j} = \frac{(1-e^{-v_0})(\beta - \alpha e^{-v_0})}{v_0(v_0 - a + 1)}\sigma^{-1},\ \tilde{c}_{13} = \frac{(1-e^{-v_1})(1-e^{v_0-v_1})}{v_1(a - v_1 - 1)}\beta\sigma^{-1},$$

$$\tilde{c}_{14} = \frac{2(1-e^{-v_0})}{v_0}\beta\sigma^{-1}$$

and as follows

$$f_{iju} = \frac{\tilde{c}_{ij}^2 U_i^2}{2v_i}e^{-v_i u},\ u \geq 0;$$

– for $(i,j) \in I_4$

$$\lim_{t\to\infty} t^{-1}e^{-v_0 t}\tilde{\Psi}(t) = \tilde{c}_0 U_0 \quad P_\vartheta - \text{a.s.},$$

$$\tilde{c}_0 = \frac{2(1-e^{-v_0})}{v_0}(\beta - \alpha e^{-v_0})\sigma^{-1}$$

and

$$f_{iju} = \frac{\tilde{c}_0^2 U_0^2}{4v_0}e^{-v_i u},\ u \geq 0;$$

– for $(i,j) \in I_5$

$$\lim_{t\to\infty} |e^{-v_i t}\tilde{\Psi}(t) - U_{ij}(t)| = 0 \quad P_\vartheta - \text{a.s.},$$

where by $(i,j) \in I_5 \setminus \{(1,6)\}$

$$U_{ij}(t) = \sigma^{-1}(X_0(0)\phi_i^*(t) + b\int_{-1}^0 \phi_i^*(t-s-1)e^{-v_i(s+1)}X_0(s)ds + \int_0^\infty \phi_i^*(t-s)e^{-v_i s}dW(s)),$$

$$U_{16}(t) = \sigma^{-1}(X_0(0)\phi_0^*(t) + b\int_{-1}^0 \phi_0^*(t-s-1)e^{-v_i(s+1)}X_0(s)ds + \int_0^\infty \phi_0^*(t-s)e^{-v_i s}dW(s))$$

and

$$f_{iju}(T) = \sigma^{-2}e^{v_i u}\int_0^\infty e^{-2v_i t}U_i(T-t)\hat{U}_i(T-t)dt,\ u \geq 0,$$

$$\hat{U}_i(t) = X_0(0)\hat{\phi}_i(t) + b\int_{-1}^{0} \hat{\phi}_i(t-s-1)e^{-v_i(s+1)}X_0(s)ds + \int_0^{\infty} \hat{\phi}_i(t-s)e^{-v_i s}dW(s),$$

$$\hat{\phi}_i(t) = \hat{A}_i \cos\xi_i t + \hat{B}_i \sin\xi_i t,$$

$$\hat{A}_i = A_i^* \cos\xi_i u - B_i^* \sin\xi_i u,\ \hat{B}_i = -A_i^* \sin\xi_i u + B_i^* \cos\xi_i u,\ i = 0, 1.$$

Note that $\hat{U}_i(t) \equiv U_i(t)$ by $u = 0$.
The relationships (16), (17) are proved.
Because the function $x_\Psi(t)$ is defined similar to the function $x_0(t)$ (its structure and properties have been investigated, for example, in [Gu/Ku]), we can see that $mes\{u \in (2,3] :\ f_{iju} = 0\} = 0$ in the cases $(i,j) \in I_1$ and it is obviously that $f_{iju} \neq 0\ P_\vartheta -$ a.s. for $(i,j) \in I_2 \cup I_3 \cup I_4$.
Put for $(i,j) \in I_5$

$$\overline{f}_{iju} = \sup_{t\in(0,\infty)} |f_{iju}(t)|,\ \ \underline{f}_{ij0} = \inf_{t\in(0,\infty)} |f_{ij0}(t)|.$$

It is clear that for $u = 0$ and $u \geq 2$ these values are positive and finite. From here and (16), (17) follows, in particular, the finiteness of the stopping times $\tau_\varepsilon(n),\ n \geq 1$ defined by (10), because for all $(i,j) \in I_1 \cup I_2 \cup I_3 \cup I_4$ the limits $f_{ij0} > 0\ P_\vartheta - a.s.$
By using (16) and the definition of $\tau_\varepsilon(n)$ we have the next limiting equalities:
– for $(i,j) \in I_1$

$$\lim_{n\to\infty} \frac{\tau_\varepsilon(n)}{\varepsilon^{-1}c_n} = \lim_{\varepsilon\to 0} \frac{\tau_\varepsilon(n)}{\varepsilon^{-1}c_n} = f_{ij0}^{-1}\ P_\vartheta - \text{a.s.}; \tag{21}$$

take into account the inequalities for all the cases

$$\int_0^{\tau_\varepsilon(n)-2-4h} \Psi^2(t)dt < \varepsilon^{-1}c_n \leq \int_0^{\tau_\varepsilon(n)-2-3h} \Psi^2(t)dt,$$

we obtain:
– for $(i,j) \in I_2 \cup I_3$

$$e^{2v_i(2+3h)} f_{ij0}^{-1} \leq \underline{\lim}_{n\to\infty} \frac{e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n} \leq \overline{\lim}_{n\to\infty} \frac{e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n} \leq e^{4v_i(1+2h)} f_{ij0}^{-1}\ P_\vartheta - \text{a.s.}, \tag{22}$$

$$e^{2v_i(2+3h)} f_{ij0}^{-1} \leq \underline{\lim}_{\varepsilon\to 0} \frac{e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n} \leq \overline{\lim}_{\varepsilon\to 0} \frac{e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n} \leq e^{4v_i(1+2h)} f_{ij0}^{-1}\ P_\vartheta - \text{a.s.}, \tag{23}$$

and as follows

$$2 + 3h - \frac{1}{2v_i}\ln f_{ij0} + \frac{1}{2v_i}\ln\varepsilon^{-1} \leq \underline{\lim}_{n\to\infty}[\tau_\varepsilon(n) - \frac{1}{2v_i}\ln c_n] \leq$$

$$\leq \overline{\lim}_{n\to\infty}[\tau_\varepsilon(n) - \frac{1}{2v_i}\ln c_n] \leq 2(1+2h) - \frac{1}{2v_i}\ln f_{ij0} + \frac{1}{2v_i}\ln\varepsilon^{-1}\ P_\vartheta - \text{a.s.}, \tag{24}$$

$$2+3h-\frac{1}{2v_i}\ln f_{ij0}+\frac{1}{2v_i}\ln c_n\le\underline{\lim}_{\varepsilon\to 0}[\tau_\varepsilon(n)-\frac{1}{2v_i}\ln\varepsilon^{-1}]\le$$

$$\le\overline{\lim}_{\varepsilon\to 0}[\tau_\varepsilon(n)-\frac{1}{2v_i}\ln\varepsilon^{-1}]\le 2(1+2h)-\frac{1}{2v_i}\ln f_{ij0}+\frac{1}{2v_i}\ln c_n\ P_\vartheta-\text{a.s.};\qquad(25)$$

– for $(i,j)\in I_4$

$$e^{2v_i(2+3h)}f_{ij0}^{-1}\le\underline{\lim}_{n\to\infty}\frac{\tau_\varepsilon^2(n)e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n}\le\overline{\lim}_{n\to\infty}\frac{\tau_\varepsilon^2(n)e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n}\le e^{4v_i(1+2h)}f_{ij0}^{-1}\ P_\vartheta-\text{a.s.},(26)$$

$$e^{2v_i(2+3h)}f_{ij0}^{-1}\le\underline{\lim}_{\varepsilon\to 0}\frac{\tau_\varepsilon^2(n)e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n}\le\overline{\lim}_{\varepsilon\to 0}\frac{\tau_\varepsilon^2(n)e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n}\le e^{4v_i(1+2h)}f_{ij0}^{-1}\ P_\vartheta-\text{a.s.}\quad(27)$$

From (17) and by the definition (10) of moments $\tau_\varepsilon(n)$ for all $(i,j)\in I_5$ we have

$$e^{2v_i(2+3h)}\overline{f}_{ij0}^{-1}\le\underline{\lim}_{n\to\infty}\frac{e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n}\le\overline{\lim}_{n\to\infty}\frac{e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n}\le e^{4v_i(1+2h)}\underline{f}_{ij0}^{-1}\ P_\vartheta-\text{a.s.}\qquad(28)$$

and

$$e^{2v_i(2+3h)}\overline{f}_{ij0}^{-1}\le\underline{\lim}_{\varepsilon\to 0}\frac{e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n}\le\overline{\lim}_{\varepsilon\to 0}\frac{e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n}\le e^{4v_i(1+2h)}\underline{f}_{ij0}^{-1}\ P_\vartheta-\text{a.s.}\qquad(29)$$

From (28) we can obtain for every $\varepsilon>0$

$$2+3h-\frac{1}{2v_i}\ln\overline{f}_{ij0}+\frac{1}{2v_i}\ln\varepsilon^{-1}\le\underline{\lim}_{n\to\infty}[\tau_\varepsilon(n)-\frac{1}{2v_i}\ln c_n]\le$$

$$\le\overline{\lim}_{n\to\infty}[\tau_\varepsilon(n)-\frac{1}{2v_i}\ln c_n]\le 2(1+2h)-\frac{1}{2v_i}\ln\underline{f}_{ij0}+\frac{1}{2v_i}\ln\varepsilon^{-1}\ P_\vartheta-\text{a.s.}\qquad(30)$$

and from (29) for $n\ge 1$

$$2+3h-\frac{1}{2v_i}\ln\overline{f}_{ij0}+\frac{1}{2v_i}\ln c_n\le\underline{\lim}_{\varepsilon\to 0}[\tau_\varepsilon(n)-\frac{1}{2v_i}\ln\varepsilon^{-1}]\le$$

$$\le\overline{\lim}_{\varepsilon\to 0}[\tau_\varepsilon(n)-\frac{1}{2v_i}\ln\varepsilon^{-1}]\le 2(1+2h)-\frac{1}{2v_i}\ln\underline{f}_{ij0}+\frac{1}{2v_i}\ln c_n\ P_\vartheta-\text{a.s.}\qquad(31)$$

Note that in cases $I_2\cup I_3\cup I_5$

$$\lim_{n\to\infty}\frac{\tau_\varepsilon(n)}{\ln c_n}=\lim_{\varepsilon\to 0}\frac{\tau_\varepsilon(n)}{\ln\varepsilon^{-1}}=\frac{1}{2v_i}\ P_\vartheta-\text{a.s.}\qquad(32)$$

Put $\delta_\varepsilon(n)=\tau_\varepsilon(n)-k_nh$.
Now we can show the finiteness of the stopping time ν_ε. From (16), (21), (22) and

(26) we have:
– for $(i,j) \in I_1$

$$\lim_{n\to\infty} \frac{1}{c_n} \int_2^{\delta_\varepsilon(n)} \Psi(t-2-3h)\Psi(t)dt = (\varepsilon f_{ij0})^{-1} f_{ij(2+3h)} \ P_\vartheta - \text{a.s.}; \tag{33}$$

– for $(i,j) \in I_2 \cup I_3 \cup I_4$

$$e^{4v_i}(\varepsilon f_{ij0})^{-1}|f_{ij(2+3h)}| \le \lim_{n\to\infty} |\frac{1}{c_n} \int_2^{\delta_\varepsilon(n)} \Psi(t-2-3h)\Psi(t)dt| \le$$

$$\le \overline{\lim_{n\to\infty}} |\frac{1}{c_n} \int_2^{\delta_\varepsilon(n)} \Psi(t-2-3h)\Psi(t)dt| \le e^{2v_i(2+3h)}(\varepsilon f_{ij0})^{-1}|f_{ij(2+3h)}| \ P_\vartheta - \text{a.s.} \tag{34}$$

Consider the cases $(i,j) \in I_5$. For all $u \ge 2$ and $(i,j) \in I_5$ functions $f_{iju}(T)$ are periodic with corresponding periods $\Delta > 1$ and each of them has maximum two roots.
Denote these roots for $u = 2+3h$ as $t_m(i,j),\ m = 1,2$ on the set $(0,\Delta]$. Then define V_{ij} the union of open disjoint neighbourhoods with the radius less then $1/6$ for all roots $t_m(i,j) + N\Delta,\ m = 1,2,\ N \ge 0$ and put

$$\mathcal{R}_{ij}^+ = (0,\infty) \setminus V_{ij}.$$

Define

$$f_{iju}^* = \inf_{t\in\mathcal{R}_{ij}^+} |f_{iju}(t)|,$$

$$Q_n(i,j) = \{k = \overline{1,3} : nh - kh \in \mathcal{R}_{ij}^+\},$$

$$r_{ij}(n) = \arg \max_{k\in Q_n(ij)} |f_{ij(2+3h)}(nh-kh)|.$$

By the definition $f_{iju}^* > 0$ for $u = 0$ and $u \ge 2$. Note that for any $h \in (0,1/3]$ and $(i,j) \in I_5$ the sets $Q_n(i,j)$ are not empty and for sufficiently large n from (17) we have

$$r_{ij}(n) = \arg \max_{k\in Q_n(i,j)} |e^{-2v_i(nh-kh)} \int_0^{nh-kh} \Psi(t-2-3h)\Psi(t)dt|,$$

besides by the definition of $Q_n(i,j)$

$$f_{ij(2+3h)}^* \le |f_{ij(2+3h)}(nh - r_{ij}(n)h)| \le \overline{f}_{ij(2+3h)},\ n >> 1$$

and

$$f_{ij(2+3h)}^* \le |e^{-2v_i(nh-r_{ij}(n)h)} \int_0^{nh-r_{ij}(n)h} \Psi(t-2-3h)\Psi(t)dt| \le \overline{f}_{ij(2+3h)},\ n >> 1.$$

Then for $(i,j) \in I_5$ we obtain

$$\overline{\lim_{n\to\infty}} |e^{-2v_i nh} \int_0^{nh-r_n h} \Psi(t-2-3h)\Psi(t)dt| =$$

$$= \overline{\lim_{n\to\infty}} e^{-2v_i r_n h} |e^{-2v_i(nh-r_n h)} \int_0^{nh-r_n h} \Psi(t-2-3h)\Psi(t)dt| \le e^{-2v_i h}\overline{f}_{ij(2+3h)},$$

$$\underline{\lim_{n\to\infty}} |e^{-2v_i nh} \int_0^{nh-r_n h} \Psi(t-2-3h)\Psi(t)dt| \ge$$

$$\ge \underline{\lim_{n\to\infty}} e^{-2v_i r_{ij}(n)h} |e^{-2v_i(nh-r_{ij}(n)h)} \int_0^{nh-r_{ij}(n)h} \Psi(t-2-3h)\Psi(t)dt| \ge e^{-6v_i h} f^*_{ij(2+3h)}$$

and as follows for all $\varepsilon > 0$

$$e^{-6v_i h} f^*_{ij(2+3h)} \le \underline{\lim_{n\to\infty}} |e^{-2v_i \tau_\varepsilon(n)} \int_0^{\tau_\varepsilon(n)-k_n h} \Psi(t-2-3h)\Psi(t)dt| \le$$

$$\le \overline{\lim_{n\to\infty}} |e^{-2v_i \tau_\varepsilon(n)} \int_0^{\tau_\varepsilon(n)-k_n h} \Psi(t-2-3h)\Psi(t)dt| \le e^{-2v_i h}\overline{f}_{ij(2+3h)}. \quad (35)$$

In such a way for the cases $(i,j) \in I_5$ from (28) and (35) we have

$$e^{4v_i}(\varepsilon\overline{f}_{ij0})^{-1} f^*_{ij(2+3h)} \le \underline{\lim_{n\to\infty}} \frac{1}{c_n} | \int_2^{\delta_\varepsilon(n)} \tilde{\Psi}(t-2-3h)\tilde{\Psi}(t)dt| \le$$

$$\le \overline{\lim_{n\to\infty}} \frac{1}{c_n} | \int_2^{\delta_\varepsilon(n)} \tilde{\Psi}(t-2-3h)\tilde{\Psi}(t)dt| \le e^{2v_i(2+3h)}(\varepsilon \underline{f}_{ij0})^{-1}\overline{f}_{ij(2+3h)} \ P_\vartheta - \text{a.s.} \quad (36)$$

The finiteness of ν_ε follows from the definition ν_ε, (33), (34), (36) and the condition (14) on the sequences (c_n) and (κ_n).
Then the finiteness of the stopping times $T(\varepsilon)$ is established.
Let us estimate the mean square deviation of θ^*_ε. From (11) and by definitions of the stopping time ν_ε and ρ it follows that for all $\vartheta \in \mathcal{R}^2$

$$E_\vartheta(\theta^*_\varepsilon - \theta)^2 = E_\vartheta G^{-2}_\varepsilon(\nu_\varepsilon)\zeta^2_\varepsilon(\nu_\varepsilon) \le \frac{\varepsilon^2}{\rho} E_\vartheta \frac{1}{\kappa^2_{\nu_\varepsilon}} \zeta^2_\varepsilon(\nu_\varepsilon) \le$$

$$\le \frac{\varepsilon^2}{\rho} \sum_{n\ge 1} \frac{1}{\kappa^2_n} E_\vartheta \zeta^2_\varepsilon(n) \le \frac{9(2+\overline{s}^2)\varepsilon}{\rho} \sum_{n\ge 1} \frac{c_n}{\kappa^2_n} = \varepsilon.$$

Then the first property of sequential plans $(T(\varepsilon), \theta^*_\varepsilon)$ is established.
In order to establish the second property note that similar to (33), (34), (36) for all $n \ge 1$ we can obtain
– for $(i,j) \in I_1$

$$\lim_{\varepsilon\to 0} \varepsilon \int_0^{\delta_\varepsilon(n)} \Psi(t-2-3h)\Psi(t)dt = f^{-1}_{ij0} f_{ij(2+3h)} c_n \ P_\vartheta - \text{a.s.}; \quad (37)$$

– for $(i,j) \in I_2 \cup I_3 \cup I_4$

$$e^{4v_i} f^{-1}_{ij0} |f_{ij(2+3h)}| c_n \le \underline{\lim_{\varepsilon\to 0}} \varepsilon \left| \int_0^{\delta_\varepsilon(n)} \Psi(t-2-3h)\Psi(t)dt \right| \le$$

$$\leq \overline{\lim_{\varepsilon\to 0}}\ \varepsilon \left| \int_0^{\delta_\varepsilon(n)} \Psi(t-2-3h)\Psi(t)dt \right| \leq e^{2v_i(2+3h)} f_{ij0}^{-1} |f_{ij(2+3h)}| c_n \ P_\vartheta - \text{a.s.}; \qquad (38)$$

– for $(i,j) \in I_5$

$$e^{4v_i} \overline{f}_{ij0}^{-1} f^*_{ij(2+3h)} c_n \leq \underline{\lim_{\varepsilon\to 0}}\, \varepsilon \left| \int_0^{\delta_\varepsilon(n)} \Psi(t-2-3h)\Psi(t)dt \right| \leq$$

$$\leq \overline{\lim_{\varepsilon\to 0}}\ \varepsilon \left| \int_0^{\delta_\varepsilon(n)} \Psi(t-2-3h)\Psi(t)dt \right| \leq e^{2v_i(2+3h)} \underline{f}_{ij0}^{-1} \overline{f}_{ij(2+3h)} c_n \ P_\vartheta - \text{a.s.} \qquad (39)$$

Similar to [Vas/Ko] by the definition of ν_ε and from (37)-(39) we can see that for sufficiently small ε and $(i,j) \in I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5$

$$\nu'_{ij} \leq \nu_\varepsilon \leq \nu''_{ij} \ \ P_\vartheta - \text{a.s.}, \qquad (40)$$

where

$$\nu'_{ij} = \max\{\inf\{n \geq 1 : \ c_n/\kappa_n > g'_{ij}\} - 1, \ 1\},$$

$$\nu''_{ij} = \inf\{n \geq 1 : \ c_n/\kappa_n > g''_{ij}\},$$

$$g'_{ij} = \begin{cases} \rho^{1/2} f_{ij0} |f^{-1}_{ij(2+3h)}|, & (i,j) \in I_1, \\ \rho^{1/2} e^{-2v_i(2+3h)} f_{ij0} |f^{-1}_{ij(2+3h)}|, & (i,j) \in I_2 \cup I_3 \cup I_4, \\ \rho^{1/2} e^{-2v_i(2+3h)} \underline{f}_{ij0} \overline{f}^{-1}_{ij(2+3h)}, & (i,j) \in I_5, \end{cases}$$

$$g''_{ij} = \begin{cases} g'_{ij}, & (i,j) \in I_1, \\ \rho^{1/2} e^{-4v_i} f_{ij0} |f^{-1}_{ij(2+3h)}|, & (i,j) \in I_2 \cup I_3 \cup I_4, \\ \rho^{1/2} e^{-4v_i} \overline{f}_{ij0} (f^*_{ij(2+3h)})^{-1}, & (i,j) \in I_5. \end{cases}$$

From (12), (21), (25), (27), (31) and (40) follows the third assertion of Theorem 1:
– for $(i,j) \in I_1$

$$f^{-1}_{ij0} c_{\nu'_{ij}} \leq \underline{\lim_{\varepsilon\to 0}}\ \varepsilon T(\varepsilon) \leq \overline{\lim_{\varepsilon\to 0}}\ \varepsilon T(\varepsilon) \leq f^{-1}_{ij0} c_{\nu''_{ij}} \ P_\vartheta - \text{a.s.};$$

– for $(i,j) \in I_2 \cup I_3$

$$2 + 3h - \frac{1}{2v_i} \ln f_{ij0} + \frac{1}{2v_i} \ln c_{\nu'_{ij}} \leq \underline{\lim_{\varepsilon\to 0}}\, [T(\varepsilon) - \frac{1}{2v_i} \ln \varepsilon^{-1}] \leq$$

$$\leq \overline{\lim_{\varepsilon\to 0}}\, [T(\varepsilon) - \frac{1}{2v_i} \ln \varepsilon^{-1}] \leq 2(1+2h) - \frac{1}{2v_i} \ln f_{ij0} + \frac{1}{2v_i} \ln c_{\nu''_{ij}} \ P_\vartheta - \text{a.s.};$$

– for $(i,j) \in I_4$

$$e^{2v_0(2+3h)} f^{-1}_{ij0} c_{\nu'_{ij}} \leq \underline{\lim_{\varepsilon\to 0}}\ \varepsilon T^2(\varepsilon) e^{2v_0 T(\varepsilon)} \leq \overline{\lim_{\varepsilon\to 0}}\ \varepsilon T^2(\varepsilon) e^{2v_0 T(\varepsilon)} \leq$$

$$\leq e^{4v_0(1+2h)} f^{-1}_{ij0} c_{\nu''_{ij}} \ P_\vartheta - \text{a.s.};$$

– for $(i,j) \in I_5$

$$2+3h-\frac{1}{2v_i}\ln \overline{f}_{ij0}+\frac{1}{2v_i}\ln c_{\nu'_{ij}} \le \underline{\lim}_{\varepsilon\to 0}[T(\varepsilon)-\frac{1}{2v_i}\ln \varepsilon^{-1}] \le$$

$$\le \overline{\lim}_{\varepsilon\to 0}[T(\varepsilon)-\frac{1}{2v_i}\ln \varepsilon^{-1}] \le 2(1+2h)-\frac{1}{2v_i}\ln \underline{f}_{ij0}+\frac{1}{2v_i}\ln c_{\nu''_{ij}} \ P_\vartheta - \text{a.s.}$$

Then the second property of sequential plans and as follows the first assertion of Theorem 1 are established.
In order to proof the second assertion of Theorem 1 note that according to (33), (34) and (36)

$$\overline{\lim_{n\to\infty}} c_n |G_\varepsilon^{-1}(n)| < \infty$$

and from (13), (14) follows that

$$\sum_{n\ge 1}\frac{1}{c_n} < \infty.$$

In view of the form for the deviation of the estimators $\theta_\varepsilon(n)$ it suffice to establish the next limiting equality

$$\lim_{n\to\infty}\frac{1}{c_n}\zeta_\varepsilon(n) = 0 \ P_\vartheta - \text{a.s.},$$

which follows from (11), as well as Chebychev's inequality and by the Borel–Cantelli lemma.
The strong consistency of estimators $\theta_\varepsilon(n),\ \varepsilon > 0$ is obtained.
Theorem 1 is proved.

Proof of Theorem 2. Now we establish the finiteness of stopping times $\tilde{T}(\varepsilon)$.
At first we find for $u = 0$ and $u \ge 1$ the limits

$$\lim_{T\to\infty}\frac{1}{T}\int_1^T \tilde{\Delta}Y(t)\tilde{\Delta}Y(t-u)dt = f(u) \ P_\vartheta - \text{a.s.} \tag{41}$$

in stationary case and

$$\lim_{T\to\infty}|\frac{1}{e^{2v_0T}}\int_1^T \tilde{\Delta}Y(t)\tilde{\Delta}Y(t-u)dt - f_u(T)| = 0 \ P_\vartheta - \text{a.s.} \tag{42}$$

in periodic case, where $f(u)$ is random function of $u \ge 0$ and $f_u(T)$ are periodic functions of T for all $u \ge 0$.
From (2) we have

$$\tilde{\Delta}Y(t) = \tilde{X}(t) + \tilde{\Delta}V(t),\ t \ge 1.$$

Processes $\tilde{X}(t)$ and $\tilde{\Delta}Y(t)$ are mutually independent. Similar to the proof of Theorem 1 we can get the limiting relationships using the definition of the process $\tilde{X}(t)$:
– in stationary case

$$f(u) = \begin{cases} \int_0^\infty \tilde{x}_0^2 dt + 1, & u = 0, \\ \int_0^\infty \tilde{x}_0(t+u)\tilde{x}_0(t)dt, & u \ge 1; \end{cases}$$

– in periodic case

$$f_u(T) = e^{-v_0 u}\int_0^\infty e^{-2v_0 t}U_0^*(T-t)U_u^*(T-t)dt,\ u \geq 0,$$

$$U_u^*(t) = X_0(0)\tilde{\phi}_u^*(t) + b\int_{-1}^0 \tilde{\phi}_u^*(t-s-1)e^{-v_0(s+1)}X_0(s)ds + \int_0^\infty \tilde{\phi}_u^*(t-s)e^{-v_0 s}dW(s),$$

$$\tilde{\phi}_u^*(t) = \tilde{A}_u^*\cos\xi_0 t + \tilde{B}_u^*\sin\xi_0 t,$$

$$\tilde{A}_u^* = \tilde{A}_0\cos\xi_0 u - \tilde{B}_0\sin\xi_0 u,\ \tilde{B}_u^* = \tilde{B}_0\cos\xi_0 u - \tilde{A}_0\sin\xi_0 u.$$

By the definition of ξ_0 we can see, that functions $f_u(T)$ are periodic with the period $\Delta > 1$. Note that $f(0) > 0$ and $0 < \underline{f}_0 = \inf\limits_T f_0(T) < \sup\limits_T f_0(T) = \overline{f}_0 < \infty$.
The relationships (41), (42) and as follows the finiteness of the moments $\tilde{\tau}_\varepsilon(n),\ n \geq 1,\ \varepsilon > 0$ are established.
From (41), (42) and by the definition of the stopping times $\tilde{\tau}_\varepsilon(n)$ we have the next limiting relationships:
– in stationary case

$$\lim_{n\to\infty}\frac{\tilde{\tau}_\varepsilon(n)}{\varepsilon^{-1}c_n} = \lim_{\varepsilon\to 0}\frac{\tilde{\tau}_\varepsilon(n)}{\varepsilon^{-1}c_n} = (2f(0))^{-1}\ P_\vartheta - \text{a.s.}; \tag{43}$$

– in periodic case for any $\varepsilon > 0$

$$e^{2v_0(2+5\tilde{h})}[\varepsilon(1+e^{2v_0\tilde{h}})\overline{f}_0]^{-1} \leq \underline{\lim}_{n\to\infty} c_n^{-1}e^{2v_0\tilde{\tau}_\varepsilon(n)} \leq$$

$$\leq \overline{\lim}_{n\to\infty} c_n^{-1}e^{2v_0\tilde{\tau}_\varepsilon(n)} \leq e^{4v_0(1+3\tilde{h})}[\varepsilon(1+e^{2v_0\tilde{h}})\underline{f}_0]^{-1}\ P_\vartheta - \text{a.s.} \tag{44}$$

and for $n \geq 1$

$$e^{4v_0(1+3\tilde{h})}[(1+e^{2v_0\tilde{h}})\overline{f}_0]^{-1}c_n \leq \underline{\lim}_{\varepsilon\to 0}\varepsilon e^{2v_0\tilde{\tau}_\varepsilon(n)} \leq \overline{\lim}_{\varepsilon\to 0}\varepsilon e^{2v_0\tilde{\tau}_\varepsilon(n)} \leq$$

$$\leq e^{2v_0(2+7\tilde{h})}[(1+e^{2v_0\tilde{h}})\underline{f}_0]^{-1}c_n\ P_\vartheta - \text{a.s.} \tag{45}$$

From (43), (44) in periodic case for $\varepsilon > 0$

$$2(1+3\tilde{h}) - \frac{1}{2v_0}\ln(1+e^{2v_0\tilde{h}}) - \frac{1}{2v_0}\ln\overline{f}_0 + \frac{1}{2v_0}\ln\varepsilon^{-1} \leq$$

$$\leq \underline{\lim}_{n\to\infty}[\tilde{\tau}_\varepsilon(n) - \frac{1}{2v_0}\ln c_n] \leq \overline{\lim}_{n\to\infty}[\tilde{\tau}_\varepsilon(n) - \frac{1}{2v_0}\ln c_n] \leq 2 + 7\tilde{h} -$$

$$-\frac{1}{2v_0}\ln(1+e^{2v_0\tilde{h}}) - \frac{1}{2v_0}\ln\underline{f}_0 + \frac{1}{2v_0}\ln\varepsilon^{-1}\ P_\vartheta - \text{a.s.} \tag{46}$$

and for $n \geq 1$

$$2(1+3\tilde{h}) - \frac{1}{2v_0}\ln(1+e^{2v_0\tilde{h}}) - \frac{1}{2v_0}\ln \overline{f}_0 + \frac{1}{2v_0}\ln c_n \leq$$

$$\leq \underline{\lim}_{\varepsilon\to 0}[\tilde{\tau}_\varepsilon(n) - \frac{1}{2v_0}\ln \varepsilon^{-1}] \leq \overline{\lim}_{\varepsilon\to 0}[\tilde{\tau}_\varepsilon(n) - \frac{1}{2v_0}\ln \varepsilon^{-1}] \leq$$

$$\leq 2 + 7\tilde{h} - \frac{1}{2v_0}\ln(1+e^{2v_0\tilde{h}}) - \frac{1}{2v_0}\ln \underline{f}_0 + \frac{1}{2v_0}\ln c_n \ P_\vartheta - \text{a.s.} \tag{47}$$

From (41), (43) we can obtain in stationary case

$$\lim_{n\to\infty}\frac{\varepsilon}{c_n}\int_1^{\tilde{\tau}_\varepsilon(n)} \tilde{\Delta}Y(t-u)\tilde{\Delta}Y(t)dt = \lim_{\varepsilon\to 0}\frac{\varepsilon}{c_n}\int_1^{\tilde{\tau}_\varepsilon(n)} \tilde{\Delta}Y(t-u)\tilde{\Delta}Y(t)dt =$$

$$= (2f(0))^{-1}f(u), \ u \geq 1 \quad P_\vartheta - \text{a.s.}$$

and

$$\lim_{n\to\infty}\frac{\varepsilon}{c_n}\tilde{G}_\varepsilon(n) = \lim_{\varepsilon\to 0}\frac{\varepsilon}{c_n}\tilde{G}_\varepsilon(n) = \tilde{G}(\tilde{h}) \ P_\vartheta - \text{a.s.}, \tag{48}$$

$$\tilde{G}(\tilde{h}) = (2f(0))^{-1}\begin{pmatrix} f(2+5\tilde{h}) & f(1+\tilde{h}) \\ f(2+6\tilde{h}) & f(1+6\tilde{h}) \end{pmatrix}.$$

Similar to the Case I we can see that $mes\{u \in [\tilde{h}_0, 1/5] : \ f(u) = 0\} = 0$ and $mes\{u \in [\tilde{h}_0, 1/5] : \det \tilde{G}(u) = 0\} = 0$. As follows $\det \tilde{G}(\tilde{h}) \neq 0 \ P_\vartheta -$ a.s. From here, (14), (15) and (48) we have the finiteness of the moments $\tilde{\nu}_\varepsilon$ in stationary case. Put

$$\tilde{G}(T,\tilde{h}) = \begin{pmatrix} f_{2+5\tilde{h}}(T) & e^{-2v_0}f_{1+5\tilde{h}}(T) \\ f_{2+6\tilde{h}}(T) & e^{-2v_0}f_{1+6\tilde{h}}(T) \end{pmatrix}.$$

The matrix function $\tilde{G}(T)$ is periodic with the period $\Delta > 1$ and the equation

$$\det \tilde{G}(T,\tilde{h}) = 0$$

have as a maximum four roots $\tilde{t}_m, \ m = \overline{1,4}$ on the set $(0,\Delta]$ for any $\tilde{h}$. Put $\tilde{\delta}_\varepsilon(n) = \tilde{\tau}_\varepsilon(n) - \tilde{k}_n\tilde{h}$. Note that in periodic case by the definition of $\tilde{G}_\varepsilon(n)$ (15) and from (42) by the way of the proof of Theorem 1 we can get the following relationships

$$\lim_{n\to\infty}|\frac{1}{e^{2v_0\tilde{\delta}_\varepsilon(n)}}\tilde{G}_\varepsilon(n) - \tilde{G}(\tilde{\delta}_\varepsilon(n),\tilde{h})| = 0 \ P_\vartheta - \text{a.s.} \tag{49}$$

and for some constants $\tilde{g}_1, \ \tilde{g}_2$

$$0 < \tilde{g}_1 = \underline{\lim}_{n\to\infty}||\tilde{G}^{-1}(\tilde{\delta}_\varepsilon(n),\tilde{h})|| \leq \overline{\lim}_{n\to\infty}||\tilde{G}^{-1}(\tilde{\delta}_\varepsilon(n),\tilde{h})|| = \tilde{g}_2 < \infty, \tag{50}$$

$$0 < \tilde{g}_1 = \underline{\lim}_{\varepsilon\to 0} ||\tilde{G}^{-1}(\tilde{\delta}_\varepsilon(n), \tilde{h})|| \le \overline{\lim}_{\varepsilon\to 0} ||\tilde{G}^{-1}(\tilde{\delta}_\varepsilon(n), \tilde{h})|| = \tilde{g}_2 < \infty. \tag{51}$$

From (14), (15), (44), (49) and (50) follows the finiteness of moments $\tilde{\nu}_\varepsilon$ in periodic case.
Then the finiteness of the stopping times $\tilde{T}(\varepsilon)$ is established.
The second property of sequential estimators $(\tilde{T}(\varepsilon), \tilde{\vartheta}(\varepsilon))$ and the strong consistency of the estimators $\tilde{\vartheta}_\varepsilon(n)$ may be proved similar to the proof of Theorem 1.
Now we fined the limiting low and upper bounds for the time of sequential estimation $\tilde{T}(\varepsilon)$. Put for $k = 1, 2$

$$\tilde{\nu}(k) = \inf\{n \ge 1 : \ c_n/\kappa_n > \tilde{g}(k)\} - 1,$$

$$\nu^*(k) = \inf\{n \ge 1 : \ c_n/\kappa_n > g^*(k)\},$$

$$\tilde{g}(1) = g^*(1) = 2f(0)\tilde{\rho}^{1/2}||\tilde{G}^{-1}(\tilde{h})||,$$

$$\tilde{g}(2) = \tilde{\rho}^{1/2}\tilde{g}_1 e^{-4v_0(1+3\tilde{h})}(1 + e^{2v_0\tilde{h}})\underline{f}_0,$$

$$g^*(2) = \tilde{\rho}^{1/2}\tilde{g}_2 e^{-2v_0(2+\tilde{h})}(1 + e^{2v_0\tilde{h}})\overline{f}_0.$$

By the definition of $\tilde{\nu}_\varepsilon$ and from (43), (45), (48), (49), (51) follows that by small ε
– in stationary case

$$\tilde{\nu}(1) \le \tilde{\nu}_\varepsilon \le \nu^*(1), \tag{52}$$

– in periodic case

$$\tilde{\nu}(2) \le \tilde{\nu}_\varepsilon \le \nu^*(2). \tag{53}$$

From (15), (43), (47), (52) and (53) follows the third assertion of Theorem 4:

–in stationary case

$$(2f(0))^{-1}c_{\tilde{\nu}(1)} \le \underline{\lim}_{\varepsilon\to 0} \ \varepsilon\tilde{T}(\varepsilon) \le \overline{\lim}_{\varepsilon\to 0} \ \varepsilon\tilde{T}(\varepsilon) \le (2f(0))^{-1}c_{\nu^*(1)} \ P_\vartheta - \text{a.s.};$$

– in periodic case

$$2(1 + 3\tilde{h}) - \frac{1}{2v_0}\ln(1 + e^{2v_0\tilde{h}}) - \frac{1}{2v_0}\ln\overline{f}_0 + \frac{1}{2v_0}\ln c_{\tilde{\nu}(2)} \le$$

$$\le \underline{\lim}_{\varepsilon\to 0}[\tilde{T}(\varepsilon) - \frac{1}{2v_0}\ln\varepsilon^{-1}] \le \overline{\lim}_{\varepsilon\to 0}[\tilde{T}(\varepsilon) - \frac{1}{2v_0}\ln\varepsilon^{-1}] \le 2 + 7\tilde{h} -$$

$$-\frac{1}{2v_0}\ln(1 + e^{2v_0\tilde{h}}) - \frac{1}{2v_0}\ln\underline{f}_0 + \frac{1}{2v_0}\ln c_{\nu^*(2)} \ P_\vartheta - \text{a.s.}$$

Theorem 2 is proved.
Remark 2. It should be pointed out that one could obtain the following limiting

equalities for $(i,j) \in I_1$ in Problem I

$$\lim_{\varepsilon \to 0} \varepsilon \tilde{T}(\varepsilon) = f_{ij0}^{-1} c_{\nu_{ij}^*} \ P_\vartheta - \text{a.s.}$$

and in stationary case in Problem II

$$\lim_{\varepsilon \to 0} \varepsilon \tilde{T}(\varepsilon) = (2f(0))^{-1} c_{\nu^*(1)} \ P_\vartheta - \text{a.s.}$$

if the magnitudes $\rho^{1/2}\varepsilon^{-1}c_n|G_\varepsilon^{-1}(n)|$ and $\tilde{\rho}^{1/2}\varepsilon^{-1}c_n|\tilde{G}_\varepsilon^{-1}(n)|$ in the definitions of ν_ε and $\tilde{\nu}_\varepsilon$ correspondingly were replaced by the nearest integer from above and the sequences (c_n) and (κ_n) were chosen in such a way that the relation c_n/κ_n were fractional for all $n \geq 1$.

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