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## Initial offerings of options

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# Initial Offerings of Options

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## Abstract

This paper considers the introduction of stock options in an (dynamically) incomplete securities market made up of a riskless bond and the stock. The stock price follows a geometric Brownian motion with constant drift. However, there is incomplete information about the unknown stochastic volatility. The option price is determined by a uniform-price auction. Thus an option pricing formula results from the interaction of market participants relying on private information on the unknown stochastic volatility under an explicit market structure. This paper incorporates market microstructure considerations into an extended Black-Scholes model with incomplete information on the underlying volatility. It relies on the growing importance of auctionlike trading rules in financial markets.

JEL classification: G13, D44, D82, C72

# 1 Introduction

Since the seminal work of Black and Scholes (1973) and Merton (1973) on the pricing of stock options their arbitrage pricing approach has been used extensively to derive pricing formulae for various kinds of options and general contingent claims. In the Black/Scholes (1973) framework an option can be priced in a preference-free way if its cash-flow at maturity can be created by a continuously adjusted self-financing portfolio in the underlying stock and a riskless asset. Thus the option is a redundant asset. This paper addresses the problem of pricing a non-redundant option, which seems by far to be the more realistic case. There are essentially three strands of research in dealing with non-redundancy. First, a consumption-based representative consumer framework can be employed (for example see Amin/Ng(1993)). Second, in models with stochastic volatility, volatility is assumed to be uncorrelated with aggregate consumption (for example see Hull/White (1987)) or a "market price of the stock's volatility risk" is assumed (for example see Heston (1993)). Third, a mean-variance framework for hedge portfolios can be employed (for example see Föllmer/Leuckert (1999)).

The purpose of this paper is two-fold. First, it presents an alternative to the outlined set-ups on pricing of non-redundant options. Second, it aims at determining the influence of private information about parameters of the underlying stock price distribution on option pricing. It starts out by a risk neutral seller who is interested in selling non-redundant stock options to investors. The seller uses a standard selling procedure for financial instrument, namely a uniform-price auction. This pricing procedure is chosen in analogy to IPO markets, where the market price is primarily determined by auction-like procedures. The option is introduced to an incomplete financial market made up of the stock and a riskless bond. The distributional assumptions are kept simple in order to allow for comparisons with the standard option pricing models. In fact, we use the Black/Scholes model with a stochastic volatility. However, in contrast to earlier models in the literature, there is incomplete information. Every investor has some private information about the unknown stochastic volatility. The distributional assumptions can be relaxed to general price processes provided the main modelling issue of private information about parameters of the stock price distribution is retained.

The paper proceeds as follows. Section 2 introduces the model and explains

price determination in the uniform-price auction. Section 3 gives two results. First, the bidding behavior of market participants is derived. It is shown that bidders submit Black/Scholes analogous bids. From this the (ex ante) expected selling price of the option follows. It is a weighted Black/Scholes formula with the weights being determined by bidding behavior. Finally, section 4 concludes. In order to make the exposition more accessible, all proofs are collected in the appendix.

## 2 Model

For ease of exposition there is only one European call option written on a risky stock with exercise price  $K$  and expiration date  $T$  in a financial market made up of a riskless bond and the stock. The option is sold at time  $t = 0$  by a risk neutral seller in a (sealed-bid) uniform-price auction<sup>1</sup>, i.e. it is allocated to the bidder with the highest price bid at a price which equals the highest unsuccessful bid. For the sake of simplicity, there are two risk neutral bidders competing with each other in the auction. These assumptions could be relaxed provided every bidder is restricted to one unit. However, the analysis gets technically more involved.

The following assumptions on security price dynamics are standard in the option pricing literature. The bond price at time  $t$ ,  $0 \leq t \leq T$ , is given by

$$R_t = \exp(rt), \quad (1)$$

where  $r$  denotes the continuously compounded rate of interest. Stock prices are given by

$$S_t = S_0 \exp(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t), \quad (2)$$

where  $S_t$  denotes (state-dependent) stock prices at time  $t$ ,  $0 \leq t \leq T$ ,  $S_0$ ,  $\mu$  positive constants and  $W$  a standardized Brownian motion. However, in contrast to the Black-Scholes model, where  $\sigma$  is constant, it is assumed that

$$\sigma = \frac{1}{2}(X_1 + X_2) \quad (3)$$

holds true.  $X_i$ ,  $i = 1, 2$ , are i.i.d., uniformly distributed on  $[0, 1]$  and independent of  $W$ .

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<sup>1</sup>For a single option, the uniform-price auction reduces to a second-price or Vickrey auction (cp. Vickrey (1961). For a general discussion on this and other auctions cp. Müller (2001).

Note that the option payoff at expiration is given by

$$V_T = \max(S_T - K, 0). \quad (4)$$

Thus the option is of common value (or of quality uncertainty), i.e. the monetary value bidders attach to it is the same for all bidders. This value, however, will generally be unknown at the time the auction takes place.

Bidder  $i$  gets to know the realization  $x_i$  of her information variable  $X_i$ . Bidders are distinguished solely by their private information; otherwise they are identical. It is assumed that the number of bidders and the joint distribution of all random variables is common knowledge. When preparing their bids bidders do not cooperate.

A rational bidder takes the bidding behavior of her competitors into account and behaves strategically. Since a bidder has only incomplete information about the relevant features of the bidding situation, the auction is to be considered a non-cooperative game with incomplete information among the bidders. We rely on the Bayes-Nash equilibrium concept due to Harsanyi (1967/68) to analyze this game. For that purpose, a bidder's strategy is defined as a real-valued function mapping possible realizations of her information variable into bids. A Bayes-Nash equilibrium  $(B_1, \dots, B_n)$  of bidders' strategies is defined as follows:  $B_i$  maximizes bidder  $i$ 's expected gains assuming bidding strategies  $B_1, \dots, B_{i-1}, B_{i+1}, \dots, B_n$  of her competitors and the assumed bidding behavior is correct.

## 3 Results

### 3.1 Bidding strategy

Intuitively, bidders consider the decision problem as one, where they form an estimate of the unknown volatility with the understanding that bids increase with an increasing volatility estimate.<sup>2</sup> Note that forming this estimate and submitting a bid on the basis of it has two opposing effects. On the one hand, a high volatility estimate increases the probability of getting the option. On the other hand, a high volatility estimate increases the loss from being the one who overvalued the volatility and thus suffering from the winner's curse (cp.

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<sup>2</sup>See also Bergman/Grundy/Wiener (1996) for general properties of option prices.

Milgrom/Weber (1982)). The optimal bid will be such that it balances those two effects.<sup>3</sup>

Since bidders differ only by their private information, bidding behavior can be described by a common bidding strategy. Note that different bids are the result of different information only. The resulting bidding strategy in the uniform-price auction is given by Theorem 1.

**Theorem 1**

*The unique symmetric Bayes-Nash equilibrium in strictly increasing and differentiable strategies is given by  $(B^*, B^*)$  with*

$$B^*(x) = S_0\Phi(g^*(x)) - K \exp(-rT)\Phi(h^*(x)), \tag{5}$$

*where  $\Phi$  is the distribution function of a standard (mean 0 and variance 1) normally distributed random variable and  $g^*(x)$  and  $h^*(x)$  are given by*

$$g^*(x) = \frac{\ln \frac{S_0}{K} + (r + \frac{1}{2}x^2T)}{x\sqrt{T}} \tag{6}$$

*and*

$$h^*(x) = g^*(x) - x\sqrt{T}. \tag{7}$$

**Proof:** See the Appendix

A bidder with information  $x$  submits a bid which equals the Black/Scholes formula with a volatility  $x$ . Note that the bidder with the highest information value is the one who gets the option.

### 3.2 Option pricing

By Theorem 1 and the rules of the sale, the seller sells the option at a price which is given by a Black/Scholes formula with a volatility  $\min(x_1, x_2)$ . This

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<sup>3</sup>It is well-known (cp. Milgrom/Weber (1982)) that in common value auctions unsophisticated bidding leads to the effect that the successful bidder tends to be the one who overvalued the good. Substantial losses are quite often attributed to this effect. Thus rational bidders shade their bids downwards when preparing their bids in order to account for the winner's curse effect.

is due to the fact that the bidder who submits the highest bid is the one with the highest information. However, she is charged the bid of the unsuccessful bidder. According to Theorem 1, this bid is given by the Black/Scholes formula with the lowest information  $\min(x_1, x_2)$  on volatility. Taking this into account, the expected selling price of the option can be derived. We have

**Theorem 2**

*The expected selling price  $C_0$  of the option is given by*

$$C_0 = S_0 \int_0^1 \Phi(g^*(x))2(1-x)dx - K \int_0^1 \Phi(h^*(x))2(1-x)dx. \quad (8)$$

**Proof:** See the Appendix

Note that formula (8) generalizes the Black/Scholes formula while retaining most of its attractive features. In particular, it is of closed form and option prices are easily calculated. This is in contrast to most formulae accounting for stochastic volatility in the literature. (cp. Hull/White (1987), Stein/Stein (1991), Heston (1993a) and Amin/Ng (1993)). The formula of Theorem 2 inherits most of the properties of the Black/Scholes formula. This includes the well-known comparative statics with respect to  $S_0$ ,  $K$ , and  $r$ . However, it might have the potential to account for the empirical biases that are documented for the Black/Scholes formula in the literature (cp., for example, Bakshi/Cao/Chen (1997), and Dumas/Fleming/Whaley (1998)).

How does (8) compare to other stochastic volatility option pricing formulae in the literature? Hull/White (1987) address the stochastic volatility problem for a diffusion model with the volatility being determined by a second diffusion. They arrive at a series not a closed-form solution. However, they argue that their formula could be interpreted as an expected Black/Scholes formula, where the expectation is taken over the distribution of "mean variances". Stein/Stein (1991) and Heston (1993a) use a diffusion model with the volatility being governed by an Orenstein-Uhlenbeck process. They succeed in deriving closed-form solutions. Again, it is argued that the admittedly rather complicated formulae could be interpreted as an "average Black/Scholes formula over different paths of volatility". Unfortunately, the mixing distributions are difficult to obtain. Similarly, Amin/Ng (1993) argue in favor of their formula as expectation of the Black/Scholes formula with respect to "the average variance" (cp. Amin/Ng

(1993), p. 890). Even setting apart the different distributional and modelling assumptions, all formulae discussed above are fundamentally different when it comes to the mixing distribution applied to the Black/Scholes formula. (8) is the only option pricing formula that uses a market microstructure set-up to determine the mixing distribution. As such, the resulting mixing distribution can be easily derived. Summing up, (8) differs both in form and derivation from the formulae given in the literature.

## 4 Conclusion

This paper considers an extended Black/Scholes framework with a stochastic volatility, where market participants have private information on certain aspects of the volatility process. It determines the revenues that a seller of a European call option can achieve in a uniform-price auction. In order to do this, the bidding behavior of market participants is derived. It is shown that bidders use Black/Scholes analogous bidding strategies. From this the (ex ante) expected selling price of the option is derived. It is a weighted Black/Scholes formula with the weights being determined by bidding behavior. This weighing scheme distinguishes the derived formula from those in the literature. The analysis is not limited to call options. In fact, any (European-style) contingent claim can be priced accordingly.

Our model uses certain simplifications. First, stochastic volatility is independent of the basic uncertainty represented by  $W$ . This assumption is made for ease of exposition and can be relaxed. Second, there are several assumptions which, when relaxed, give rise to technical problems. Certainly, risk neutrality matters. Furthermore, results for an endogenous quantity to be sold cannot easily be achieved. Also, allowing for price-quantity bids on the bidders part will lead to problems.

The considered set-up gives rise to an incomplete financial market. In an incomplete financial market there is a plethora of alternative option prices consistent with an arbitrage-free environment. In order to sort out these prices, different approaches have been used. In that respect, this paper adds to the on-going discussion on this topic by using a market microstructure approach.



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## 6 Proofs

As usual (cp. Black/Scholes (1973), Merton (1973), and Müller (1985)), the proof will be given for the case  $\mu = 0$  and  $r = 0$ . This is done in order to save on notation. The arguments will exactly go through for arbitrary parameters. Let  $n(x)$  denote the normal density

$$n(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

Furthermore, the following notation is used.

$$g(x, y) = \frac{\ln \frac{S_0}{K} + \frac{1}{2} \left(\frac{x+y}{2}\right)^2 T}{\frac{x+y}{2} \sqrt{T}}$$

$$h(x, y) = g(x, y) - \frac{x+y}{2} \sqrt{T}$$

$$v(x, y) = S_0 \Phi(g(x, y)) - K \Phi(h(x, y))$$

$$v^*(x) = v(x, x)$$

First, regularity conditions will be proved.

### Lemma 1

(a)

$$E[\max(S_T - K, 0) \mid X_1 = x, X_2 = y] = S_0 \Phi(g(x, y)) - K \Phi(h(x, y)) \quad (9)$$

(b)  $v(x, y)$  is strictly increasing in  $x$  and  $y$  and  $v^*(x)$  is strictly increasing in  $x$ .

### Proof of Lemma 1:

(a) We will use the fact that the following holds true for a normally distributed random variable  $Y$  with mean  $a$  and variance  $\tau^2$ .

$$E[\exp(Y) \times 1_{\{\exp Y > M\}}] = \exp\left(a + \frac{\tau^2}{2}\right) \Phi\left(\frac{a + \tau^2 - \ln M}{\tau}\right)$$

$$P(\{\exp Y > M\}) = \Phi\left(\frac{a - \ln M}{\tau}\right)$$

Let

$$Z = \frac{x+y}{2} \times W_T - \frac{1}{2}\left(\frac{x+y}{2}\right)^2 T.$$

$Z$  is normally distributed with mean

$$E[Z] = -\frac{1}{2}\left(\frac{x+y}{2}\right)^2 T$$

and variance

$$\text{Var} Z = \left(\frac{x+y}{2}\right)^2 T.$$

Then we get

$$\begin{aligned} E \left[ \max(S_T - K, 0) \mid X_1 = x, X_2 = y \right] &= E[S_T \times 1_{\{S_T > K\}} \mid X_1 = x, X_2 = y] \\ &- KE[1_{\{S_T > K\}} \mid X_1 = x, X_2 = y] \\ &= S_0 E\left[\exp\left(\frac{x+y}{2} \times W_T - \frac{1}{2}\left(\frac{x+y}{2}\right)^2 T\right) \times 1_{\{\exp\left(\frac{x+y}{2} \times W_T - \frac{1}{2}\left(\frac{x+y}{2}\right)^2 T\right) > \frac{K}{S_0}\}}\right] \\ &- KE[1_{\{\exp\left(\frac{x+y}{2} \times W_T - \frac{1}{2}\left(\frac{x+y}{2}\right)^2 T\right) > \frac{K}{S_0}\}}] \\ &= S_0 E[\exp(Z) \times 1_{\{\exp Z > \frac{K}{S_0}\}}] - KP(\{\exp Z > \frac{K}{S_0}\}) \\ &= S_0 \Phi(g(x, y)) - K \Phi(h(x, y)) \end{aligned}$$

(b) Since

$$\frac{\partial h}{\partial x} = \frac{\partial g}{\partial x} - \frac{\sqrt{T}}{2} \tag{10}$$

and

$$Kn(h(x, y)) = S_0 n(g(x, y)), \tag{11}$$

$$\begin{aligned}
\frac{\partial v}{\partial x} &= S_0 n(g(x, y)) \frac{\partial g}{\partial x} - Kn(h(x, y)) \frac{\partial h}{\partial x} \\
&= S_0 n(g(x, y)) \frac{\partial g}{\partial x} - Kn(h(x, y)) \frac{\partial g}{\partial x} + Kn(h(x, y)) \frac{\sqrt{T}}{2} \\
&= Kn(h(x, y)) \frac{\sqrt{T}}{2} > 0
\end{aligned}$$

holds true, i.e.  $v(x, y)$  is strictly increasing in  $x$ . Similarly,

$$\frac{\partial v}{\partial y} = Kn(h(x, y)) \frac{\sqrt{T}}{2}$$

and

$$(v^*)'(x) = Kn(h(x, x)) \sqrt{T},$$

which proves the claim.

### Proof of Theorem 1:

The proof is given in two steps. First, it is shown that  $B^*$  of the form (5) is a Bayes-Nash equilibrium. Second, uniqueness is established.

#### (a) $(B^*, B^*)$ is a Bayes-Nash equilibrium<sup>4</sup>

We will show that the optimal bid  $b_i^*$  of a bidder  $i$  is given by  $B^*(x_i)$  if her competitor bids according to  $B^*(X_j)$ ,  $j \neq i$ , and  $x_i$  is the realisation of bidder  $i$ 's information variable. By symmetry, the analysis can be restricted to the decision problem of bidder 1. Bidder 1 is successful with her bid  $b$ , if

$$b > (B^*(X_2))$$

holds, and in that case she pays

$$B^*(X_2).$$

Thus bidder 1's conditional expected gain from a bid  $b$  is given by

$$\begin{aligned}
&E [(V_T - B^*(X_2)) 1_{\{B^*(X_2) < b\}} \mid X_1 = x_1] \\
&= \int_0^{(B^*)^{-1}(b)} (E[V_T \mid X_1 = x_1, X_2 = y] - B^*(y)) f(y \mid x_1) dy \\
&= \int_0^{(B^*)^{-1}(b)} (v(x_1, y) - v(y, y)) f(y \mid x_1) dy \tag{12}
\end{aligned}$$

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<sup>4</sup>The proof mimics the one of Milgrom/Weber (1982), p. 1101.

by Lemma 1 (a), where  $f$  denotes the corresponding density. The integrand is positive for  $v(x_1, y) > v(y, y)$  and negative for  $v(x_1, y) < v(y, y)$ . Since  $v(\cdot, y)$  is strictly increasing for fixed  $y$ , the integral is maximized by choosing  $b$  such that

$$(B^*)^{-1}(b) = x_1$$

holds true. Consequently, the optimal bid  $b^*$  is of the form  $b^* = B^*(x_1)$ .

**(b) Uniqueness**

To show uniqueness, let  $(\tilde{B}, \dots, \tilde{B})$  be a symmetric Bayes-Nash equilibrium in strictly increasing and differentiable strategies. If bidder 2 bids according to  $\tilde{B}(X_2)$ , bidder 1's conditional expected gain from a bid  $b$  is given by

$$\begin{aligned} E & [(V - \tilde{B}(X_2))1_{\{\tilde{B}(X_2) < b\}} | X_1 = x_1] \\ &= \int_0^{\tilde{B}^{-1}(b)} (v(x_1, y) - \tilde{B}(y))f(y | x_1)dy. \end{aligned} \quad (13)$$

The necessary condition for a maximum is given by

$$v(x_1, \tilde{B}^{-1}(b^*)) - b^* = 0$$

for the optimal bid  $b^*$ . Since  $b^* = \tilde{B}(x_1)$  holds true by the definition of equilibrium

$$\tilde{B}(x_1) = v^*(x_1) = B^*(x_1)$$

follows. Q.E.D.

**Proof of Theorem 2:**

Using the bidding strategy derived in Theorem 1 we get

$$\begin{aligned} C_0 &= E[B^*(X_1)1_{\{X_2 > X_1\}}] + E[B^*(X_2)1_{\{X_1 > X_2\}}] \\ &= 2E[B^*(X_1)1_{\{X_2 > X_1\}}] \\ &= 2 \int_{x_1=0}^1 \int_{x_1}^1 B^*(x_1)dx_2dx_1 \\ &= 2 \int_{x_1=0}^1 B^*(x_1)(1 - x_1)dx_1 \\ &= S_0 \int_0^1 \Phi(g^*(x_1))2(1 - x_1)dx_1 - K \int_0^1 \Phi(h^*(x_1))2(1 - x_1)dx_1. \end{aligned}$$