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# Hotelling Games with Three, Four, and More Players\*

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## Abstract

In this paper the standard Hotelling model with quadratic transport costs is extended to the multi-firm case. The sequential game consists of a location choice stage and a price setting stage. Considering locational equilibria it is shown that neither holds the Principle of Maximum Differentiation - as in the duopoly model - nor does the Principle of Minimum Differentiation hold - as in the multiple firms game with linear transport cost. This result is in line with recent research which shows that the extreme differentiation patterns are often not stable if one adds flexibility to the model.

For games with up to nine players explicit subgame perfect equilibrium solutions are calculated. They are characterized by a U-shaped price structure and interior corner firms locations. Welfare considerations show that the level

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of differentiation is almost at the socially optimal level if the number of firms is larger than three. If it is smaller then there is too little differentiation.

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*Key words:* spatial competition, multi-firm competition, interval

## 1 Introduction

The literature on Horizontal Product Differentiation focuses on the extent to which competing firms should give their products a similar design interpreted as a location in the space of product characteristics. For the two-stage model where firms choose locations in the first stage and set prices in the second stage Hotelling (1929) acclaimed the Lemma 2 of Minimum Differentiation. According to this principle firms approach each other as closely as possible and share the market equally. D'Aspremont et al. (1979) corrected him by showing that in his model neither this strategy nor any other location choices were subgame perfect since they fail to imply an equilibrium in prices for each subgame. By altering the utility function from a linear to a quadratic form, resulting in a tougher second stage price competition, the Principle of Maximum Differentiation could be established where firms maximize the distance to the opposite player.<sup>1</sup>

Influenced by this result some researchers believed that "this [maximum] differentiation behavior could be fairly general." (Neven, 1985: 322). However, subsequent results indicate that relaxing certain model assumptions shifts the balance away from the centrifugal towards the centripetal forces within the model leading to less differentiation. For example, Economides (1986) considers a whole family of utility functions which lie between the linear and the quadratic form. He shows that some equilibria exist where differentiation is not at maximum. Böckem (1994), Hinloopen and van Marrewijk (1999), Smithies (1941), and Wang and Yang (1999), among others, generalize the model on the demand side. More general distributions of the consumers are introduced into the model by Neven (1986), Xiangzhu

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<sup>1</sup>Note that the quadratic term in the utility function measures a loss of utility.

(1997), Tabuchi and Thisse (1995), and Baake and Oechssler (1997). Others analyze markets without boundaries (Lambertini, 1994), the impact of demand uncertainty (Balvers and Szerb, 1996), the introduction of information exchange through communication (Mai and Peng, 1999), and endogenous household locations (Fujita and Thisse, 1986). In all those cases, introducing flexibility into the model may destroy the equilibrium of maximum differentiation and may lead to in-between solutions. This kind of equilibrium even appears in the model considered by D'Aspremont et al. (1979) if one permits mixed strategy Nash equilibria (Bester et al., 1996).

However, the major part of the literature concentrates on duopoly markets only. Widely ignored is the question of how the number of firms affects the equilibrium outcome. Exceptions comprise de Palma et al. (1987) who treat the three-firm case in a probabilistic framework, and Lancaster (1979), Salop (1979), Novshek (1980), and Economides (1989, 1993) who do not restrict the number of firms. Linear utility models with multiple firms located on a circumference were analyzed by Salop (1979) and Economides (1989). It has been shown by Salop that in contrast to its unit interval duopoly counterpart considered by Hotelling, price equilibria exist in the symmetric subgame where symmetry refers to an overall equal price and equidistance of succeeding firms. Nevertheless, a perfect equilibrium could not be found for every subgame. Economides elaborated on the similar  $n$ -firms interval model with linear utility. In contrast to the circular model it supports a noncooperative equilibrium in every price subgame but fails to have an equilibrium for the stage of the location choices.

Within the class of multi-firm models with a quadratic utility function only the circular model was considered. Economides (1989) proved the existence of a price equilibrium for each pattern of locations and further the existence of a subgame perfect equilibrium with equidistantly located firms. This equilibrium can be interpreted as the Principle of Maximum Differentiation since firms try to maximize the minimum distance to each adjacent competitor. The circular model is seen as an approximation to the interval model which is sometimes less favored because it exhibits the interval limit problem. This introduces asymmetry into the model since

it usually matters if a firm is an inside or a corner firm. Furthermore, the circumference might be appropriate as a representation of characteristics such as color but it fails when it comes to other characteristics which are ordered like a convex set of real numbers (height and weight, for example). This paper fills the research gap by analyzing the multi-firm interval model with quadratic utility. Its main objective is to examine if the results deduced from the circular multi-firm model are sufficiently general to be extended to the interval model.

Essentially, behavior of Hotelling firms is driven by a trade-off between the short-run and the long-run effects of relocation. In the short-run firms may attract new customers by moving towards a competitor's position. The strategic effect is a lower price for both the aggressor as well as for the stationary firm because of the increased competition. Which effect dominates depends on the model setup. Features which weaken the second stage price competition such as a small curvature of the utility function (Economides, 1986), a high elasticity of total demand (Böckem, 1994; Hinloopen and Marrewijk, 1999; Smithies, 1941; Wang and Yang, 1999), adopting price-matching policy (Zhang, 1995), and the possibility to collude on prices (Jehiel, 1992 ; Friedman and Thisse, 1993) may even destroy the locational equilibria or lead to minimum differentiation.

In the Hotelling game with two firms and a quadratic utility function in equilibrium firms locate at the interval borders. By differentiating, players avoid the tough price competition of the second stage. One may expect a similar behavior when the number of firms increases, i.e. the corner firms are located at the interval boundaries and the remaining firms are equidistant in order to maximize the minimum distance to their neighbors. One of the main results of the paper is that the Principle of Maximum Differentiation does not hold for the interval model. Considering this symmetric configuration I show that the corner firms have incentives to move inwards. Furthermore, weakening this Lemma 2 by allowing corner firms to squeeze their (equidistantly located) inside competitors is not a valid equilibrium strategy either if the number of firms is larger than three. At the other extreme we can exclude minimum differentiation in the sense that firms do not have incentives

to move towards the central firm(s) in every given locational pattern as is the case in the related linear utility model.

Due to numerical difficulties, explicit location and price equilibria could only be computed for a maximum number of nine firms. In those cases we obtain symmetric equilibria where corner firms move considerably towards the center firm(s) and prices are U-shaped. The maximum differentiation results from the circular model are destroyed because of the monopoly power that benefits the corner firms. In the linear utility case this leads to a strong tendency towards the market center which prevents the existence of locational equilibria (Economides, 1993). In the model considered here, the greater curvature of the utility function secures the existence of locational equilibria.

The paper is organized as follows. In the next section the model is described. In Section 3 it is demonstrated that a price equilibrium exists for all locational patterns. Symmetric equilibrium price structures are examined in Section 4. Analyses of equidistant locational configurations and explicit perfect equilibria for games with up to nine players follow in Section 5 and 6, respectively, which represent the core of this work. Section 7 concludes the paper. Some of the proofs are contained in Appendix A.

## 2 The model

We examine a generalized Hotelling-game with quadratic utility of customers. While d'Aspremont et al. (1979) consider a duopoly model we allow for an arbitrary (but fixed) number of firms. The game proceeds in two stages. In the first stage  $n$  firms choose locations  $\bar{x} = (x_1, x_2, \dots, x_n)$  on the unit interval  $[0, 1]$ . At the second stage, prices  $\bar{p} = (p_1, p_2, \dots, p_n)$  are fixed simultaneously by the firms. The interval can be seen as a street in which firms represent shops. The firms produce a homogeneous product and sell one unit to each consumer. Consumers living in the street are equally distributed over the interval. Firms use the same constant returns to scale production technology. Marginal costs are normalized to zero.

Since products are homogeneous only transport costs matter for the decision from which firm to buy. Consumers are endowed with the following utility function which is separable with respect to the products:

$$u_w(x_j, p_j) = k - p_j - (x_j - w)^2$$

where  $x_j$  represents the location of firm  $j$ . Consumer utility  $u_w$  has a peak where its location  $w$  and the firm's location coincide. The term  $-(x_j - w)^2$  can be interpreted as the quadratic disutility which consumers incur through the distance of transport.  $k > 0$  is the reservation price. Only if  $k$  exceeds the sum of price and transport costs does the consumer buy.

We are looking for perfect Nash equilibria in pure strategies and assume the coordination problem away. Thus, firms are exogenously assigned numbers which represent their position in the spatial ordering. We neglect them not only in view of the analytical challenges they introduce but also because mixed strategies are not played by people in complex situations (Rapoport and Amaldoss, 2000). Nevertheless, ignoring the coordination problem in the multi-firm setting can be seen as a shortcoming of the model since no plausible explanations exist why and how such an ordering could appear, in particular if the number of firms is large.

### 3 Price equilibrium

In order to solve for a subgame perfect equilibrium, we first consider the last stage of the game. At this stage locations are already chosen and prices are to be posted simultaneously and non-cooperatively. To establish results on the existence and uniqueness of the price equilibrium we start by examining the demand of the firms.

The purchasing decision of a consumer is determined by her position and reservation price, the position of the firms, and the prices they charge. In the duopoly case when the reservation price constraint is binding, i.e., if  $u_w$  of some individuals located between the shops becomes negative for the price-location combination of both firms, these customers will not buy any products. Then firms become local

monopolists. Let us consider the case where the reservation price is sufficiently high such that every consumer buys a unit of the product. Assuming further that each firm chooses a different location, the duopoly demand can easily be calculated by finding the consumer who is indifferent between buying from the left or from the right firm, i.e. where  $u_w(x_1, p_1) = u_w(x_2, p_2)$ . This individual divides the set of consumers into two convex subsets where the left subset will be supplied by the left firm and vice versa. In the multi-firm case, demand for an inside firm  $j$  ( $1 < j < n$ ) is not necessarily the set between the marginal consumers with respect to firm  $j - 1$  and  $j + 1$ . This holds because the direct neighbors might attract no demand at all by charging prohibitively high relative prices. In this case firm  $j$  competes with two of the more distant firms. However, this could never be an equilibrium because a firm with no demand would be better off lowering its price until it attracts some customers.

Assuming each firm has a positive market share,<sup>2</sup> demand can be expressed as:

$$\begin{aligned} D_{1,0}(\bar{p}|\bar{x}) &= \frac{p_2 - p_1}{2(x_2 - x_1)} + \frac{x_1 + x_2}{2}, \\ D_{j,0}(\bar{p}|\bar{x}) &= \frac{p_{j+1} - p_j}{2(x_{j+1} - x_j)} - \frac{p_j - p_{j-1}}{2(x_j - x_{j-1})} + \frac{x_{j+1} - x_{j-1}}{2}, \text{ if } 1 < j < n, \\ D_{n,0}(\bar{p}|\bar{x}) &= 1 - \frac{p_n - p_{n-1}}{2(x_n - x_{n-1})} - \frac{x_n + x_{n-1}}{2}. \end{aligned}$$

Obviously, demand  $D_{j,0}$  is linear in price  $p_j$ .<sup>3</sup> Starting from a price  $p_j^{prohib}$  at which the marginal consumer is indifferent between buying from firms  $j - 1$ ,  $j$ , and  $j + 1$ , by successively decreasing its price an inside firm  $j$  will reach a value  $p'_j$  at which it drives one of the neighbors out of business and starts to compete with the next firm. Then the above relationship does not hold anymore. If for example the whole demand of firm  $j + 1$  would have been withdrawn by firm  $j$ , i.e. for the marginal consumer with  $u_w(p_{j+2}, x_{j+2}) = u_w(p_j, x_j)$  inequality  $u_w(p_j, x_j) > u_w(p_{j+1}, x_{j+1})$  holds, the corresponding demand relationship would be

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<sup>2</sup>This corresponds to the following condition:  $\forall j \ (j = 1, \dots, n - 1) \ \exists w \mid u_w(p_j, x_j) = u_w(p_{j+1}, x_{j+1}) = \max_i [u_w(p_i, x_i)] > 0$

<sup>3</sup>The second subscript denotes the number of neighbors whose demand is totally withdrawn by firm  $j$ .



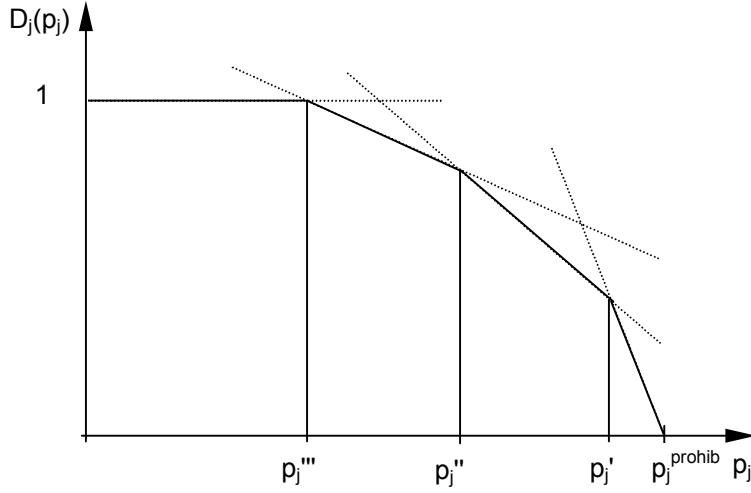


Figure 1: Demand curve for an inside firm with three neighbors

$$D_{j,1}(\bar{p}|\bar{x}) = \frac{p_{j+2}-p_j}{2(x_{j+2}-x_j)} - \frac{p_j-p_{j-1}}{2(x_j-x_{j-1})} + \frac{x_{j+1}-x_{j-1}}{2}, \text{ if } 1 < j < n.$$

This subsequent piece of the demand curve is linear too but less steep. Decreasing price  $p_j$  further leads to the undercutting of more and more competitors until the firm has attracted all customers or  $p_j = 0$ . Figure 1 shows the resulting demand curve. Undercutting of neighbors leads to kinks in this curve at prices  $p_j'$ ,  $p_j''$ , and  $p_j'''$  where the whole demand of the respective neighbors is withdrawn. The resulting demand function of firm  $j$  corresponds to the lower envelope of all  $D_{j,k}$  ( $k = n - 1$ ). Obviously it is concave. Similarly, one obtains a concave demand curve for the corner firms. Hence, we have established:

**Proposition 1** *The demand function  $D_j$  is concave with respect to the firm's price  $p_j$ .*

Now we can apply Economides' (1989) reasoning for the circular model. By applying Kakutani's fixed point theorem we are able to prove the existence of a non-cooperative price equilibrium (Friedman, 1977). Moreover, uniqueness of this equilibrium can be shown. Economides' proof for the circular model is easily extended to the interval model.

Note that up to this point configurations were neglected where some firms choose the same location. If this is the case then these firms would price their products at marginal cost because it would always pay off to undercut the rival's price to attract some demand given prices are higher than marginal costs. This logic follows the well known Bertrand result. Together with the above considerations this leads to:

**Proposition 2** *For every given pattern of locations there exists a unique equilibrium of the price setting stage.*

## 4 Symmetric price equilibria

Before we turn to the first stage of the game, firms' pricing behavior is analyzed for some symmetric configurations as they might be candidates for a subgame perfect equilibrium. Given that each firm has a positive market share maximizing the profit function  $\Pi_j(\bar{p}|\bar{x}) = p_j D_{j,0}(\bar{p}|\bar{x})$  with respect to  $p_j$  leads to the following price reaction functions:

$$\begin{aligned} p_1^*(\bar{p}|\bar{x}) &= [p_2 + (x_1 + x_2)(x_2 - x_1)] / 2, \\ p_j^*(\bar{p}|\bar{x}) &= \frac{p_{j+1}(x_j - x_{j-1})}{2(x_{j+1} - x_{j-1})} + \frac{p_{j-1}(x_{j+1} - x_j)}{2(x_{j+1} - x_{j-1})} + \frac{(x_{j+1} - x_j)(x_j - x_{j-1})}{2}, \text{ if } 1 < j < n, \\ p_n^*(\bar{p}|\bar{x}) &= [p_{n-1} + (2 - x_{n-1} - x_n)(x_n - x_{n-1})] / 2. \end{aligned}$$

Obviously, the system of prices is identified since the number of independent equations equals the number of variables. It is difficult to calculate prices explicitly without restricting locations to simple patterns. Hence, let us consider symmetric configurations where corner firms locate at  $\alpha\delta$  and  $1 - \alpha\delta$ , respectively and inside firms positions are at equal distance  $\delta = \frac{1}{2\alpha+n-1}$  away from their direct neighbors. In the case of  $\alpha = 0$  this configuration corresponds to the Principle of Maximum Differentiation in the sense that firms maximize the minimum distance to their direct neighbors.

Resulting prices can be expressed as:

$$p_1^*(\bar{p}|\alpha\delta, \alpha\delta + \delta, \alpha\delta + 2\delta, \dots, 1 - \alpha\delta) = [p_2 + \delta^2 + 2\alpha\delta^2] / 2,$$

$$p_j^*(\bar{p} | (\alpha\delta, \alpha\delta + \delta, \alpha\delta + 2\delta, \dots, 1 - \alpha\delta)) = \left[ \frac{p_{j+1}}{2} + \frac{p_{j-1}}{2} + \delta^2 \right] / 2, \text{ if } 1 < j < n,$$

$$p_n^*(\bar{p} | (\alpha\delta, \alpha\delta + \delta, \alpha\delta + 2\delta, \dots, 1 - \alpha\delta)) = [p_{n-1} + \delta^2 + 2\alpha\delta^2] / 2.$$

These are equilibrium prices only if it would not be profitable for any firm  $j$  to drive others out of the market. If this would be the case then we should insert a demand term  $D_{j,l}(\bar{p} | \bar{x})$ ,  $l > 0$  into the above profit function instead of  $D_{j,0}(\bar{p} | \bar{x})$ . However, for the symmetric configurations as considered here we can show that driving neighbors out of business is not a credible strategy because this would require to set a price below zero. Thus, we conclude:

**Proposition 3** *Given the symmetric equidistant market structure  $\bar{x} = (\alpha\delta, \alpha\delta + \delta, \alpha\delta + 2\delta, \dots, 1 - \alpha\delta)$  where  $\alpha \geq 0$  the equations  $p_1^*(\bar{p} | \bar{x}) = [p_2 + \delta^2 + 2\alpha\delta^2] / 2$ ,  $p_j^*(\bar{p} | \bar{x}) = \left[ \frac{p_{j+1}}{2} + \frac{p_{j-1}}{2} + \delta^2 \right] / 2$ , if  $1 < j < n$ , and  $p_n^*(\bar{p} | \bar{x}) = [p_{n-1} + \delta^2 + 2\alpha\delta^2] / 2$  represent equilibrium prices.*

**Proof.** See Appendix A. ■

This and most of the following proofs require explicit price functions, i.e. prices which only depend on location choices. From the above derivations we just get equations in which the price a firm charges in equilibrium depends on the prices of the firm's direct neighbors which in turn depend on prices of other firms and so forth. In general, it is a difficult task to solve this equation system. Fortunately, in equidistant configurations the structure of the system allows simplifications such that the neighbor's prices are replaced by terms for which upper and lower boundaries can be calculated. For example, considering the implicit price function of an inside firm  $j$ ,  $p_j^* = \left[ \frac{p_{j+1}}{2} + \frac{p_{j-1}}{2} + \delta^2 \right] / 2$ , Lemma 2 in the Appendix A tells us that  $p_j^*$  can also be stated in the following two ways:

$$p_j = a_j p_{j-1} + (1 - a_j) \delta^2 + b_j \alpha \delta^2$$

$$p_j = c_j p_{j+1} + (1 - c_j) \delta^2 + d_j \alpha \delta^2$$

The parameters of the first equation incorporate prices and positions of firms which are located on the left of firm  $j$  while the parameters of the second equation capture those located on the right side. As shown in Lemma 2 upper and lower

boundaries can be calculated for the parameters which only depend on the number of firms on the right respectively on the left and on the size of  $\alpha$ . Coefficient  $\alpha$  corresponds to the ratio of the distances between the corner firm and its interval boundary and the distance between two direct neighbors. Substituting  $p_{j-1}$  and  $p_{j+1}$  in the equation of  $p_j^*$  and transforming readily leads to an equation without prices on the right hand side. For this price we can calculate upper and lower boundaries as well which are sufficiently close for our proofs to work.

Returning to the equidistant locational configuration it is easily tested that a solution for the case that  $\alpha = 0$  would be  $p_j = \frac{1}{(n-1)^2}$  ( $j = 1, \dots, n$ ). Hence, prices of all firms are equal where prices decrease quadratically with the number of firms. If corner firms possess some market power because they compete only with rivals on one side then they might take advantage of it by moving inwards. Considering the Hotelling model with linear transport costs, Economides (1993) shows that such a symmetric configuration leads to a U-shaped curve of prices. As stated in Proposition 4 this is the case in our model too.

**Proposition 4** *Given a symmetric and equidistant market structure  $\bar{x} = (\alpha\delta, \alpha\delta + \delta, \alpha\delta + 2\delta, \dots, 1 - \alpha\delta)$  the equilibrium prices are the higher the closer a firm's location is to one of the border firms with respect to the spatial ordering of firms. Further, equilibrium prices are equal between opposite firms, i.e. firms  $i$  and  $n + 1 - i$ , for  $i = 1, \dots, n$ .*

**Proof.** See Appendix A. ■

Indeed corner firms are able to charge higher prices than their competitors in the interior. This is attributable to the fact that they only have to compete at one side. Corner firms are not as much affected by price decisions of their competitors as inside firms since their distance to other firms is maximal with respect to the given spatial ordering. At the other extreme the center firm is closest to its rivals in this respect and therefore in a position in which it is difficult to raise its prices. Not only corner firms benefit from the advantage of being remote to competition but also their closer neighbors. The corner firm's market power decreases continuously

towards the central firm(s). This result is equivalent to the one Economides (1993) deduced from the  $n$ -firm interval model with linear utility.

Having characterized equilibrium price structures on the basis of some simple locational patterns, we are now ready to turn to the first stage of the game.

## 5 Symmetric locational patterns

In this section we are concerned about the existence of symmetric locational equilibria given the price reaction functions of the last stage. Symmetry in this context means that the configuration of locations in the left half of the interval is mirrored in the right half. Attention is restricted to locational patterns with equal distance between direct neighbors. We show that in contrast to the circular model no symmetric equidistant locational equilibrium can be sustained in the interval model when the number of firms is larger than three. Thus, *a fortiori* the Principle of Maximum Differentiation is not valid in this case. Moreover, we show that firms do not have an incentive to move towards the middle firm(s) regardless of how firms are distributed along the interval. This destroys the possibility for the (strong) Principle of Minimum Differentiation to hold.

Following Economides (1989) we argue that two firms would not choose the same location since this would drive prices down to the level of the marginal costs. Deviating from this position increases profits. Moreover, one could think of an equilibrium in which all firms make equal profits. In this model such an equilibrium cannot occur. Equal profits require equal prices since prices and demand are positively related.<sup>4</sup> As shown in the last section, equal prices appear in a symmetric configuration where corner firms are located at the interval boundaries. In this case inside firms attract demand twice as high as that of corner firms. Let us, nevertheless, elaborate on this pattern. It is a candidate equilibrium since it corresponds to the Principle of Maximum Differentiation. This principle holds for the duopoly and

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<sup>4</sup>This follows because the higher the distance of a firm's location to its neighbors the higher the demand and the lower the price competition.

might thus be also an equilibrium outcome of the multi-firm case.

In order to represent an equilibrium the symmetric choices of locations have to maximize the profit function  $\Pi_j(p^*(\bar{x}), x_j | \bar{x}_{-j})$  of each firm  $j$  where  $\bar{x}_{-j}$  are the symmetric locations of all firms except firm  $j$ . Changes of profits as a result of a marginal movement arise from two different effects. The first one is called the demand effect. Relocating firms could gain (lose) additional demand by approaching (moving away from) one of their neighbor firms. The second effect is called the strategic price effect and is induced by the second stage price competition. Since prices are fixed sequentially after the choice of the location, rivals can react on an approach by lowering their prices.

The incentives to deviate from the given configuration might be greatest for the corner firms since their demand effect is positive.<sup>5</sup> Indeed, according to the first order conditions which turn out to be negative for the right and positive for the left firm while the second order conditions are negative, there are incentives for the corner firms to move inwards. In the Appendix A it is shown that the symmetric market structure  $\bar{x} = (0, \delta, 2\delta, \dots, 1)$  where corner firms are located at the interval boundaries and the inside firms are located equidistantly does not support a perfect equilibrium (see Appendix A). Hence, we conclude that

**Proposition 5** *The Principle of Maximum Differentiation does not hold in the multi-firm unit interval Hotelling game with quadratic disutility.*

As expected the corner firms have a substantial interest to change their locations from the boundaries towards their competitors' positions which is due to the direct demand effect overcompensating the price effect. This establishes a market power of the corner firms which drives them to squeeze their rivals in-between.

Another potential symmetric equilibrium might be suggested which accounts for the market power of the corner firms. In this configuration the distance between direct neighbors equals  $\delta$ . Corner firms are located  $\alpha\delta$  ( $0 < \alpha$ ) away from the interval

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<sup>5</sup>For the symmetric pattern considered here one can easily show that the demand effect is zero for every inside firm since all prices are equal.

boundaries. Hence, the two corner firms squeeze all inside firms which themselves maximize the minimum distance to their neighbors.<sup>6</sup> Let us first consider the corner players. We have shown that their profits increase when they deviate from the maximum differentiation pattern. This does not generally hold if one allows the border players to choose a position inside the interval (without changing their relative position). Concluding from the following Lemma 1 the incentives to relocate decrease proportionally with  $\alpha$  such that for a certain value of  $\alpha$  the first order conditions of the border firms become zero while the second order conditions are negative providing that this location choice represents a local maximum.<sup>7</sup>

**Lemma 1** *Given the symmetric equidistant market structure  $\bar{x} = (\alpha\delta, \alpha\delta + \delta, \alpha\delta + 2\delta, \dots, 1 - \alpha\delta)$  where  $\alpha > 0$  the profit maximizing first order condition of firms  $j = 1$  ( $j = n$ ),  $\frac{\partial \Pi_j(p^*(\bar{x}), x_j | \bar{x}_{-j})}{\partial x_j}$  is positive (negative) if  $\alpha < \frac{(3-2a_3)(7-2a_3)}{(7-2a_3)^2 - (1-2a_3)(1+b_3)}$ , zero if  $\alpha = \frac{(3-2a_3)(7-2a_3)}{(7-2a_3)^2 - (1-2a_3)(1+b_3)}$ , and negative (positive) otherwise, where  $a_n = \frac{1}{2}$ ,  $a_{i-1} = \frac{1}{4-a_i}$  and  $b_n = 1$ ,  $b_{n-1} = \frac{b_n}{4-a_n}$ . Furthermore,  $\Pi_j(p^*(\bar{x}), x_j | \bar{x}_{-j})$  is concave in  $x_j$ .*

**Proof.** See Appendix A. ■

Lemma 1 states that in the symmetric configuration corner firms use their market power to move inwards as long as the price reduction from the increased second stage competition does not exceed the gains of new demand. Given the parameter boundaries of Lemma 1 the critical value of  $\alpha$  can be calculated as one third in the case of three firms. If the number of firms goes to infinity it approaches 0.385. Parameter  $\alpha$  can be interpreted as an indicator of market power possessed by the corner firms. It reflects the best response of corner firms to an equidistant configuration. Although the market power increases with the number of firms it remains on a moderate level.

From Lemma 1 we can readily construct an equilibrium if the number of firms equals three.

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<sup>6</sup>The case of  $\alpha = \frac{1}{2}$  corresponds to the welfare maximizing configuration.

<sup>7</sup>The conditions could also be stated in terms of  $c_{n-2}$  and  $d_{n-2}$ . Since they are equal to  $a_3$  and  $b_3$ , respectively, by construction this would not make a difference.

**Proposition 6** *If the number of firms equals three  $\bar{x} = (\frac{1}{8}, \frac{1}{2}, \frac{7}{8})$  represents subgame perfect equilibrium location choices.*

It is easy to show that the center firm has no incentive to relocate in a symmetric configuration. Further, to prove Proposition 6 insert  $a_3 = \frac{1}{2}$  and  $b_3 = 1$  into the above  $\alpha$ -term which makes the first order condition zero. The resulting value of  $\alpha$  equals one third, i.e. in this case the distance between the corner firms and their respective interval boundary is one third of the distance between the corner firm and the market center. To complete the proof that this configuration is to establish a perfect equilibrium,  $\alpha$  must not exceed the value where it would be profitable for the center firm to jump to a border position. This condition is clearly fulfilled. Thus, the equidistant configuration represents a perfect equilibrium in the three-firm case.<sup>8</sup>

For the case that the number of firms is larger than three, the value of  $\alpha$  for which the first order conditions of the border players become zero should also provide no incentive for the inside firms to relocate, i.e. at this value their respective first order conditions should be zero too provided the profit function is quasi-concave. However, calculating the first order conditions of the inside firms for  $n = 4$  leads to a negative value of  $\alpha$  and to values of  $\alpha$  larger than one if  $n$  is greater than five.<sup>9</sup> Taking into account that the optimal level of  $\alpha$  for a corner firm is not larger than 0.385 this means that if  $n \geq 5$  inside firms are more attracted by the central firm(s) than corner firms. This can be attributed to the weakness of the central firms to respond to price changes of their neighbors as described above. Hence, we conclude

**Theorem 1** *No symmetric equidistant market structure  $\bar{x} = (\alpha\delta, \alpha\delta + \delta, \alpha\delta + 2\delta, \dots, 1 - \alpha\delta)$  where  $\alpha \geq 0$  is supported in the multi-firm unit interval Hotelling*

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<sup>8</sup>Together with the existence of the price equilibrium (see Section 4) it can be concluded that the corresponding location choices represent a subgame perfect equilibrium.

<sup>9</sup>Similar as for Proposition 6 it can be shown that  $\alpha = \frac{(26-7a_4)}{(2-6b_4-a_4)}$  makes the first order condition of the second or the second to last player, respectively, zero where  $2 - \sqrt{3} < a_4 \leq \frac{1}{2}$  and  $b_4 \leq 1$ . Obviously, the latter constraints permit negative solutions for  $\alpha$ . This would be the case if the number of firms is four because then  $a_4 = \frac{1}{2}$  and  $b_4 = 1$  (see Lemma 2). Increasing the number of firms leads to decreasing values of  $a_4$  and  $b_4$ , such that  $\alpha$  becomes larger than one in the case of five competitors.



game if the number of firms is larger than three.

Up to this point we have shown that the strong Principle of Maximum Differentiation does not represent equilibrium behavior in oligopoly. Further, even the weak version of this principle where only corner firms do not differentiate maximally is an equilibrium if at least four firms are competing. In the following we will focus on the opposite strategy, the Principle of Minimum Differentiation. Examining the unit interval multi-firm Hotelling model with linear transport costs, Economides (1993) has shown that this principle is prevalent in its strong version. The strong version states that whatever spatial pattern firms start there are always incentives to move towards the central firm.

However, in this section we have shown that the two outside-right and the two outside-left firms would not move inwards in each (symmetric equidistant) configuration because this would not maximize their respective profit functions. In other words, there exist values of  $\alpha \in R$  for which the first order conditions of the quasi-concave profit functions become zero. Thus we conclude:

**Theorem 2** *The strong Principle of Minimum Differentiation does not hold in the multi-firm unit interval Hotelling game with quadratic disutility.*

We have shown that some prominent locational structures can be excluded from the set of potential equilibrium strategies. Unfortunately, we still do not know which if any equilibria appear in the case of more than three players. This question is addressed in the following section.

## 6 Locational equilibria for three to nine players

In order to obtain explicit locational equilibrium solutions for the multi-firm case one first needs the explicit price reaction functions of the last stage of the game. As derived in Section 3 the price of a firm depends on its location, the location of its direct neighbors, and the prices they charge. Hence, prices depend indirectly on all other prices and locations. Transforming this system of implicit price functions into

a system of functions with only locations on the right hand side is a difficult task. In the previous section this was accomplished by restricting the location choices to symmetric equidistant configurations. Without imposing this structural constraint the equation system could not be solved when the number of firms is large. Hence only those cases are considered where the market consists of not more than nine firms.

Equilibria were found by calculating the first and second order conditions of the profit function providing that the solution represents a local maximum. To exclude an instability of the pattern with respect to larger steps of relocation we made sure that it would not pay off for any firm to change the position in the exogenously given ordering.<sup>10</sup>

The solutions for the three- to nine-player cases are shown in Figure 2. As we might expect from the results of the previous section the corner firms move considerably towards the inside firms. In the case of three competitors the distance to the interval boundary is 0.125. Adding firms to the market leads to corner firm locations which are closer to the interval end points in absolute terms. Thus, it seems that competition for the corner firms becomes tougher as they are pushed towards the borders. However, relating the absolute distance between the corner firm and its respective interval boundary to the average distance between the direct neighbors shows a reverse picture. The relative position of the corner firms will be strengthened the more players are in the game. In the four-player case this ratio is about one half as opposed to one third in the three firms case. The oligopoly cases of five to nine players lead to only weakly increasing values of about 0.53. Thus apparently the pressure for the corner firms to differentiate decreases in relative terms with the number of firms since price competition is softened at the same time.

Viewed from the welfare perspective this is an interesting result. The socially optimal outcome requires that all succeeding firms have equal distance. Corner firms however locate one half of this stretch away from the interval boundaries

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<sup>10</sup>The computations were run using the Mathematica software package. The programs are available from the author on request.

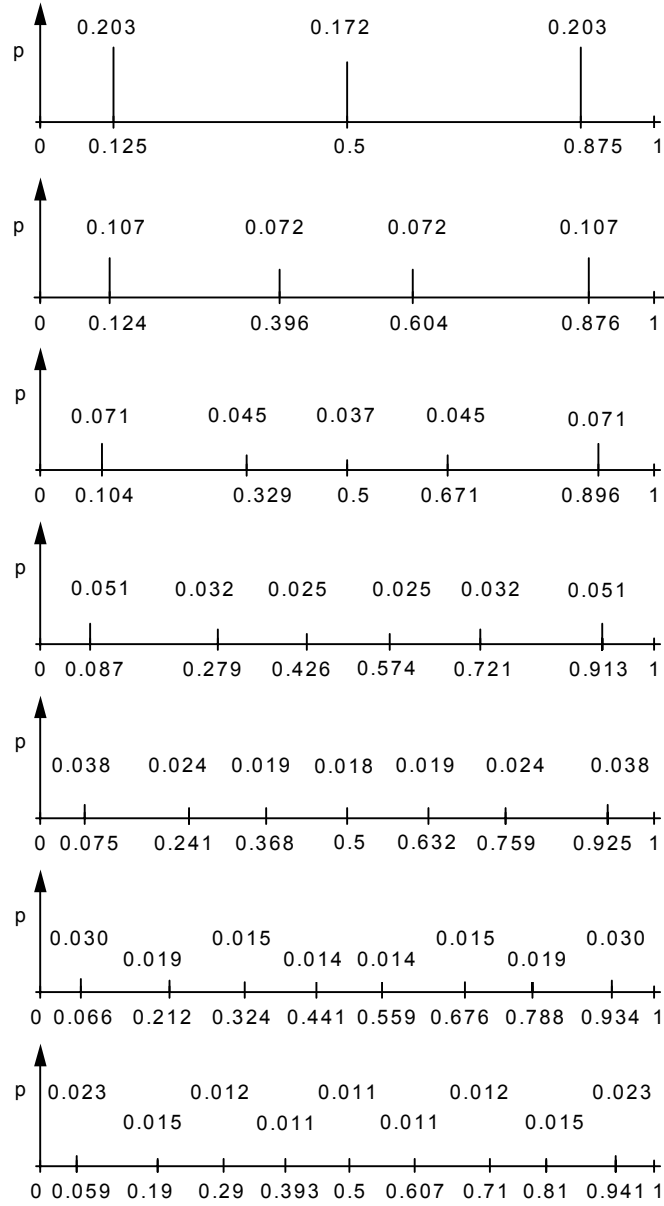


Figure 2: Equilibrium outcome in pure strategies of locations and prices for the three to nine firms Hotelling game (from top to bottom)

firms	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5	Firm 6	Firm 7	Firm 8	Firm 9
three	0.0550	0.0787	0.0550	-	-	-	-	-	-
four	0.0209	0.0219	0.0219	0.0209	-	-	-	-	-
five	0.0113	0.0105	0.008	0.0105	0.0113	-	-	-	-
six	0.0068	0.0063	0.0043	0.0043	0.0063	0.0068	-	-	-
seven	0.0044	0.0041	0.0028	0.0025	0.0028	0.0041	0.0044	-	-
eight	0.0030	0.0028	0.0024	0.0017	0.0017	0.0024	0.0028	0.0030	-
nine	0.0022	0.0020	0.0013	0.0012	0.0012	0.0012	0.0013	0.0020	0.0022

Table 1: Equilibrium profits

since this pattern would minimize transport costs. Although it was shown that this configuration is not supported as an equilibrium in this model the patterns of four and more players come rather close to the optimal solution. In games with two and three players firms could increase welfare by choosing locations closer to each other than in equilibrium. In games with more than four firms there seems to be a (slight) tendency of too few differentiation of the corner firms.

Surprisingly, the market outcome becomes more and more similar to the optimal social solution if the number of firms is large. This is attributable to the change in the price competition. Asymmetry in price competition is large between firms close to the boundaries. This applies in particular to the firms 1, 2, and  $n - 1$ ,  $n$ , respectively. However, as the asymmetry decreases between firms near the market center the incentives to differentiate become similar leading to similar distances between them.

As shown in Table 1 equilibrium profits decrease with the number of firms. This is due to the increase of price competition induced by additional competitors. More interestingly, the relative increase of market power enjoyed by the corner firms following an increase of the number of rivals is also reflected in the firms' profits. In the three-firm case the equilibrium profit of the center firm considerably exceeds the corner firms' profits. In this case a higher number of consumers attracted by the center firm compensates for the lower price. When four players compete, profits are

almost equally distributed while in the oligopoly with five or more players central firms earn much less than their corner rivals. Although the center firm finds itself in an uncomfortable position there is no higher profitable market niche to relocate. It cannot be excluded that this might change as the number of firms grows. Positions between the boundary and the corner firm or between the corner and its neighbor firm may provide a region large enough to be more profitable than the center location if the number of firms exceeds nine. Then subgame perfect equilibria may not be identified by first and second order conditions alone which makes the search for them more difficult or even destroys the possibility for a perfect equilibrium.

## 7 Conclusions

We analyzed the multi-firm unit interval Hotelling model assuming quadratic transport costs. Firms choose locations in the first stage and prices in the second stage of the game. Existence and uniqueness of the short run price game is established. In contrast to the results for the similar circular model and the duopoly interval model it has been shown that the Principle of Maximum Differentiation does not hold. In the corresponding configuration where corner firms are located at the interval boundaries and adjacent firms are located equidistantly corner firms would benefit from moving marginally towards the market center. Other symmetric equidistant configurations do not represent perfect equilibria either. Equally the (strong) Principle of Minimum Differentiation does not hold as it is prevalent in the multi-firm interval model with linear transport costs. Hence, if a perfect equilibrium exists it must correspond to an in-between differentiation configuration.

Explicit symmetric perfect equilibria are calculated for games with up to nine firms. They are characterized by an U-shaped price structure and corner firms which are located away from the interval boundaries. The distance between the corner firms and their direct neighbors is larger than the distances between the inside firms. This follows because of an asymmetric price competition. Furthermore, it seems that corner firms are in a favorable position since they are able to squeeze the

market to a certain extent and to charge the highest prices. In fact the profits exceed their rivals' profits only when  $n \geq 5$ . If the number is smaller then they suffer from having a demand limit at one side. Increasing the number of firms leads to a change in the stage of price competition. Center firms' positions are weakened because more rivals positioned close by increase the price competition by which corner firms are less affected. The resulting increase of "relative" market power together with an increase of overall competition leads to lower absolute but higher relative profits of the corner firms. Further, the competitive advantage of the corner firms is partially transferred to their neighbors which is reflected in higher prices between neighbors towards the boundaries.

Viewed from the welfare perspective we find too much differentiation if the number of competing firms is small ( $n \leq 3$ ). Increasing this number shifts relatively more power to the corner firms which use this power to squeeze their inside rival. This leads to an approximately optimal level of differentiation in the equilibrium case of more than three players.

How does the number of firms influence the equilibrium outcome of Hotelling games? Considering our results and reviewing Salop (1979) and Economides (1989, 1993), it seems that increasing the number of firms has three effects. First, price equilibrium is not endangered. For the quadratic transport costs model, a Nash equilibrium exists for the duopoly as well as for the oligopoly. Assuming linear transport costs price equilibria exist for the multi-firm case in symmetric configurations while it does not exist in the duopoly case. Second, the threat of (second stage) competition is reduced if a market boundary is present. For the multi-firm interval models under linear transport costs this leads to the strong Principle of Minimum Differentiation while it destroys the Principle of Maximum Differentiation under quadratic transport costs. Third, given a market boundary corner firms enjoy a greater market power as their inside competitors. This is reflected in an U-shaped price structure in symmetric and equilibrium configurations and larger than average distances between corner firms and their direct neighbors.

## A Appendix

**Proof of Proposition 3.** Since prices maximize the profit function if each firm has a positive market share we only have to show that driving neighbor firms out of the market does not pay off. For this purpose we show that given a symmetric and equidistant market structure  $\bar{x} = (\alpha\delta, \alpha\delta + \delta, \alpha\delta + 2\delta, \dots, 1 - \alpha\delta)$ ,  $\alpha \geq 0$ , the price  $p'_j$  ( $1 < j \leq n$ ) for which  $D_{j-1,0}(p_{-j}^*, p'_j | \bar{x}) = 0$  is negative. Likewise the price  $p'_j$  ( $1 \leq j < n$ ) for which  $D_{j+1,0}(p_{-j}^*, p'_j | \bar{x}) = 0$  is negative.

If firm  $j$ 's neighbors are inside firms their demand in the equilibrium configuration can be expressed as  $D_{j-1,0}(p_{-j}^*, p_j | \bar{x}) = \frac{p_{j+1}^* - p_j}{2\delta} - \frac{p_j - p_{j-1}^*}{2\delta} + \delta$  and  $D_{j+1,0}(p_{-j}^*, p_j | \bar{x}) = \frac{p_{j+1}^* - p_j}{2\delta} - \frac{p_j - p_{j-1}^*}{2\delta} + \delta$ , respectively (see Section 4). Setting these equations to zero and solving for  $p_j$  leads to  $p_j = 2p_{j-1}^* - p_{j-2}^* - 2\delta^2$  and  $p_j = 2p_{j+1}^* - p_{j+2}^* - 2\delta^2$ . By applying Lemma 2 it is easy to show that the equilibrium prices assuming each firm has a positive market share cannot be larger than  $\delta^2$ . Of course, the best response prices will not increase if firm  $j$  lowers its price. Then it is clear that  $p_j$  must be negative. If firm  $j$ 's neighbor is a corner firm the demand of the corner firm in the equilibrium configuration can be expressed as  $D_{j-1,0}(p_{-j}^*, p_l | \bar{x}) = \frac{p_j - p_l^*}{2\delta} + \frac{\delta}{2}$ ,  $l \in \{0, n\}$ . Setting this equation to zero and solving for  $p_j$  leads to  $p_j = p_l - \delta^2$ . In this case the same argumentation applies as before. ■

**Proof of Proposition 4.** Let us consider two arbitrarily chosen succeeding firms  $i$  and  $i+1$  ( $1 \leq i \leq n-1$ ,  $n > 2$ ) where  $n$  is the number of firms. If firm  $i$  is the left corner firm ( $i = 1$ ) then we obtain the following implicit price reaction functions (see Section 4):  $p_1^* = \frac{p_2}{2} + \frac{\delta^2}{2} + \alpha\delta^2$  and  $p_2^* = \frac{p_1}{4} + \frac{p_3}{4} + \frac{\delta^2}{2}$ . According to Lemma 2 the price function of firm 3 can be represented as:  $p_3^* = a_3 p_2 + (1 - a_3)\delta^2 + b_3 \alpha \delta^2$  where  $2 - \sqrt{3} < a_3 \leq \frac{1}{2}$ , and  $0 < b_3 \leq 1$ . Solving this system for  $p_1^*$  and  $p_2^*$  and subtracting leads to  $p_2^* - p_1^* = \frac{\delta^2 \alpha (-6 + 2a_3 + b_3)}{(7 - 2a_3)}$  which is less than zero. Similarly we can show that  $p_n^* - p_{n-1}^* > 0$ .

If firm  $i$  is one of the inside firms we obtain the following system of price functions according to Lemma 2 which is proven thereafter:

$$p_i = c_i p_{i+1} + (1 - c_i)\delta^2 + d_i \alpha \delta^2$$

$$p_{i+1} = a_{i+1}p_i + (1 - a_{i+1})\delta^2 + b_{i+1}\alpha\delta^2,$$

where  $2 - \sqrt{3} < a_{i+1} \leq \frac{1}{2}$ ,  $2 - \sqrt{3} < c_i \leq \frac{1}{2}$  and  $0 < b_{i+1} \leq 1$ ,  $0 < d_{i+1} \leq 1$ . After some calculations we obtain

$$(*) \quad p_i - p_{i+1} = \delta^2 \alpha \frac{(1-a_{i+1})d_i - b_{i+1}(1-c_i)}{1-a_{i+1}c_i}$$

Now let us assume that firm  $i$  is located on the left part of the interval, i.e.  $i < \frac{n}{2}$ . In this case it follows from Lemma 2 (which is proved below) that  $c_i > a_{i+1}$  and  $b_{i+1} < d_i$  and hence  $(*)$  is positive. Similarly this difference would be negative if  $i \geq \frac{n+1}{2}$ . Further  $p_i$  and  $p_{i+1}$  are equal if firm  $i$  and  $i+1$  are the two central firms.

■

**Lemma 2** *Given  $a_n = \frac{1}{2}$  and  $b_n = 1$  then (i) the sequences  $a_i = \frac{1}{4-a_{i+1}}$  and  $b_i = \frac{b_{i+1}}{4-a_{i+1}}$  are increasing in  $i$  ( $i = 1, \dots, n$ ). Equivalently, given  $c_1 = \frac{1}{2}$  and  $d_1 = 1$  then (ii) the sequences  $c_i = \frac{1}{4-c_{i-1}}$  and  $d_i = \frac{d_{i-1}}{4-c_{i-1}}$  are decreasing in  $i$  ( $i = 1, \dots, n$ ).*

**Proof.** (i) Given  $a_n = \frac{1}{2}$  it is easily shown by complete induction that  $2 - \sqrt{3} < a_i < 2 + \sqrt{3}$  ( $i = 1, \dots, n-1$ ). Then it is clear that  $a_i < a_{i+1}$ . Further, it follows immediately that  $b_i = \frac{b_{i+1}}{4-a_{i+1}} < b_{i+1}$  given the constraints of  $a_i$ . Proof of (ii) works analogously. ■

**Proof of Proposition 5.** The profit maximizing first order condition of firm  $j = 1$  is  $\frac{\partial \Pi_1(p^*(\bar{x}), x_1 | \bar{x}_{-1})}{\partial x_1} = \frac{\partial D_1}{\partial x_1} + \frac{\partial D_1}{\partial p_2} \frac{\partial p_2^*}{\partial x_1} = \frac{1}{2} + \frac{p_2^* - p_1^*}{2(x_2 - x_1)^2} + \frac{1}{2(x_2 - x_1)} \frac{\partial p_2^*}{\partial x_1} \stackrel{!}{=} 0$ . To solve this equation we need explicit price reaction functions. Again we take advantage of Lemma 2. Thus we consider implicit functions only for  $p_1^*$  and  $p_2^*$ . Price  $p_3^*$  we represent in the following way where  $a_3$ , and  $b_3$  have the properties according to Lemma 2.

$$\begin{aligned} p_1^* &= \frac{1}{2}p_2 + \frac{(x_1+x_2)(x_2-x_1)}{2} \\ p_2^* &= \frac{p_3(x_2-x_1)}{2(x_3-x_1)} + \frac{p_1(x_3-x_2)}{2(x_3-x_1)} + \frac{(x_3-x_2)(x_2-x_1)}{2} \\ p_3^* &= a_3p_2 + (1-a_3)\delta^2 + b_3\alpha\delta^2 \end{aligned}$$

Note that corner firms are located at the interval boundaries and hence  $\alpha = 0$ . This system can easily be translated into a system of explicit price reaction functions. After some calculations we obtain  $\frac{\partial \Pi_1(p^*(\bar{x}), x_1 | \bar{x}_{-1})}{\partial x_1} = \frac{1}{2} - \frac{2}{[7-2a_3]}$  and  $\frac{\partial^2 \Pi_1(p^*(\bar{x}), x_1 | \bar{x}_{-1})}{\partial x_1^2} = -\frac{6(5-2a_3)}{\delta(7-2a_3)^2}$ . From Lemma 2 we know that  $2 - \sqrt{3} < a_3 \leq \frac{1}{2}$  and hence  $\frac{\partial \Pi_1(p^*(\bar{x}), x_1 | \bar{x}_{-1})}{\partial x_1} >$



0 and  $\frac{\partial^2 \Pi_1(p^*(\bar{x}), x_1 | \bar{x}_{-1})}{\partial x_1^2} < 0$ . ■

**Proof of Lemma 1.** We proceed as in the previous proof which leads us to the first order condition of firm  $i = 1$  :  $\frac{\partial \Pi_1(p^*(\bar{x}), x_1 | \bar{x}_{-1})}{\partial x_1} = \frac{1}{2} - \frac{2}{(7-2a_3)} + \frac{\alpha}{2} \left( \frac{(1-2a_3)(b_3+1)}{(7-2a_3)^2} - 1 \right)$  where  $2 - \sqrt{3} < a_3 \leq \frac{1}{2}$  and  $0 < b_3 \leq 1$ . It is easy to show that the first order condition becomes zero at  $\alpha = \frac{(3-2a_3)(7-2a_3)}{(7-2a_3)^2 - (1-2a_3)(1+b_3)}$  and for all values of  $\alpha$  below (above) the first order condition would be positive (negative).

$$\text{Now consider the second order condition } \frac{\partial^2 \Pi_1(p^*(\bar{x}), x_1 | \bar{x}_{-1})}{\partial x_1^2} =$$

$$\underbrace{-\frac{6(5-2a_3)}{\delta(7-2a_3)^2}}_{<0} + \underbrace{\frac{\alpha}{2\delta} \left( \frac{(1+b_3)(11-4a_3)}{(7-2a_3)^2} - 1 \right)}_{<0} - \underbrace{\frac{9\alpha(1+b_3)(5-2a_3)}{4(7-2a_3)^3}}_{<0} < 0$$

It is obvious that the left and the right terms are negative. The middle term is also smaller than zero. Taking into account the above restrictions for  $a_3$  and  $b_3$  its value cannot exceed  $\frac{\alpha}{2\delta} \left( \frac{2(3+4\sqrt{3})}{6^2} - 1 \right) < 0$ . Hence, the second order condition is negative irrespective of  $\alpha$ . Further, the profit function is concave and thus quasi-concave in  $x_1$ . The proof for the opposite corner firm  $i = n$  works analogously.

■

**Lemma 3** *If corner firms' distance to their respective interval boundary is  $\alpha\delta$  ( $\alpha \geq 0$ ) and the distance between direct neighbors equals  $\delta$  then the price reaction function of firm  $j$  ( $1 < j \leq n$ ) can be represented as  $p_j = a_j p_{j-1} + (1 - a_j)\delta^2 + b_j \alpha \delta^2$ , where  $a_n = \frac{1}{2}$ ,  $a_{j-1} = \frac{1}{4-a_j}$ , and  $b_n = 1$ ,  $b_{j-1} = \frac{b_j}{4-a_j}$ . Further, for every  $i$ ,  $0 \leq i \leq n$  parameters are constrained as follows:  $2 - \sqrt{3} < a_j \leq \frac{1}{2}$  and  $0 < b_j \leq 1$ . Equivalently, price reaction function of firm  $j$  ( $1 \leq j < n$ ) can be represented as  $p_j = c_j p_{j+1} + (1 - c_j)\delta^2 + d_j \alpha \delta^2$ , where  $c_1 = \frac{1}{2}$ ,  $c_{j+1} = \frac{1}{4-c_j}$ , and  $d_1 = 1$ ,  $d_{j+1} = \frac{d_j}{4-c_j}$ . For every  $0 \leq i \leq n$  parameters are constrained as follows:  $2 - \sqrt{3} < c_j \leq \frac{1}{2}$ , and  $0 < d_j \leq 1$*

**Proof.** The first part of Lemma 2 will be proven by complete induction. The proof of the last part can be completed analogously.

As shown in Section 4 the price reaction function of firm  $j$  is  $p_j^*(\bar{p}) (\alpha\delta, \alpha\delta + \delta, \alpha\delta + 2\delta, \dots, 1 - \alpha\delta) = [p_{j-1} + \delta^2] / 2 + \alpha\delta^2$ , if  $j = n$ . Hence, Lemma 2 holds in this case.

*Induction hypothesis:* We will show that whenever Lemma 2 holds for firm  $j$  it

holds for firm  $j - 1$ , too.

*Induction proof:* For an inside firm  $j - 1$  the price reaction function derived from the first order conditions is  $p_{j-1}^*(\bar{p}(\alpha\delta, \alpha\delta + \delta, \alpha\delta + 2\delta, \dots, 1 - \alpha\delta)) = \left[\frac{p_j}{2} + \frac{p_{j-2}}{2} + \delta^2\right] / 2$ . Given  $p_j = a_j p_{j-1} + (1 - a_j)\delta^2 + b_j \alpha \delta^2$  and inserting this expression into the former equation leads to  $p_{j-1} = a_{j-1} p_{j-2} + (1 - a_{j-1})\delta^2 + b_{j-1} \alpha \delta^2$  where  $a_{j-1} = \frac{1}{4-a_j}$ , and  $b_{j-1} = \frac{b_j}{4-a_j}$ . If  $2 - \sqrt{3} < a_j \leq \frac{1}{2}$ , and  $0 < b_j \leq 1$  hold, one can easily check that also  $2 - \sqrt{3} < a_{j-1} \leq \frac{1}{2}$ , and  $0 < b_{j-1} \leq 1$  hold. ■

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