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Simultaneuos-Equations Models

Axel Werwatz and Christian Müller

1 Introduction

A **simultaneuos-equations model** (SEQM) consists of several interdependent equations. Typically, these equations are not standard regression equations with an **endogenous variable** on the left-hand side and one or several **exogenous regressors** on the right-hand side that are independent of the error term. Rather, endogenous variables may also appear on the right hand side of the equations that comprise the SEQM.

But SEQMs are not merely a collection of equations with endogenous regressors. They are truly systems of equations in the sense that there are cross-equation relationships between the variables.

The well-known macroeconometric model of Klein (1950) is a good example to illustrate these points. The Klein's model consists of six equations, three **statistical equations** and three **identities**. The three statistical equations look like standard regression equations:

$$\begin{array}{lclclclclcl} C_t & = & \alpha_0 & + & \alpha_1 P_t & + & \alpha_2 P_{t-1} & + & \alpha_3 (W_t^p + W_t^g) & + & \epsilon_{1t} \\ I_t & = & \beta_0 & + & \beta_1 P_t & + & \beta_2 P_{t-1} & + & \beta_3 K_{t-1} & + & \epsilon_{2t} \\ W_t^p & = & \gamma_0 & + & \gamma_1 Y_t & + & \gamma_2 Y_{t-1} & + & \gamma_3 A_t & + & \epsilon_{3t} \end{array} \tag{1}$$

Here, the αs, βs and γs are unknown regression coeffcients, ϵ_{1t}, ϵ_{2t} and ϵ_{3t} are unobservable error terms and all capital letters denote observable variables, whose meaning will be described below as necessary.

Klein's model is completed by the following three identities:

$$\begin{array}{lclclclcl} Y_t & = & C_t & + & I_t & + & G_t \\ P_t & = & Y_t & + & T_t & - & W_t^p \\ K_t & = & K_{t-1} & + & I_t & & \end{array} \tag{2}$$

These identities neither include unknown coeffcients nor error terms. They hold "by definition". Nonetheless, they are an integral part of the model.

The first equation of (2), for instance, says that total spending Y_t in an economy in year t is the sum of private consumption spending C_t, investment spending I_t and government spending G_t. This is an accounting relationship. Similarly, the second equation of (2) states that we obtain private profits P_t if we subtract from total spending Y_t indirect taxes T_t and the total wage bill of private enterprises W_t^p. These identities introduce interdependencies between the variables of the statistical equations (1).

Note, for instance, that C_t depends on W_t^p via the first equation of (1) and that W_t^p depends on Y_t via the third equation of (1). But Y_t depends on C_t though the first identity in (2) which implies that W_t^p depends on Y_t. In this way, the first and third equation of (1) are interdependent or simultaneous. This relatedness is not a result of some relationship between the error terms ϵ_{1t} and ϵ_{3t} of these equations. Rather, it is a result of the equations in (1) and (2) truly being a system of equations with various cross-equation relationships between the observable variables of the system. This simulateneity has important consequences if we want to consistently estimate the unknown coeffients in (1).

2 Estimating Simultaneous-Equations Models

2.1 Identification

The equations of Klein's SEQM defined by (1) and (2) are motivated by economic theory. They are an attempt to write down in a parsimonious way the "structure" of the economy from an aggregate, macro perspective. The parameters of these equations are thus the proper targets of estimation: after all, they supposedly tell us "how the economy works". In the following sections, we will briefly discuss two approaches of estimating the parameters of a structural SEQM such as Klein's model.

A necessary condition for consistent estimation is identification. Identification in SEQMs is a serious issue and warrants an extended treatment that can't be provided in the format of this chapter. But any good econometrics textbook features an extensive discussion of this matter. See, for instance, Greene (1998) or Goldberger (1991).

Intuitively speaking, identification is difficult because of the interrelatedness of the variables, which makes it hard to disentangle their various relationships. A necessary and easily verified condition for identification in SEQMs is that the number of exogenous variables excluded from any equation must be as least as large as the number of endogenous variables included in that equation ("order condition").

In Klein's model of equations (1) and (2), the endogenous variables are C_t, I_t, W_t^p, Y_t, K_t and P_t (note that the latter three are endogenous despite not being a left-hand side variable of any statistical equation). All other variables are exogenous or predetermined. You may verify that in the first equation of (1) there are two endogenous (C_t and P_t) and six excluded exogenous variables (K_{t-1}, Y_{t-1}, A_t, G_t, T_t and W_t^g). Hence, the order equation is satisfied for this equation. Greene (1998) shows that the condition is satisfied for the entire system (as well as other, sufficient conditions for identification).

2.2 Some Notation

Let us write down the M statistical equations (i.e, the equations that have unknown coefficients and error terms) of the structural form (i.e., the equations as they are suggested by economics) of a SEQM :

$$\begin{array}{ccccccc} \mathrm{y}_1 & = & \mathrm{Y}_1^T\ \beta_1 & + & \mathrm{X}_1^T\ \gamma_1 & + & \varepsilon_1 \\ \vdots & & & \vdots & & & \\ \mathrm{y}_M & = & \mathrm{Y}_M^T \beta_M & + & \mathrm{X}_M^T \gamma_M & + & \varepsilon_M. \end{array}$$

Focusing on the first equation, y_1 is the left-hand side variable, Y_1 is the vector of endogenous right-hand side variables, x_1 is the vector of exogenous or predetrmined right-hand side variables and β_1 and γ_1 are the objects of interest: the unknown structural parameters of the model.

We can illustrate the notation by using, say, the M-th equation of Klein's model introduced above: $M = 3$ (the coefficients of the identities in (2) are known and need not be estimated), $\mathrm{y}_M = W_t^p$, $\mathrm{Y} = Y_t$, $\beta_M = \gamma_1$, $\mathrm{X}_M^T = [1 \quad Y_{t-1} \quad A_t]$, $\gamma_M = [\gamma_0 \quad \gamma_2 \quad \gamma_3]^T$ and $\varepsilon_M = \epsilon_{3t}$.

Suppose that we have T observations of all the variables of the model. Then we can write down the SEQM in terms of the data matrices and vectors as

follows:

$$\begin{array}{lllll} y_1 & = & Y_1\beta_1 & + & X_1\gamma_1 & +\epsilon_1 \\ \vdots & & & & \vdots & \\ y_M & = & Y_M\beta_M & + & X_M\gamma_M & +\epsilon_M. \end{array}$$

Using again the M-th equation to illustrate the notation, y_M and ϵ_M are $T \times 1$ vectors whereas Y_M and X_M are matrices with n rows that have as many columns as Y_M^T and X_M^T, respectively.

Defining $Z_m = (Y_1, X_1)$ and $\delta_m = (\beta_M, \gamma_M)$ for $m = 1, \ldots, M$, we can write the system even more compactly as

$$\begin{array}{lllll} y_1 & = & Z_1\ \delta_1 & + & \epsilon_1 \\ \vdots & & \vdots & & \\ y_M & = & Z_M\,\delta_M & + & \epsilon_M. \end{array} \tag{3}$$

Estimating $\delta_1, \ldots, \delta_M$ by applying ordinary least squares to each equation is not a consistent estimation procedure as some of the right-hand side variables are endogenous and therefore correlated with the error terms.

For instance, in the second equation of (1), P_t is correlated with the error term. This can be seen by substituting the first identity of (2) for X_t into the second identity of (2) and observing that P_t is a linear function of I_t.

2.3 Two-Stage Least Squares

Endogenous regressors can be treated by instrumenting them and this is precisely what the method of two-stage least squares does.

1. In the first stage, the endogenous regressor is regressed by ordinary least squares on all exogenous regressors and the predicted values of this regression are obtained. Denoting the matrix of observations of **all** exogenous or predetermined variables of the SEQM as X we can write the matrix of predicted values of the endogenous regressors of the m-th equation as

$$\hat{Y}_m = X\{(X^TX)^{-1}X^TY_m\} \tag{4}$$

2. In the second stage, the two-stage least squares estimator for equation m is obtained by regressing y_m on the fitted values $\hat{Y}_m$ and the matrix of exogenous or predetermined regressors of equation m, X_m.

Using the notation of (3), Greene (1998) shows that the two-stage least squares estimator of the parameters of equation m can be written as

$$\begin{aligned} \widehat{\delta}_{m,2SLS} &= \{(Z_m^T X)(X^T X)^{-1}(X^T Z_m)\}^{-1} Z_m X (X^T X)^{-1} X^T y_m. \\ &= \{\widehat{Z}_m^T \widehat{Z}_m\}^{-1} \widehat{Z}_m^T y_m \end{aligned} \tag{5}$$

This procedure, applied to all equations $m = 1, \ldots, M$, is consistent but not efficient. This is because it does not exploit the fact that the equations form a system or interdependent equations but rather estimates the equations one-by-one. In the following section, we will describe the three-stage least squares estimation procedure which builds on but also improves upon two-stage least squares.

2.4 Three-Stage Least Squares

Let us rewrite (3) such that the system nature of the SEQM is even more evident:

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} Z_1 & 0 & \cdots & 0 \\ 0 & Z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Z_M \end{pmatrix} + \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_M \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_M \end{pmatrix} \tag{6}$$

(where y, Z and ϵ all have $T \times M$ rows) or simply as

$$y = Z\delta + \epsilon \tag{7}$$

Using this "system" notation, we can write the two-stage least squares estimator (5) for all M equations very compactly as

$$\widehat{\delta}_{2SLS} = \{\widehat{Z}^T \widehat{Z}\}^{-1} \widehat{Z}^T y, \tag{8}$$

where

$$\widehat{Z} = \begin{pmatrix} X(X^TX)^{-1}X^TZ_1 & 0 & \cdots & 0 \\ 0 & X(X^TX)^{-1}X^TZ_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X(X^TX)^{-1}X^TZ_M \end{pmatrix}$$

$$= \begin{pmatrix} \widehat{Z}_1 & 0 & \cdots & 0 \\ 0 & \widehat{Z}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widehat{Z}_M \end{pmatrix} \tag{9}$$

Written in this notation, the two-stage least-squares estimator looks like a "system" estimator but it is really equation-by-equation instrumental variables regression.

As remarked earlier, this estimator is consistent but it is not efficient. A more efficient estimator is suggested by looking at the covariance matrix of the long residual vector ϵ of (7).

$$E[\epsilon\,\epsilon^T] = \bar{\Sigma} = \begin{pmatrix} \sigma_{11}I & \sigma_{12}I & \cdots & \sigma_{1M}I \\ \sigma_{21}I & \sigma_{22}I & \cdots & \sigma_{2M}I \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{M1}I & \sigma_{M2}I & \cdots & \sigma_{MM}I \end{pmatrix} = \Sigma \otimes I, \tag{10}$$

where I is an $T \times T$ identity matrix, $\otimes$ denotes the Kronecker product and Σ is the covariance matrix of the error terms $\epsilon_1, \epsilon_2, \ldots, \epsilon_M$. This non-spherical covariance matrix suggestes to improve upon (8) by using generalized least squares. This is precisely the idea behind the three-stage least squares estimator.

Suppose we knew the covariance matrix Σ. Then a generalized least squares, instrumental variables estimator could be computed as

$$\delta_{GLS} = \{\widehat{Z}^T(\Sigma^{-1} \otimes I)\widehat{Z}\}^{-1}\widehat{Z}^T(\Sigma^{-1} \otimes I)y. \tag{11}$$

In practice, Σ is unknown and has to be estimated. A consistent estimator of element (i, j) of Σ is based on the sum of squares of the residuals of the i-th and j-th equation after estimating each equation by two-stage least squares:

$$\hat{\sigma}_{ij} = \frac{(y_i - Z_i\widehat{\delta}_i)^T(y_j - Z_j\widehat{\delta}_j)}{T}, \tag{12}$$

where $\widehat{\delta}_i$ and $\widehat{\delta}_j$ are the two-stage least squares estimator for equations i and j as defined in (5). Estimating each element of Σ in this way defines the estimator $\widehat{\Sigma}$ of Σ and allows to compute the feasible generalized lest squares estimator:

$$\widehat{\delta}_{3SLS} = \{\widehat{Z}^T(\widehat{\Sigma}^{-1} \otimes I)\widehat{Z}\}^{-1}\widehat{Z}^T(\widehat{\Sigma}^{-1} \otimes I)y. \tag{13}$$

An estimator of the asymptotic covariance matrix of (13) is given by

$$\widehat{\text{Asy. Var}}(\widehat{\delta}_{3SLS} = \{\widehat{Z}^T(\widehat{\Sigma}^{-1} \otimes I)\widehat{Z}\}^{-1} \tag{14}$$

Summing up, the three-stage least squares estimator of δ in (7) is obtained by carrying out the following three-steps:

1. The first stage is identical to the two-stage procedure: instruments for the endogenous regressors are computed as the predicted values of an ordinary least squares regression of each endogenous regressor on all exogenous regressors.
2. In the second stage, the two-stage least squares estimator for each equation m is computed and the residuals are used according to (12) to obtain an estimate of Σ, the covariance matrix of the error terms of the SEQM.
3. In the third stage, the estimate of Σ is used to calculate the generalized least squares estimator defined in (13) and an estimate of its covariance matrix as described in (14)

2.5 Computation

```
{d3sls, cov3, d2sls} = seq(seqlist1, seqlist2)
```
estimates a SEQM via two-stage and three-stage least squares

The quantlet `seq` is all you need to estimate a SEQM with XploRe. It belongs to the `metrics` quantlib which you have to load before you can use `seq`:

```
library("metrics")
```

`seq` requires two inputs and will return two vectors and a matrix along with an ANOVA-type table in the output window. The two inputs are lists. The first list (`seqlist1`) is a list of matrices that contain the data. The second list (`seqlist2`) is a list of string vectors that contain variable names.

`seqlist1`

If there are M statistical equations (i.e., equations with unknown coefficents and an error term) in the system then `seqlist1` is a list of $M+2$ matrices. The first matrix consists of the cocatenated vectors of observations of the left-hand side variables. The next M matrices contain the observations of the right-hand side variables of the M statistical equations of the system. The last and $(M+2)$nd matrix contains the observations of all exogenous and predetermined variables of the system. That is, the $M+2$ matrices are:

$$\begin{array}{ll} 1. & (y_1, y_2, \ldots, y_M) \\ 2. & Z_1 \\ 3. & Z_2 \\ \vdots & \vdots \\ (M+1). & Z_M \\ (M+2). & X \end{array}$$

`seqlist2`

is a list of $M+2$ string vectors. Each vector contains the names of the variables that form the corresponding matrix in `seqlist1`. For instance, the first string vector of `seqlist2` contains the names of the left-hand side variables of the system whereas the last string vector contains the names of all exogenous and predetermined variables of the system.

The XploRe code for the Klein's model will serve as an example:

```
; reading in the data
data=read("klein.dat")

; getting rid of the missing values of the lagged variables
data=data[2:rows(data),]

; assigning the columns of "data" to the variables
C=data[,2]
P=data[,3]
Wp=data[,4]
I=data[,5]
K1=data[,6]
Y=data[,7]
Wg=data[,8]
```

```
G=data[,9]
T=data[,10]
W=data[,11]
P1=data[,12]
Y1=data[,13]
A=data[,14]

; preparing the matrices that form seqlist1
; first matrix: concatenated vectors
;                                 of left-hand side variables
y=C~I~Wp
; a column of "1"s for constant term
ONE=matrix(rows(data),1)
; second through (M+1)st matrix:
;                      matrices of right-hand side variables
z1=P~P1~W~ONE
z2=P~P1~K1~ONE
z3=Y~Y1~A~ONE
; (M+2)nd matrix: matrix of instruments
;                     (exogenous and predetermined variables)
x=ONE~G~T~Wg~A~P1~K1~Y1

; forming a list of matrices
seqlist1=list(y,z1,z2,z3,x)

; now we will prepare the string vectors that will form
; seqlist2 vector with names of left-hand side variables
yl="C"|"I"|"Wp"
; vectors with names of right-hand side variables
;                                    of equations 1 though M
l1="P"|"P1"|"W"|"ONE"
l2="P"|"P1"|"K1"|"ONE"
l3="E"|"E1"|"A"|"ONE"

; vector with names of instruments
xl="ONE"|"G"|"T"|"Wg"|"A"|"P1"|"K1"|"Y1"

; putting the string vectors into a list
seqlist2=list(yl,l1,l2,l3,xl)
```

```
; finally, we call the seq-quantlet
{d1,s,d2}=seq(seqlist1,seqlist2)
```

seq01.xpl

seq produces two kinds of output: ANOVA-style tables in the output window and coefficient estimates and their estimated standard errors as vectors or matrices.

There is a table of estimation results both for the two-stage (shown first) and three-stage least squares procedures (shown last). Both tables are designed to be easily readable and self-explanatory. We hope that you are going to agree after looking at the three-stage least squares table for the Klein's model:

```
[ 1,] "======================================================"
[ 2,] "   3stage Least-squares estimates"
[ 3,] "======================================================"
[ 4,] "EQ   dep. var.              R2"
[ 5,] "======================================================"
[ 6,] " 1   C                      0.980     "
[ 7,] " 2   I                      0.826     "
[ 8,] " 3   Wp                     0.986     "
[ 9,] "======================================================="
[10,] "VARIABLE    Coef.Est.        Std.Err.          t"
[11,] "-----------------------------------------------------"
[12,] "P              0.125          0.108          1.155"
[13,] "P1             0.163          0.100          1.624"
[14,] "W              0.790          0.038         20.826"
[15,] "ONE           16.441          1.305         12.603"
[16,] "-----------------------------------------------------"
[17,] "P             -0.013          0.162         -0.081"
[18,] "P1             0.756          0.153          4.942"
[19,] "K1            -0.195          0.033         -5.990"
[20,] "ONE           28.178          6.794          4.148"
[21,] "-----------------------------------------------------"
[22,] "E              0.400          0.032         12.589"
[23,] "E1             0.181          0.034          5.307"
[24,] "A              0.150          0.028          5.358"
[25,] "ONE            1.797          1.116          1.611"
[26,] "======================================================="
[27,] "INSTRUMENTS     Mean        Std.Dev.          "
[28,] "-----------------------------------------------------"
```

```
[29,] "ONE            1.000          0.000"
[30,] "G              9.914          3.910"
[31,] "T              6.805          2.032"
[32,] "Wg             5.119          1.957"
[33,] "A              0.000          6.205"
[34,] "P1            16.376          4.028"
[35,] "K1           200.495          9.919"
[36,] "Y1            57.986          8.919"
[37,] "==================================================="
```

Besides showing the estimation results in the output window you can also access some of them as vectors or matrices. Specifically, the following quantities are returned by `seq`:

`d3sls`
: the vector of three-stage least squares estimates of the elements of δ in (7). It is computed according to (13).

`cov3`
: the estimated covariance matrix of the three-stage least squares estimator, computed according to (14).

`d2sls`
: the vector of two-stage least squares estimates of the elements of δ in (7). It is obtained by computing (5) for each equation and stacking-up the resulting vectors.

In the following section, we will use `seq` to estimate a money-demand system.

3 Application: Money Demand

Economists often refer to the money stock as one important determinant of the price level. Therefore, the evolution of the amount of money in the economy is also a focus of monetary policy analysis. A convenient tool for this is to look at the so called demand for money. The recent economic literature analyzes the long-run demand for money (denoted m) as a function of aggregated income (y,) short and long term interest (i^s, i^l) and inflation rates within a cointegration framework. If there is more than one cointegration relationship and/or the relationship of interest helps to explain more than just one variable, these relationships are estimated more efficiently within a system rather than as a single equation (see (Ericsson 1999) on that). The explanatory variables considered in the money demand equation might cointegrate not only to a money demand function but also to a stationary spread between long and short term interest rates and a stationary real interest rate. Therefore, in a study about European money demand, Müller and Hahn (2000) applied a system specification to determine whether or not there exists a stationary relationship between the money stock, aggregated income, 3-months interest rates, government bond yield and a measure of European inflation. All data are weighted sums of the series of each of the eleven countries except for the price measure which has been obtained as the ratio of nominal and real income. In case of the interest rates the weights are real income shares and in the cases of money and income the official EURO rates have been used.

Using a system approach suggests to consider a reduced form regression, where no endogenous variables may enter any of the equations on the right hand side. In contrast to that, the change in money stock is often considered to depend e.g. on the current change in inflation (Lütkepohl and Wolters 1998) and the same is true for relationship between the short and the long term interest rates. That's why the reduced form regression is used to identify the long-run relationships (cointegration relationships) while in a second step the model is re-written to yield a structural form, as described above. Thus, the structural SEQM is:

$$
\begin{aligned}
\triangle(m-p)_t &= \alpha_{1,1} + \alpha_{1,2}\triangle(m-p)_{t-1} + \alpha_{1,3}\triangle y_{t-1} + \alpha_{1,4}\triangle i^l_{t-1} + \alpha_{1,5}\triangle i^s_{t-1} \\
&\quad + \alpha_{1,6}\triangle^2 p_{t-1} + \alpha_{1,7}\triangle^2 p_t + \alpha_{1,8} ec1_{t-1} + \varepsilon_{1,t} \qquad (15) \\
\triangle y_t &= \alpha_{2,1} + \alpha_{2,2}\triangle(m-p)_{t-1} + \alpha_{2,3}\triangle y_{t-1} + \alpha_{2,4}\triangle i^l_{t-1} + \alpha_{2,5}\triangle i^s_{t-1} \\
&\quad + \alpha_{2,6}\triangle^2 p_{t-1} + \alpha_{1,8} ec1_{t-1} + \alpha_{2,9} ec2_{t-1} + \varepsilon_{2,t} \qquad (16) \\
\triangle i^l_t &= \alpha_{3,1} + \alpha_{3,2}\triangle(m-p)_{t-1} + \alpha_{3,3}\triangle y_{t-1} + \alpha_{3,4}\triangle i^l_{t-1} + \alpha_{3,5}\triangle i^s_{t-1} \\
&\quad + \alpha_{3,6}\triangle^2 p_{t-1} + \varepsilon_{3,t} \qquad (17) \\
\triangle i^s_t &= \alpha_{4,1} + \alpha_{4,2}\triangle(m-p)_{t-1} + \alpha_{4,3}\triangle y_{t-1} + \alpha_{4,4}\triangle i^l_{t-1} + \alpha_{4,5}\triangle i^s_{t-1} \\
&\quad + \alpha_{4,6}\triangle^2 p_{t-1} + \alpha_{4,10}\triangle i^l_t + \varepsilon_{4,t} \qquad (18) \\
\triangle^2 p_t &= \alpha_{5,1} + \alpha_{5,2}\triangle(m-p)_{t-1} + \alpha_{5,3}\triangle y_{t-1} + \alpha_{5,4}\triangle i^l_{t-1} + \alpha_{5,5}\triangle i^s_{t-1} \\
&\quad + \alpha_{5,6}\triangle^2 p_{t-1} + \alpha_{5,9} ec2_{t-1} + \varepsilon_{1,t} \qquad (19) \\
ec1_t &= (m-p)_t - 1.574 y_t + 3.405 i^l_t - 2.061 i^s_t \qquad (20) \\
ec2_t &= i^l - 4\triangle p \qquad (21)
\end{aligned}
$$

Here, $\triangle = 1 - L$ where L is the backshift operator (i.e., $\triangle y_{t-1} = (1 - L)y_{t-1} = y_{t-1} - y_{t-2}$). It can be easily verified that the order condition holds and all parameters are identified. In our case the endogenous variables are $\triangle m - p, \triangle y_t, \triangle i^l_t, \triangle i^s_t$ and $\triangle^2 p_t$ (all variables except the interest rates in logarithms). There are no purely exogenous variables but predetermined (lagged endogenous) variables only. The error correction terms are given by the identities (20) and (21). They are the results of the first step, the reduced form regression. Thus, we re-estimate the model including the short-run adjustment parameters by 3SLS under more general assumptions about the contemporaneous structure than in the reduced rank model.

We have used the following XploRe code to estimate the parameters of equations (15) to (19):

```
; reading in the data
z=read("eu.raw")
; getting rid of missing values due to lagged variables
z=z[4:rows(z),]
; assigning columns of z to variable names
dmp       =  z[,4]
```

```
dmp1      =  z[,5]
dy        =  z[,7]
dy1       =  z[,8]
d2p       =  z[,12]
d2p1      =  z[,13]
dil       =  z[,15]
dil1      =  z[,16]
dik       =  z[,18]
dik1      =  z[,19]
ec11      =  z[,21]
ec21      =  z[,23]
; creating the matrices for seqlist1
lhs=dmp~dy~dil~dik~d2p
one= matrix(rows(z),1)
z1=one~dmp1~dy1~dil1~dik1~d2p1~d2p~ec11
z2=one~dmp1~dy1~dil1~dik1~d2p1~ec11~ec21
z3=one~dmp1~dy1~dil1~dik1~d2p1
z4=one~dmp1~dy1~dil1~dik1~d2p1~dil
z5=one~dmp1~dy1~dil1~dik1~d2p1~ec21
x=one~dmp1~dy1~dil1~dik1~d2p1~ec11~ec21
; forming seqlist1 as a list of matrices
seqlist1=list(lhs,z1,z2,z3,z4,z5,x)
; creating list of string vectors
yl="dmp"|"dy"|"dzl"|"dzk"|"d2p"
zl1="one"|"dmp1"|"dy1"|"dil1"|"dik1"|"d2p1"|"d2p"|"ec11"
zl2="one"|"dmp1"|"dy1"|"dil1"|"dik1"|"d2p1"|"ec11"|"ec21"
zl3="one"|"dmp1"|"dy1"|"dil1"|"dik1"|"d2p1"
zl4="one"|"dmp1"|"dy1"|"dil1"|"dik1"|"d2p1"|"dil"
zl5="one"|"dmp1"|"dy1"|"dil1"|"dik1"|"d2p1"|"ec21"
xl="one"|"dmp1"|"dy1"|"dil1"|"dik1"|"d2p1"|"ec11"|"ec21"
; forming seqlist2 as a list of string vectors
seqlist2=list(yl,zl1,zl2,zl3,zl4,zl5,xl)
; finally, calling seq to estimate the model
{d3sls,cov3,d2sls}=seq(seqlist1,seqlist2)
```

seq02.xpl

These lines of XploRe code yield the following table in the output window (table of two-stage least squares estimates has been omitted). It shows for each equation the left-hand side variable (top panel), three-stage least squares coefficent estimates, estimated standard errors and t-values. The bottom panel

of the table lists the predetermined and exogenous variables that were used as instruments in the estimation procedure, along with their sample means and standard deviations.

```
[ 1,] "======================================================"
[ 2,] "   3stage Least-squares estimates"
[ 3,] "======================================================"
[ 4,] " EQ  dep. var.              R2"
[ 5,] "======================================================"
[ 6,] " 1   dmp                    0.643     "
[ 7,] " 2   dy                     0.506     "
[ 8,] " 3   dzl                    0.279     "
[ 9,] " 4   dzk                    0.079     "
[10,] " 5   d2p                    0.299     "
[11,] "======================================================"
[12,] "VARIABLE    Coef.Est.        Std.Err.          t"
[13,] "-----------------------------------------------------"
[14,] "one           -0.050          0.074         -0.684"
[15,] "dmp1           0.680          0.113          6.023"
[16,] "dy1            0.124          0.114          1.089"
[17,] "dil1          -0.126          0.153         -0.821"
[18,] "dik1           0.029          0.149          0.195"
[19,] "d2p1          -0.897          0.541         -1.658"
[20,] "d2p           -1.421          1.078         -1.319"
[21,] "ec11          -0.018          0.026         -0.669"
[22,] "-----------------------------------------------------"
[23,] "one            0.184          0.054          3.403"
[24,] "dmp1           0.270          0.087          3.082"
[25,] "dy1            0.393          0.106          3.694"
[26,] "dil1           0.016          0.129          0.124"
[27,] "dik1           0.041          0.098          0.418"
[28,] "d2p1           0.922          0.242          3.809"
[29,] "ec11           0.061          0.019          3.262"
[30,] "ec21          -0.159          0.054         -2.928"
[31,] "-----------------------------------------------------"
[32,] "one           -0.002          0.001         -1.980"
[33,] "dmp1           0.148          0.098          1.516"
[34,] "dy1            0.192          0.121          1.590"
[35,] "dil1           0.425          0.138          3.075"
[36,] "dik1          -0.092          0.103         -0.894"
```

```
[37,] "d2p1          0.136          0.261          0.522"
[38,] "-----------------------------------------------------"
[39,] "one          -0.003          0.002         -1.698"
[40,] "dmp1          0.120          0.174          0.686"
[41,] "dy1           0.365          0.221          1.656"
[42,] "dil1          0.400          0.374          1.069"
[43,] "dik1          0.108          0.157          0.687"
[44,] "d2p1         -0.022          0.371         -0.060"
[45,] "dil          -0.270          0.762         -0.354"
[46,] "-----------------------------------------------------"
[47,] "one          -0.004          0.002         -2.472"
[48,] "dmp1          0.030          0.047          0.646"
[49,] "dy1           0.009          0.057          0.154"
[50,] "dil1         -0.077          0.068         -1.144"
[51,] "dik1          0.062          0.048          1.296"
[52,] "d2p1         -0.323          0.125         -2.574"
[53,] "ec21          0.072          0.028          2.596"
[54,] "======================================================"
[55,] "INSTRUMENTS     Mean        Std.Dev.          "
[56,] "-----------------------------------------------"
[57,] "one           1.000          0.000"
[58,] "dmp1          0.000          0.005"
[59,] "dy1           0.006          0.005"
[60,] "dil1         -0.001          0.004"
[61,] "dik1         -0.002          0.006"
[62,] "d2p1          0.000          0.002"
[63,] "ec11         -2.813          0.027"
[64,] "ec21          0.052          0.010"
[65,] "======================================================"
```

Interpreting the results, two groups of estimators are of particular interest. These are first the structural or contemporaneous explanatory variables' parameters ($\alpha_{4,10}$ and $\alpha_{1,7}$) and, second the parameters of the error correction terms ($\alpha_{.,8}$ and $\alpha_{.,9}$), the so called short-run adjustment parameters.

Within the first group, we obtain a negative relationship between real money growth and inflation growth, which is indicated by the coefficient of -1.42. Its sign does not come as a surprise since an increase in inflation will naturally depreciate the value of nominal money stock. The coefficient does not seem to be statistically significant however, which is indicated by the marginal probability

of 0.187. Similarly the changes in the long-term interest rate do not seem to have a significant impact on short rate movements of the same period.

The second group of interest provides some insight into the feasibility and effects of monetary policy as well as to some extent into some basic economic relationships. To start with, the error correction term which is given in (20) and labeled as the long-run money demand enters the money and income growth equations (15) and (16). This term indicates what effect money demand has on the respective variables in excess of the long-run equilibrium. In the first equation we assumed it to lead to a slow-down in money growth. This feature should be present if one expects money to be demanded in quite the same way as many other ordinary commodities. Thus, in such a case we would observe an inherent tendency to restore equilibrium. The estimation results suggests however that this adjustment does not take place. This is because although the corresponding coefficient yields the correct sign it has too large a standard error compared to its magnitude. When the true coefficient is zero then the we would also have to assume that there is no money demand in Europe altogether and the cointegrating relationship should better be rewritten in such a way that it is normalised on a variable in whose equation the error correction term enters significantly. This could be the income equation for example. Sticking to the interpretation of a long-run money demand equilibrium we notice that excess demand of real money will lead to higher income growth in the next period as indicated by $\alpha_{2,8}$. Of course, re-formulating the $ec1$ term does not affect the significance of the coefficient but it could change the sign and will change magnitude and economic interpretation, which will not be done here since we are investigating the hypothesis of the existence of a money demand.

The second error correction term has an interpretation as a real interest rate. When real interest rates are high, the respective coefficient in the income equation indicates that income growth will be less in the following period. This, too, is economically reasonable, because credits are more expensive in that case. Already in the first step, in the reduced form estimation, we found no evidence of an endogenous tendency for the long-term interest rate to adjust to deviations from the long-run real interest equilibrium level. Therefore the error correction term for real interest rates have not been included in eq. (19). Instead, as the corresponding coefficient of the current estimation ($\alpha_{2,9}$) implies also, these deviations may help to predict future inflation.

Since we used the error correction terms obtained in the reduced form regression and applied the zero adjustment coefficient restrictions identified in this first step, there was not much more to learn about the effect of excess money

demand on prices, say. Therefore the additional insight from this 3SLS estimation is mainly the sensitivity of the short-run adjustment estimates when explicit structural assumptions enter the model. It has to be pointed out however, that no final conclusions can be drawn yet because as the t-statistic of the additional structural explanatory variables indicate, their inclusion might not have contributed much to explain the underlying data generating process. That's why it is not quite clear which of variables are really part of this process and which are not. In order to find out more about that some model selection procedures could be applied. A natural extension in that direction would be e.g. to systematically exclude unnecessary variables due to some criteria like t-values, F-statistics, Akaike or Schwartz criteria to obtain more efficient estimates of the remaining **true** model.

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