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Working Paper

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SFB 373 Discussion Paper, No. 2000,92

Provided in Cooperation with:
Collaborative Research Center 373: Quantification and Simulation of Economic Processes, Humboldt University Berlin

Suggested Citation: Platen, Eckhard (2000) : Risk premia and financial modelling without measure transformation, SFB 373 Discussion Paper, No. 2000,92, Humboldt University of Berlin, Interdisciplinary Research Project 373: Quantification and Simulation of Economic Processes, Berlin, https://nbn-resolving.de/urn:nbn:de:kobv:11-10048184

This Version is available at:
https://hdl.handle.net/10419/62246

Risk Premia and Financial Modelling Without Measure Transformation

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September 29, 2000

Abstract. This paper describes a financial market modelling framework that exploits the notion of a deflator . The denominations of the deflator measured in units of primary assets form a minimal set of basic financial quantities that completely specify the overall market dynamics, where deflated asset prices appear as martingales. A specific form for the risk premia is obtained as a natural consequence of the approach. Contingent claim prices are computed under the real world measure both in the case of complete and incomplete markets avoiding the use of an equivalent risk neutral measure transformation.

1991 *Mathematics Subject Classification:* primary 90A12; secondary 60G30, 62P20.
JEL Classification: G10, G13
Key words and phrases: financial market modelling, deflator, risk premium, contingent claim pricing, incomplete market.

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1 Introduction

There exists a rich literature on asset price modelling and no-arbitrage pricing. For a recent account, see, for instance, Karatzas & Shreve (1998). The standard Black-Scholes model, see Black & Scholes (1973), has several deficiencies, for instance, it does not model stochastic volatility or stochastic interest rates. An improved financial market model should certainly be arbitrage free but in addition should correct for some of the deficiencies in the Black-Scholes model. It should preferably involve only a minimal set of factors that are able to reflect major stylised empirical facts and permit the efficient calibration and computation of key financial quantities. Furthermore, it should exploit natural symmetries in the exchange prices of assets.

This paper establishes a financial market modelling framework that can be conveniently used to construct models that fulfill the above requirements. It relies on the characterisation of a market deflator in the denominations of primary assets. A deflator used as numeraire makes deflated primary assets to local martingales. As a consequence the form of the risk premia for primary asset prices is obtained. The paper also shows that one cannot freely choose the appreciation rates of asset prices under the real world measure. It derives in a simple manner in the deflated world the well-known results on local-risk minimisation in incomplete markets, as described in Föllmer & Sondermann (1986), Föllmer & Schweizer (1991) and Hofmann, Platen & Schweizer (1992).

Essentially, the paper suggests the modelling of a financial market by specification of the denominations of the deflator in units of primary assets. This avoids the use of an equivalent risk neutral measure and establishes a symmetry for the structure of all deflated savings accounts of primary assets. The exchange prices for primary assets are then formed by ratios of the corresponding denominations of the deflator. This approach is in line with the view expressed by Bühlmann, Delbaen, Embrechts & Shiryaev (1998) who in a discrete time setting underline the importance of the deflator in financial modelling. We refer also to Duffie (1996) for the use of a deflator in a continuous time setting. Assuming positive nominal interest rates, Rogers (1997) proposed a potential approach to the modelling of the term structure of interest rates and exchange rates that models the state price density which is equivalent to the inverse of the deflator.

For pricing and hedging of contingent claims in an incomplete market our approach will lead us to the minimisation of the quadratic variation and thus the risk intensity of the corresponding profit and loss process. Finally, we will show that uniqueness of contingent claim prices in a Markovian multi-factor model requires deflated asset prices to be martingales and not only local martingales.

The paper introduces in Section 2 primary assets and the deflator. Section 3 suggests a multi-factor model for which in Section 4 examples are given and in Section 5 contingent claim prices are derived. Section 6 specifies complete and

incomplete financial market models as sets of contingent claims in a multi-factor model.

2 Deflator and Risk Premia

To obtain a useful financial market model it is important to formulate a small set of essential working principles. This guarantees the model to be in itself consistent, see Platen (1999a). In the following we will construct a financial market model on the basis of three such assumptions which we will introduce in this section.

Let us consider *primary assets* that are assets which are income or loss producing such as shares with dividend payments or currencies with interest payments. The first assumption is verbally expressed as follows.

(i) *In a financial market model each primary asset has its own time value.*

This means, the time value of the domestic currency shall be expressed via the corresponding *savings account process* $B^0 = \{B^0(t),\, t \in [0,T]\}$, where

$$dB^0(t) = B^0(t)\, f^0(t)\, dt \tag{2.1}$$

for $t \in [0,T]$, $T \in (0,\infty)$, with initial value $B^0(0) = 1$. The savings account accrues continuously the interest f^0 which specifies the *income rate* from holding the domestic currency. Similarly, the time value of the jth primary asset, $j \in \{0,\ldots,d\}$, which can be, for instance, a share or a currency, is analogously characterised by the jth *savings account process* $B^j = \{B^j(t),\, t \in [0,T]\}$, where

$$dB^j(t) = B^j(t)\, f^j(t)\, dt \tag{2.2}$$

for $t \in [0,T]$, with $B^j(0) = 1$. Here f^j is the jth *income rate* which can be, for instance, a dividend rate or domestic or foreign interest rate. In the case of shares, B^j is a share savings account with all dividends reinvested. The jth savings account measures accumulated income or loss generated by the jth asset in units of the jth asset, $j \in \{0,\ldots,d\}$. Note, the above savings accounts are modelled as bounded variation processes since accumulated income does not fluctuate as strongly as a market index, exchange rate or asset price.

We model uncertainty in our financial market by an m-dimensional standard Wiener process $W = \{W(t) = (W^1(t),\ldots,W^m(t))^\top,\, t \in [0,T]\},\ m \geq d$, defined on a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$. Here the filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0,T]}$ is assumed to be right-continuous and complete, fulfilling the usual conditions, where $\mathcal{A}_0$ is the trivial σ-algebra, see Karatzas & Shreve (1988). As usual, the increments $W(t+h) - W(t)$ are assumed to be independent of $\mathcal{A}_t$ for $t \in [0,T]$ and $h > 0$.

To formulate the following second assumption we note that a deflator is a strictly positive $\underline{\mathcal{A}}$-adapted stochastic process that when used as numeraire makes the resulting deflated price processes to $(\underline{\mathcal{A}}, P)$-local martingales.

(ii) *In a financial market model there exists a deflator.*

In a standard risk neutral setting it has been shown that the deflator is the optimal growth portfolio, see Karatzas & Shreve (1998). The deflator is thus a specific index. In this sense deflating of primary asset prices can be interpreted as index benchmarking which is a fundamental concept in portfolio optimisation. For $j \in \{0, \ldots, d\}$ let us denote by $D^j = \{D^j(t),\, t \in [0,T]\}$ the jth *denomination of the deflator*, which is the number of units of the jth asset that must be paid for the deflator.

The i,jth *exchange price* $X^{i,j}(t)$ at time t, that is the price for one unit of the jth primary asset expressed in units of the ith primary asset, is obviously given by the ratio

$$X^{i,j}(t) = \frac{D^i(t)}{D^j(t)} \tag{2.3}$$

for $t \in [0,T]$ and $i,j \in \{0,\ldots,d\}$. Thus an exchange price is the ratio of two corresponding denominations of the deflator. Obviously, $X^{0,j}(t)$ is the price at time t for one unit of the jth asset when measured in units of the domestic currency. If the jth asset is another currency, then $X^{0,j}(t)$ is the corresponding exchange rate. $X^{0,j}(t)$ could also denote, for instance, a dividend adjusted share price or a commodity spot price at time t.

By assumption (ii), the jth *deflated savings account process* $\hat{S}^j = \{\hat{S}^j(t),\, t \in [0,T]\}$ with

$$\hat{S}^j(t) = \frac{B^j(t)}{D^j(t)} \tag{2.4}$$

for $t \in [0,T]$ and $j \in \{0,\ldots,d\}$ must form an $(\underline{\mathcal{A}}, P)$-local martingale. To be more specific we assume that $\hat{S}^j$ follows the stochastic differential equation (SDE)

$$d\hat{S}^j(t) = \hat{S}^j(t) \sum_{k=1}^{d} \sigma^{j,k}(t)\, dW^k(t) \tag{2.5}$$

for $t \in [0,T]$ with initial value

$$\hat{S}^j(0) = \frac{1}{D^j(0)} > 0, \tag{2.6}$$

$j \in \{0,\ldots,d\}$. Here the j,kth *volatility process* $\sigma^{j,k} = \{\sigma^{j,k}(t),\, t \in [0,T]\}$ is assumed to be predictable, see Protter (1990), with

$$\int_0^T (\hat{S}^j(s)\, \sigma^{j,k}(s))^2\, ds < \infty \tag{2.7}$$

a.s. for $j \in \{0, \ldots, d\}$ and $k \in \{1, \ldots, d\}$. Furthermore, we assume that the jth income rate f^j is a predictable process with

$$\int_0^T |f^j(s)|\, ds < \infty, \tag{2.8}$$

a.s. for $j \in \{0, \ldots, d\}$. Note, the deflated savings account market has a symmetric structure. This fundamental observation has important consequences as we will see below.

From (2.4), (2.2) and (2.5) it follows by the Itô formula, see Protter (1990), that the jth denomination of the deflator satisfies the SDE

$$dD^j(t) = D^j(t) \left[\left(f^j(t) + \sum_{k=1}^{d} (\sigma^{j,k}(t))^2 \right) dt - \sum_{k=1}^{d} \sigma^{j,k}(t)\, dW^k(t) \right] \tag{2.9}$$

for $t \in [0, T]$ and $j \in \{0, \ldots, d\}$. Note, D^j is fully characterised by the corresponding income rate and volatilities. By application of the Itô formula to (2.3) and (2.9) we can now derive for the i, jth exchange price the SDE

$$\begin{aligned} dX^{i,j}(t) &= X^{i,j}(t) \Bigg[\left(f^i(t) - f^j(t) \right) dt \\ &\quad + \sum_{k=1}^{d} (\sigma^{j,k}(t) - \sigma^{i,k}(t)) \left\{ -\sigma^{i,k}(t)\, dt + dW^k(t) \right\} \Bigg] \end{aligned} \tag{2.10}$$

for $t \in [0, T]$ with initial value $X^{i,j}(0) = \frac{D^i(0)}{D^j(0)}$ and $i, j \in \{0, \ldots, d\}$.

For investment purposes savings accounts of primary assets are important. The jth *savings account price* $S^{i,j}(t)$ at time t when measured in units of the ith asset is given by the product

$$S^{i,j}(t) = X^{i,j}(t)\, B^j(t) \tag{2.11}$$

and it follows by the Itô formula according to (2.10) and (2.2) that

$$dS^{i,j}(t) = S^{i,j}(t) \left[f^i(t)\, dt + \sum_{k=1}^{d} (\sigma^{j,k}(t) - \sigma^{i,k}(t)) \left\{ -\sigma^{i,k}(t)\, dt + dW^k(t) \right\} \right] \tag{2.12}$$

for $t \in [0, T]$ with $S^{i,j}(0) = \frac{D^i(0)}{D^j(0)}$ and $i, j \in \{0, \ldots, d\}$. Note, the symmetry of the market, when expressed in deflated savings accounts, leads to the specific structure of the exchange prices in (2.10) and the savings account prices in(2.12).

There has been a long standing debate between theorists and practitioners how risk premia for asset prices should be modelled. This is referred to as the so-called risk premium puzzle. In our framework we can directly read off from the SDE (2.12) the i, jth *risk premium*

$$p^{i,j}(t) = \sum_{k=1}^{d} \left[(\sigma^{i,k}(t))^2 - \sigma^{i,k}(t)\, \sigma^{j,k}(t) \right] \tag{2.13}$$

for $t \in [0,T]$ and $i,j \in \{1,\dots,d\}$. For $i=0$ this is the risk premium for the denomination of the jth savings account in domestic currency. In Rogers (1997) such risk premia have been discussed for currency markets. We have derived in this paper risk premia for the savings accounts of a wide range of primary assets including stocks, currencies, commodities, etc. We underline, under the assumptions (i) and (ii) it turns out that there is a definitive form for the risk premia. This fact is typically ignored in the standard literature. It clearly restricts the possible structure of a financial market model and is of particular interest in portfolio optimisation and value at risk analysis that both must be performed under the real world measure.

If we set in equation (2.12) $i=0$, then this SDE describes the real world dynamics, for instance, of a dividend paying share price or interest paying foreign or domestic savings account expressed in units of the domestic currency. Note, if the different denominations of the deflator are almost independent, then the risk premium is close to the square of the volatility of the corresponding denomination of the deflator. However, in the case when the different denominations of the deflator are highly correlated, then the risk premia are small.

It will follow from our approach that a different risk premium as given in (2.13) provides an arbitrage opportunity in a financial market model. This leads us to the verbal formulation of our third assumption.

(iii) *Price processes in a financial market model must not allow arbitrage.*

In particular, to exclude obvious arbitrage opportunities we assume first that price processes must be unique. A second more subtle but rather standard no-arbitrage property will be discussed at the end of Section 5. It says that there should not exist a self-financing deflated portfolio with nonnegative deflated gains process that with strictly positive probability turns out to be strictly positive. This means one should not be able to create wealth out of nothing with strictly positive probability.

To construct arbitrage free, sufficiently flexible and computationally tractable financial market models that fulfill the above three working assumptions we introduce in the following section a *multi-factor model*.

3 Multi-Factor Model

We consider the Markovian factors $Z^0,\dots,Z^n$ which are assumed to be continuous, adapted diffusion processes that are given by a system of SDEs of the form

$$dZ^\ell(t) = \alpha^\ell(t,Z^0(t),\dots,Z^n(t))\,dt + \sum_{k=1}^{m} \beta^{\ell,k}(t,Z^0(t),\dots,Z^n(t))\,dW^k(t) \quad (3.1)$$

for $t \in [0,T]$ with given initial value $Z^\ell(0) \in (a_\ell, b_\ell)$, $-\infty \leq a_\ell < b_\ell \leq \infty$, $\ell \in \{0,\ldots,n\}$. Let $\Gamma = (a_0,b_0) \times \ldots \times (a_n,b_n)$ denote the set on which the factor process $Z = \{Z(t) = (Z^0(t),\ldots,Z^n(t))^\top,\ t \in [0,T]\}$ resides. Furthermore, we introduce the stopping time τ as the first time t when $Z(t)$ leaves the set Γ or the time t reaches T, that is

$$\tau = \inf\{t \in [0,T] :\ (t, Z(t)) \notin [0,T) \times \Gamma\}. \tag{3.2}$$

We assume that the functions α^ℓ and $\beta^{\ell,k}$ are such that a strong unique solution of (3.1) exists up until the stopping time τ and that this solution never hits the boundary of Γ, that is

$$P(\tau = T) = 1. \tag{3.3}$$

For computational tractability it is convenient to use diffusion processes with explicitly known transition densities. However, other choices are also possible and more general factors that reside on more complicated domains could, in principle, be used.

In a *multi-factor model* for each $j \in \{0,\ldots,d\}$ the jth denomination D^j of the deflator, see (2.9), is assumed to be a function $F^j : [0,T] \times \Gamma \to (0,\infty)$ which is differentiable with respect to time $t \in [0,T]$ and twice differentiable with respect to the factors $(Z^0,\ldots,Z^n) \in \Gamma$. Furthermore, the model must be such that the assumptions (i) - (iii) are satisfied. A rich set of continuous financial market models can be generated by such a multi-factor model.

To derive the income rates and volatilities that follow in a multi-factor model from the assumptions (i) - (iii) we introduce for a sufficiently smooth function $h : [0,T] \times \Gamma \to \Re$ the operators

$$\begin{aligned} L^0\, h(t,Z^0,\ldots,Z^n) \;=\; & \frac{\partial h(t,Z^0,\ldots,Z^n)}{\partial t} \\ & + \sum_{\ell=0}^{n} \alpha^\ell(t,Z^0,\ldots,Z^n)\, \frac{\partial h(t,Z^0,\ldots,Z^n)}{\partial Z^\ell} \\ & + \frac{1}{2} \sum_{\ell,r=0}^{n} \sum_{k=1}^{m} \beta^{\ell,k}(t,Z^0,\ldots,Z^n)\, \beta^{r,k}(t,Z^0,\ldots,Z^n) \\ & \quad \cdot \frac{\partial^2 h(t,Z^0,\ldots,Z^n)}{\partial Z^\ell\, \partial Z^r} \end{aligned} \tag{3.4}$$

and

$$L^k\, h(t,Z^0,\ldots,Z^n) = \sum_{\ell=0}^{n} \beta^{\ell,k}(t,Z^0,\ldots,Z^n)\, \frac{\partial h(t,Z^0,\ldots,Z^n)}{\partial Z^\ell} \tag{3.5}$$

for $(t,Z^0,\ldots,Z^n) \in [0,T) \times \Gamma$ and $k \in \{1,\ldots,m\}$. By application of the Itô

formula we then obtain from (3.1) with (3.4) and (3.5) the equation

$$\begin{aligned} D^j(t) &= F^j(t, Z^0(t), \ldots, Z^n(t)) \\ &= D^j(0) + \int_0^t L^0 F^j(s, Z^0(s), \ldots, Z^n(s))\, ds \\ &\quad + \sum_{k=1}^m \int_0^t L^k F^j(s, Z^0(s), \ldots, Z^n(s))\, dW^k(s) \end{aligned} \tag{3.6}$$

for $t \in [0, T]$, $j \in \{0, \ldots, d\}$. By comparison of (3.6) with (2.9) it follows the j,kth volatility of the deflated savings account $\hat{S}^j$ for $k \in \{1, \ldots, d\}$ in the form

$$\begin{aligned} \sigma^{j,k}(t) &= -\frac{L^k F^j(t, Z^0(t), \ldots, Z^n(t))}{F^j(t, Z^0(t), \ldots, Z^n(t))} \\ &= -\sum_{\ell=0}^n \beta^{\ell,k}(t, Z^0(t), \ldots, Z^n(t)) \frac{\partial \ln F^j(t, Z^0(t), \ldots, Z^n(t))}{\partial Z^\ell} \end{aligned} \tag{3.7}$$

and for $k \in \{d+1, \ldots, m\}$ with $\sigma^{j,k}(t) = 0$. Furthermore, we obtain the jth income rate

$$f^j(t) = \frac{L^0 F^j(t, Z^0(t), \ldots, Z^n(t))}{F^j(t, Z^0(t), \ldots, Z^n(t))} - \sum_{k=1}^d (\sigma^{j,k}(t))^2 \tag{3.8}$$

for $j \in \{0, \ldots, d\}$. This means, by choosing the factors $Z^0, \ldots, Z^n$ and the functions $F^0, \ldots, F^d$ that determine the different denominations of the deflator we specify all income rates and volatilities. That is, we fully characterise a financial market model on the basis of a set of factor processes and functions. We thus have now a rich framework that allows us to search for a suitable multi-factor model that with a minimal number of factors reflects established empirical facts. The second example that we will describe in the following section generates with such a minimal number of factors a realistic behaviour of stochastic volatilities together with important stylised facts.

4 Two Examples

For illustration, we will now consider two examples of multi-factor models that allow us to discuss typical features of the above framework. In the first example we assume that the different denominations of the deflator are geometric Brownian motions. This leads us to a standard Black-Scholes model (BSM).

Example 1: BSM
In this case we set

$$D^j(t) = F^j(t, Z^0(t), \ldots, Z^d(t)) = D^j(0) \exp(Z^j(t)) \tag{4.1}$$

for $t \in [0,T]$. Here we keep the volatilities $\sigma^{j,k}$, see (3.7), and the income rates f^j, see (3.8), constant for $k \in \{1,\dots,d\}$ and $j \in \{0,\dots,d\}$, and set $m = n = d$. According to (2.9) and by application of the Itô formula we note that we have to choose

$$dZ^j(t) = \left[f^j + \frac{1}{2} \sum_{k=1}^{d} (\sigma^{j,k})^2 \right] dt - \sum_{k=1}^{d} \sigma^{j,k}\, dW^k(t) \tag{4.2}$$

for $t \in [0,T]$ with $Z^j(0) = 0$ and $j \in \{0,\dots,d\}$. Here we have $a_j = -\infty$, $b_j = \infty$ and the relations (3.3), (2.7) and (2.8) are fulfilled. The jth savings account at time t, when expressed in units of the domestic currency, satisfies according to (2.12) the SDE

$$dS^{0,j}(t) = S^{0,j}(t) \left[f^0\, dt + \sum_{k=1}^{d} (\sigma^{j,k} - \sigma^{0,k}) \left\{ -\sigma^{0,k}\, dt + dW^k(t) \right\} \right] \tag{4.3}$$

for $t \in [0,T]$ and $j \in \{0,\dots,d\}$. This BSM when expressed in domestic currency yields constant risk premia of the form, see (2.13),

$$p^{0,j}(t) = \sum_{k=1}^{d} \left[(\sigma^{0,k})^2 - \sigma^{0,k}\, \sigma^{j,k} \right] \tag{4.4}$$

for $j \in \{0,\dots,d\}$. When a BSM with constant volatilities is not specified with the above risk premia, then there exist primary assets with deflated savings account price processes that are not driftless. In such a case it will follow from our approach that arbitrage opportunities arise. This fact does not seem to be acknowledged in the standard literature, where real world appreciation rates for stock prices under a BSM are often freely chosen.

The BSM does not reflect a number of important stylised empirical facts on asset prices. For instance, in practice volatilities appear to be stochastic and log-returns are leptokurtic. To overcome some of these and other deficiencies the *minimal market model* (MMM) has been proposed in Platen (1999b, 2000).Here the different denominations of the deflator are essentially constructed by functions of square root processes. This leads us to a second example for a multi-factor model which is a standard version of the MMM.

Example 2: MMM
Let us specify the factors $Z^0,\dots,Z^d$ as square root processes that are given by the SDE

$$dZ^\ell(t) = \frac{\nu}{4} \sum_{k=1}^{d} (\gamma^{\ell,k})^2 \,(1 - Z^\ell(t))\, dt - \sqrt{Z^\ell(t)} \sum_{k=1}^{d} \gamma^{\ell,k}\, dW^k(t) \tag{4.5}$$

for $t \in [0,T]$ with initial value $Z^\ell(0) > 0$ and $\ell \in \{0,\dots,d\}$. Here we set $m = n = d$ and assume that the scaling parameters $\gamma^{\ell,k}$ and dimension $\nu > 2 + 4q$ are given constants. According to Revuz & Yor (1999), a square root process with

dimension $\nu > 2$ has a strong unique solution, stays always positive and does not explode. Thus we have $a_\ell = 0$ and $b_\ell = \infty$ for $\ell \in \{0, \ldots, n\}$ and $P(\tau = T) = 1$. The deflator in the denomination of the ℓth primary asset has the form

$$D^\ell(t) = F^\ell(t, Z^0(t), \ldots, Z^d(t)) = (Z^\ell(t))^q \exp(\eta\, t)\, \xi_\ell \tag{4.6}$$

for $t \in [0, T]$ and $\ell \in \{0, \ldots, d\}$. The exponent $q > 0$, average growth rate η and initial parameter $\xi_\ell > 0$ are assumed to be given constants. By application of (3.7) we obtain from (4.6) the j,kth volatility

$$\sigma^{\ell,k}(t) = \frac{q\, \gamma^{\ell,k}}{\sqrt{Z^\ell(t)}} \tag{4.7}$$

for $k \in \{1, \ldots, d\}$ and from (3.8) the ℓth income rate

$$f^\ell(t) = \eta + \frac{q}{4} \sum_{k=1}^{d} (\gamma^{\ell,k})^2 \left\{ \frac{1}{Z^\ell(t)} (\nu - 2(q+1)) - \nu \right\} \tag{4.8}$$

for $t \in [0, T]$ and $\ell \in \{0, \ldots, d\}$.

For comparison with the typically studied risk neutral setting let us consider the denomination D^0 of the deflator in the domestic currency when discounted by the domestic savings account B^0. It satisfies according to (2.9), (4.7) and (4.6) the SDE

$$d\left(\frac{D^0(t)}{B^0(t)}\right) = -\frac{D^0(t)}{B^0(t)} \sum_{k=1}^{d} \frac{q\, \gamma^{0,k}}{\sqrt{Z^0(t)}}\, d\tilde{W}^k(t) \tag{4.9}$$

with

$$d\tilde{W}^k(t) = dW^k(t) - \frac{q\, \gamma^{0,k}}{\sqrt{Z^0(t)}}\, dt \tag{4.10}$$

for $t \in [0, T]$ and $k \in \{1, \ldots, d\}$. Thus we have from (4.5) with (4.10) the SDE

$$dZ^0(t) = \sum_{k=1}^{d} (\gamma^{0,k})^2 \left(\frac{\nu - 4\, q}{4} - \frac{\nu}{4}\, Z^0(t) \right) dt - \sqrt{Z^0(t)} \sum_{k=1}^{d} \gamma^{0,k}\, d\tilde{W}^k(t) \tag{4.11}$$

for $t \in [0, T]$. If we would now interpret $\tilde{W}^1, \ldots, \tilde{W}^d$ as independent standard Wiener processes under a probability measure $\tilde{P}$, then Z^0 given in (4.1) would represent under this measure a square root process of dimension $\nu - 4q$. For a dimension $\nu - 4q < 2$ the square root process Z^0 would reach under $\tilde{P}$ the level zero with strictly positive probability. On the other hand for $\nu > 2$ the process Z^0 given by (4.5) is not absorbed at zero under P w.p.1. Thus for $\nu - 4\, q < 2$ the probability measures P and $\tilde{P}$ would not be equivalent for the given MMM and an equivalent risk neutral measure in the standard form, see Karatzas & Shreve (1998), does not exist. However, for $\nu - 4q > 2$ it can be shown that the measures P and $\tilde{P}$ are equivalent and the deflated savings accounts in the MMM are $(\underline{\mathcal{A}}, P)$-martingales, see Heath & Platen (2000).

This example demonstrates that one cannot simply construct a multi-factor model under the real world probability measure and then hope that there exists some equivalent risk-neutral measure. To avoid such complications we do not involve an equivalent risk neutral measure in our approach and have assumed so far only that deflated savings accounts are local martingales. However, below we will learn that the required uniqueness of prices will force deflated savings account prices and other prices to become martingales. Thus we demand in the MMM for the dimension ν the lower bound $2 + 4q$.

5 Hedging European Contingent Claims

A practicable financial market model must generate an appropriate price system for a set of reasonable payoff structures. We will now identify in a multi-factor model the prices and hedging strategies for a nonnegative, deflated European payoff $H = H(\bar{T}, Z^0(\bar{T}), \ldots, Z^n(\bar{T})) \in [0, \infty)$ that matures at a deterministic time $\bar{T} \in (0, T]$. To achieve this we assume that there exists a function $u : [0, \bar{T}] \times \Gamma \to [0, \infty)$ such that

$$H = u(\bar{T}, Z^0(\bar{T}), \ldots, Z^n(\bar{T})) \tag{5.1}$$

and $u(t, Z^0, \ldots, Z^n)$ is differentiable with respect to time t and twice differentiable with respect to the factors $Z^0, \ldots, Z^n$ in $[0, \bar{T}) \times \Gamma$. We interpret the function u as a candidate for a deflated pricing function for the European payoff H. By hedging arguments we will specify the appropriate form of this function.

In the above framework let a hedger form a portfolio of primary assets with the goal to hedge the payoff H that matures at $\bar{T}$. Suppose that the hedger follows a *strategy* $\delta = \{\delta(t) = (\delta^0(t), \ldots, \delta^d(t))^\top,\ t \in [0, \bar{T}]\}$, where $\delta^j(t)$ denotes the number of units of the jth primary asset held at time $t \in [0, \bar{T}]$, $j \in \{0, \ldots, d\}$. For such a strategy the associated *deflated portfolio value* at time t is given by the sum

$$\hat{V}_\delta(t) = \sum_{j=0}^{d} \delta^j(t)\, \hat{S}^j(t) \tag{5.2}$$

for $t \in [0, \bar{T}]$. To hedge the European payoff H using the candidate deflated pricing function u we consider a strategy δ with δ^0 satisfying the relation

$$\delta^0(t) = \frac{1}{\hat{S}^0(t)} \left(u(t, Z^0(t), \ldots, Z^n(t)) - \sum_{j=1}^{d} \delta^j(t)\, \hat{S}^j(t) \right) \tag{5.3}$$

for $t \in [0, \bar{T}]$. Thus the associated deflated portfolio process $\hat{V}_\delta$ equals the process $u = \{u(t, Z^0(t), \ldots, Z^n(t)),\ t \in [0, \bar{T}]\}$, that is

$$\hat{V}_\delta(t) = u(t, Z^0(t), \ldots, Z^n(t)) \tag{5.4}$$

for $t \in [0, \bar{T}]$. The associated *deflated gains* are then given in the form

$$\begin{aligned} \hat{G}_\delta(t) &= \sum_{j=0}^{d} \int_0^t \delta^j(s)\, d\hat{S}^j(s) \\ &= \sum_{k=1}^{d} \int_0^t \sum_{j=0}^{d} \delta^j(s)\, \sigma^{j,k}(s)\, \hat{S}^j(s)\, dW^k(s) \end{aligned} \tag{5.5}$$

for $t \in [0, \bar{T}]$, see (2.5).

Let us introduce the associated *deflated profit and loss (P&L) process* $\hat{C}_\delta = \{\hat{C}_\delta(t),\, t \in [0, \bar{T}]\}$ with

$$\hat{C}_\delta(t) = \hat{V}_\delta(t) - \hat{G}_\delta(t) - \hat{V}_\delta(0) \tag{5.6}$$

for $t \in [0, \bar{T}]$. Note, at time $t = 0$ the P&L process is zero. We remark that $\frac{(\hat{C}_\delta(t) + \hat{V}_\delta(0))\, D^0(t)}{B^0(t)}$ corresponds to the discounted cost introduced in Föllmer & Sondermann (1986). Then it follows from (5.6), (5.4), (5.1), (5.5) and the Itô formula that

$$\begin{aligned} \hat{C}_\delta(t) &= u(t, Z^0(t), \ldots, Z^n(t)) - \hat{G}_\delta(t) - u(0, Z^0(0), \ldots, Z^n(0)) \\ &= \int_0^t L^0\, u(s, Z^0(s), \ldots, Z^n(s))\, ds \\ &\quad + \sum_{k=1}^{d} \int_0^t \left(L^k\, u(s, Z^0(s), \ldots, Z^n(s)) - \sum_{j=0}^{d} \delta^j(s)\, \sigma^{j,k}(s)\, \hat{S}^j(s) \right) dW^k(s) \\ &\quad + \sum_{k=d+1}^{m} \int_0^t L^k\, u(s, Z^0(s), \ldots, Z^n(s))\, dW^k(s) \end{aligned} \tag{5.7}$$

for $t \in [0, \bar{T}]$, where L^0 and L^k are defined in (3.4) and (3.5).

To hedge the European payoff H that matures at time $\bar{T}$ let us now bring the deflated P&L process $\hat{C}_\delta$ as close as possible to zero. One possible mathematical formulation of this objective, that is relatively technical, has been given in Föllmer & Sondermann (1986), known as local-risk minimisation. We suggest a more intuitive and rather simple formulation that we call *risk intensity minimisation*. It simply requires the first and second term for the deflated P&L process in (5.7) to vanish. The first term in (5.7) vanishes according to (5.1) and (5.7) if the function u is such that it solves the partial differential equation (PDE)

$$L^0\, u(t, Z^0, \ldots, Z^n) = 0 \tag{5.8}$$

for $(t, Z^0, \ldots, Z^n) \in (0, \bar{T}) \times \Gamma$ with terminal condition

$$u(\bar{T}, Z^0, \ldots, Z^n) = H(\bar{T}, Z^0, \ldots, Z^n) \tag{5.9}$$

for $(Z^0, \ldots, Z^n) \in \Gamma$. To fulfill also our assumption (iii) we must assume that the PDE (5.8) - (5.9) has a unique solution. This identifies a unique deflated pricing function u for H.

To make the second term in (5.7) to zero we consider for the deflated P&L process $\hat{C}_\delta$ its quadratic variation process $\langle \hat{C}_\delta \rangle = \left\{ \langle \hat{C}_\delta \rangle_t = \int_0^t \varepsilon_\delta(s)\, ds,\ t \in [0, \bar{T}] \right\}$, with *risk intensity*

$$\begin{aligned} \varepsilon_\delta(t) &= \sum_{k=1}^{d} \left(L^k\, u(t, Z^0(t), \ldots, Z^n(t)) - \sum_{j=0}^{d} \delta^j(t)\, \sigma^{j,k}(t)\, \hat{S}^j(t) \right)^2 \\ &\quad + \sum_{k=d+1}^{m} \left(L^k\, u(t, Z^0(t), \ldots, Z^n(t)) \right)^2 \end{aligned} \tag{5.10}$$

at time $t \in [0, \bar{T}]$. A perfect hedge would be equivalent to zero risk intensity. It is clear that we do not have any freedom to minimise the second sum of squares in (5.10). However, by an appropriate choice of δ we can make the first sum in (5.10), that is the second term in (5.7), to zero. This means we have to solve the system of equations

$$L^k\, u(t, Z^0(t), \ldots, Z^n(t)) = \sum_{j=0}^{d} \delta^j(t)\, \sigma^{j,k}(t)\, \hat{S}^j(t) \tag{5.11}$$

for all $k \in \{1, \ldots, d\}$ and $t \in [0, \bar{T})$. With the notation

$$c^k(t) = L^k\, u(t, Z^0(t), \ldots, Z^n(t)) - u(t, Z^0(t), \ldots, Z^n(t))\, \sigma^{0,k}(t) \tag{5.12}$$

and

$$b^{k,j}(t) = \hat{S}^j(t)\, (\sigma^{j,k}(t) - \sigma^{0,k}(t)) \tag{5.13}$$

for $j, k \in \{1, \ldots, d\}$ we introduce the vectors

$$\delta^*(t) = (\delta^1(t), \ldots, \delta^d(t))^\top, \tag{5.14}$$

$$c(t) = (c^1(t), \ldots, c^d(t))^\top \tag{5.15}$$

and the diffusion coefficient matrix

$$b(t) = [b^{k,j}(t)]_{k,j=1}^{d} \tag{5.16}$$

of the primary assets when denominated in the domestic currency. Thus the linear system of equations (5.11) can be written in the form

$$c(t) = b(t)\, \delta^*(t) \tag{5.17}$$

for all $t \in [0, \bar{T})$. To obtain a unique solution of (5.17) we assume from now on that the diffusion coefficient matrix $b(t)$ is nonsingular for each $t \in [0, T]$. Thus we obtain

$$\delta^*(t) = b^{-1}(t)\, c(t) \tag{5.18}$$

for $t \in [0, \bar{T})$. The vector $\delta^*(t)$ determines according to (5.14) the hedge ratios $\delta^1(t), \ldots, \delta^d(t)$ and by (5.3) also $\delta^0(t)$. The corresponding hedging strategy $\delta = (\delta^0(t), \ldots, \delta^d(t))^\top$ generates then according to (5.7) at time t the deflated P&L

$$\hat{C}_\delta(t) = \sum_{k=d+1}^{m} \int_0^t L^k \, u(s, Z^0(s), \ldots, Z^n(s)) \, dW^k(s) \tag{5.19}$$

for $t \in [0, \bar{T}]$. Note from (2.5) that the primary assets $\hat{S}^0, \ldots, \hat{S}^d$, which are for us the only available instruments for hedging, are driven by the Wiener processes $W^1, \ldots, W^d$. Since we have no tradeable instruments to hedge against the uncertainty generated by the Wiener processes $W^{d+1}, \ldots, W^m$ the risk intensity (5.10) is minimised.

Note, the above results are obtained by allowing deflated price processes to be local martingales and no equivalent risk neutral measure was involved. We call the obtained strategy δ the *risk intensity minimising strategy*. Under suitable assumptions this strategy can be shown to coincide with the local-risk minimising strategy obtained in Föllmer & Sondermann (1986), Föllmer & Schweizer (1991) and Hofmann, Platen & Schweizer (1992).

Note, in the case $m = d$ the deflated P&L process is according to (5.19) zero and the corresponding risk intensity minimising strategy δ is *self-financing* that is

$$\hat{G}_\delta(t) = \hat{V}_\delta(t) - \hat{V}_\delta(0) \tag{5.20}$$

for all $t \in [0, \bar{T}]$.

We will now show under mild assumptions that the uniqueness of the solution of the PDE (5.8) - (5.9) forces us to allow only martingales and exclude strict local martingales as deflated asset price processes in a financial market model. We assume that the transition density $p(t, x; s, y)$ for the factor process Z exists with

$$P(Z(s) \in A \mid Z(t) = x) = \int_A p(t, x; s, y) \, dy \tag{5.21}$$

for $A \in \Gamma$, fulfilling for fixed $(s, y) \in [0, T] \times \Gamma$ the PDE

$$L^0 \, p(t, x; s, y) = 0 \tag{5.22}$$

for $(t, x) \in (0, s) \times \Gamma$ with terminal condition

$$p(s, x; s, y) = \begin{cases} 1 & \text{for } x = y \\ 0 & \text{otherwise} \end{cases}, \tag{5.23}$$

where L^0 is defined in (3.4). Then under mild integrability assumptions on the payoff function H, which have simply to secure the finiteness of the terms involved, the function $\bar{u} : [0, \bar{T}] \times \Gamma \to [0, \infty)$ with

$$\bar{u}(t, Z^0, \ldots, Z^n) = \int_\Gamma H(\bar{T}, y^0, \ldots, y^n) \, p(t, (Z^0, \ldots, Z^n); \bar{T}, y) \, dy \tag{5.24}$$

satisfies the PDE

$$L^0\,\bar{u}(t,Z^0,\ldots,Z^n)=\int_\Gamma H(\bar{T},y^0,\ldots,y^n)\,L^0\,p(t,(Z^0,\ldots,Z^n);\bar{T},y)\,dy=0 \tag{5.25}$$

for $(t,Z^0,\ldots,Z^n)\in(0,\bar{T})\times\Gamma$ with terminal condition

$$\bar{u}(\bar{T},Z^0,\ldots,Z^n)=H(\bar{T},Z^0,\ldots,Z^n) \tag{5.26}$$

for $(Z^0,\ldots,Z^n)\in\Gamma$. Note, the operator L^0 does not refer to the variables $\bar{T}$ and y on the right hand side of (5.25). The PDE (5.25) - (5.26) is identical to the PDE (5.8) - (5.9) which we already assumed to have a unique solution u. Therefore, the functions u and $\bar{u}$ must be identical. Due to (5.24) and the Markov property of Z we have then the conditional expectation

$$u(t,Z^0(t),\ldots,Z^n(t))=E\left(H(\bar{T},Z^0(\bar{T}),\ldots,Z^n(\bar{T}))\,\middle|\,\mathcal{A}_t\right) \tag{5.27}$$

for $t\in[0,\bar{T}]$ and the process $u=\{u(t,Z^0(t),\ldots,Z^n(t)),\,t\in[0,\bar{T}]\}$ is an $(\underline{\mathcal{A}},P)$-martingale. Consequently, with the choice $H(T,Z^0(T),\ldots,Z^n(T))=\hat{S}^j(T)$ the deflated savings account process $\hat{S}^j$ must form, according to (5.27), an $(\underline{\mathcal{A}},P)$-martingale and thus cannot be a strict local martingale. This is an important consequence of assumption (iii). It underlines also the fundamental role of martingales in financial modelling and questions the use of local martingales in asset price modelling.

Let us now restrict for a given multi-factor model our attention to those European payoffs for which the above analysis is valid and call these payoffs *European contingent claims*. We then say, a *financial market model* is formed by the set of European contingent claim price processes on the given multi-factor model. Thus we exclude asset price dynamics and payoffs for which any of the above assumptions or arguments do not hold.

We finally show that such a financial market model fulfills also the second no-arbitrage property that we formulated in the context of assumption (iii). More precisely, it follows by the above established martingale property of deflated gains, see (5.5) and (5.27), with a standard martingale argument, see, e.g. Karatzas & Shreve (1998), that there does not exist a self-financing strategy δ and a nonnegative associated deflated gains process $\hat{G}_\delta$ that with strictly positive probability turns out to be strictly positive at time $\bar{T}$ because

$$E\left(\hat{G}_\delta(\bar{T})\right)=E\left(\hat{G}_\delta(\bar{T})\,\middle|\,\mathcal{A}_0\right)=\hat{G}_\delta(0)=0.$$

This is a strong version of the standard no-arbitrage property that excludes the possibility to create wealth from nothing, see, e.g. Karatzas & Shreve (1998).

6 Complete and Incomplete Market Models

We call the above defined financial market model a *complete* market model, when all European contingent claims can be hedged by a risk intensity minimising and self-financing strategy with zero deflated P&L process. Otherwise we call it an *incomplete* market model. Summarising the previous results we can then formulate the following theorem.

Theorem 6.1 *In the above financial market model, with $b(t)$ nonsingular for all $t \in [0,T]$, the risk intensity minimising strategy δ for a European contingent claim H that matures at time $\bar{T} \in (0,T]$ is given by* (5.18) *and* (5.3). *The corresponding deflated pricing function u solves the PDE* (5.8) *with terminal condition* (5.9) *and the conditional expectation process $\hat{V} = \{\hat{V}_\delta(t),\, t \in [0,T]\}$ with*

$$\hat{V}_\delta(t) = u(t, Z^0(t), \ldots, Z^n(t)) = E(H \mid \mathcal{A}_t) \tag{6.1}$$

for $t \in [0,\bar{T}]$ forms an $(\underline{\mathcal{A}}, P)$-martingale. For $m = d$ the financial market model is complete.

Most important is the property of the above financial market model that for a given European contingent claim H with maturity $\bar{T} \in (0,T]$ the corresponding deflated price process $u = \{u(t, Z^0(t), \ldots, Z^n(t)),\, t \in [0,\bar{T}]\}$, see (6.1), forms an $(\underline{\mathcal{A}}, P)$-martingale. This was not clear when we started our analysis allowing also strict local martingales as deflated price processes. The fundamental martingale property provides access to the computation of deflated European contingent claim prices as conditional expectations under the real world measure. The BSM and the MMM give examples for models with known transition densities for the factors that result in almost explicit expressions for standard contingent claim prices, see Heath & Platen (2000).

Let us now also characterise the deflator as a portfolio of primary assets in the above financial market model. We have simply to recognise the fact that the deflated deflator is the constant $u(t, Z^0(t), \ldots, Z^n(t)) = 1$ and the corresponding hedge ratios follow from (5.18) and (5.3) with the specification $u = 1$ and thus $c^k(t) = -\sigma^{0,k}(t)$. Therefore the number of units of primary assets in the portfolio that represents the deflator is known and depends only on the volatilities of the deflated primary assets.

In the standard literature one typically uses an equivalent risk neutral measure for contingent claim pricing, see Karatzas & Shreve (1998). The deflated domestic savings account process $\hat{S}^0$ is in the above financial market model an $(\underline{\mathcal{A}}, P)$-martingale and the Radon-Nikodym derivative process $\Lambda = \{\Lambda(t) = \frac{\hat{S}^0(t)}{\hat{S}^0(0)},\, t \in [0,T]\}$ is then also an $(\underline{\mathcal{A}}, P)$-martingale. Λ characterises an equivalent risk neutral martingale measure $\tilde{P}$ by the relation $\frac{d\tilde{P}}{dP} = \Lambda(T)$. Thus by Bayes's rule we obtain from deflated contingent claim prices the well-known risk neutral contingent

claim prices for complete and incomplete markets as given in Karatzas & Shreve (1998), Föllmer & Sondermann (1986), Föllmer & Schweizer (1991) and Hofmann, Platen & Schweizer (1992). To illustrate the risk neutral valuation let us consider for a European contingent claim H, that matures at time $\bar{T} \in [0, T]$ the random variable $h = h(\bar{T}, Z^0(\bar{T}), \ldots, Z^n(\bar{T})) = D^0(\bar{T})\, H(\bar{T}, Z^0(\bar{T}), \ldots, Z^n(\bar{T}))$ as the corresponding payoff that is denominated in the domestic currency. Then we have from (6.1)

$$\hat{V}_\delta(t) = u(t, Z^0(t), \ldots, Z^n(t)) = E\left(\frac{h}{D^0(\bar{T})} \,\middle|\, \mathcal{A}_t\right)$$

for $t \in [0, \bar{T}]$. Thus in the denomination of the domestic currency the price for the payoff h at time t must be

$$V^0_\delta(t) = D^0(t)\, \hat{V}_\delta(t) = E\left(\frac{\Lambda(\bar{T})}{\Lambda(t)}\, \frac{B^0(t)}{B^0(\bar{T})}\, h \,\middle|\, \mathcal{A}_t\right) = \tilde{E}\left(\frac{B^0(t)}{B^0(\bar{T})}\, h \,\middle|\, \mathcal{A}_t\right),$$

where $\tilde{E}$ can be shown to represent the expectation under the minimal equivalent martingale measure $\tilde{P}$, see Hofmann, Platen & Schweizer (1992).

Finally, let us discuss the case $m > d$, where not all driving Wiener processes appear in the noise terms of the primary assets. For instance, some Wiener processes might only generate uncertainty related to interest and income rates. Such an incomplete model can be often completed by forming, for instance, portfolios or buckets of zero coupon bonds. These buckets are then income producing and can be again interpreted as primary assets. Taking such constructions of primary assets into account one can fully concentrate on the modelling of complete markets. One type of incompleteness can then be constructed from such a complete market by simply restricting the permitted trading strategies allowing, for instance, trading only in specific assets or piecewise constant allocation strategies. An important type of incompleteness arises when the hedging strategies can be only based on incomplete information that is by allowing for the pricing and hedging of contingent claims only a subfiltration of $\underline{\mathcal{A}}$. The above framework allows a systematic study of such types of incompleteness, however this goes beyond the scope of this paper and will be considered elsewhere.

Conclusion

The different denominations of a deflator have been shown to be sufficient to characterise a financial market model. As a consequence risk premia for primary asset prices have been identified. A financial market modelling framework has been suggested that avoids equivalent measure transformations and leads naturally to martingales as deflated price processes. The presented risk intensity minimising approach is consistent with the local-risk minimisation framework

and can be naturally extended in various ways, for instance, to cover asset price dynamics with jumps, general contingent claim classes and different types of incompleteness.

Acknowledgement

The author likes to thank David Heath, Chris Heyde and Martin Schweizer for their interest in this research and stimulating and fruitful discussions on the subject. He gratefully acknowledges the support of the SFB 373 at the Humboldt University in Berlin.

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