Karlsen, Hans Arnfinn; Myklebust, Terje; Tjøstheim, Dag

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NONPARAMETRIC ESTIMATION IN A NONLINEAR
COINTEGRATION TYPE MODEL

BY HANS ARNFINN KARLSEN, TERJE MYKLEBUST AND DAG TIØSTHEIM

University of Bergen
Department of Mathematics, Johannes Bruns gt.12, 5008 Bergen, Norway

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Abstract

We derive an asymptotic theory of nonparametric estimation for a nonlinear transfer function model \( Z_t = f(X_t) + W_t \), where \( \{X_t\} \) and \( \{Z_t\} \) are observed nonstationary processes, and \( \{W_t\} \) is a stationary process. In econometrics this can be interpreted as a nonlinear cointegration type relationship, but we believe that our result have wider interest. The class of nonstationary processes allowed for \( \{X_t\} \) is a subclass of the class of null recurrent Markov chains. This subclass contains the random walk model and the unit root processes. We derive the asymptotics of a nonparametric estimate of \( f(x) \) under two alternative sets of assumptions on \( \{W_t\} \): i) \( \{W_t\} \) is a linear process. ii) \( \{W_t\} \) is a Markov chain satisfying some mixing conditions. The latter requires considerably more work but also holds larger promise for further developments. The finite sample properties of \( f(x) \) are studied via a set of simulation experiments.

Key words and phrases. Cointegration, Nonstationary time series models, null recurrent Markov chain, nonparametric kernel estimators, transfer function model.

1 Introduction

Two time series \( \{X_t\} \) and \( \{Z_t\} \) are said to be linearly cointegrated if they are both nonstationary (persistent) of unit root type, and if there exists a linear combination \( aX_t + bZ_t = W_t \) such that \( \{W_t\} \) is stationary. This means that the series \( \{X_t, Z_t\} \) move together in the long run. The concept of cointegration was introduced by Granger and further developed by Engle and Granger (1987), and since its introduction there have been numerous papers in econometrics exploring various aspects of it. Some main results are given in Johansen (1996).

The long term relationship between two econometric time series may not necessarily be linear, however, and the processes \( \{X_t\} \) and \( \{Z_t\} \) may not be linearly generated unit root processes. This has lead to a search for nonlinear cointegration type relationship such as \( Z_t = f(X_t) + W_t \) for some nonlinear function \( f \) and some possibly nonlinearly generated input process \( \{X_t\} \). Indeed, functional relationship of this type have been fitted to economic data (see e.g. Granger and Hallman 1991 and Aparicio and Escribano 1997), but to our knowledge the properties of the resulting nonparametric estimates have not been established. (See Xia 1998 for a consistency property in a simplified situation, however).

There are at least two difficulties (cf. Granger 1995 and others): which class of processes should be chosen as a basic class of persistent processes, and how should an estimation theory for an estimate of \( f \) be constructed? The main goal of this paper is to try to answer these questions; i.e. we wish to establish a nonparametric estimation theory for the function \( f \) in the nonlinear transfer function model

\[
Z_t = f(X_t) + W_t
\]

where \( \{W_t\} \) is a non-observed stationary process, and \( \{X_t\} \) and \( \{Z_t\} \) are observed processes which are nonstationary in a sense to be made more precise. The connection to the nonlinear cointegration problem is obvious, but we would like to point out that the estimation of the transfer function \( f \) in the general context we are considering should also be of interest in other areas. In a traditional transfer function problem some sort of mixing condition is often assumed for \( \{X_t\} \) to obtain a central limit theorem for \( \hat{f}(x) \). However, mixing assumptions on \( \{X_t\} \) are ruled out in the general situation we look at. A minimal condition for doing asymptotic analysis for \( \hat{f}(x) \) is that, as the number of observations on \( \{X_t\} \) increases, there must be infinitely many observations in any neighborhood of \( x \). This means \( \{X_t\} \) must return to a neighborhood of \( x \) infinitely often, which in turn implies that the framework of a recurrent Markov chain is especially convenient. Since \( \{X_t\} \) may be nonstationary, null recurrent processes have to be included, and it should be noted that the class of null recurrent processes contain unit root processes (cf. Myklebust et al 2000). Unlike the parametric situation, where a unit root speeds up the convergence of (global) estimates due to the large spread of the observations, in the nonparametric case, which is concerned with local estimates, the nonstationarity slows down the convergence, because the time until the process returns to the local neighborhood increases. (The expected time being infinite in the null recurrent case).

In Karlsen and Tjøstheim (1998) (hereafter referred to as KT) we developed an asymptotic theory for nonparametric estimation for a nonstationary univariate nonlinear
ear autoregressive model in the framework of so-called $\beta$-null recurrent processes. This
is a subclass of the null recurrent processes which contains the random walk. For an
alternative theoretical approach in the random walk case we refer to Phillips and Park

We will rely on central parts of the theory of Karlsen and Tjøstheim (1998) for
our derivations in this paper. But a host of new problems emerges in the transfer
function case as will be made clear in the following. We also present several examples
of the theory. In particular it will be seen that the verification of asymptotic theory via
simulation on null recurrent series presents some special difficulties, which are absent
in the ordinary stationary mixing situation.

2 Notation and some basic conditions

We will follow the notation of KT since our proofs and results will be closely based on
that paper. Thus, we denote by $\{X_t, t \geq 0\}$ a $\phi$-irreducible Markov chain on a general
state space $(E, \mathcal{E})$ with transition probability $P$. In this paper we take $E \subset \mathbb{R}$, and
we denote the class of non-negative measurable functions with $\phi$-positive support by
$\mathcal{E}^+$. For a set $A \in \mathcal{E}$ we write $A \in \mathcal{E}^+$ if the indicator function $1_A \in \mathcal{E}^+$. The process
$\{X_t, t \geq 0\}$ will be assumed to be Harris recurrent. This essentially implies that given
a neighborhood $\mathcal{N}_x$ of $x$ with $\phi(\mathcal{N}_x) > 0$, $\{X_t\}$ will return to $\mathcal{N}_x$ with probability one,
and this is what makes asymptotics for a nonparametric estimation possible. The chain
is positive recurrent if there exists an invariant probability measure such that $\{X_t \geq 0\}$
is strictly stationary, and it is null recurrent otherwise. In this paper we are primarily
interested in the null recurrent situation, in which case there exists a (unique up to a
constant, non-probability) invariant measure, which will be denoted by $\pi$.

We assume aperiodicity and that the transition probability for the $X_t$-process sat-
sifies the so-called minorization inequality

$$P \geq s \otimes \nu$$ (2.1)

where $s$ is a small function and $\nu$ a small measure with $\nu(E) = 1$. The pair $(s, \nu)$ is called
an atom. This is not a strong restriction (cf. KT and Nummelin, 1984). Moreover, the
minorization condition may be relaxed. Clearly, (2.1) implies that $0 \leq s(x) \leq 1$ for all
$x \in E$.

If a set $C \in \mathcal{E}^+$ is such that $1_C$ is a small function, $C$ itself is said to be small.
Under quite wide conditions (cf. KT ) a compact set will be small. From (2.1) we get the
identity

$$P(x, A) = (1 - s(x)) \left\{ \frac{P(x, A) - s(x)\nu(A)}{1 - s(x)} \right\} 1(s(x) < 1) + 1_A(x)1(s(x) = 1)$$

$$+ s(x)\nu(A)$$

$$= (1 - s(x))Q(x, A) + s(x)\nu(A)$$

so that the transition probability $P$ can be thought of as a mixture of the transition
probability $Q$ and the small measure $\nu$. Since $\nu$ is independent of $x$ this means that
the chain regenerates each time \( \nu \) is chosen. This occurs with probability \( s(x) \). The reasoning can be formalized by introducing the split chain \( \{X_t, Y_t\} \) where the auxiliary chain, \( \{Y_t\} \), can only take values 0 and 1. Given that \( X_t = x \), \( Y_{t-1} = y_{t-1} \), \( Y_t \) takes the value one with probability \( s(x) \) so that \( \alpha = E \times \{1\} \) is a proper atom (cf. Nummelin, 1984, p 51) for the split chain. We denote by

\[
S_\alpha = \min\{t \geq 1: Y_t = 1\}
\]

the corresponding recurrence time. We will also make use of the consecutive sequence of recurrence times starting at time \( t = 0 \);

\[
\tau_k = \min\{t > \tau_{k-1}: Y_t = 1\}, \quad \tau_{-1} = -1 \quad \text{for} \quad k \geq 0, \quad \tau = \tau_\alpha = \tau_0 \quad (2.2)
\]

and the number of regenerations in the time interval \([0, n]\), i.e.,

\[
T(n) = \max_k \{k: \tau_k \leq n\} \vee 0.
\]

An invariant measure \( \pi_s \) can be defined in terms of the atom \((s, \nu)\) of (2.1). In fact (KT, Section 2.3)

\[
\pi_s \equiv \nu G_{s, \nu}, \quad G_{s, \nu} \equiv \sum_{\ell = 0}^{\infty} (P - s \otimes \nu)\ell. \quad (2.3)
\]

If the measure \( \pi_s \) is absolutely continuous with respect to Lebesgue measure, we denote by \( p_s \) the corresponding density so that \( p_s(x)dx = \pi_s(dx) \). Similarly, we define the density \( p_C(x) = p_s(x)/\pi_s(C) \). For a \( \pi \)-integrable function \( g \) on \( R \) we use the notation \( \pi_s g \) for

\[
\pi_s g = \pi_s(g) = \int g(x)\pi_s(dx)
\]

Corresponding to \( T(n) \) for a set \( C \in \mathcal{E}^+ \), the number of times \( \{X_t\} \) is visiting \( C \) up to time \( n \) is denoted by

\[
T_C(n) = \sum_{t = 0}^{n} 1_C(X_t)
\]

From KT (Remark 2.7) we have that \( \lim_{n \to \infty} T_C(n)/T(n) = \pi_s 1_C \).

The kernel \( G_{s, \nu} \) of (2.3) plays an important role in Section 4 and it easily follows from the above that for a \( \pi_s \)-integrable \( g \) defined on \( E \),

\[
E_x \sum_{t=0}^{\tau} g(X_t) = G_{s, \nu} g(x). \quad (2.4)
\]

The minorization condition and the accompanying split chain permit decomposing the chain into separate and identical parts defined by the regeneration points. We have for a function \( g \),

\[
S_n(g) \equiv \sum_{t=0}^{n} g(X_t) = U_0 + \sum_{k=1}^{T(n)} U_k + U_{(n)} \quad (2.5)
\]

where

\[
U_k = \begin{cases} 
\sum_{t=\tau_{k-1}+1}^{\tau_k} g(X_t), & \text{when } k \geq 0; \\
\sum_{t=\tau_{(n)}}^{n} g(X_t), & \text{when } k = (n) 
\end{cases} \quad (2.6)
\]
The sequence \( \{ (U_k, (\tau_k - \tau_{k-1}), k \geq 1 \} \) consists of independent identically distributed (iid) random variables. This partition of the chain is basic for the subsequent asymptotic analysis.

We have to make a restriction on the way the process regenerates: the chain \( \{ X_t \} \) is \( \beta \)-null recurrent if there exists a small non-negative function \( h \), an initial measure \( \lambda \), a constant \( \beta \in (0, 1) \) and a slowly varying function \( L_h \) so that

\[
E_{\lambda} \left[ \sum_{t=0}^{n} h(X_t) \right] \sim \frac{1}{\Gamma(1 + \beta)} n^\beta L_h(n)
\]

as \( n \to \infty \). This condition is equivalent with (cf. KT) a restriction on the tail distribution of the recurrence time \( S_\alpha \) in that

\[
P_\alpha(S_\alpha > n) = \frac{1}{\Gamma(1 - \beta) n^\beta L_\alpha(n)} \left( 1 + o(1) \right)
\]

where \( L_\alpha \) is a slowly varying function depending on \( s \). In the sequel (2.8) will be referred to as the tail condition.

A random walk process is \( \beta \)-null recurrent with \( \beta = 1/2 \).

### 2.1 Basic conditions

In all of the proofs we use \( c_1, c_2, \ldots \) as a sequence of generic constants in our proofs, and if \( \{ a_n \} \) and \( \{ b_n \} \) are two real-valued strictly positive sequences, then we write \( a_n \ll b_n \) if \( a_n = o(b_n) \).

If \( \eta \) is a non-negative measurable function and \( \lambda \) is a measure, then the kernel \( \eta \otimes \lambda \) is defined by

\[
\eta \otimes \lambda(x, A) = \eta(x) \lambda(A), \quad (x, A) \in (E, \mathcal{E}) .
\]

If \( H \) is a general kernel, the function \( H \eta \), the measure \( \lambda H \) and the number \( \lambda H \eta \) are defined by

\[
H \eta(x) = \int H(x, dy) \eta(y), \quad \lambda H(A) = \int \lambda(dx) H(x, A), \quad \lambda H \eta = \int \lambda(dx) H(x, dy) \eta(y) .
\]

The convolution of two kernels \( H_1 \) and \( H_2 \) gives another kernel defined by

\[
H_1 H_2(x, A) = \int H_1(x, dy) H_2(y, A) .
\]

Due to associative laws the number \( \lambda H_1 H_2 \eta \) is uniquely defined. If \( A \in \mathcal{E} \) and \( 1_A \) is the corresponding indicator variable, then \( H1_A(x) = H(x, A) \). The kernel \( I_\eta \) is defined by \( I_\eta(x, A) = \eta(x) 1_A(x) \).

We denote by \( h = h_n \) the bandwidth used in the nonparametric estimation. It is assumed to satisfy \( h_n \to 0 \), and with no loss of generality we also assume \( h_n \leq 1 \). Let \( K: R \to R \) be a kernel function, and for a fixed \( x \), let \( K_{x,h}(y) = h^{-1} K((y - x)/h) \), \( \mathcal{N}_x(h) = \{ y : K_{x,h}(y) \neq 0 \} \) and \( \mathcal{N}_x = \mathcal{N}_x(1) \). In our context a locally bounded function will be taken to mean bounded in a neighborhood of \( x \), and a locally continuous function is continuous at the point \( x \). Without loss of generality we may assume that this
neighborhood equals $N_x$, and that local continuity implies local boundness. This is since $N_x(h) = x \oplus hN_0$.

In the analysis of a kernel estimator of the transfer function $f$ of (1.1) we will look at a slight generalization in that we allow an instantaneous transformation of $W_t$ as well, resulting in

$$Z_t = f(X_t) + g_W(W_t).$$

(2.9)

Here, $g_W$ is assumed to be known, e.g. $g_W(w) \equiv w$, and $f$ is to be estimated. We will consider this problem under two sets of conditions on $\{W_t\}$. In Section 3 $\{W_t\}$ will essentially be assumed to be a linear process, whereas a Markov assumption will be adopted in Section 4.

The following set of conditions is always assumed:

K$_0$: The kernel $K$ is non-negative, $\int K(u) du < \infty$ and $\int K^2(u) du < \infty$.

P$_0$: The $\{X_t\}$ process is a Harris recurrent Markov chain.

F$_0$: The transfer function $f$ is continuous at the point $x$.

Subsets of the conditions listed below are used according to need. The concepts of $\phi$-mixing and $\alpha$-mixing are defined in Hall & Heyde (1980, p. 277).

K$_1$: The support $N_0$ of the kernel is contained in a compact set.

K$_2$: The kernel is bounded and $N_x$ is a small set.

K$_3$: The kernel is normalized so that $\int K(u) du = 1$.

K$_4$: The kernel satisfies $\int uK(u) du = 0$.

P$_1$: The invariant measure $\pi_s$ has a locally bounded density $p_s$.

P$_2$: The density $p_s$ is locally continuous.

P$_3$: The density $p_s$ possesses locally continuous partial derivatives of a given specified order.

P$_4$: The density is locally strictly positive, i.e., $\lim_{y \to x} p_s(y) > 0$.

P$_5$: For all $h > 0$, the function: $y \sim P(y, N_x(h))$ is continuous

P$_6$: The tail condition, (2.8), holds.

P$_6'$: $\beta \overset{\text{def}}{=} \sup\{p \geq 0: E_{\alpha} S_{n}^{p} < \infty\} \wedge 1 > 0$.

T$_1$: The set $C$ in $T_C$ is a small set.

F$_1$: The function $f$ possesses locally continuous partial derivatives of a given order.

W$_1$: The process $\{W_t\}$ is strictly stationary, $Eg_W(W_0) = 0$, $Eg^4_W(W_0) < \infty$, and $\phi$-mixing with a finite mixing rate; $\sum_{t} \phi_{t}^{1/2} < \infty$.

W$_2$: The process $\{W_t\}$ is a strictly stationary linear process, $W_t = \sum_k a_k \varepsilon_{t-k}$, with $\sum_k |a_k| < \infty$, $\{\varepsilon_t\}$ satisfies W$_1$ and $g_W(w) \equiv w$. 
W₃: The process \( \{W_t\} \) is an irreducible ergodic Markov chain which satisfies (2.1).

W₄: In addition to W₃, \( \{W_t\} \) is strongly \( \alpha \)-mixing with mixing rate

\[
\sum \ell \ell^{2m-1+\frac{2m}{k}} \alpha \ell < \infty, \quad \pi g_W = 0 \quad \text{and} \quad \pi g_W^{2m(k+1)} \quad \text{is finite for some} \ k, m \geq 1,
\]

where \( \pi \) is the unique stationary measure of \( \{W_t\} \).

## 3 Nonparametric estimation of \( f \) when \( \{W_t\} \) is linear

The kernel estimate of \( f \) in (2.9) is given by

\[
\tilde{f}(x) = \frac{\sum_{t=0}^{n} K_{x,h}(X_t)Z_t}{\sum_{t=0}^{n} K_{x,h}(X_t)}. \tag{3.1}
\]

We get a central limit theorem by an adaptation of Th. 4.4 in KT and its proof.

**Theorem 3.1** Assume that the processes \( X \) and \( W \) are independent in (2.9) and that \( \{X_t, t \geq 0\} \) is started with an arbitrary initial measure \( \lambda \). Moreover, assume that \( K_1 \) - \( K_4, P_1, P_2, P_4-P_6 \) and \( W_2 \) hold. Let \( \sigma_W^2 = \mathbb{E} g_W^2(W_0), \ \epsilon > 0 \) and \( \psi_x(y) = f(y) - f(x) \).

If \( h_n^{-1} \ll n^{\beta-\epsilon} \) with \( \beta \) defined in (2.7), then

\[
\left\{ h_n \sum_{t=0}^{n} K_{x,h_n}(X_t) \right\}^{1/2} \left\{ \tilde{f}(x) - f(x) - \frac{\pi s I_{K_{x,h_n}} \psi_x}{\pi s K_{x,h_n}} \right\} \xrightarrow{d} \mathcal{N}(0, \sigma_W^2 \int K^2(u)du). \tag{3.2}
\]

If \( P_3 \) holds with order 2 and \( f \) possesses continuous derivatives of second order, then the bias term \( \pi s I_{K_{x,h_n}} \psi_x / \pi s K_{x,h_n} \) is negligible when \( h_n^{-1} \gg n^{\beta/5+\epsilon} \).

**Proof:** Denote the left hand side of (3.2) by \( \Lambda_{n,h_n} \). Since \( \tilde{f}(x) \) is a ratio, we get some additional bias terms.

Write,

\[
Z_t = \{Z_t - f(X_t)\} + (f(X_t) - f(x)) + f(x)
\]

\[
Z_t K_{x,h}(X_t) = g_h(X_t, W_t) + \psi_x(X_t) K_{x,h}(X_t) + f(x) K_{x,h}(X_t)
\]

where \( g_h(z,u) = g_W(u) \cdot K_{x,h}(z) \). In the \( S_n \)-notation of (2.5) this gives

\[
\tilde{f}(x) - f(x) = S_n^{-1}(K_{x,h}) \left\{ S_n(g_h) + S_n(\psi_x \cdot K_{x,h}) \right\}.
\]

The last term on the right hand side represents the bias. It contains a stochastic quantity, and we want to replace it by a deterministic bias term. Let

\[
a_h = \frac{\pi s I_{K_{x,h}} \psi_x}{\pi s I_{K_{x,h}} 1}.
\]

Then

\[
\tilde{f}(x) - f(x) - a_h = S_n^{-1}(K_{x,h}) \left\{ S_n(g_h) + S_n(\psi_x \cdot K_{x,h}) - a_h S_n(K_{x,h}) \right\}
\]

\[
= S_n^{-1}(K_{x,h}) \left\{ S_n(g_h) + S_n(b_h) \right\};
\]
where
\[ b_h = I_{K_x,h}(\psi_x - a_h). \]
We note that \( \mu_{b_h} = \pi_{a,b_h} = 0 \). We can write
\[ \Lambda_{n,h} = \Delta_{n,h} + \Delta'_{n,h} \]
where
\[ \Delta_{n,h} = S_n^{-1/2}(K_{x,h}) h^{1/2} S_n(g_h) \]
\[ \Delta'_{n,h} = \left\{ \tilde{p}_C(x) \right\}^{-1/2} T_{C}^{-1/2}(n) h^{1/2} S_n(b_h) \]
\[ \tilde{p}_C(x) = T_{C}^{-1}(n) S_n(K_{x,h_n}), \]
and where \( C \) is a purely auxiliary set chosen such that \( T_1 \) holds. In the proof of Th. 4.4. in KT we replace \( P \xi \) with \( f \), then using \( K_1, K_2, P_2 \) and \( P_6 \),
\[ \Delta'_{n,h_n} = o_P(1) \tag{3.4} \]
and by the second part of Th. 4.3 (cf. also the proof of Th. 4.4.) in KT,
\[ \tilde{p}_C(x) = p_C(x) + o_P(1) \tag{3.5} \]
using \( K_1, K_2, P_1, P_4, P_6 \).

It remains to verify that \( \Delta_{n,h_n} \) satisfies the prescribed CLT. We assume first that \( W_1 \) holds and then generalize to \( W_2 \) towards the end of the proof.

Let \( c_W(k) = \text{Cov}(gw(W_0), gw(W_k)) \), \( \sigma^2 = c_W(0) \int K^2(u)du \) and
\[ \xi_{h,t} = \frac{h^{1/2} K_{x,h}(X_t)}{\left[ \sum_{j=0}^{n} K_{x,h}(X_j) \right]^{1/2}} = \frac{h^{1/2} K_{x,h}(X_t)}{S_n^{1/2}(K_{x,h})}. \]
Then
\[ \Delta_{n,h} = \frac{h^{1/2} S_n(g_h)}{S_n^{1/2}(K_{x,h})} = \sum_{t=0}^{n} \xi_{h,t} gw(W_t). \tag{3.6} \]

Let \( \mathcal{F}^X \) be the \( \sigma \)-algebra generated by \( \{X_t\} \). It is sufficient to prove that the conditional distribution of \( \Delta_{n,h_n} \) given \( \mathcal{F}^X \) converges in distribution to \( \mathcal{N}(0, \sigma^2) \) where \( \sigma^2 \) is independent of \( x \). Indeed, let \( z \) be fixed. Then if
\[ P(\Delta_{n,h_n} \leq z \mid \mathcal{F}^X) \xrightarrow{n} \Phi(z/\sigma) \quad \text{a.s.} \ [P_\lambda] \tag{3.7} \]
it follows from
\[ P_\lambda(\Delta_{n,h_n} \leq z) = E_\lambda \left[ P(\Delta_{n,h_n} \leq z \mid \mathcal{F}^X) \right] \]
and by the dominated convergence theorem that
\[ \lim_{n \to \infty} P_\lambda(\Delta_{n,h_n} \leq z) = E_\lambda \left[ \lim_{n \to \infty} P(\Delta_{n,h_n} \leq z \mid \mathcal{F}^X) \right] = \Phi(z/\sigma). \tag{3.8} \]

In order to prove (3.7) we first show that
\[ \sigma^2_{n,h_n} \overset{\text{def}}{=} E(\Delta_{n,h_n}^2 \mid \mathcal{F}^X) \xrightarrow{\text{pln}} \sigma^2 \tag{3.9} \]
and then verify that conditional on $\mathcal{F}_{\mathcal{X}}$, $\Delta_{n,h_{n}}/\sigma_{n,h_{n}}$ has an asymptotic standard normal distribution.

By (3.6),

$$
\sigma_{n,h}^{2} = cW(0) \sum_{t=0}^{n} \xi_{s,t}^{2} + \sum_{0<|k|\leq n} cW(k) \sum_{t=-k}^{n-k} \xi_{s,t+k}
$$

$$
= cW(0) \eta_{n,h,0} + \sum_{0<|k|\leq n} cW(k) \eta_{n,h,k}, \quad \text{say}
$$

(3.10)

and by $K_{2}$

$$
|\eta_{n,h,k}| \leq \sum_{t=0}^{n} \xi_{s,t}^{2} \leq \frac{h}{n} \sum_{t=0}^{n} \frac{K_{2}^{2}(X_{t})}{\sum_{t=0}^{n} K_{2}^{2}(X_{t})} \leq \sup_{u} \{K(u)\} .
$$

Hence by the dominated convergence theorem it is enough to prove that

$$
\eta_{n,h,n,k} \xrightarrow{n \to \infty} \begin{cases} 0, & \text{for all } k \neq 0; \\
\int K^{2}(u)du, & \text{when } k = 0
\end{cases}
$$

since $W_{1}$ implies that

$$
\sum_{k} |cW(k)| < \infty .
$$

Let $C$ be small. We have

$$
\eta_{n,h,n,k} = \{\hat{p}_{C}(x)\}^{-1}\left\{T_{C}^{-1}(n) \sum_{t=-k}^{n-k} h_{n} K_{x,h_{n}}(X_{t}) K_{x,h_{n}}(X_{t+k})\right\}
$$

$$
\leq \{\hat{p}_{C}(x)\}^{-1}\left\{T_{C}^{-1}(n) \sum_{t=0}^{n} h_{n} K_{x,h_{n}}(X_{t}) K_{x,h_{n}}(X_{t+k})\right\}
$$

$$
\leq c_{1} \{\hat{p}_{C}(x)\}^{-1}\left\{T_{C}^{-1}(n) \sum_{t=0}^{n} K_{x,h_{n}}(X_{t}) 1_{N_{x}(h_{n})}(X_{t+k})\right\}
$$

since by $K_{1}-K_{3}$,

$$
K_{x,h}(u) \leq h^{-1} c_{1} 1_{N_{x}(h)}(u) .
$$

By $P_{1}$ and $P_{3}$ we have that $P^{k}(x, \{x\}) = 0$ (cf. the proof of Lemma 4.2 in KT) for all $k \geq 1$. Hence

$$
\lim_{k \to \infty} P^{k}(x, N_{x}(h)) = 0 .
$$

(3.11)

Let $h_{0}$ be fixed and arbitrary and consider $n$ so large that $h_{n} \leq h_{0}$. Then

$$
\eta_{n,h,n,k} \leq c_{1} \{\hat{p}_{C}(x)\}^{-1}\theta_{n,h,0,h_{0},k}
$$

(3.12)

where

$$
\theta_{n,h,0,h_{0},k} = T_{C}^{-1}(n) \sum_{t=0}^{n} K_{x,h_{n}}(X_{t}) 1_{N_{x}(h_{0})}(X_{t+k}) .
$$

Then by extending the proof of Th. 4.2 in KT using $K_{1}$, $K_{2}$, $P_{1}$, $P_{5}$, $P_{6}'$ (which is implied by $P_{6}$)

$$
\lim_{h \to 0} \lim\sup \{\theta_{n,h,0,h_{0},k}\} \leq c_{2} P^{k}(x, N_{x}(h_{0})) \quad \text{a.s.}
$$

(3.13)
where $c_2$ is a constant independent of $h_0$. By (3.11), (3.12) and (3.13) it follows that
\[
\eta_{n,h_0,k} = o(1) \quad \text{a.s.} 
\tag{3.14}
\]
when $k > 0$. Exactly the same arguments can be used to show that (3.14) holds for $k < 0$. When $k = 0$ it follows by KT (Th. 4.1) using K$_{T}$-K$_{3}$, P$_{1}$, P$_{2}$, P'$_{0}$ and Bochners theorem,
\[
\eta_{n,h,0} = \int K^2(u)du + o(1) \quad \text{a.s.} 
\]
Thus we have proved (3.9).

To prove (3.7) (and hence (3.8)) it remains to verify that a fourth order Liapunov condition holds. By $W_1$

\[
\sum_{t=0}^{n} E(\xi_{h,n,t}^4 g_{W_t}^4(\omega) \mid \mathcal{F}^X) \leq c_1 \sum_{t=0}^{n} \xi_{h,n,t}^4 
= c_1 \frac{h_n^2 \sum_{t=0}^{n} K_{h,n}^4(X_t)}{\left[\sum_{t=0}^{n} K_{h,n}^2(X_t)\right]^2} 
= c_0 \left\{ \frac{1}{h_n} \frac{\sum_{t=0}^{n} h_n^3 K_{h,n}^4(X_t)}{\sum_{t=0}^{n} K_{h,n}^2(X_t)} \right\} \left\{ \frac{\sum_{t=0}^{n} h_n^3 K_{h,n}^4(X_t)}{\sum_{t=0}^{n} K_{h,n}^2(X_t)} \right\} 
= o(1) \quad \text{a.s.}
\]

since
\[
\frac{\sum_{t=0}^{n} h_n^3 K_{h,n}^4(X_t)}{\sum_{t=0}^{n} K_{h,n}^2(X_t)} \leq \left\{ \sup_u |K(u)|^3 \right\}
\]

and
\[
\left\{ T_C(n)h_n \right\} \hat{p}_C(x) = h_n \sum_{t=0}^{n} K_{h,n,t} \uparrow \infty \quad \text{a.s.} \quad \lim_{n \to \infty} T_C(n)h_n = \infty \quad \text{a.s.} \quad (3.15)
\]

Thus the array $\{Z_{n,t}\}_{t=0}^{n}$ satisfies the conditions in a dependent central limit theorem (Bergström, 1982, Th. 1., p 161) for given $\mathcal{F}^X$ where $Z_{n,t} \overset{\text{def}}{=} \xi_{h,n,t} g_{W_t} / \sigma_{n,h}$. 

It is a standard procedure to generalize from condition $W_1$ to condition $W_2$. The linear representation is truncated
\[
W_t = W_{t,m} + W_{t}^{(m)}, \quad W_{t,m} = \sum_{|j| < m} a_j e_{t-j}, \quad m \geq 1,
\]
so that
\[
\Delta_{n,h} = \Delta_{n,h,m} + \Delta_{n,h}^{(m)}.
\]

It is enough to prove that for all $z$,

i): \forall m \geq 1: \quad P(\Delta_{n,h,m} \leq z \mid \mathcal{F}^X) \xrightarrow{n} \Phi(z/\sigma_m) \quad \text{a.s. [P]} 

ii): \quad \sigma_m^2 \xrightarrow{m} \sigma^2 

iii): \quad \lim_{n} E\left[ (\Delta_{n,h}^{(m)})^2 \mid \mathcal{F}^X \right] \xrightarrow{m} 0 \quad \text{a.s. [P]}


For all fixed $m$ the process $W_{(m)} = \{W_{k,m}, k \geq 0\}$ satisfies $W_1$. Hence part i) holds. Let $c_{W_{(m)}}$ denote the covariance function that belongs to $\{W_{k,m}, k \geq 0\}$. Then since $c_W$ and the sequence $\{a_k\}$ by assumption $W_2$ are summable, it follows that

$$
\lim_{m} \sum_{k} |c_{W_{(m)}}(k) - c_W(k)| = 0 \quad \text{which implies that part ii) holds by (3.10).}
$$

Moreover, we have that

$$
E[|\Delta_{n,h_n,m} - \Delta_{n,h_n}|^2 \mid \mathcal{F}^X] = E[(\Delta_{n,h_n}^{(m)})^2 \mid \mathcal{F}^X] = \sum_{|k|<n} c_{W_{(m)}}(k) \eta_{n,k,h_n}.
$$

By the above reasoning,

$$
\sum_{|k|<n} c_{W_{(m)}}(k) \eta_{n,k,h_n} = c_{W_{(m)}}(0) \int K^2(u) du \quad \text{a.s.}
$$

Since $\lim_{m \to \infty} c_{W_{(m)}}(0) = 0$, we have verified part iii).

The bias term is negligible if $h_n$ goes to zero sufficiently fast. This is shown as in KT (proof of Th. 4.4).

The tail condition (2.8) is difficult to verify since it almost requires exact knowledge of the first order asymptotics of the rate of $P^n(x, A)$. We believe that $P'_6$ is considerably weaker, and it may be easier to verify than $P_6$, since it is fulfilled if $E_{\nu} \tau^p < \infty$ for some $p > 0$. It turns out that it is possible to incorporate this modification of the tail condition in the proof of Theorem 3.1: (Note that we use the notation $a_n \ll b_n$ to denote $a_n = o(b_n)$ as $n \to \infty$ for 2 sequences $\{a_n\}$ and $\{b_n\}$)

**Corollary 3.1** Assume that the conditions of Theorem 3.1 hold and that the tail condition, $P_6$, is replaced by $P'_6$. We also assume $F_1$ of order 1 and for some $c, \delta > 0$,

$$
n^\delta \ll h_n^{-1} \ll n^{\beta/2 - \epsilon}.
$$

(3.16)

Then the conclusion in Theorem 3.1 still holds.

**Proof of Corollary 3.1:** We have to prove that (3.4), (3.5) and (3.15) still hold when we replace $P_6$ by $P'_6$. By KT (Th. 4.1) (3.5) is true using $K_1$-$K_3$, $P_1$, $P_2$, $P'_6$ and the right hand side of (3.16). By KT (Lemma 3.1, second part),

$$
n^{\beta - \epsilon} \ll T(n)
$$

which entails that (3.15) holds.

It remains to verify (3.4). It is enough to show that

$$
T_C^{-1/2}(n) h_n^{1/2} S_n(b_{h_n}) \xrightarrow{\text{a.s.}} 0
$$

holds without the tail condition. Recall that

$$
\psi_{x}(y) = f(y) - f(x), \quad a_{h} = \frac{\pi_{x} I_{K_{x,h}} \psi_{x}}{\pi_{x} I_{K_{x,h}} 1}, \quad b_{h}(y) = K_{x,h}(y)(\psi_{x}(y) - a_{h})
$$

(3.17)
so that \( \pi_s b_h = 0 \). Smoothness of order 1 for the function \( f \) implies by (3.17)
\[
|a_h| \leq c_0 h, \quad |b_h| \leq c_1 \mathcal{N}_x(h).
\]
In accordance with (2.5) and (2.6), let
\[
S_n(b_h) = U_{0,h_n} + V_{T(n),h_n} + U_{n,h_n}, \quad V_{T(n),h_n} = \sum_{k=1}^{T(n)} U_{k,h_n}
\]
which implies that
\[
|U_{k,h_n}| \leq U_{k,h_n}^0 \leq U_{k,1}^0, \quad k \in \{0, 1, \ldots, \infty, \ (n)\}
\]
(3.18) where the \( U_{k,h_n}^0 \)'s are generated by \( g_h = c_1 \mathcal{N}_x(h) \) instead of \( b_h \). We find that
\[
|U_{0}^0_T(n) | \leq U_{T(n)+1}^0 = T^{1/2}(n) F_T^{1/2}
\]
where \( F_n \equiv U_T^{1/2} n^{-1}/n \). Since \( \mathbb{E} U_{k,1}^2 < \infty \) then by the strong law of the large numbers \( F_n = o(1) \) a.s. Likewise we have that \( U_{0,h_n} = o(T^{1/2}(n)) \) a.s. Let \( \delta \) be defined by (3.16). Then
\[
\left| T(n)^{-1/2} h_n^{1/2} V_{T(n),h_n} \right| = \left| \{ T^\delta(n) h_n \}^{1/2} T^{-p}(n) V_{T(n),h_n} \right|, \quad p = 1/2 + \delta/2
\]
(3.19) where, by assumption, \( n^\delta h_n = o(1) \) a.s. Rephrasing notation, we
\[
T^{-p}(n) V_{T(n),h_n} = n^{-p} V_{n_1,h_n}, \ (T(n), n) = (n_1, n_2).
\]
Let \( a = \beta - \epsilon > 0 \). Then \( T(n) \gg n^a \) a.s. Hence by (3.19), (3.4) follows if
\[
\lim_{n_2 \geq n_2^a} \frac{\sum_{n_1=1}^{n_1} \sum_{n_2=1}^{n_2} E \left| n^{-p} \sum_{k=1}^{n_1} U_{k,h_n} \right|^{2m}}{n_2^{2m}} < \infty.
\]
(3.20) By the Borel Cantelli lemma (3.20) is true if for some \( m \geq 1 \)
\[
\sum_{n_1=1}^{n_1} \sum_{n_2=1}^{n_2} E \left| n^{-p} \sum_{k=1}^{n_1} U_{k,h_n} \right|^{2m} < \infty.
\]
Let \( m > \delta^{-1}(1 + a^{-1}) \). Then by (3.18)
\[
E \left| n^{-p} \sum_{k=1}^{n_1} U_{k,h_n} \right|^{2m} = n^{-m\delta} E \left| n^{-1/2} \sum_{k=1}^{n_1} U_{k,h_n} \right|^{2m} \leq c_2 n^{-m\delta} E \left| U_{0,1}^0 \right|^{2m} \leq c_3 n^{-m\delta}
\]
where \( c_3 \) is finite since \( \mathcal{N}_x \) is small (cf. KT, proof of Lemma 4.1).

\[ \square \]

**Remark 3.1** It is possible to allow for a stochastic bandwidth, i.e., \( h_n \) is replaced by \( q_{T_c(n)} \) where the function \( q \) is independent of \( \beta \) (cf. KT, Th. 3.4).

\[ \square \]
4 Nonparametric estimation of $f$ when $\{W_i\}$ is Markov process

In this section the linear process assumption $W_2$ is replaced by the Markov assumption $W_3$. This also makes it possible to relax the $\phi$-mixing of $W_1$ to $\alpha$-mixing as in $W_4$. In the preceding section we did not directly use a central limit theorem for the $X$-process. But now we have to rely on a null recurrent central limit theorem, more precisely Theorem 4.1 based upon Corollary 3.2, of KT. However, because we cannot in general restrict $g_W$ to be a small function (consider e.g., $g_W(w) \equiv w$), Lemma 4.1 and Lemma 4.2 of KT cannot be used and this complicates matters considerably.

Throughout Sections 4.3 to 4.6 we make the assumption that $\{X_i\}$ and $\{W_i\}$ are independent. However, we believe that it is possible to avoid this assumption, and that this type of generalization is easier to achieve in the framework of Markov theory.

Before embarking on a proof of the asymptotic properties of $f$ we need a series of lemmas on the regeneration structure of the compound Markov process $\{(X_i, W_i)\}$. We think that these results are of independent interest, and that they are potentially useful in other situations.

We denote by $g$ a general measurable functions defined on $R^2$, $g_X$, $g_W$ are measurable functions defined on $R$, and $g = g_X \otimes g_W$ means that $g(x,w) = g_X(x)g_W(w)$.

4.1 A nonparametric CLT for null recurrent processes

Assume that $\{X_i\}$ is a Markov chain which satisfies the minorization condition (2.1) and the tail condition (2.8). Let

$$V_{g_n} = \sum_{j=0}^{\tau} g_h(X_j), \quad \mu_{g_n} = E\mu V_{g_n}, \quad \sigma^2_{g_n} = E\sigma^2 V_{g_n} - \mu_{g_n}^2$$

where $g_h$ is a real-valued function defined on $E$ for all $h > 0$, and $\tau$ is defined in (2.2). Consider the following conditions where $-\infty < \mu, \mu' < \infty$, $0 < \sigma, \sigma' < \infty$, $v \in [0, 1]$, $m \geq 2, \epsilon > 0, 0 < d_m, d_m' < \infty$, $\beta$ is defined in (2.8), $\lambda$ is an initial measure and $h \downarrow 0$.

$$\mu_{g_n} = \mu + o(1), \quad \mu_{g_n} = \mu' + o(1). \quad (4.1)$$

$$h \sigma^2_{g_n} = \sigma^2 + o(1). \quad (4.2)$$

$$h \sigma^2_{[g_n]} = \sigma^2 + o(1). \quad (4.3)$$

$$E\sigma^2 | V_{g_n} - \mu_{g_n}|^2 \leq d_m h^{-2m+\epsilon}. \quad (4.4)$$

$$E\sigma^2 | V_{g_n} - \mu_{g_n}|^2 \leq d_m' h^{-2m+\epsilon}. \quad (4.5)$$

$$h^{-1} \ll n^{\beta \delta m - \epsilon}, \quad \delta_m = \frac{m - 1}{m - v}. \quad (4.6)$$

$$\exists g_0: h |g_h| \leq g_0 \quad \text{and} \quad P_{\lambda}(V_{g_0} < \infty). \quad (4.7)$$
The following theorem is essentially a translation of a CLT-result in KT. It will be used to prove the main CLT-results of the present paper.

**Theorem 4.1** Let \( C \) be a small set. Assume that (4.1) - (4.6) hold for an \( m \geq 2 \) and a \( v \in [0,1] \). Then for any initial measure \( \lambda \) for \( X_0 \) so that (4.7) holds,

\[
h_n^{1/2} S_n^{-1/2}(1_C) S_n(g_n) \sim n \mathcal{N}(0, \sigma^2 \pi_s^{-1} 1_C).
\]

**Proof of Theorem 4.1:** The proof is essentially based upon KT (Th. 3.2). Since \( g_h \) is a function in one variable the conditions in that theorem simplifies. Moreover, (4.1) - (4.3) are conditions implied by the corresponding conditions in Th. 3.2. In the conditions (4.1) - (4.7) the quantity \( v \) is allowed to vary everywhere in the interval \([0,1]\) and that represent a minor extension of Th. 3.2. It is permitted by a trivial extension of the proof of Lemma 3.3 of KT.

\[\square\]

Before we can put Theorem 4.1 to use we need to analyze the regeneration structure of \( \{X_t, W_t\} \) closer. This is done in Sections 4.2 - 4.7. We then state our main result in Section 4.8.

### 4.2 Decomposition of \( S_n(g) \)

We assume that the compound chain, \( \{(X_t, W_t)\} \) satisfies (2.1), so that it can be extended by the split chain method, with \( \{(X_t, W_t, Y_t)\} \) being a split chain. Note that if \( \{X_t\} \) and \( \{W_t\} \) separately satisfy the minorization inequality condition (2.1), it is not obvious that the compound chain \( \{(X_t, W_t)\} \) does. If \( \{X_t\} \) and \( \{W_t\} \) are independent, then it is trivial to verify (2.1) as is shown in the beginning of Section 4.3. Let

\[ \eta_k = \inf\{t > \eta_{k-1}: Y_t = 1\}, \quad k \geq 0, \quad \eta_{-1} = -1. \]

Then, the sequence \( \{\eta_k\} \) represents the regeneration times for the compound process. The basic decomposition, (2.5), gives

\[
S_n(g) = V_0 + \sum_{k=1}^{T(n)} V_k + V(n), \quad T(n) = \sup\{k: \eta_k \leq n\} \vee 0
\]

(4.8)

where

\[
V_k = \begin{cases} 
\sum_{t=\eta_{k-1}+1}^{\eta_k} g(X_t, W_t), & \text{for } k \geq 0 \\
\sum_{t=\eta_{T(n)}+1}^{n} g(X_t, W_t), & \text{for } k = (n).
\end{cases}
\]
According to the general theory (4.8) represents essentially a decomposition into independent variables where \( \{(V_k, \eta_k - \eta_{k-1})\}, k \geq 1 \) are iid and where \( V_k \) has expectation \( \mu_g \), variance \( \sigma_g^2 \), i.e.

\[
\mu_g = E_\nu (V_0), \quad \sigma_g^2 = \text{var}_\nu (V_0) = E_\nu (V_0^2) - \mu_g^2
\]

where \( \nu \) refers to the compound chain \( \{(X_t, W_t)\} \).

Our first problem is to find conditions which ensure that \( \mu_g \) and \( \sigma_g^2 \) are finite. Again, by reference to general theory (cf. Appendix A, (A.10)) we have that, with \( s \) referring to the compound chain,

\[
\mu_g = \pi_s g, \quad \sigma_g^2 = \pi_s (g^2) + 2\pi_s I_g H G_{s,\nu} g - \pi_s^2 (g)
\]

(4.9)

where

\[
H = P - s \otimes \nu, \quad G_{s,\nu} = \sum_{j=0}^{\infty} H^j .
\]

(4.10)

The conditions ensuring \( \mu_g = 0 \) and \( \sigma_g^2 < \infty \) are not evident from (4.10) if we want to avoid the strong restriction that \( g_W \) is a small function. If \( g_W(w) \equiv w \), requiring \( g_W \) to be small is equivalent to \( \phi \)-mixing which is not satisfied for an autoregressive process, say. The problem is linked to the term \( G_{s,\nu} \). In fact, we also need to show existence of higher moments and to verify conditions connected to the bandwidth as exemplified in (4.1) - (4.7).

In Sections 4.3 - 4.6, we make the assumption that \( \{X_t\} \) and \( \{W_t\} \) are independent.

### 4.3 \( \beta \) null recurrence for the compound process

Let \( P \) denote the transition probability for the Markov process \( \{(X_t, W_t)\} \). We label quantities associated with the \( X \)-process by one and with the \( W \)-process by two. The transition probability \( P \) satisfies (2.1) when \( P_1 \) and \( P_2 \) do, since

\[
P = P_1 \otimes P_2 \geq (s_1 \otimes s_2) \otimes (\nu_1 \otimes \nu_2) = s \otimes \nu .
\]

**Lemma 4.1** Assume that \( \{X_t\} \) and \( \{W_t\} \) are independent and that the tail condition (2.8) holds for \( \{X_t\} \). Then the compound process \( \{(X_t, W_t)\} \) is \( \beta \)-null recurrent, i.e. the tail condition holds for the compound process.

**Proof.** Let \( C_1 \) and \( C_2 \) be small sets and \( \nu = \nu_1 \otimes \nu_2 \).

\[
E_\nu \sum_{t=0}^{n} 1_{C_1}(X_t) 1_{C_2}(W_t) = \sum_{t=0}^{n} \left[ \nu_1 P_1^t 1_{C_1} \right] \left[ \nu_2 P_2^t 1_{C_2} \right] = \left[ \pi_2 1_{C_2} \right] \sum_{t=0}^{n} \left[ \nu_1 P_1^t 1_{C_1} \right] + \sum_{t=0}^{n} \left[ \nu_1 P_1^t 1_{C_1} \right] b_t
\]
where \( b_t = v_t P_{1C_2} - \pi_2 1_{C_2} \) and where \( \pi_2 \) is the stationary measure for \( \{W_t\} \). Since \( \{W_t\} \) is ergodic \( b_t = o(1) \). Since \( \{X_t\} \) is \( \beta \)-null (cf. KT (Lemma 2.2 and (2.31) )), we have that

\[
\sum_{t=0}^{n} [\nu_t P_1 1_{C_1}] = [\pi_{s1} 1_{C_1}] \psi_1(n)(1 + a_n), \quad \psi_1(n) = n^\beta L_{s1}(n), \quad a_n = o(1) .
\]

Let \( \psi_M = \sup_{t \leq M} \psi_1(t) \), \( A = \sup_t |a_t| \), \( B = \sup_t |b_t| \), \( B^{(M)} = \sup_{t>M} |b_t| \). Then, for all \( M > 0 \),

\[
\left| \sum_{t=0}^{n} [\nu_t P_1 1_{C_1}] b_t \right| \leq B \left\{ [\pi_{s1} 1_{C_1}] \psi_M(1 + A) \right\} + B^{(M)} \left\{ [\pi_{s1} 1_{C_1}] \psi_1(n)(1 + |a_n|) \right\}
\]

so that

\[
\lim_{n \to \infty} \frac{\sum_{t=0}^{n} [\nu_t P_1 1_{C_1}] b_t}{\psi_1(n)} = \lim_{n \to \infty} B \left\{ [\pi_{s1} 1_{C_1}] \frac{\psi_M}{\psi_1(n)}(1 + A) \right\} + \lim_{M \to \infty} \frac{B^{(M)}}{n} \left\{ [\pi_{s1} 1_{C_1}](1 + |a_n|) \right\} = 0 . \quad (4.11)
\]

Hence, \( \sum_{t=0}^{n} [\nu_t P_1 1_{C_1}] b_t = d_n \psi_1(n) \), \( d_n = o(1) \), and therefore

\[
E_u \left\{ \sum_{t=0}^{n} 1_{C_1}(X_t) 1_{C_2}(W_t) \right\} = [\pi_{s1} 1_{C}] \psi(n)(1 + d_n), \quad \psi(n) \overset{\text{def}}{=} \psi_1(n) \pi_2(s_2) .
\]

\[\square\]

### 4.4 Decomposition structure

In the following \( H = P - s \otimes \nu \) and \( H_j = P_j - s_j \otimes \nu_j \) for \( j = 1, 2 \).

We extend both chains with the split chain method so that we have \( \{(X_t, Y_t^1)\} \) and \( \{(W_t, Y_t^2)\} \). Due to independence \( \{X_t, W_t, Y_t\} \) is the split chain for the compound process \( (X, W) \) where \( Y_t = Y_t^1 Y_t^2 \). Thus

\[
\eta_k = \inf\{t > \eta_{k-1}; Y_t^1 = Y_t^2 = 1\}, \quad k \geq 0, \quad \eta_{-1} = -1 .
\]

We look at the decomposition structure in a different way where we try to benefit from the marginal decomposition of the \( X \)-process, i.e., the regenerations defined by \( \{\tau_{1k}^1\} \).

\[
\tau_{1k}^1 = \inf\{t > \tau_{1k-1}^1; Y_t^1 = 1\}, \quad k \geq 0, \quad \tau_{1}^1 = \tau_{-1}^1 = -1 .
\]

Let

\[
U_j = U_{g,j} = \sum_{t=\tau_{j-1}^1+1}^{\tau_j^1} g(X_t, W_t), \quad s \geq 0 . \quad (4.12)
\]
We note that in general the \( U_j \)'s are neither unconditionally nor conditionally independent. The simultaneous regeneration times are also marginal regeneration times, i.e.,

\[
\{ \eta_k, \ k \geq 0 \} \subseteq \{ \tau_j^1, \ j \geq 0 \}
\]

Let \( T_{k-1} = -1 \) and

\[
T_k = \inf \{ k > T_{k-1}^1: \ Y_{\tau_k^1}^2 = 1, \ k \geq 0, \ T = T_0. \}
\]

Then

\[
\eta_k = \tau_k^1, \ k \geq 0
\]

which gives

\[
V_0 = V_\theta = \sum_{t=0}^{\tau_1^T} g(X_t, W_t) = \sum_{j=0}^{T} \sum_{t=\tau_j^1 + 1}^{\tau_j^1} g(X_t, W_t) = \sum_{j=0}^{T} U_j
\]

and in general

\[
V_k = \sum_{j=\tau_{k-1} + 1}^{\tau_k} U_j, \ k \geq 0.
\]

The following lemma contains necessary information about the process \( \{ W_{\tau_k^1}, k \geq 0 \} \)

Lemma 4.2 The process \( \{ W_{\tau_k^1}, k \geq 0 \} \) is a Markov process with transition probability 
\[
P = P_\lambda \phi_v, \text{ where } \phi_v = \sum_{t=0}^{\infty} \{ \nu_1 H_s^t \} P_s^t. \text{ Moreover,}
\]

\[
P \geq (s \otimes \nu)
\]

with

\[
(s, \nu) = \left( s_2, \nu_2 \phi_v \right).
\]

Let \( \lambda = \lambda_1 \otimes \lambda_2 \) be the initial measure for \( \{(X_t, W_t)\} \). Let \( W^* = \{(W_t^*, Y_t^2), t \geq 0\} \) be the split chain generated by \( P \) and \( (s, \nu) \) and let \( W_{\tau}^* = \{(W_{\tau_k^1}^*, Y_{\tau_k^1}^2), k \geq 0\} \). Then

\[
W_{\tau}^* = W^*
\]

when the initial measure for \( W^* \) is \( \tilde{\lambda} = \lambda_2 \phi_{\lambda_1} \). In particular, let \( T \) denote the first regeneration time for \( W^* \). Then the occupation time formula is given by

\[
E_\lambda \sum_{k=0}^{T} 1_A(W_{\tau_k^1}) = E^{T}_{\tilde{\lambda}} \sum_{k=0}^{T} 1_A(W_{\tau_k^1}) = \begin{cases} \tilde{\lambda} G_{s, \nu} 1_A, \text{ in general;} \\ \pi_2 G_{s, \nu} 1_A, \text{ if } \lambda_2 = \pi_2; \\ \pi_{s_2} 1_A, \text{ if } \lambda = \nu. \end{cases}
\]

where \( G_{s, \nu} = \sum_{t=0}^{\infty} (P_{s} - s \otimes \nu)^t. \)
Proof of Lemma 4.2: See Appendix B. □

The following result indicates that the rate of convergence of the transition probability towards the stationary measure is at least as good for the \( \tilde{W} \)-process as for the \( W \)-process.

**Lemma 4.3** Suppose that \( W \) is geometric ergodic. Then this is also true for \( \tilde{W} \). If \( W \) is strongly mixing with mixing rate defined by \( \alpha = \{ \alpha_j \} \), then \( \tilde{W} \) is strongly mixing with mixing rate \( \alpha \) which is equal to or faster than \( \alpha \). In particular for \( k \geq 0 \)

\[
\sum_{\ell=1}^{\infty} \ell^k \alpha_\ell < \infty \implies E_{\pi_2} T_{\tilde{W}}^{k+1} < \infty. \tag{4.17}
\]

**Proof of Lemma 4.3:** See Appendix B. □

### 4.5 Upper bounds for \( E|V_g|^m \)

In this subsection we assume that (cf. (3.3))

\[
g(x, w) = (g_X \otimes gw)(x, w) = g_X(x)gw(w). \tag{4.18}
\]

In the proof of the CLT we need the moments of \( V_0 \) of (4.13). The following theorem is our main result in this direction.

**Theorem 4.2** Let \( m \geq 1 \) and \( U_k \) is defined by (4.12) and (4.18), then

\[
E_{\nu} \left\{ \sum_{k=0}^{T} |U_k|^m \right\} \leq \left\{ \pi_{s_1} |gW|^m \right\} E_{\nu_1} \left\{ U_{|s_1|}^m \right\}. \tag{4.19}
\]

We use the notation

\[
\delta_j = \tau_j - 1 - \tau_{j-1}, \quad j \geq 0, \quad \mathcal{H}_j = \mathcal{F}_{\tau_j}^X \vee \mathcal{F}_{\tau_j}^{Y_1} \vee \mathcal{F}_{\tau_j}^W \vee \mathcal{F}_{\tau_j}^{Y_2}.
\]

Then \( U_j \) is measurable \( \mathcal{H}_j \) and \( \{ T \geq j \} \in \mathcal{H}_{j-1} \). By (4.13), \( V_0 = \sum_{j=0}^{\infty} U_j 1( T \geq j ) \) and for \( m \geq 1 \),

\[
E_\lambda \left[ U_j^m 1(T \geq j) \right] = E_\lambda \left[ U_j^m 1(T \geq j) \mid \mathcal{H}_{j-1} \right] \\
= E_\lambda \left[ 1(T \geq j) E_\lambda \left[ U_j^m \mid \mathcal{H}_{j-1} \right] \right].
\]
4.5 Upper bounds for \( \mathbb{E}[V'_\delta]^m \)

The following technical result, which is the first step in the proof of Theorem 4.2, use the independence of \( X \) and \( W \) together with the regeneration property of \( X \).

Lemma 4.4 [Decoupling]

Let \( \lambda = \lambda_1 \otimes \lambda_2 \). Let \( j \geq 0 \) be fixed and let \( \{X'_i\} \) be an independent copy of \( \{X_i\} \) so that \( \{X'_i\} \) is independent of both \( \{X_i\} \) and \( \{W_i\} \). Let \( \xi_W \) be a real-valued function defined on \( R \times \{0, 1\} \) and for fixed \( j \) let

\[
a_\ell = \xi_W(W_{\tau_j^{i-1}+\ell}, Y_{\tau_j^{i-1}}^{j-1}), \quad \ell \geq 0, \quad Y_{\tau_j^{i-1}}^{j-1} = y
\]

and \( U_{\xi,j} \) be an extension of (4.12) given by

\[
U_{\xi,j} = \sum_{t=\tau_j^{i-1}+1}^{\tau_j^j} g_X(X_t)\xi_W(W_t, Y_{\tau_j^{i-1}}^{j-1}), \quad s \geq 0 .
\]  

(4.20)

Then for \( m \geq 1 \),

\[
\mathbb{E}_\lambda [U_{\xi,j}^m | \mathcal{H}_{j-1}] = \begin{cases} 
\mathbb{E}_\lambda [U_{\alpha}^m], & \text{for } j = 0 \\
\mathbb{E}_\alpha [U_{\alpha}^m], & \text{for } j \geq 1 
\end{cases}
\]

where \( U_{\alpha} = \sum_{t=0}^{\tau_0} g_X(X'_t) a_\ell \).

Proof of Lemma 4.4:

Let \( j \geq 1 \). By (4.20),

\[
U_{\xi,j} = \sum_{t=\tau_j^{i-1}+1}^{\tau_j^j} g_X(X_t)\xi_W(W_t, Y_{\tau_j^{i-1}}^{j-1}) = \sum_{t=\tau_j^{i-1}+1}^{\tau_j^j} g_X(X_{\tau_j^{i-1}+\ell})\xi_W(W_{\tau_j^{i-1}+\ell}, Y_{\tau_j^{i-1}}^{j-1})
\]

so that

\[
\mathbb{E}_\lambda [U_{\xi,j}^m | \mathcal{H}_{j-1}] = \mathbb{E}_\lambda \left[ \left\{ \sum_{t=1}^{\delta_j} g_X(X_{\tau_j^{i-1}+\ell})\xi_W(W_{\tau_j^{i-1}+\ell}, Y_{\tau_j^{i-1}}^{j-1}) \right\}^m | \mathcal{H}_{j-1} \right]
\]

\[
= \mathbb{E}_\alpha \left[ \sum_{t=1}^{\delta_j} g_X(X'_t) a_\ell \right]^m
\]

\[
= \mathbb{E}_\alpha \left[ \sum_{t=0}^{\tau_0} g_X(X''_t) a_\ell \right]^m
\]

\[
= \mathbb{E}_\alpha U_{\alpha}^m,
\]

where we have used that

\[
\mathcal{L}_\lambda \left\{ X_{\tau_j^{i-1}+\ell}, 1 \leq \ell \leq \delta_j \right\} = \mathcal{L}_\alpha \left\{ X'_\ell, 1 \leq \ell \leq \delta_0 \right\} = \mathcal{L}_\nu \left\{ X''_{\ell}, 0 \leq \ell \leq \tau_0 \right\}
\]

where \( \mathcal{L}_\lambda \) denote the simultaneous distribution with initial measure \( \lambda \) and \( X'' \) has the same distribution as \( X' \).
If $j = 0$, then
\[
E_{\lambda} \left[ U_{\xi,j}^m \mid \mathcal{H}_{j-1} \right] = E_{\lambda} \left[ \left\{ \sum_{t=0}^{\tau^*_0} g(X_t) \xi(W_t, y) \right\}^m \mid \mathcal{F}^W \vee \mathcal{F}^{Y^2} \right] = E_{\lambda_1} U_{2}^m.
\]

Using the previous lemma and a general moment formula given in Corollary A.1 in Appendix A we obtain a useful exact formula.

**Lemma 4.5** Let $U_k$ be defined by (4.12). Let $m \geq 1$. Then
\[
E_{\nu} \left\{ \sum_{k=0}^{T} U_k^m \right\} = \sum_{r=1}^{m} \sum_{\ell \in \Delta_{m,r}} \binom{m}{\ell} \sum_{j^{(2)} \in N^{r-1}_{+}} \left\{ \pi_{s_1} f_{j^{(2)},\ell}^{X} \right\} \left\{ \pi_{s_2} f_{j^{(2)},\ell}^{W} \right\}
\]
where $N^{r-1} = \{ (j_1, \ldots, j_r) \in N^r : j_1 \geq 0, j_2 \geq 1 \cdots j_r \geq 1 \}$, $\Delta_{m,r} = \{ \ell \in N^r : \sum_{j=1}^{r} \ell_j = m \}$, $j^{(2)} = (j_2, \ldots, j_r)$ and
\[
f_{j^{(2)},\ell}^{X} = I_{g_X^1} H_{g_X^2}^{j_2} \cdots H_{g_X^r}^{j_r} I_{g_X^1}, \quad f_{j^{(2)},\ell}^{W} = I_{g_W^1} P_{g_W^2}^{j_2} \cdots P_{g_W^r}^{j_r} I_{g_W^1}.
\]
More generally we have for arbitrary $\lambda = \lambda_1 \otimes \lambda_2$ with $f_{j,\ell}^{X} = H_{j_1}^{j_2} f_{j^{(2)},\ell}$ and $f_{j,\ell}^{W} = P_{j_2} f_{j^{(2)},\ell}$
\[
E_{\lambda} \left\{ \sum_{k=0}^{T} U_k^m \right\}
= \sum_{r=1}^{m} \sum_{\ell \in \Delta_{m,r}} \binom{m}{\ell} \sum_{j^{(2)} \in N^{r-1}_{+}} \left\{ (\lambda - \nu)(f_{j^{(2)},\ell}^{X} \otimes f_{j^{(2)},\ell}^{W}) + \left\{ \nu_1 f_{j^{(2)},\ell}^{X} \right\} \left\{ \tilde{\lambda}_G \right\} \left\{ \nu_2 f_{j^{(2)},\ell}^{W} \right\} \right\}.
\]

**Remark 4.1** If $\lambda = \nu$, then
\[
\tilde{\lambda}_G \sim S, \quad P_{2} f_{j,\ell}^{W} = \nu G \sim S, \quad P_{2} P_{2}^{j_2} f_{j^{(2)},\ell}^{W} = \pi_{s_2} P_{2}^{j_2+1} f_{j^{(2)},\ell}^{W} = \pi_{s_2} f_{j^{(2)},\ell}^{W}
\]
and
\[
\sum_{j_1=0}^{\infty} \nu_1 f_{j^{(2)},\ell}^{X} = \nu_1 \left\{ \sum_{j_1=0}^{\infty} H_{j_1}^{j_2} \right\} f_{j^{(2)},\ell}^{X} = \pi_{s_1} f_{j^{(2)},\ell}^{X}.
\]
Thus (4.22) reduces to (4.21) when $\lambda = \nu$. 

Proof of Lemma 4.5:
We rewrite the first term on the left hand side of (4.21) using that \( \{ T \geq k \} = \{ T \geq k - 1 \} \cap \{ Y_{\tau_h}^2 = 0 \} \), so that
\[
E_\nu \left( \sum_{k=0}^{T} U_k^m \right) = E_\nu (U_0^m) + \sum_{k=1}^{\infty} E_\nu \left[ 1(T \geq k - 1) E_\nu (U_k^m | Y_{\tau_h}^2 = 0) \mid H_{k-1} \right].
\]
Let \( U_{\xi,k} \) be defined by (4.20) where \( \xi_W(w,y) = g_W(w)(1-y) \). By Lemma 4.4 and Corollary A.1 we find that for \( k \geq 1 \)
\[
\begin{align*}
E_\nu (U_k^m | Y_{\tau_h}^2 = 0) &= E_\nu (U_{\xi,k}^m | H_{k-1}) \\
&= \sum_{r=1}^{m} \sum_{\ell \in \Delta_{m,\ell}} \left( \begin{array}{c} m \\ \ell \end{array} \right) \sum_{j \in N \times N_h^{\tau_h}} \{ \nu_1 f_{j,\ell}^X \} \left\{ \prod_{i=1}^{r} g_W(W_{\tau_h}+t_i) \right\} 1(Y_{\tau_h}^2 = 0).
\end{align*}
\]
where \( t_i = j_1 + \cdots + j_i \). Let
\[ G_{k} = \mathcal{F}_{\tau_h}^X \cup \mathcal{F}_{\tau_h}^{Y_1} \cup \mathcal{F}_{\tau_h}^{W} \cup \mathcal{F}_{\tau_h}^{Y_2}. \]
Then by conditioning with respect to \( G_{k-1} \) we find that
\[
E_\nu \left\{ 1(T \geq k - 1) \left\{ \prod_{i=1}^{r} g_W(W_{\tau_h}+t_i+1) \right\} 1(Y_{\tau_h}^2 = 0) \right\} = E_\nu \left\{ 1(T \geq k - 1) H_2 f_{j,\ell}^W(W_{\tau_h}) \right\}. \]
Hence
\[
\begin{align*}
E_\nu \left\{ \sum_{k=1}^{T} U_k^m \right\} &= \sum_{k=1}^{\infty} E_\nu \left[ 1(T \geq k - 1) E_\nu (U_k^m | Y_{\tau_h}^2 = 0) \mid H_{k-1} \right] \\
&= \sum_{r=1}^{m} \sum_{\ell \in \Delta_{m,\ell}} \left( \begin{array}{c} m \\ \ell \end{array} \right) \sum_{j \in N \times N_h^{\tau_h}} \{ \nu_1 f_{j,\ell}^X \} \left\{ \sum_{k=1}^{\infty} 1(T \geq k - 1) H_2 f_{j,\ell}^W(W_{\tau_h}) \right\} \\
&= \sum_{r=1}^{m} \sum_{\ell \in \Delta_{m,\ell}} \left( \begin{array}{c} m \\ \ell \end{array} \right) \sum_{j \in N \times N_h^{\tau_h}} \{ \nu_1 f_{j,\ell}^X \} \{ \pi_{s_2} H_2 f_{j,\ell}^W \}.
\end{align*}
\]
Similarly, we find that
\[
E_\nu \left\{ U_0^m \right\} = \sum_{r=1}^{m} \sum_{\ell \in \Delta_{m,\ell}} \left( \begin{array}{c} m \\ \ell \end{array} \right) \sum_{j \in N \times N_h^{\tau_h}} \{ \nu_1 f_{j,\ell}^X \} \{ \nu_2 f_{j,\ell}^W \} \quad \text{ (4.26)}
\]
and combining (4.25), (4.26) and inserting \( \pi_{s_2} H_2 = \pi_{s_2} - \nu_2 \) we get
\[
E_\nu \left\{ \sum_{k=0}^{T} U_k^m \right\} = \sum_{r=1}^{m} \sum_{\ell \in \Delta_{m,\ell}} \left( \begin{array}{c} m \\ \ell \end{array} \right) \sum_{j \in N \times N_h^{\tau_h}} \{ \nu_1 f_{j,\ell}^X \} \{ \pi_{s_2} f_{j,\ell}^W \}.
\]
Now
\[ \pi_{s_2, f_{j, \ell}^X} = \pi_{s_2, f_{j, \ell}^W}. \]

\[
\sum_{j_1=0}^{\infty} \nu_1 f_{j_1, (j_2, \ell)}^X = \sum_{j_1=0}^{\infty} \nu_1 H_{j_1}^{y_1} I_{g_X}^{y_1} H_{j_2}^{y_2} I_{g_X}^{y_2} \cdots H_{j_\ell}^{y_\ell} I_{g_X}^{y_\ell} \\
= \pi_{s_1} I_{g_X}^{y_1} H_{j_1}^{y_1} I_{g_X}^{y_2} \cdots H_{j_\ell}^{y_\ell} I_{g_X}^{y_\ell} \\
= \pi_{s_1} f_{j, \ell}^{X, (j_2, \ell)}
\]

and (4.21) is proved.

The proof of (4.22) is similar. Instead of (4.25) we get

\[
E_{\lambda} \left\{ \sum_{k=1}^{T} U_k^m \right\} = \sum_{r=1}^{m} \sum_{\ell \in \Delta_{m, r}} \binom{m}{\ell} \sum_{j \in N \times N_{+}^{r-1}} \left\{ \nu_1 f_{j, \ell}^X \right\} E_{\lambda} \left\{ \sum_{k=0}^{T} H_f f_{j, \ell}^W (W_k^1) \right\} \\
= \sum_{r=1}^{m} \sum_{\ell \in \Delta_{m, r}} \binom{m}{\ell} \sum_{j \in N \times N_{+}^{r-1}} \left\{ \nu_1 f_{j, \ell}^X \right\} \left\{ \tilde{\lambda} G_{s, \nu}^{j, \ell} H_f f_{j, \ell}^W \right\}
\]

and (4.26) is changed to

\[
E_{\lambda} \left\{ U_0^m \right\} = \sum_{r=1}^{m} \sum_{\ell \in \Delta_{m, r}} \binom{m}{\ell} \sum_{j \in N \times N_{+}^{r-1}} \left\{ \lambda_1 f_{j, \ell}^X \right\} \left\{ \lambda_2 f_{j, \ell}^W \right\}.
\]

Combining (4.27), (4.28) and inserting that \( \tilde{\lambda} G_{s, \nu}^{j, \ell} H_f = \tilde{\lambda} G_{s, \nu}^{j, \ell} P_2 - \nu_2 \) we obtain (4.22).

\[ \square \]

**Remark 4.2** If \( m = 1 \), then by (4.21) we find that

\[
\mu_g = E_{\nu} \left\{ \sum_{j=0}^{T} U_j \right\} = \left\{ \pi_{s_1} g_X \right\} \left\{ \pi_{s_2} g_w \right\}
\]

which is consistent with (4.9).

**Remark 4.3** If \( m = 2 \). Then

\[
E_{\nu} \left\{ \sum_{j=0}^{T} U_j^2 \right\} = \left\{ \pi_{s_1} g_X^2 \right\} \left\{ \pi_{s_2} g_w^2 \right\} + 2 \sum_{\ell=1}^{\infty} \left\{ \pi_{s_1} I_{g_X}^{\ell} H_{g_X}^{\ell} I_{g_X}^{\ell} \right\} \left\{ \pi_{s_2} I_{g_w}^{\ell} P^{\ell} I_{g_w}^{\ell-1} \right\}.
\]
Remark 4.4 By (4.10), (4.13) and (4.22) we find that for general $\lambda = \lambda_1 \otimes \lambda_2$, $g = g_X \otimes gw$,

$$E_\lambda V_0 = E_\lambda V_g = \sum_{j=0}^\infty (\lambda - \nu)(P_j g_X \otimes P_j^g gw) + \sum_{j=0}^\infty \left\{ \nu_1 H_1^j g_X \right\} \left\{ P_2^j gW \right\}.$$

If $g_X$ is small, $\lambda_2 = \pi_2$ and $\sup_j \pi_2 G\sim S_\nu^G P_2^{j+1} gW < \infty$ then $E_\lambda V_g$ is finite. By Lemma A.1 taking $p = 1 + \delta$, $f = P_2^{j+1} gW$ and $\lambda = \pi_2$ we have that

$$\pi_2 \left( \frac{G}{\sim S_\nu^G} P_2^{j+1} gW \right)^{1+\delta} \leq c_2 \left\{ \frac{1}{\pi_2} \left( \frac{1}{T} \right)^{1+2\delta} \right\} \left\{ \frac{1}{\pi_2} \left( \frac{1}{T} \right)^{1+\delta} \right\}$$

with $\eta \in (0, 1)$ and $\delta > 0$ arbitrary.

---

Proof of Theorem 4.2: Assume that $g_X \geq 0$. By Cauchy-Schwartz, recalling that $\sum_{j=1}^m \ell_j = m$,

$$|\pi_2 f^{W}_{j^{(2)}, \ell}| = \frac{|\pi_2 I^{gW}_{gW} P_2^{j_2} I^{gW}_{gW} \cdots P_2^{j_r} I^{gW}_{gW} |}{m} \leq \frac{m}{E_\pi_2} |gW|^{\ell_1} (W_0) \cdots |gW|^{\ell_r} (W_{j_r}) \leq \prod_{r=1}^m \frac{1}{E_\pi_2} |gW|^{\ell_r} (W_0) = \pi_2 |gW|^m.$$

Inserting (4.29) into (4.21) we obtain

$$E_\nu \left\{ \sum_{k=0}^T U^m_k \right\} \leq \left\{ \pi_2 |gW|^m \right\} \sum_{r=1}^m \sum_{\ell_1} \left( \frac{m}{\ell_1} \right) \sum_{j^{(2)} \in N^r_+} \left\{ \pi_1 I^{gX}_{gX} H^{j_2} I^{gX}_{gX} \cdots H^{j_r} I^{gX}_{gX} \right\}$$

from which (4.19) follows trivially.

---

4.6 Moment bounds of $V_{gh}$ expressed in terms of bandwidth

The following results describe how higher order moments of $V_0$ behave as a function of the bandwidth. This is directly related to (4.4) and (4.5).
**Theorem 4.3** Let \( g_X = K_{x,h} \) and assume that the conditions \( K_1, K_2, P_1 \) and (4.18) hold. Then for all \( k, m \geq 1 \),

\[
E_{\nu}|V_0|^{2m} = E_{\nu}|V_{gX}|^{2m} \leq d_{m,k} h^{-2m+1/(k+1)}
\]  
(4.30)

where

\[
d_{m,k} \triangleq \left\{ \frac{1}{\pi s_1^{2m(k+1)}} \left\{ E_{\nu} \frac{k}{k+1} \left( T + 1 \right)^{2m \left( \frac{k+1}{k} \right)} \right\} \right\} \left\{ \frac{1}{d_{2m}} \right\}
\]

and the sequence of constants \( \{d'_{m}\} \) is only dependent upon \( \mathcal{N}_x \) and \( \sup_{u} K(u) \).

First we need the following lemma which gives a more direct upper bound of \( E|V_0|^p \).

**Lemma 4.6** For all \( p > 0 \) and \( \delta \in (0, \infty) \)

\[
E_{\nu}|V_0|^p \leq E_{\nu} \frac{1}{(1+\delta)} \left\| \sum_{j=0}^{T} |U_j|^p (1+\delta) \left| E_{\nu} \frac{\delta}{(1+\delta)} \left| T + 1 \right|^{p(1+\delta^{-1})} \right. \right.
\]

**Proof of Lemma 4.6:** Let \( r = 1 + \delta \) and \( q = 1 + \delta^{-1} \)

\[
E_{\nu}|V_0|^p = E_{\nu} \left| \sum_{j=0}^{T} U_j \right|^p
\]

\[
\leq E_{\nu} \left| \max_{0 \leq j \leq T} |U_j| (T + 1) \right|^p
\]

\[
= E_{\nu} \left| \max_{0 \leq j \leq T} |U_j|^p |T + 1|^p \right.
\]

\[
\leq E_{\nu} \frac{1}{r} \left| \max_{0 \leq j \leq T} |U_j|^{pr} \right| E_{\nu} \frac{1}{q} \left| T + 1 \right|^{pq}
\]

\[
\leq E_{\nu} \frac{1}{r} \left| \sum_{j=0}^{T} |U_j|^{pr} \right| E_{\nu} \frac{1}{q} \left| T + 1 \right|^{pq}
\]

\[
\square
\]

**Proof of Theorem 4.3:**

By Lemma 4.6 with \( p = 2m \) and \( \delta = k \),

\[
E_{\nu}|V_0|^{2m} \leq E_{\nu} \frac{1}{k+1} \left| \sum_{j=0}^{T} |U_j|^{2m(k+1)} \right| E_{\nu} \frac{k}{k+1} \left| T + 1 \right|^{2m \left( \frac{k+1}{k} \right)}
\]  
(4.31)

From Theorem 4.2 we have

\[
E_{\nu} \left| \sum_{j=0}^{T} U_j^{2m(k+1)} \right| \geq \left\{ \frac{1}{\pi s_1} |gW|^{2m(k+1)} \right\} E_{\mu} \left| U_{gX}^{2m(k+1)} \right|
\]  
(4.32)
and by KT (Lemma 4.1 with $\xi_0 \equiv 1$),

$$E_\nu |U_{gX}|^{2m(k+1)} \leq d_m' h^{-2m(k+1)+1}$$  \hspace{1cm} (4.33)

where it is also shown that the sequence of constants \{d_m\} is only dependent upon $N_\nu$ and sup \nu $K(u)$.

Inserting (4.32) and (4.33) into (4.31) we get (4.30). \hfill \Box

### 4.7 Asymptotic variance

Exact information about the first order properties of the asymptotic variance is important (cf. (4.2) and (4.3)). This is the content of the next result. Our method of proof uses a truncation technique based on the notion of a generalized autocovariance function. We believe that the latter concept has some independent interest.

**Theorem 4.4** Assume that $g_{X,h} = K_{x,h}$ and that the conditions $K_1$, $K_2$, $P_1$, $P_5$, and $W_4$ with $m = 1$ and $k \geq 1$ hold. Then, if $\mu_{gW} = 0$, as $h \downarrow 0$

$$h \sigma_{g_{X,h}\otimes gW}^2 = p_s(x) \left\{ \int K^2(u) du \right\} \left\{ \pi_s g_W^2 \right\} + o(1).$$

We also have that

$$h \sigma_{g_{X,h}\otimes gW}^2 = p_s(x) \left\{ \int K^2(u) du \right\} \left\{ \pi_s g_W^2 \right\} + o(1). \hspace{1cm} (4.34)$$

In the proof of Theorem 4.4 we use an extension of the cross covariance function defined for ergodic processes. Let $\mathcal{E}$ denote set of integers and $g$, $f$ real valued measurable functions defined on $E$. Then

$$\gamma_{g,f}(\ell) \overset{\text{def}}{=} \pi_s I_g P_{f1} I_{f1}, \quad \ell \in \mathcal{E}$$  \hspace{1cm} (4.35)

and $\gamma_g = \gamma_{g,g}$. Note that $\gamma_{gW} = \{\pi_s s_2\}^{-1} c_W$ where $c_W$ is the covariance function for the stationary process \{g_W(W_t), t \geq 0\} (cf. Section 3).

**Lemma 4.7** Assume that $g(x,w) = g_X(x) g_W(w)$, $g_X$ is small, $\mu_{gW} = 0$, and the $W$-process satisfies $W_4$ with $m = 1$ and $k \geq 1$. Then

$$\sigma_g^2 = \sum_{\ell = -\infty}^{\infty} \gamma_{gX}(\ell) \gamma_{gW}(\ell). \hspace{1cm} (4.36)$$

We also have that

$$\sigma_{|g|}^2 = \sum_{\ell = -\infty}^{\infty} \gamma(|g|,|g| - s \mu_{|g|})(\ell) - \pi_s |g| \pi_s |g| s. \hspace{1cm} (4.37)$$
Proof of Lemma 4.7:
Our proof is based upon Theorem A.2. By (4.31) and (4.32) with \( m = 1 \) and the smallness of \( g_X \),

\[
\pi^2(\sigma^2) + 2\pi^2 I_g P G_{s,\nu} g = E_0 V_0^2 \leq c_0 \pi_{2+1}^{1/2} |g_W|^{2(k+1)} E_{0}^{k+1} T^{2k+2} < \infty.
\]

To prove that the last term is finite note first that

\[
E_0 T^{2k+2} = E_0 T^{2k+2}.
\]

By (2.1), \( \pi_2 \geq \{ \pi_2 \nu_2 \} \nu_2 \). Hence \( \pi_2 \geq c_1 \nu_2 \) and the right hand side of (4.38) is finite if

\[
E_{0} T^{2k+2} < \infty.
\]

By Lemma 4.3 this is true if

\[
\sum_{\ell=1}^{\infty} \ell^{(k+1)/\ell} \alpha_{\ell} < \infty
\]

But (4.39) follows from W4 and hence (A.14) is fulfilled. Assume that \( g_{W,m} \) is defined by (A.13). Let

\[
g_m(x, w) = g_X(x) g_{W,m}(w)
\]

and \( \xi_m = g_{W,m} - s_2 \mu_m \) with \( \mu_m = \pi_2 g_{W,m} \). Then, using that for general functions \( g \) and \( g' \)

\[
\gamma_{g_X \otimes g_W \otimes \gamma_{g_X}} = \gamma_{g_X \otimes g'_X} \gamma_{g_W \otimes g'_W},
\]

we obtain

\[
\left| \gamma_{g_m, g_m - g_X \otimes \gamma_{s_2 \mu_m}}(\ell) \right| \\
\leq \left| \gamma_{g_X \otimes g_{W,m}, g_X \otimes \xi_m}(\ell) \right| \\
= \left| \gamma_{g_X}(\ell) \right| \left| \gamma_{g_{W,m}, \xi_m}(\ell) \right| \\
= \left| \gamma_{g_X}(\ell) \right| \left\{ \pi_2(s_2) \right\}^{-1} \left| \pi_2 I_{g_{W,m}} P_{\ell} \xi_m \right| \\
\leq \left\{ \pi_2(s_2) \right\}^{-1} \gamma_X(0) \left| \text{Cov}_{\pi_2} \left( g_{W,m}(W_0), \xi_m(W_\ell) \right) \right| + E_{\pi_2} \left[ g_{W,m}(W_0) \right] E_{\pi_2} \left[ \xi_m \right] \\
\leq \left\{ \pi_2(s_2) \right\}^{-1} \gamma_X(0) 8 \left\| g_{W,m} \right\|_{1/2(k+1)} \left\| \xi_m \right\|_{1/2(k+1)} \alpha_{\ell}^{k/(k+1)} \\
\leq \left\{ \pi_2(s_2) \right\}^{-1} \gamma_X(0) 16 \left\| g_{W,m} \right\|_{1/2(k+1)} \alpha_{\ell}^{k/(k+1)}
\]

where we have used a strong mixing inequality (cf. Hall & Heyde, 1980, Corollary A.2, p. 278),

\[
\left| \text{Cov}_{\pi_2} \left( g_{W,m}(W_0), \xi_m(W_\ell) \right) \right| \leq 8 \left\| g_W \right\|_{1/2(k+1)} \left\| \xi_m \right\|_{1/2(k+1)} \alpha_{\ell}^{k/(k+1)}
\]

and that \( E_{\pi_2} \left[ \xi_m \right] = 0 \). By W4

\[
\sum_{\ell=1}^{\infty} \ell^{(k+1)/\ell} \alpha_{\ell} < \infty.
\]
Since for $p > 0 \sum_{t=1}^{\infty} \ell^p \alpha_\ell < \infty$ implies that $\sum_{t=1}^{\infty} \alpha_\ell^{1/p} < \infty$, it follows that

$\sum_{\ell=1}^{\infty} \alpha_\ell^{k/(k+2)} < \infty$.

Together with

$\sum_{\ell=1}^{\infty} \alpha_\ell^{k/(k+1)} \leq \sum_{\ell=1}^{\infty} \alpha_\ell^{k/(k+2)} < \infty$

this shows that (A.16) holds and (4.36) follows from Theorem A.2.

The last part of the lemma follows in the same way.

\[\square\]

**Remark 4.5** The formulae (4.36) and (4.37) can be viewed as generalizations of the formulae $\text{Var}(n^{-1/2} \sum_{j=0}^{n-1} X_j) = \sum_{t=\infty}^{-\infty} \text{Cov}(X_t, X_{t-\ell}) + o(1)$ in case $\{X_i\}$ is a stationary process with an absolutely summable covariance function.

**Proof of Theorem 4.4:**

By Lemma 4.7

$$h \sigma_{gX,h \otimes gw}^2 = \sum_{\ell} \gamma_{gX,h}(\ell) \gamma_{gw}(\ell)$$

and for $\ell > 0$, $h \gamma_{gX,h}(\ell) = h \pi_{s_1} I_{K_{x,h}} P^\ell_1 K_{x,h} = o(1)$ by P5 (cf. KT proof of Lemma 4.2). Since

$$|h \gamma_{gX,h}(\ell)| \leq \int p_s(x + hu) K(u) P^\ell (x + hu, \mathcal{N}_x(h)) du \leq c_0,$$

we can apply the dominated convergence theorem, i.e.,

$$\lim_{h \to 0} h \sigma_{gX,h \otimes gw}^2 = \sum_{\ell} \left( \lim_{h \to 0} h \gamma_{gX,h}(\ell) \right) \gamma_{gw}(\ell)$$

$$= \left( \lim_{h \to 0} h \gamma_{gX,h}(0) \right) \gamma_{gw}(0)$$

$$= \left( \lim_{h \to 0} h \pi_{s_1} K_{x,h}^2 \right) \left\{ \pi_{s_2} g_{gw}^2 \right\}$$

$$= p_s(x) \left\{ \int K^2(u) du \right\} \left\{ \pi_{s_2} g_{gw}^2 \right\}.$$

The proof of (4.34) is similar since

$$\gamma_{|g|_{|g|-s_1s_2}^2}(\ell) = \gamma_{gX}(\ell) \gamma_{gw}(\ell) - \gamma_{gX,s_1}(\ell) \gamma_{gw,s_2}(\ell) \{ \pi_{s_1} K_{x,h} \} \{ \pi_{s_2} g_{gw} \}$$

and

$$\pi_{s_1} g \pi_{s_2} |g \cdot s| = \{ \pi_{s_1} K_{x,h} \} \{ \pi_{s_1} K_{x,h} \cdot s_1 \} \{ \pi_{s_2} g_{gw} \} \{ \pi_{s_2} g_{gw} \cdot s_2 \}.$$  

\[\square\]
4.8 Main results

**Theorem 4.5** Assume that \( \{X_t\} \) and \( \{W_t\} \) are independent and that \( K_1, K_2, K_4, P_1, P_2, P_4, P_5, P_6 \) and \( W \) hold. Let \( k \geq 1 \) and \( m \geq 2 \). Assume that for some \( \epsilon > 0 \),

\[
h_n^{-1} \ll n^{\beta_m-\epsilon}, \quad \delta_m = \frac{m-1}{m-1/(k+1)}.
\]

Then for all \( \lambda = \lambda_1 \otimes \pi_2 \)

\[
\Delta_{n,h_n} = S_{n}^{-1/2}(K_{x,h_n})h_n^{1/2}S_n(K_{x,h_n} \otimes gW) \xrightarrow{d} \mathcal{N}\left( 0, \left\{ \int K^2(u)du \right\} \{\pi_2g_W^2\} \right).
\]

**Proof of Theorem 4.5:** Recall that for \( C = C_1 \times C_2, C_i \in \mathcal{E}_i, i = 1, 2, \)

\[
T_C(n) = \sum_{t=0}^{n} 1_{C_1}(X_t)1_{C_2}(W_t)
\]

represents the number of visits to \( C \) of \( \{(X_t, W_t)\} \) up to time \( n \). We choose \( C_1 \) and \( C_2 \) so that both sets are small. Then by KT (second part of Th. 4.3) using \( K_1, K_2, P_1, P_2, P_4, P_6 \)

\[
\hat{p}_{C_1}(x) \overset{\text{def}}{=} \frac{S_n(K_{x,h_n})}{T_C \times E_2(n)} = p_{C_1}(x) + o_F(1)
\]

with \( E = E_1 \times E_2 \), and where \( p_{C_1}(x) \) is \( p_{h_1}(x)/\pi_{s_1}1_{C_1} \). By KT (Corollary 2.1)

\[
\frac{T_C(n)}{T_{C_1 \times E_2}(n)} = \frac{\pi_{s_1}1_{C_1 \times E_2}}{\pi_{s_1}1_{C_1 \times E_2}} + o(1) \quad \text{a.s.} = \pi_2(C_2) + o(1) \quad \text{a.s.} \cdot
\]

We can write

\[
\Delta_{n,h_n} = \{\hat{p}_{C_1}(x)\}^{-1/2}\left\{ \frac{T_C(n)}{T_{C_1 \times E_2}(n)} \right\}^{1/2}\left\{ T_C^{-1/2}(n)h_n^{1/2}S_n(g_{h_n}) \right\} = A_{n,h_n}^0 \Delta_{n,h_n}^0, \quad \text{say.}
\]

We have that

\[
A_{n,h_n} = \{ p_{C_1}^{-1}(x)\pi_2(C_2) \} + o_F(1).
\]

Hence, it is enough to prove that

\[
\Delta_{n,h_n}^0 \xrightarrow{d} \mathcal{N}\left( 0, p_{C_1}(x)\left\{ \int K^2(u)du \right\} \{\pi_2g_W^2\} \right)
\]

where \( g_{h}(z,w) = K_{x,h}(z)gW(w) \).

By \( P_1 \) and \( P_2 \) and Bochner theorem, (4.1) is satisfied. From Theorem 4.3 and Theorem 4.4 the conditions (4.2) - (4.5) are fulfilled.

It remains to verify (4.7). Let \( g_0 = c_01_{N_1}gW \) where \( c_0 \) is an appropriate constant. Then \( |h_{g_{h}}| \leq g_0 \). We have to prove that

\[
P_\lambda(V_{g_0} < \infty) = 1
\]
But this is fulfilled if $E_{\lambda} V_{\delta_0} < \infty$. By Remark 4.4 this is true if

$$E_{\pi_2} \left[ T_\sim^{1+2\delta} \right] \pi_2 |g_W|^{\frac{1+\delta}{\gamma}} < \infty$$

(4.42)

for some $\delta > 0$ and $\eta \in (0, 1)$. By $W_4$ and Lemma 4.3

$$\pi_2 |g_W|^{2m(k+1)} < \infty, \quad E_{\pi_2} T_{\sim}^{2m+\frac{2m}{k}} < \infty .$$

Thus (4.42) holds.

Hence

$$\Delta^0_{n,h_n} \xrightarrow{d} N (0, \sigma_C^2)$$

and

$$\sigma_C^2 = \{\pi_{\delta_1} 1_{C_1}\}^{-1} \{\pi_{\delta_2} 1_{C_2}\}^{-1} p_{\delta_1}(x) \left\{ \int K^2(u) du \right\} \left\{ \pi_{\delta_2} g_W \right\} .$$

It follows that (4.41) holds.

5 Examples and simulations

The purpose of this section is to illustrate the small sample properties of the estimator $\hat{f}(x)$ defined by (3.1). As can be seen from Myklebust et al (1999) it is not simple to find examples of null recurrent and $\beta$-null recurrent processes. The trivial example is the random walk given by

$$X_t = X_{t-1} + \epsilon_t$$

(5.1)

where $\{\epsilon_t\}$ consists of independent identically distributed variables with strictly positive density function with respect to the Lebesgue measure almost everywhere on $R$. This process is $\beta$-null recurrent with $\beta = 1/2$. Another class of null recurrent processes is the process given by

$$X_t = 1(|X_{t-1}| \leq M)g(X_{t-1}) + 1(|X_{t-1}| > M)X_{t-1} + \epsilon_t$$

for some finite $M > 0$ and some measurable function $g$ finite on $|x| \leq M$. This process behaves as a random walk for large $X_t$'s. Other examples of null recurrent processes are the first order threshold model studied by Meyn and Tweedie (1994) and the exponential autoregressive process looked at by Cline and Pu (1997).

We will look at some of these examples in the context of (1.1) and (3.1). We will examine the finite sample properties of our estimators by means of simulations. A difficult and largely unresolved problem is that of choosing a proper bandwidth. Theorem 3.1 and Theorem 4.4 of KT only give the allowable rate as $n$ tends to infinity. It should be noted that these rates are different from the stationary case, $n$ effectively being replaced by $n^{\beta}$. It practice we have found it useful to use cross-validation and to let the bandwidth $h$ depend on $x$. In fact we have typically let $h_n$ be proportional to $n^{-1/5} \{\hat{p}_C(x)\}^{1/5}$ where $\hat{p}_C(x)$ could be thought of as the locally estimated density.

We will look at two aspects of the estimation problem. In Figures 1-4 we concentrate on the accuracy of the estimation of the function $f(x)$ for the model (1.1). Only
single realizations are shown, but they are representative for the quality of the estimates. In Figure 5 we turn to the finite sample approximation of the asymptotic normal distribution derived in Theorems 3.1 and 4.5.

In the sequence of plots given in Figures 1-3 we look at the estimate of \( f(x) \) defined in (3.1). In all cases \( X_t \) is the random walk process given in (5.1), with \( \{ \varepsilon_t \} \) consisting of iid standard normal variables. In Figure 2 \{ \( W_t \) \} has been chosen to be the stationary Gaussian AR process

\[
W_t = 0.8W_{t-1} + \varepsilon_t
\]

where \( \{ \varepsilon_t \} \) consists of iid standard normal variables independent of \( \{ \varepsilon_t \} \), whereas in Figures 1 and 3 \{ \( W_t \) \} is simply identical to \( \{ \varepsilon_t \} \). The function \( f(x) \) has taken to be \( f(x) = x, \) \( f(x) = \sqrt{|x|} \) and \( f(x) = x^2 \) in Figures 1-3, respectively. Both \( X_t \) and \( Z_t \) of (1.1) have been plotted in the a)-parts of the figures. (The traces are virtually coinciding in Figure 1a due to the scale). In all of the cases \( f(x) \) is reasonably well estimated for the region for which we have data – and in all cases with the moderate sample size of 500 (Note that Figure 3c gives a magnified picture in the region \(-10 \leq x \leq 10\)), The traces of Figures 2a and 3a do not look particularly well cointegrated perhaps, but this is due to the nonlinear nature of the cointegrating relationship, where \( \{ Z_t - f(X_t) \} \) is stationary, whilst both \( \{ X_t \} \) and \( \{ Z_t \} \) are nonstationary. In Figures 2c and 2d we have also included plots of the trace and of the estimated \( \hat{M}_{\hat{W}}(x) = E(\hat{W}_t \mid \hat{W}_{t-1} = x) \) for the estimated residual process

\[
\hat{W}_t = Z_t - \hat{f}(X_t).
\]

It is seen from Figure 2 that the stationary AR behavior of \( \{ W_t \} \) is well recovered, and thus gives a very informal verification of the cointegrating property. In a more stringent approach the properties of \( \hat{M}_{\hat{W}}(x) \) would have to be evaluated along the lines of Theorem 4.4 of KT. The plots of \( \hat{M}_{\hat{W}}(x) \) for the other two cases were similar.

Our last single realization example is for the same case as for Figure 1, except that \( W_t \) is now taken to be a random walk \( W_t = W_{t-1} + \varepsilon_t \), which would imply that \( \{ X_t \} \) is not cointegrated with \( \{ Z_t \} \). It is not difficult to show theoretically that we get inconsistent estimates of \( f(x) \) in this situation, and this is clearly revealed in Figure 4b; it is also seen that the two traces drift apart in Figure 4a.

Estimates similar to those in Figures 1-4 have been presented in the cointegration literature. Our contribution, which we believe to be new, is that we have singled out classes of processes and assumptions for which an asymptotic theory of these estimates can be constructed, such that it should be possible to work out confidence intervals and bands (and possibly stringent tests of nonlinear cointegration in the sense discussed in this paper). Our last experiment indicates the finite sample approximation to the asymptotic distribution of Theorem 3.1.

The quality of the sample approximation has to be judged using a multitude of realizations. A problem not encountered in the stationary case is that the simulated realizations may cover very different \( x \)-regions. (This is one reason for using only one realization in Figures 1-4.) Hence, for a fixed \( x = x' \), close to the starting value \( X_0 = 0 \), say, of each realization, some realizations may have many observations in the neighborhood of \( x' \), whereas other realizations may have none observations in the vicinity of \( x' \) for the sample size we are considering. This kind of behavior does not
occur in the stationary case, where the expected time until the process reaches \( x' \) is always finite and in practice small, when \( |x - x'| \) is small. This means that in the verification of Theorem 3.1 we can either keep \( x \) fixed and wait until we have sufficiently many observations close to \( x \); the other realizations being discarded, or we can choose a central realization-dependent value of \( x \); e.g., the modal value of the sample, for studying the normalized ratio (3.2) of Theorem 3.1.

We have chosen to adopt both procedures, although clearly we introduce some extraneous stochastics into the problem in the latter case. The approximation to normality as a function of sample size, for the quantity

\[
\left[ \frac{h_n \sum K_{x,h_n}}{\int K^2(u)du} \right]^{1/2} [\tilde{f}(x) - x]
\]
derived from the simple cointegrated system

\[X_t = X_{t-1} + \varepsilon_t, \quad Z_t = X_t + \varepsilon_t,\]
at the point \( x = 7.5 \) is shown in Figure 5a. 1000 realizations have been used, and a particular realization is admitted into the evaluation as respectively 100, 200, 300, 500 and 800 observations are accumulated in the interval (5,10). For Figure 7b, on the other hand, a fixed point \( x \) has not been used; rather \( x \) has been taken to be the modal value and is thus varying from one realization to another. In this case the length of the time series has been 500, 1000 and 3000, respectively.

The quality of the approximation seems to be quite good with 500 observations in the "relevant" region surrounding \( x \).

\[2x\]

### A Appendix

In this section we assume that \( \{X_t\} \) is an aperiodic \( \phi \)-irreducible Markov chain with state space \((E, \mathcal{E})\) where \( \mathcal{E} \) is countably generated. We also assume that the transition probability \( P \) satisfies a minorization inequality, i.e., \( P \geq s \otimes \nu \) and that \( \{X_t\} \) is Harris recurrent.

#### A.1 Higher order moments

**Theorem A.1** Let \( g = \{g_j\} \) be a sequence of real-valued measurable functions defined on \( E \). Let

\[
U_g = U = \sum_{j=0}^{\tau} g_j(X_j).
\]

Then

\[
E_x U^m = \sum_{r=1}^{m} \sum_{\ell \in \Delta_m, r} \binom{m}{\ell} \psi_{r,\ell}(x)
\]

(A.1)
where $\Delta_{m,r} = \{\ell \in \mathcal{N}^r; \sum_{j=1}^r \ell_j = m\}$, \((m)_{(\ell)} = (\ell_1, \ldots, \ell_r)\),

$$
\psi_{r,\ell} = \sum_{j \in \mathcal{N} \times \mathcal{N}_r^{r-1}} H^{j_1} I_{g_{j_1}} H^{j_2} I_{g_{j_1+j_2}} \cdots H^{j_r} I_{g_{j_1+\cdots+j_r}} 1
$$

(A.2)

with $H = P - s \otimes \nu$.

**Remark A.1** If $g_j \equiv g$ then

$$
\psi_{r,\ell} = \sum_{j \in \mathcal{N}^r} H^{j_1} I_{g_{j_1}} H^{j_2+1} I_{g_{j_1+j_2}} \cdots H^{j_r+1} I_{g_{j_1+\cdots+j_r}} 1
= G_{s,\nu}(I_g H) \cdots G_{s,\nu}(I_g H)G_{s,\nu} g.
$$

Hence this result partly generalize Theorem A.1. in KT.

**Proof of Theorem A.1:**

Let

$$
s_k = j_1 + \cdots + j_k, \quad k = 1, \ldots, r
$$

and

$$
B_s = \mathcal{F}_s^X \vee \mathcal{F}_{s-1}^Y, \quad A_s = (1 - Y_s).
$$

By KT (Lemma A.3) we can write

$$
U^m = \sum_{r=1}^m \sum_{\ell \in \Delta_{m,r}} {m \choose \ell} J_{r,\ell}, \quad J_{r,\ell} = \sum_{j \in \mathcal{N} \times \mathcal{N}_r^{r-1}} Z_{j,\ell}
$$

(A.3)

with

$$
Z_{j,\ell} \overset{\text{def}}{=} g_{s_1}^{j_1}(X_{s_1}) \cdots g_{s_r}^{j_r}(X_{s_r}) \prod_{k=0}^{s_r-1} A_k.
$$

(A.4)

It is sufficient to prove that

$$
E_x J_{r,\ell} = \psi_{r,\ell}, \quad \ell \in \Delta_{m,r}, \quad r = 1, \ldots, m.
$$

We will prove this by induction on $r$. When $r = 1$, then

$$
J_{1,\ell} = \sum_{j_1=0}^\infty g_{j_1}^{j_1}(X_{j_1}) \prod_{k=0}^{j_1-1} A_k = \sum_{j_1=0}^\infty g_{j_1}^{j_1}(X_{j_1})1(\tau \geq j)
$$

and

$$
E_x J_{1,\ell} = \sum_{j_1=0}^\infty E_x \left[ g_{j_1}^{j_1}(X_{j_1})1(\tau \geq j) \right] = \sum_{j_1=0}^\infty H^{j_1} I_{g_{j_1}} 1 = \psi_{1,\ell}.
$$

Assume that these formulae are correct for $r - 1$. Let $\ell$ be fixed, $Z_\ell = Z_{j,\ell}$ and $j_{(r)} = (j_1, \ldots, j_r)$. Then

$$
Z_\ell = Z_{j_{(r-1)}} A_{s_{r-1}} D_{s_{r-1},j_r}
$$
where
\[
D_{s_{r-1}, j_r} \overset{\text{def}}{=} \sum_{j_{r-1}}^{\infty} g_{s_{r-1} + j_{r-1}}^r (X_{s_{r-1} + j_{r-1}}) \left\{ \prod_{k=s_{r-1} + 1}^{s_{r-1} + j_{r-1} - 1} A_k \right\}
\]
so that
\[
J_r, \ell = \sum_{j \in \mathcal{N} \times \mathcal{N}^{r-1}} \mathbf{Z}_{j(r-1)} A_{s_{r-1}} D_{s_{r-1}, j_r}.
\]

Taking conditional expectation with respect to $\mathcal{B}_{s_{r-1}+1}$ gives
\[
\mathbf{E}\left[ \mathbf{Z}_{j(r-1)} A_{s_{r-1}} D_{s_{r-1}, j_r} \mid \mathcal{B}_{s_{r-1}+1} \right] = \mathbf{Z}_{j(r-1)} A_{s_{r-1}} \mathbf{E}\left[ D_{s_{r-1}, j_r} \mid \mathcal{B}_{s_{r-1}+1} \right]
\]
and
\[
\mathbf{E}\left[ D_{s_{r-1}, j_r} \mid \mathcal{B}_{s_{r-1}+1} \right] = \mathbf{E}\left[ g_{s_{r-1} + j_{r-1}}^r (X_{s_{r-1} + j_{r-1}}) \left\{ \prod_{k=s_{r-1} + 1}^{s_{r-1} + j_{r-1} - 1} A_k \right\} \mid \mathcal{B}_{s_{r-1}+1} \right] = \mathbf{E}_{X_{s_{r-1} + 1}} \left[ g_{s_{r-1} + j_{r-1}}^r (X_{s_{r-1} + 1}) \left\{ \prod_{k=1}^{s_{r-1} - j_{r-1} - 1} A_k \right\} \right]
\]
so that
\[
\sum_{j_{r-1}}^{\infty} \mathbf{E}\left[ D_{s_{r-1}, j_r} \mid \mathcal{B}_{s_{r-1}+1} \right] = \phi_{s_{r-1}}^{j_{r-1}, 0} (X_{s_{r-1} + 1})
\]
where
\[
\phi_{s_{r-1}}^{j_{r-1}, 0} = \sum_{j_{r-1}}^{\infty} H_{j_{r-1}} g_{s_{r-1} + j_{r-1}}^r
\]
and
\[
\phi_{s_{r-1}}^j (X_{s_{r-1}}) \overset{\text{def}}{=} \mathbf{E}\left[ A_{s_{r-1}} \phi_{s_{r-1}}^{j_{r-1}, 0} (X_{s_{r-1} + 1}) \mid \mathcal{B}_{s_{r-1}} \right] = \mathbf{E}_{X_{s_{r-1}}} \left[ \phi_{s_{r-1}}^{j_{r-1}, 0} (X_{s_{r-1}}) \right] = H \phi_{s_{r-1}}^{j_{r-1}, 0} (X_{s_{r-1}})
\]
The product (A.4) is reduced from $r$ to $r-1$ since
\[
g_{s_{r-1}}^{j_{r-1}} (X_{s_{r-1}}) \cdot H \phi_{s_{r-1}}^j (X_{s_{r-1}}) = \left[ I_{g_{s_{r-1}}} H \phi_{s_{r-1}}^j \right] (X_{s_{r-1}}) = g_{s_{r-1}}^{j_{r-1}, 0} (X_{s_{r-1}}), \text{ say.}
\]

Hence, by (A.3), (A.5), (A.6), (A.7) we obtain
\[
\mathbf{E}_x J_{r, \ell} = \sum_{j(r-1) \in \mathcal{N}^{r-2}} \sum_{j_{r-1}}^{\infty} \mathbf{E}\left[ D_{s_{r-1}, j_r} \mid \mathcal{B}_{s_{r-1}+1} \right] = \mathbf{E}_x \left\{ \sum_{j(r-1) \in \mathcal{N}^{r-2}} \mathbf{Z}_{j(r-1), \ell} \phi_{s_{r-1}}^{j_{r-1}, 0} (X_{s_{r-1}}) \right\} = \mathbf{E}_x \left\{ \sum_{j(r-1) \in \mathcal{N}^{r-2}} \mathbf{Z}_{j(r-1), \ell}' \right\}
\]
where
\[ Z^r_{j(r-1), \ell} = g^r_1(X_{x_1}) \cdots g^r_{s_{r-1}}(X_{x_{s_{r-1}}}) g^r_{s_{r-1}+1}(X_{x_{s_{r-1}+1}}) \prod_{k=0}^{s_{r-1}-1} A_k. \]

By the induction hypotheses we are through since
\[ I_{r, \ell} 1(x) = g^r_{s_{r-1}+\ell}(x) = I_{s_{r-1}} \left[ \sum_{j_{r}=1}^{\infty} H^{j_{r}} g^r_{s_{r-1}+j_{r}}(x) \right]. \]

\[ \square \]

**Corollary A.1** Let \( U_a = \sum_{j=0}^{\infty} a_j g(X_j) \). Then \( E_x [U_a^m] \) is given by (A.1) with
\[ \psi_{r, \ell} = \sum_{j \in \mathcal{N} \times \mathcal{N}_{r-1}} d_{j, \ell}(x) \left\{ a^0_{j_1} a^1_{j_1+j_2} \cdots a^r_{j_1+\cdots+j_r} \right\} \tag{A.8} \]

where
\[ d_{j, \ell} = H^{j_1} I_{g^{j_1}} \cdots H^{j_r} I_{g^{j_r}}. \]

In particular for \( m = 1, 2 \) we have that
\[ E_x [U_a] = \sum_{j=0}^{\infty} d_j(x) a_j, \quad d_j = H^j I_g 1 \]
\[ E_x [U_a^2] = \sum_{j=0}^{\infty} d_j a_j^2 + 2 \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} d_{j, \ell}(x) \left\{ a_j a_{j+\ell} \right\}, \quad d_{j, \ell} = H^j I_g H^\ell I_g 1. \tag{A.9} \]

**Proof of Corollary A.1:**

We obtain (A.8) from (A.2) with \( g_j = a_j g \). With \( m = 2 \) we have
\[ E_x [U_a^2] = \sum_{\ell \in \mathcal{N}_{r-1}} \left( \begin{array}{c} 2 \\ \ell \end{array} \right) \psi_{r, \ell}(x) + \sum_{\ell \in \mathcal{N}_{r-1}} \left( \begin{array}{c} 2 \\ \ell \end{array} \right) \psi_{r, \ell}(x) \]
\[ = \psi_{1, 2}(x) + 2 \psi_{2, (1, 1)}(x) \]
\[ = \left[ \sum_{j_1=0}^{\infty} H^{j_1} I_{g^{j_1}} \right] 1(x) + 2 \left[ \sum_{j_1=0}^{\infty} H^{j_1} I_{g^{j_1}} \right] \left[ \sum_{j_2=1}^{\infty} H^{j_2} I_{g^{j_1+j_2}} \right] 1(x) \]
\[ = \sum_{j=0}^{\infty} H^j g_j^2(x) + 2 \sum_{j=0}^{\infty} \sum_{s=1}^{\infty} H^j I_g H^s g_{j+s}(x). \]

Hence
\[ E_v [U_a^2] = \sum_{j=0}^{\infty} a_j^2 \{ \nu H^j g^2 \} + 2 \sum_{j=0}^{\infty} \sum_{s=1}^{\infty} a_j a_{j+s} \{ \nu H^j I_g H^s g \}. \]

\[ \square \]

**Remark A.2** In particular if \( a_j \equiv 1 \) then (A.9) gives the formulae
\[ E_v [U_a] = \pi_s g, \quad E_v [U_a^2] = \pi_s g^2 + 2 \pi_s I_g H G \pi_s g. \tag{A.10} \]
A.2 Moment inequality

Lemma A.1 Assume that (2.1) holds. Let $p > 1$ and $\eta \in (0, 1)$ and $f$ a real-valued measurable function defined on $E$. Then for any probability measure $\lambda$

$$\lambda \left[ G_{s,v} f \right]^t \leq c_2 E_\lambda^{t/p} \left[ \tau^{1+2(p-1)} \right] \sup_{j \geq 0} \lambda^{t/q} \| f \|^q,$$

$$t = p/(1 + \eta(p - 1)), \quad q = p/\eta(p - 1),$$

and where $c_2$ is a universal constant only dependent upon $p$ and $\eta$.

Proof of Lemma A.1:

Let $q' = p/(p - 1)$, $r = q'/(1 - \eta)$, $w = 1/q'$, $v = 1/q'$, $u = 2/q'$. Then $u = v + w$, $p^{-1} + q^{-1} + r^{-1} = 1$, $1/t = 1/p + 1/q$, $pu = 2(p - 1)$ and $qv = \eta^{-1}$.

From the right hand side of (2.4) and by the Cauchy Schwartz inequality, we get

$$\left[ G_{s,v} f \right]^t(x) = \left[ \sum_j E_x \{ 1(\tau \geq j) f(X_j) \} \right]^t$$

$$\leq \left[ \sum_j P_x^{1/p} (\tau \geq j) E_x^{1/q} \{ |f|^q(X_j) \} \right]^t$$

$$\leq \left[ \sum_j P_x^{1/p} (\tau \geq j) \left\{ P^j \| f \|^q(x) \right\}^{1/q} \right]^t$$

$$= \left( \sum_j \left( j^u P_x^{1/p} (\tau \geq j) \right) \left( j^{-v} \left\{ P^j \| f \|^q(x) \right\}^{1/q} \right)^{1/r} \right)^{t/p} \left( \sum_j \left( j^{-w} \right)^{t/q} \right)^{1/r}$$

$$= c_1 V^{t/p} Z^{t/q}, \text{ say.}$$

We apply the Cauchy Schwartz inequality with $p_1 = p/t$ and $q_1 = q/t$. This gives

$$\lambda \left[ G_{s,v} f \right]^t \leq c_1 \lambda V^{t/p} Z^{t/q}$$

$$\leq c_1 \lambda^{t/p} V \lambda^{t/q} Z$$

$$= c_1 \left[ \sum_{j=0}^{\infty} j^{p-1} P_\lambda (\tau \geq j) \right]^{t/p} \lambda^{t/q} \left[ \sum_{j=0}^{\infty} j^{-\eta-1} \lambda P^j \| f \|^q \right]^{1/q}$$

$$\leq c_2 \left[ \sum_{j=0}^{\infty} j^{2(p-1)} P_\lambda (\tau \geq j) \right]^{t/p} \sup_{j \geq 0} \lambda^{t/q} P^j \| f \|^q$$

$$\leq c_2 E_\lambda^{t/p} \tau^{1+2(p-1)} \sup_{j \geq 0} \lambda^{t/q} (P^j \| f \|^q)$$

and

$$c_2 = \left[ \sum_{j=0}^{\infty} j^{-w} \right]^{1/r} \left[ \sum_{j=0}^{\infty} j^{-\eta-1} \right]^{1/q}.$$
A.3 Asymptotic formulae

Let \( k \) be an integer. Then (cf. (4.35))
\[
\gamma_{g,f}(k) = \pi_s I_g P[k] I_f 1
\]
and \( \gamma_g(k) = \gamma_{g,g}(k) \).

**Lemma A.2** Assume that \( g \) is a small function in one variable such that \( \mu_g = \pi_s g = 0 \). Then
\[
\sigma_g^2 = \sum_{\ell=-\infty}^{\infty} \gamma_g(\ell)
\]
and when \( \mu_g = \pi_s g \neq 0 \), then
\[
\sigma_g^2 = \pi_s(g)\pi_s(g - g \cdot s) + \sum_{\ell=-\infty}^{\infty} \gamma_{(g,g - s\mu_g)}(\ell) . \tag{A.11}
\]

**Proof of Lemma A.2:**

We have by (A.10) that
\[
\sigma_g^2 = \pi_s g^2 - \pi_s^2 g + 2\pi_s I_g P G_{s,v} g - 2\mu_g \pi_s(sg)
\]
\[
= \pi_s g^2 + \mu_g^2 + 2\pi_s I_g P G_{s,v} (g - s\mu_g) - 2\mu_g \pi_s(sg) .
\]

Let \( G^{(n)} = \sum_{j=0}^{n-1} P^j \). Then
\[
G_{s,v} = G^{(n)} + P^n G_{s,v} - G^{(n)} \otimes \pi_s . \tag{A.12}
\]

Let \( \lambda = \pi_s I_g \) and \( f = G_{s,v}(g - s\mu_g) \). Then \( |\lambda| = \pi_s I_g \) (the total variation of \( \lambda \)) and by (A.12),
\[
\pi_s I_g P G_{s,v} (g - s\mu_g) = \sum_{j=1}^{n} \pi_s I_g P^j I_{(g-s\mu_g)} 1 + \lambda P^n f = \sum_{j=1}^{n} \gamma_{(g,g - s\mu_g)}(j) + \lambda P^n f .
\]

Since \( g \) is small, the function \( f \) is bounded. By Nummelin(1984, Theorem 6.10, p 112),
\[
|\lambda| P^n |f| = o(1) .
\]

Hence
\[
2\pi_s I_g P G_{s,v} (g - s\mu_g) = \sum_{\ell=-\infty}^{\infty} \gamma_{(g,g - s\mu_g)}(k) - \pi_s g^2 + \mu_g \pi_s(g s) .
\]

Thus (A.11) holds. \( \square \)
Lemma A.3 Let \( g_0, f_0 \geq 0 \) be real-valued functions such that \( \pi_s I_{g_0} PG_{s, \nu} f_0 < \infty \). Moreover, let \( \{g_m\} \) and \( \{f_n\} \) be sequences of real-valued functions such that \( g_m \xrightarrow{m} g \) a.s. \( \pi_s \), \( f_n \xrightarrow{n} f \) a.s. \( \pi_s \) and \( \sup_m |g_m| \leq g_0, \sup_n |f_n| \leq f_0 \). Then

\[
\lim_{m,n} \pi_s I_{g_m} PG_{s, \nu} f_n = \pi_s I_g PG_{s, \nu} f .
\]

Proof of Lemma A.3: Let \( \xi_n = PG_{s, \nu} f_n \) and \( \xi_0 = PG_{s, \nu} f_0 \) so that \( |\xi_n| \leq \xi_0 \). We have to prove that

\[
\xi_n \xrightarrow{n} \xi = PG_{s, \nu} f \quad \text{a.s.} \ [\pi_s]
\]

Let \( D \) be the set of points where \( f_n \) fails to converge towards \( f \). Then \( \pi_s \{ G(1_D > 0) \} = 0 \) with \( G = \sum_{t=0}^{\infty} P^t \) since \( \pi_s \) is a maximal irreducible measure. Hence \( \pi_s \{ PG_{s, \nu} 1_D > 0 \} = 0 \). The rest of the proof follows directly from the standard dominated convergence theorem since

\[
\pi_s I_{g_m} PG_{s, \nu} f_n = \pi_s I_{g_m} \xi_n = \pi_s (g_m \cdot h_n),
\]

where \( |(g_m \cdot \xi_n)| \leq (g_0 \cdot \xi_0) \) and

\[
(g_m \cdot \xi_n) \xrightarrow{m,n} (g \cdot h) \quad \text{a.s.} \ [\pi_s] .
\]

A candidate for an approximating sequence \( \{g_m, \} \) is given by

\[
g_m = g 1_{C_m} \tag{A.13}
\]

where \( \{C_m\} \) be an increasing sequence of small sets so that \( C_m \uparrow E \) and \( g \) is bounded on \( C_m \) for all \( m \).

Theorem A.2 Let \( L^p(\pi_s) \) be the space of \( L^p \) \( \pi_s \)-integrable functions and assume that

\[
g \in L^1(\pi_s) \cap L^2(\pi_s), \quad \pi_s g^2 + 2\pi_s I_{|g|} PG_{s, \nu} |g| < \infty . \tag{A.14}
\]

Let \( \{g_m\} \) be given by (A.13) and \( \mu_m = \mu_{g_m} \). Then for each \( \ell \)

\[
\gamma_{g_m, g_m - s \mu_m}(\ell) \xrightarrow{m} \gamma_{g, g - s \mu}(\ell) \tag{A.15}
\]

and

\[
E_{\nu} U_g^2 = \pi_s(g) \pi_s(g - g \cdot s) + \lim_{m \to \infty} \left\{ \sum_{\ell = -\infty}^{\infty} \gamma_{g_m, g_m - s \mu_m}(\ell) \right\} .
\]

Suppose that for all \( m \geq 1 \) and \( \ell > 0 \),

\[
|\gamma_{g_m, g_m - s \mu_m}(\ell)| \leq a\ell, \quad \sum_{\ell = 1}^{\infty} a\ell < \infty . \tag{A.16}
\]
Then with absolutely convergence,

$$E_{\nu} U_{g}^{2} = \sum_{\ell=1}^{\infty} \gamma_{\ell}(\ell)$$

(A.17)

when $\pi_{s}g = 0$ and in general

$$E_{\nu} U_{g}^{2} = \pi_{s}(g)\pi_{s}(g - s) + \sum_{\ell=1}^{\infty} \gamma_{g,\cdot-s_{\ell}}(\ell).$$

Proof of Theorem A.2:
Let $f_{m} = g_{m} - s_{\pi_{s}g_{m}}$, $g_{0} = |g|$ and $f_{0} = |g| + s$. By Lemma A.2 and Lemma A.3

$$E_{\nu} U_{g}^{2} = \pi_{s}g^{2} + \pi_{s}^{2}g - 2\pi_{s}g\pi_{s}g \cdot s + 2\pi_{s}I_{g\cdot}PG_{s\cdot}(g - s\cdot \mu_{g})$$

$$= \pi_{s}g^{2} + \pi_{s}^{2}g - 2\pi_{s}g\pi_{s}g \cdot s + \lim_{m} \{2\pi_{s}I_{g\cdot}mPG_{s\cdot}(g_{m} - s\cdot \mu_{m})\}$$

$$= \pi_{s}g^{2} + \pi_{s}^{2}g - 2\pi_{s}g\pi_{s}g \cdot s + \lim_{m} \{2\sum_{\ell=1}^{\infty} \gamma_{g_{m},g_{m}-s_{\ell}}(\ell)\}$$

$$= \pi_{s}g\pi_{s}(g - s \cdot \cdot) + \lim_{m} \{\sum_{\ell=1}^{\infty} \gamma_{g_{m},g_{m}-s_{\ell}}(\ell)\}.$$ 

It is obvious that (A.15) holds and if (A.16) holds then

$$\sum_{\ell=1}^{\infty} \gamma_{g_{m},g_{m}-s_{\ell}}(\ell) \xrightarrow{m} \sum_{\ell=1}^{\infty} \gamma_{g,\cdot-s_{\ell}}(\ell)$$

(A.18)

by the dominated convergence theorem. From (A.14) and (A.18) we can conclude that (A.17) is true.

\[ \square \]

B Appendix

Proof of Lemma 4.2:
Let $W_{r} = \{W_{r_{k}}, k \geq 0\}$. Then, we start by showing that

$$W_{r} \overset{d}{=} W \text{ when } \lambda = \tilde{\lambda}$$

(B.1)

where

$$\tilde{\lambda} = \lambda_{2} \Phi_{\lambda_{1}}, \quad \Phi_{\lambda_{1}} = \sum_{\ell=0}^{\infty} \{\lambda_{1} H_{1}^{\ell}s_{1}\} P_{2}^{\ell}, \quad P = P_{2} \Phi_{\nu_{1}}.$$ 

(B.2)

In order to prove (B.1) it is enough to show that for all $r \geq 0$ and for all $A_{i} \in \mathcal{E}_{r}$,

$$P_{\tilde{\lambda}}(W_{r_{1}} \in A_{0}, \ldots, W_{r_{r}} \in A_{r}) = P_{\lambda}(W_{0} \in A_{0}, \ldots, W_{r} \in A_{r}).$$

(B.3)
Let \( k_0 = j_0 \) and \( k_\ell = j_0 + j_1 + \cdots + j_\ell \) for \( \ell = 0, \ldots, r \). We have
\[
P_\lambda \left( W_{r, q} \in A_0 \cdots W_{r, r} \in A_r \right)
= \sum_{j_0=0}^{\infty} \sum_{j_1=1}^{\infty} \cdots \sum_{j_r=1}^{\infty} P_{\lambda_2} \left( W_{k_0} \in A_0 \cdots W_{k_r} \in A_r \right) P_{\lambda_1} \left( \tau_0 = j_0, \ldots, \tau_r = j_r \right)
= \sum_{j_0=0}^{\infty} \sum_{j_1=1}^{\infty} \cdots \sum_{j_r=1}^{\infty} \left\{ \lambda_2 P_{\lambda_2}^{j_0} I_{A_0} \cdots P_{\lambda_2}^{j_r} I_{A_r} \right\} \left\{ \lambda_1 H_{\lambda_1}^{j_0} s_1 \right\} b_{j_1} \cdots b_{j_r}
= \tilde{\lambda} I_{A_0} \tilde{P} I_{A_1} \cdots \tilde{P} I_{A_r} 1 \tag{B.4}
\]
where
\[
b_{\ell} = \nu_1 H_{\lambda}^{\ell-1} s_1, \quad \ell \geq 0.
\]
Hence (B.3) holds.

From
\[\tilde{P} \geq (s_2 \otimes \nu_2) \Phi_{\nu_1} = s_2 \otimes \nu_2 \Phi_{\nu_1}\]
we obtain the minorization inequality (4.14). Let \( \tilde{H} = \tilde{P} - s \otimes \nu \). Then
\[\tilde{H} = P_2 \Phi_{\nu_1} - s_2 \otimes \nu_2 \Phi_{\nu_1} = (P_2 - s_2 \otimes \nu_2) \Phi_{\nu_1} = H_2 \Phi_{\nu_1}, \quad \tilde{Q} = Q_2 \Phi_{\nu_1}\]
with \( Q_2(x, A) = (1 - s_2(w))^{-1} H_2(w, A)(s_2(w) < 1) + 1(s_2(w) = 1) \). The next task is to prove (4.15), i.e.,
\[W_{r}'' = (W_{r}, Y_{r}^2) \overset{d}{=} \tilde{W}^* \quad \text{when} \quad \lambda = \lambda \tilde{\lambda}\]
where \( \tilde{W}^* \) denotes the split chain generated by \( \tilde{P} \) and \((\tilde{s}, \tilde{\nu})\). Let \( \tilde{P}' \) be the transition probability function for this split chain and let \( \tilde{P}' \) be the transition probability for \( W_{r}'\).

We have to prove that
\[\tilde{P}' = \tilde{P}^*\]

First we recall the structure of a split chain. Suppose that \( P \) is a transition probability which satisfies \( P \geq s \otimes \nu \). Then the corresponding the split chain has transition probability \( \tilde{P} \) which satisfies,
\[P^n(x_0 \times y_0, dx \times y) = y_0 \nu P^{n-1}(dx) \left[ y_0 s(x) + (1 - y)(1 - s(x)) \right] + (1 - y_0) Q P^{n-1}(x_0, dx) \left[ y_0 s(x) + (1 - y)(1 - s(x)) \right], \quad n \geq 1 \tag{B.5}\]
In our case this gives for \( n = 1 \);
\[P'(w_0 \times y_0, dw \times y) = y_0 \nu (dw) \left[ \tilde{s}(w) + (1 - y)(1 - \tilde{s}(w)) \right] + (1 - y_0) Q(w_0, dw) \left[ \tilde{s}(w) + (1 - y)(1 - \tilde{s}(w)) \right] \tag{B.6}\]
by definition of the split chain. We look closer at \( \tilde{P}' \) which by (B.2) satisfies
\[\tilde{P}' = \sum_{\ell=1}^{\infty} b_{\ell} P_2^\ell\]
We replace $P_n^\ell$ by the right hand side of the following expression

$$P_n^\ell(w_0 \times y_0, dw \times y) = y_0 \nu_2 P_2^{\ell-1}(dw) \left[ ys(w) + (1 - y) (1 - s_2(w)) \right]$$

$$+ (1 - y_0) Q_2 P_2^{\ell-1}(w_0, dw) \left[ ys(w) + (1 - y) (1 - s_2(w)) \right]$$

(B.7)

and

$$\sum_{\ell=1}^\infty b_\ell \nu_2 P_2^{\ell-1} = \nu_2 \Phi_{\nu_1} = \nu, \quad \sum_{\ell=1}^\infty b_\ell Q_2 P_2^{\ell-1} = Q_2 \Phi_{\nu_1} = \tilde{\nu}.$$  

(B.8)

Then we obtain (4.15) from (B.6) - (B.8). The first equality in (4.16) follows from (4.15) and the second one is the occupation formulae given by (2.4). When $\lambda = \lambda_1 \times \pi_2$ we get

$$\tilde{\lambda} = \pi_2 \Phi_{\lambda_1} = \sum_{\ell=0}^\infty \{ \lambda_1 H_1^\ell s_1 \} \pi_2 P_\ell = \pi_2 \{ \lambda_1 G_{s_1, \nu_1 s_1} \} = \pi_2$$

If $\lambda = \nu = \nu_1 \times \nu_2$, then $\tilde{\lambda} = \nu$ and $\nu G_{\tilde{\lambda}, \nu \pi_{\tilde{\lambda}}} = \pi_2$ since

$$\tilde{\lambda} = \pi_{\tilde{\lambda}} \quad \text{and} \quad \nu G_{\tilde{\lambda}, \nu \pi_{\tilde{\lambda}}} = \pi_{\tilde{\lambda}}.$$  

\[\square\]

**Proof of Lemma 4.3:** The waiting times, $\{\delta_j, j \geq 0\}$ are given by $\delta_j = \tau_j^1 - \tau_{j-1}^1$. Let

$$b_{n,k} = P(\delta_1 + \cdots + \delta_n = k), \quad k \geq n$$

and $b_{1,k} = b_k$. Then

$$b_{n,k} = \begin{cases} \nu_1 H_1^{k-1} s_1, & \text{for } n = 1 \text{ and } k \geq 1 \\ b_k, & \text{for } n \geq 1 \text{ and } k \geq 1 \end{cases}$$

where “$*n$” denotes $n$-times convolution. The $n$-steps transition probability $P_n^n$ is given by

$$P_n^n = \left\{ \sum_{j=0}^\infty b_{n,n+j} P_2^{n+j} \right\}, \quad n \geq 1.$$  

(B.9)

Since $W$ is geometric ergodic, Nummelin (1984, Th.6.14) thereby there exists a non-negative function $M$ so that $\pi_2(M) < \infty$ and a constant $\rho \in (0, 1)$ so that

$$\|P_2^n(x, \cdot) - \pi_2\| \leq M(x) \rho^n, \quad x \in E, \quad n \geq 0.$$  

Thus by (B.9)

$$\|P_2^n(x, \cdot) - \pi\| \leq \sum_{j=0}^\infty b_{n,n+j} \|P_2^n(x, \cdot) - \pi\|$$

$$\leq \sum_{j=0}^\infty b_{n,n+j} M(x) \rho^n$$

$$\leq M(x) \rho^n \sum_{j=0}^\infty b_{n,n+j} \rho^j$$

$$\leq M(x) \rho^n.$$  

(B.10)
Hence by (B.10) the \( \mathcal{W} \)-process is geometric ergodic.

For the ergodic \( \mathcal{W} \)-process, we have that

\[
\alpha_\ell = \sup_{A, B \in E} \theta_\ell(A, B), \quad \theta_\ell(A, B) = \pi I_A P^\ell B 1 - \pi 1_A \pi 1_B .
\]

Here,

\[
\theta_\ell(A, B) = \pi I_A P^\ell B 1 - \pi 1_A \pi 1_B = \sum_{j=\ell}^\infty b_{\ell, j} \left\{ I_A P^j B_0 - \pi 1_A \pi 1_B \right\} = \sum_{j=\ell}^\infty b_{\ell, j} \theta_j(A, B) .
\]

That is

\[
\alpha_\ell \leq \sum_{j=\ell}^\infty b_{\ell, j} \sup_{A, B \in E} \theta_j(A, B) = \sum_{j=\ell}^\infty b_{\ell, j} \alpha_j \leq \alpha_\ell . \tag{B.11}
\]

By Bolthausen (1982) in general

\[
\sum_{\ell=1}^\infty \ell^k \alpha_\ell < \infty \implies E_{\pi_0} \tau_0^{k+1} < \infty .
\]

By (B.11) it follows that

\[
\sum_{\ell=1}^\infty \ell^k \alpha_\ell < \infty \implies \sum_{\ell=1}^\infty \ell^k \alpha_\ell < \infty .
\]

Hence (4.17) is true. \( \square \)

References


