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QUANTIFYING THE VALUE OF INITIAL INVESTMENT INFORMATION*

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We consider an investor maximizing his expected utility from terminal wealth with portfolio decisions based on the available information flow. This investor faces the opportunity to acquire some additional initial information $\mathcal{G}$. The subjective fair value of this information for the investor is defined as the amount of money that he can pay for $\mathcal{G}$ such that this cost is balanced out by the informational advantage in terms of maximal expected utility. We calculate this value for common utility functions in the setting of a complete market modeled by general semimartingales. The main tools are results of independent interest, namely a martingale preserving change of measure and a martingale representation thorem for initially enlarged filtrations.

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1 Introduction

In this paper we consider an investor who trades in a financial market so as to maximize his expected utility of wealth at a prespecified time. The investor faces the opportunity to acquire, in addition to the common information flow $\mathbb{F}$, some extra information $\mathcal{G}$ at a certain cost, e.g. by hiring a good analyst or by doing more research about companies that he can invest

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in. Acquiring the information $\mathcal{G}$ reduces the initial capital but at the same time enlarges the information flow to $\mathbb{G} = \mathbb{F} \vee \mathcal{G}$ on which the investor can then base his portfolio decisions. Our basic question is then: At what cost is the reduction of the investor's initial wealth offset by the increase in the set of available portfolio strategies? To be more precise, let $u^{\mathbb{F}}(y)$ and $u^{\mathbb{G}}(y)$ be the maximal expected utility of terminal wealth that can be obtained with initial capital y and portfolio decisions based on the information flow $\mathbb{F}$ and $\mathbb{G}$, respectively. Suppose our investor has initial capital x. In this paper we concentrate on the *utility indifference value* π of the additional information $\mathcal{G}$, defined as the solution $\pi = \pi(x)$ of the equation

$$u^{\mathbb{F}}(x) = u^{\mathbb{G}}(x - \pi) \ .$$

π can be interpreted as the investor's subjective fair (purchase) value of the additional information $\mathcal{G}$. Our aim is to calculate π for common utility functions in the situation of a complete market and to study the dependence of π on $\mathcal{G}$, x and on the utility function.

The first rigorous mathematical study of the utility maximization problem under additional initial information is the article of Pikovsky and Karatzas (1996). Subsequent works include Elliott, Geman and Korkie (1997), Amendinger, Imkeller and Schweizer (1998), Grorud and Pontier (1998) and Amendinger (2000). They examined the maximal expected utility under additional information $u^{\mathbb{G}}(x)$ or the expected utility gain from additional information $u^{\mathbb{G}}(x) - u^{\mathbb{F}}(x)$. In comparison to this, the present indifference approach quantifies the informational advantage in terms of money, not utility. Similar methods have been previously applied by various authors to the valuation of options instead of information: Hodges and Neuberger (1989) and Davis, Panas and Zariphopoulou (1993) used utility indifference arguments for the pricing of options in the presence of transaction costs. Indifference arguments based on quadratic criteria were applied by Mercurio (1996), Schweizer (1997) and Møller (1999) to pricing problems in incomplete markets.

The outline of the paper is as follows. In Section 2 we provide the basic notation and the central assumptions underlying this article. Basically, we need the existence of the so-called martingale preserving probability measure (MPPM) $\widetilde{P}$ corresponding to the original probability P. This measure was introduced by Föllmer and Imkeller (1993) and extensively used by Amendinger, Imkeller and Schweizer (1998), Grorud and Pontier (1998) and Amendinger (2000). The main property of the MPPM is that it decouples $\mathbb{F}$ and $\mathcal{G}$ in such a way that $\mathbb{F}$-martingales under P remain $\mathbb{G}$-martingales under $\widetilde{P}$: Hence the martingale property is preserved under an initial enlargement and a simultaneous change of measure. Section 3 is exclusively concerned with the theory of initial enlargements of filtrations. We show first that our assumptions guarantee the existence of the MPPM. By means of the MPPM we then transfer the strong predictable representation property for general local martingales from $\mathbb{F}$ to the initially enlarged filtration $\mathbb{G}$. This extends prior work of Grorud and Pontier (1998) and Amendinger (2000). In Section 4 we solve the utility maximization problem in a complete general semimartingale model for the case that the initial information is non-trivial. We then combine this in Section 5 with the results of Section 3 to calculate the utility indifference value for some common utility functions.

Finally, we compute closed form solutions for this value in a diffusion model if the additional information basically consists of a noisy signal about the terminal stock price.

2 General framework and preliminaries

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions and let $T > 0$ be a finite time horizon. For simplicity we assume $\mathcal{F}_0$ to be trivial. The expectation of a random variable Y with respect to a measure Q on $\mathcal{F}$ is denoted by $E^Q[Y]$. If $\mathcal{H}$ is a sub-σ-field of $\mathcal{A}$, the usual conditional expectation $E^Q[Y|\mathcal{H}]$ is well-defined whenever Y^+ or Y^- are Q-integrable. As in Jacod and Shiryaev (1987) we use a generalized notion of conditional expectations which is defined for all real-valued variables Y and by

$$E^Q[Y|\mathcal{H}] = \begin{cases} E^Q[Y^+|\mathcal{H}] - E^Q[Y^-|\mathcal{H}] & \text{on the set where } E^Q[|Y|\,|\mathcal{H}] < \infty, \\ +\infty & \text{elsewhere.} \end{cases} \tag{2.1}$$

All semimartingales adapted to a complete and right-continuous filtration are taken to have right-continuous paths with left limits. For unexplained terminology from stochastic calculus we refer to Dellacherie and Meyer (1980) or He, Wang and Yan (1992).
Let $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ be a filtration initially enlarging $\mathbb{F}$ by some σ-algebra $\mathcal{G} \subseteq \mathcal{A}$, i.e.

$$\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{G}\,, \quad t \in [0,T]\,. \tag{2.2}$$

We assume that $\mathcal{G}$ is generated by some random variable G taking values in a general measurable space $(X, \mathcal{X})$, i.e. $\mathcal{G} = \sigma(G)$. This causes no loss of generality since we could always choose $(X, \mathcal{X}) := (\Omega, \mathcal{G})$ and take $G : (\Omega, \mathcal{A}) \to (\Omega, \mathcal{G})$, $\omega \mapsto \omega$. In comparison to related work by Amendinger et al. (1998,1999,2000) and Grorud and Pontier (1998), we do not assume that $\mathcal{G}$ is generated by a random variable G taking values in a Polish space. In most parts of this article we shall assume that $\mathcal{G}$ satisfies the following decoupling condition.

Assumption 2.1 (D):
There exists a probability measure $R \sim P$ such that $\mathcal{F}_T$ and $\mathcal{G} = \sigma(G)$ are R-independent.

As the following lemma shows, this assumption implies that $\mathbb{G}$ satisfies the usual conditions.

Lemma 2.2 *Suppose Assumption 2.1 (D) is satisfied. Then $\mathbb{G}$ satisfies under P the usual conditions of completeness and right-continuity.*

Proof:
Theorem 1 in He, Wang and Yan (1982) shows that if $(\mathcal{F}^1_t)_{t\in[0,T]}$ and $(\mathcal{F}^2_t)_{t\in[0,T]}$ are mutually independent filtrations satisfying the usual conditions then also $(\mathcal{F}^1_t \vee \mathcal{F}^2_t)_{t\in[0,T]}$ satisfies these conditions. Assumption 2.1 (D) implies that $\mathbb{F}$ and the constant filtration given by the P-completion of $\mathcal{G}$ are independent under some $R \sim P$. Hence the claim follows. □

We shall see that Assumption 2.1 (D) implies the existence of a regular conditional distribution of G given $\mathcal{F}_T$. Moreover, Corollary 3.5 will even show that Assumption 2.1 (D) is equivalent to

Assumption 2.3 (E):
The regular conditional distribution of G given $\mathcal{F}_T$ exists and is P-a.s. equivalent to the law of G, i.e.

$$P[G \in \cdot \,|\, \mathcal{F}_T](\omega) \sim P[G \in \cdot\,] \quad \text{for } P\text{-a.a. } \omega \in \Omega\,.$$

Assumption 2.3 (E) implies by Théorème V.58 in Dellacherie and Meyer (1980) the existence of a strictly positive $\mathcal{X} \otimes \mathcal{F}_T$-measurable function $p : X \times \Omega \to (0, \infty)$ such that for P-a.a. ω and for all $B \in \mathcal{X}$

$$P[G \in B \mid \mathcal{F}_T](\omega) = \int_B p(x, \omega) P[G \in dx]\,. \tag{2.3}$$

We define $p^G(\omega) := p(G(\omega), \omega)$ and $p^x(\omega) := p(x, \omega)$ for $\omega \in \Omega$, $x \in X$. Note that each p^x is $\mathcal{F}_T$-measurable and p^G is $\mathcal{G}_T$-measurable.

3 Results on initially enlarged filtrations

This section contains results on initially enlarged filtrations that satisfy Assumption 2.1 (D) or Assumption 2.3 (E). In Section 3.1 we show that Assumption 2.1 (D) and Assumption 2.3 (E) are equivalent. Furthermore we introduce the martingale preserving measure which plays a key role in the sequel. Section 3.2 shows that the strong predictable representation property with respect to the filtration $\mathbb{F}$ implies the same property with respect to the initially enlarged filtration $\mathbb{G}$ and the corresponding martingale preserving measure.

3.1 The martingale preserving measure

Proposition 3.1 *Let Q be a probability measure equivalent to P. If Assumption 2.1 (D) is satisfied then there is a probability measure $\widetilde{Q} \sim P$, called* martingale preserving probability measure (corresponding to Q)*, with the following properties:*

1. $\widetilde{Q} = Q$ *on* $\mathcal{F}_T$,
2. $\widetilde{Q} = P$ *on* $\mathcal{G} = \sigma(G)$,
3. $\mathcal{F}_T$ *and* $\mathcal{G} = \sigma(G)$ *are* $\widetilde{Q}$*-independent.*

It is clear that these three properties uniquely determine a probability measure $\widetilde{Q}$ on $\mathcal{G}_T = \mathcal{F}_T \vee \sigma(G)$. In the sequel the term "martingale preserving measure" refers to this unique measure on $\mathcal{G}_T$.

Proof:
By Assumption 2.1 (D) there exists $R \sim P$ such that $\mathcal{F}_T$ and $\mathcal{G}$ are R-independent. Denote by Z_T the density of Q with respect to R on $(\Omega, \mathcal{F}_T)$ and by f the density of $P[G \in \cdot]$ with respect to $R[G \in \cdot]$ on $(X, \mathcal{X})$. $Z_T f(G)$ is then strictly positive and independence gives $E^R[Z_T f(G)] =$

$E^R[Z_T]E^R[f(G)] = 1$. Hence $d\widetilde{Q} := Z_T f(G)\, dR$ defines a probability measure $\widetilde{Q} \sim P$. By the R-independence of $\mathcal{F}_T$ and $\mathcal{G}$ we obtain for $A \in \mathcal{F}_T$ and $B \in \mathcal{X}$ that

$$E^{\widetilde{Q}}[1_A 1_{\{G\in B\}}] = E^R[Z_T 1_A]E^R[f(G)1_{\{G\in B\}}] = Q[A]P[G \in B]\,.$$

Properties 1 and 2 follow by choosing $B = X$ and $A = \Omega$ respectively, and then we obtain property 3. □

An immediate consequence of the equality $Q = \widetilde{Q}$ on $\mathcal{F}_T$ is that any integrability property of an $\mathcal{F}_T$-measurable random variable with respect to Q is inherited by $\widetilde{Q}$. In particular, $(Q, \mathbb{F})$-martingales remain $(\widetilde{Q}, \mathbb{F})$-martingales. But more importantly, the martingale property is even preserved under $\widetilde{Q}$ in $\mathbb{G}$, i.e., under an initial enlargement of the filtration and a simultaneous change to $\widetilde{Q}$. This result was extensively exploited in Amendinger, Imkeller and Schweizer (1998) and motivated the terminology *martingale preserving probability measure*. We now build on results from Amendinger, Imkeller and Schweizer (1998) to show

Corollary 3.2 *Suppose Assumption 2.1 (D) is satisfied. Let Q be any probability measure equivalent to P and denote by $\widetilde{Q}$ the corresponding martingale preserving measure. Then*

1. *We have*

$$\mathcal{M}_{(\mathrm{loc})}(Q, \mathbb{F}) = \mathcal{M}_{(\mathrm{loc})}(\widetilde{Q}, \mathbb{F}) \subseteq \mathcal{M}_{(\mathrm{loc})}(\widetilde{Q}, \mathbb{G})\,. \tag{3.1}$$

2. *Any $\mathbb{F}$-adapted process L has the same distribution under $\widetilde{Q}$ and Q. If L has $(Q, \mathbb{F})$-independent increments, i.e. $L_t - L_s$ is Q-independent of $\mathcal{F}_s$ for $0 \le s \le t \le T$, then L has also $(\widetilde{Q}, \mathbb{G})$-independent increments and the characteristics of L are the same for $(Q, \mathbb{F})$ and $(\widetilde{Q}, \mathbb{G})$. In particular, a $(Q, \mathbb{F})$-Brownian motion (Poisson process) is a $(\widetilde{Q}, \mathbb{G})$-Brownian motion (Poisson process).*

3. *Every $(Q, \mathbb{F})$-semimartingale is a $(Q, \mathbb{G})$-semimartingale.*

Proof:
1.: This was shown in Amendinger, Imkeller and Schweizer (1998) (proof of Theorem 2.5).
2.: The first statement is clear since $\widetilde{Q} = Q$ on $\mathcal{F}_T$ by part 1 of Proposition 3.1. If the $\mathcal{F}_T$-measurable random variable $L_t - L_s$ is Q-independent from $\mathcal{F}_s$, it is by part 1 of Proposition 3.1 $\widetilde{Q}$-independent from $\mathcal{F}_s$ and thus part 3 of Proposition 3.1 implies that it is also $\widetilde{Q}$-independent from $\mathcal{G}_s = \mathcal{F}_s \vee \mathcal{G}$. The distribution of a process with independent increments is determined by its characteristics which are unique and non-random (Jacod and Shiryaev (1987), Th. II.5.2). Hence these characteristics remain unaltered if we go from $(Q, \mathbb{F})$ to $(\widetilde{Q}, \mathbb{G})$. This implies in particular that a $(Q, \mathbb{F})$-Brownian motion (Poisson process) is a $(\widetilde{Q}, \mathbb{G})$-Brownian motion (Poisson process) since these processes are uniquely characterized by their characteristics (Jacod and Shiryaev (1987), Theorems II.5.10, II.4.4 and II.4.5).

3.: This follows from (3.1) and $\widetilde{Q} \sim Q$. □

A priori the definition of the stochastic integral involves the filtration. However, the following result shows that we don't have to differentiate between stochastic integrals with respect to $\mathbb{F}$ and $\mathbb{G}$ in our setting. The proof of Corollary 3.3 is based on (3.1), Théorème 7 in Jacod (1980) and Corollaire 9.21 in Jacod (1979) and was given in Amendinger (2000) (proof of Proposition 3.4).

Corollary 3.3 *Let S be a multidimensional $(P, \mathbb{F})$-semimartingale and suppose Assumption 2.1 (D) is satisfied. Then an $\mathbb{F}$-predictable process H is S-integrable with respect to $\mathbb{F}$ if and only if H is S-integrable with respect to $\mathbb{G}$. Moreover, the stochastic integrals of H with respect to S coincide for both filtrations.*

In comparison with part 3 of Corollary 3.2, Théorème 2.5 of Jacod (1985) shows that if the range $(X, \mathcal{X})$ of G is a Borel space, the absolute continuity

$$P[G \in \cdot \,|\, \mathcal{F}_T](\omega) \ll P[G \in \cdot\,] \tag{3.2}$$

for P-a.a. $\omega \in \Omega$ implies that every $(P, \mathbb{F})$-semimartingale on $[0,T]$ is a $(P, \mathbb{G})$-semimartingale on $[0,T]$ and explicitly gives its canonical decomposition by means of a conditional density process $(q_t^x(\omega))_{t\in[0,T]}$. The relation to our setup is given by $p^G(\omega) = q_T^{G(\omega)}(\omega)/q_0^{G(\omega)}(\omega)$. (Actually, Jacod works on the open intervall $[0,T)$ and therefore assumes that (3.2) holds P-a.s. not with T, but for every $t \in [0,T)$. We use here the closed interval $[0,T]$ and one can show that if (3.2) holds P-a.s. with T, then (3.2) also holds P-a.s. simultaneously for all $t \in [0,T]$ instead of T.)

Provided the stronger equivalence Assumption 2.3 (E) is satisfied, Amendinger, Imkeller and Schweizer (1998) showed by extending arguments of Föllmer and Imkeller (1993) that a change to the martingale preserving measure can be defined via $1/p^G$.

Theorem 3.4 *Let Q be a probability measure equivalent to P and denote by Z_T its $\mathcal{F}_T$-density with respect to P. If Assumption 2.3 (E) is satisfied then*

$$d\widetilde{Q} := \frac{Z_T}{p^G} dP \tag{3.3}$$

defines the martingale preserving probability measure corresponding to Q.

The proof of this result uses (2.3) to show that Z_T/p^G is a density with respect to P. Then one can argue almost exactly as in the proof of Proposition 2.3 in Amendinger, Imkeller and Schweizer (1998), just replacing p_t^G there by p^G. In comparison to Amendinger et al. (1998,2000) and Grorud and Pontier (1998) this construction via p^G is simpler since it avoids the use of Jacod's conditional density process (q_t^x). The decoupling property of $\widetilde{Q}$ defined by (3.3) was obtained independently by Jeulin (see Lemma 3.1 in Grorud and Pontier (1998)).

Theorem 3.4 allows us to clarify the relation between Assumption 2.1 (D) and Assumption 2.3 (E) as follows:

Corollary 3.5 *Assumption 2.1 (D) and Assumption 2.3 (E) are equivalent.*

A version of this result was proved by Grorud and Pontier (1998) (Lemmata 3.1 and 3.4). For the reader's convenience we give the proof in our general framework.

Proof:
By choosing $R = \widetilde{P}$ as the martingale preserving measure corresponding to P, i.e. $dR := 1/p^G\, dP$, Theorem 3.4 yields that Assumption 2.3 (E) implies Assumption 2.1 (D). Conversely, assume that there exists a measure $R \sim P$ such that $\mathcal{F}_T$ and $\mathcal{G} = \sigma(G)$ are R-independent. Since the distributions of (ω, G) on $\mathcal{F}_T \otimes \mathcal{X}$ under P and R are equivalent, there is a strictly positive Radon-Nikodým derivative f of $P \circ (\omega, G)^{-1}|_{\mathcal{F}_T \otimes \mathcal{X}}$ with respect to $R \circ (\omega, G)^{-1}|_{\mathcal{F}_T \otimes \mathcal{X}} = R|_{\mathcal{F}_T} \otimes R[G \in \cdot\,]$; the last equality uses the decoupling property of R. The function

$$f(x|\omega) := \frac{f(\omega, x)}{\int_X f(\omega, x)\, R[G \in dx]}\,, \quad x \in X,\, \omega \in \Omega,$$

is strictly positive and it is straightforward to verify that for $A \in \mathcal{F}_T$ and $B \in \mathcal{X}$

$$E^P\left[1_A P[G \in B|\mathcal{F}_T]\right] = E^P\left[1_A 1_{\{G \in B\}}\right] = E^P\left[1_A \int_B f(x|\cdot)\, R[G \in dx]\right].$$

Hence a regular conditional distribution of G under P given $\mathcal{F}_T$ exists and is given by

$$P[G \in B|\mathcal{F}_T](\omega) = \int_B f(x|\omega)\, R[G \in dx]\,, \quad B \in \mathcal{X}\,, \omega \in \Omega.$$

This implies that for P-a.a. ω we have

$$P[G \in \cdot\,|\mathcal{F}_T](\omega) \sim R[G \in \cdot\,] \sim P[G \in \cdot\,].$$

□

3.2 Strong predictable representation property

Throughout this section let $S = (S^1, \ldots, S^d)^{tr}$ be a d-dimensional $\mathbb{F}$-semimartingale. Our aim is to show that under Assumption 2.1 (D) the martingale representation property of S with respect to $\mathbb{F}$ and some measure Q implies the same property with respect to the initially enlarged filtration $\mathbb{G}$ and the corresponding martingale preserving measure $\widetilde{Q}$. For this we first recall a version of a classical martingale representation result. For $d = 1$ this is almost Theorem 13.9 in He, Wang and Yan (1992); the multidimensional case can be proved as in Jacod (1979), XI.1.a, with obvious modifications for the situation of a non-trivial initial σ-field.

Proposition 3.6 *Suppose the filtration $\mathbb{H}$ satisfies the usual conditions and there is a probability measure $Q^{\mathbb{H}} \sim P$ such that $S \in \mathcal{M}_{\mathrm{loc}}(Q^{\mathbb{H}}, \mathbb{H})$. Denote*

$$\Gamma^{\mathbb{H}} := \left\{ Q \sim Q^{\mathbb{H}} \,\middle|\, \frac{dQ}{dQ^{\mathbb{H}}} \text{ is } \mathcal{H}_T\text{-measurable},\ Q = Q^{\mathbb{H}} \text{ on } \mathcal{H}_0,\ S \in \mathcal{M}_{\mathrm{loc}}(Q, \mathbb{H}) \right\}. \tag{3.4}$$

Then the following statements are equivalent:

1. *If* $Q \in \Gamma^{\mathbb{H}}$ *then* $Q = Q^{\mathbb{H}}$.

2. *The set* $\mathcal{M}_{0,\mathrm{loc}}(Q^{\mathbb{H}}, \mathbb{H})$ *of local* $(Q^{\mathbb{H}}, \mathbb{H})$ *martingales null at* 0 *is equal to the set*

$$\left\{ \theta \cdot S \;\middle|\; \theta \text{ is } S\text{-integrable w.r.t. } (Q^{\mathbb{H}}, \mathbb{H}) \text{ in the sense of local martingales} \right\}$$

of stochastic integrals with respect to S.

We say that S has the *strong predictable representation property with respect to* $(Q^{\mathbb{H}}, \mathbb{H})$ *(for short* $(Q^{\mathbb{H}}, \mathbb{H})$*-PRP)* if one of these statements is valid.

Lemma 3.7 *Suppose* R *is a probability measure such that* $\mathcal{F}_T$ *and* $\mathcal{G} = \sigma(G)$ *are* R*-independent. Let* $f : \Omega \times X \to \mathbb{R}$ *be* $\mathcal{F}_T \otimes \mathcal{X}$*-measurable. If* $f(\cdot, G)$ *is* R*-integrable or nonnegative then*

$$E^R[f(\cdot, G) \mid \mathcal{F}_t \vee \mathcal{G}] = E^R[f(\cdot, x) \mid \mathcal{F}_t]\Big|_{x=\mathcal{G}} \quad , \; t \in [0, T]\,. \tag{3.5}$$

Proof:
First we show (3.5) for functions of the form $f(\omega, x) = 1_A(\omega)1_B(x)$ where $A \in \mathcal{F}_T$ and $B \in \mathcal{X}$. The R-independence of $\mathcal{F}_T$ and $\sigma(G)$ implies that $\mathcal{F}_T$ and $\sigma(G)$ are conditionally R-independent given $\mathcal{F}_t$. We therefore obtain

$$\begin{aligned} E^R[1_A 1_B(G) \mid \mathcal{F}_t \vee \sigma(G)] &= 1_B(G) E^R[1_A \mid \mathcal{F}_t \vee \sigma(G)] \\ &= 1_B(x)\Big|_{x=G} E^R[1_A \mid \mathcal{F}_t] \\ &= E^R[1_A 1_B(x) \mid \mathcal{F}_t]\Big|_{x=G}\,. \end{aligned}$$

By monotone class arguments it follows that (3.5) is valid for all bounded $\mathcal{F}_T \otimes \mathcal{X}$-measurable functions f. The dominated (monotone) convergence theorem for conditional expectations then yields the claim for R-integrable (nonnegative) $f(\cdot, G)$. □

Now we are ready to prove

Theorem 3.8 *Suppose Assumption 2.1 (D) is satisfied and* S *has the strong predictable representation property with respect to* $(Q^{\mathbb{F}}, \mathbb{F})$ *for some* $Q^{\mathbb{F}} \sim P$. *Let* $Q^{\mathbb{G}} = \widetilde{Q^{\mathbb{F}}}$ *denote the martingale preserving probability measure corresponding to* $Q^{\mathbb{F}}$. *Then* S *has the strong predictable representation property with respect to* $(Q^{\mathbb{G}}, \mathbb{G})$. *For short:* $\Gamma^{\mathbb{F}} = \{Q^{\mathbb{F}}\}$ *implies* $\Gamma^{\mathbb{G}} = \{Q^{\mathbb{G}}\}$, *or*

$$(Q^{\mathbb{F}}, \mathbb{F})\text{-}PRP \quad \textit{implies} \quad (Q^{\mathbb{G}}, \mathbb{G})\text{-}PRP. \tag{3.6}$$

Proof:
Let $Q' \in \Gamma^{\mathbb{G}}$. Without loss of generality we can assume that $dQ'/dQ^{\mathbb{G}} \in L^\infty(Q^{\mathbb{G}})$ (see He, Wang and Yan (1992), Th. 13.9). We prove that $Q' = Q^{\mathbb{G}}$ and thus show the claim by Proposition 3.6. Define $Z'_t := \left(dQ'/dQ^{\mathbb{G}}\right)\big|_{\mathcal{G}_t}$, $t \in [0, T]$. For all $t \in [0, T]$ the density Z'_t is $\mathcal{F}_t \vee \sigma(G)$-measurable

and thus of the form $Z'_t(\cdot) = z_t(\cdot, G(\cdot))$ for an $\mathcal{F}_t \otimes \mathcal{X}$-measurable function $z_t(\omega, x)$. If ν denotes the distribution of G under $Q^{\mathbb{G}}$, the process $(z_t(\cdot, x))_{t\in[0,T]}$ is RCLL in t for ν-a.a. x because $(Z'_t(\cdot, G))_{t\in[0,T]}$ is an RCLL-process. Since $S \in \mathcal{M}_{\rm loc}(Q^{\mathbb{F}}, \mathbb{F})$ there is a localizing sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-stopping times such that for all n the stopped process S^{τ_n} is a uniformly integrable $(Q^{\mathbb{F}}, \mathbb{F})$-martingale, hence also a uniformly integrable $(Q^{\mathbb{G}}, \mathbb{G})$-martingale by Corollary 3.2. Because Z' is bounded we conclude that the local $(Q^{\mathbb{G}}, \mathbb{G})$-martingale $Z'S^{\tau_n}$ is also uniformly integrable and therefore a uniformly integrable $(Q^{\mathbb{G}}, \mathbb{G})$-martingale. Expectations with respect to $Q^{\mathbb{G}}$ are denoted by $E^{\mathbb{G}}[\cdot]$. Lemma 3.7 then implies for $t \in [0, T]$ and $n \in \mathbb{N}$

$$\begin{aligned} z_t(\cdot, G)S^{\tau_n}_t &= E^{\mathbb{G}}\left[z_T(\cdot, G)S^{\tau_n}_T \mid \mathcal{F}_t \vee \sigma(G)\right] \\ &= E^{\mathbb{G}}\left[z_T(\cdot, x)\, S^{\tau_n}_T \mid \mathcal{F}_t\right]\big|_{x=G}\,. \end{aligned}$$

Since $z_t(\cdot, \cdot)$ is $\mathcal{F}_t \otimes \mathcal{X}$-measurable and $\mathcal{F}_T$ and $\sigma(G)$ are $Q^{\mathbb{G}}$-independent this implies that $\left(Q^{\mathbb{G}} \otimes \nu\right)$-a.e.

$$z_t(\omega, x)S^{\tau_n}_t(\omega) = E^{\mathbb{G}}\left[z_T(\cdot, x)\, S^{\tau_n}_T \mid \mathcal{F}_t\right](\omega)\,. \tag{3.7}$$

Analogously we obtain for each $t \in [0, T]$ that $\left(Q^{\mathbb{G}} \otimes \nu\right)$-a.e.

$$z_t(\omega, x) = E^{\mathbb{G}}[z_T(\cdot, x) \mid \mathcal{F}_t](\omega)\,. \tag{3.8}$$

Thus (3.7) and (3.8) hold $\left(Q^{\mathbb{G}} \otimes \nu\right)$-a.e. simultaneously for all rational $t \in [0, T]$ and then by right-continuity in t of $z_t(\cdot, x)$ and $z_t(\cdot, x)S^{\tau_n}_t$ even simultaneously for all $t \in [0, T]$. Hence both $(z_t(\cdot, x))_{t\in[0,T]}$ and $(z_t(\cdot, x)S^{\tau_n})_{t\in[0,T]}$ are $(Q^{\mathbb{G}}, \mathbb{F})$-martingales and thus $(Q^{\mathbb{F}}, \mathbb{F})$-martingales by Corollary 3.2 for ν-a.a. x and $n \in \mathbb{N}$. But now $(Q^{\mathbb{F}}, \mathbb{F})$-PRP implies by Proposition 3.6 for ν-a.a. x that $z_T(\cdot, x) = 1$ $Q^{\mathbb{G}}$-a.s. and the $Q^{\mathbb{G}}$-independence of $\mathcal{F}_T$ and $\sigma(G)$ then yields $Z'_T = z_T(\cdot, G(\cdot)) = 1$. □

The preceding theorem is a generalization of Theorem 4.7 in Amendinger (2000), where S is assumed to be locally in $\mathcal{H}^2(Q^{\mathbb{F}}, \mathbb{F})$. But the proof is different: In Amendinger (2000) an L^2-approximation argument shows that any (local) $(Q^{\mathbb{G}}, \mathbb{G})$-martingale null at 0 can be represented as a stochastic integral of a $\mathbb{G}$-predictable integrand with respect to S if this is possible for (local) $(Q^{\mathbb{F}}, \mathbb{F})$-martingales and $\mathbb{F}$-predictable integrands. Our argument proves that the uniqueness of the equivalent $\mathbb{F}$-martingale measure for S implies the uniqueness of the equivalent $\mathbb{G}$-martingale measure in the sense of Proposition 3.6. So the result is obtained by working in Proposition 3.6 on part 1 instead of part 2.

4 Utility maximization with non-trivial initial information

In this section let $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$ be a generic filtration describing the information flow of an investor who maximizes his expected utility by dynamically trading in a complete continuous-time security market with several risky assets. We assume that $\mathbb{H}$ satisfies the usual conditions

and emphasize that $\mathcal{H}_0$ need not be trivial. The results are applied in Section 5 to the ordinary information flow $\mathbb{F}$ and the initially enlarged information flow $\mathbb{G}$ in order to quantify the informational advantage that the investor receives from the additional information $\mathcal{G}$.
Let the discounted price process of the risky assets be given by a d-dimensional $\mathbb{H}$-semimartingale $S = (S^1, \dots, S^d)^{tr}$. We assume throughout this section that the financial market given by S is $\mathbb{H}$-complete and free of arbitrage in the following sense:

Assumption 4.1 ($\mathbb{H}$-C): *There is a* unique *probability measure $Q^{\mathbb{H}} \sim P$ with $dQ^{\mathbb{H}}/dP \in \mathcal{H}_T$ and $Q^{\mathbb{H}} = P$ on $\mathcal{H}_0$ such that $S \in \mathcal{M}_{\mathrm{loc}}(Q^{\mathbb{H}}, \mathbb{H})$.*

We denote by $Z^{\mathbb{H}} = (Z_t^{\mathbb{H}})_{t\in[0,T]}$ the $\mathbb{H}$-density process of $Q^{\mathbb{H}}$ with respect to P and by $E^{\mathbb{H}}[\cdot]$ the expectation with respect to $Q^{\mathbb{H}}$.
A measure Q is called *equivalent local martingale measure for S* if $S \in \mathcal{M}_{\mathrm{loc}}(Q, \mathbb{H})$ and $Q \sim P$. The existence of an equivalent local martingale measure for the discounted asset price process is related to the absence of arbitrage (see Delbaen and Schachermayer (1998)). Furthermore, Proposition 3.6 shows that under Assumption 4.1 S has the predictable representation property with respect to $\left(Q^{\mathbb{H}}, \mathbb{H}\right)$. Such a financial market is called *$\mathbb{H}$-complete* because every contingent future payoff can be replicated in the following sense: For every $H \in L^1\left(Q^{\mathbb{H}}, \mathcal{H}_T\right)$ there is a process ϑ in the space $L(S, \mathbb{H})$ of $\mathbb{R}^d$-valued $\mathbb{H}$-predictable processes integrable with respect to S, such that $\vartheta \cdot S$ is a $\left(Q^{\mathbb{H}}, \mathbb{H}\right)$-martingale and

$$H = E^{\mathbb{H}}[H \mid \mathcal{H}_0] + (\vartheta \cdot S)_T = E^{\mathbb{H}}[H \mid \mathcal{H}_0] + \int_0^T \vartheta \, dS \,. \tag{4.1}$$

Remark 4.2 For the informed reader, we mention here that our results could also be proved under the more general assumption that S satisfies NFLVR and that there is an equivalent σ-martingale measure for S that is unique in the sense of Assumption 4.1. Since this only involves rewriting some arguments without providing additional insight, we stick with Assumption 4.1 here.

To define the investor's optimization problem we first introduce admissible trading strategies.

Definition 4.3 *$\vartheta \in L(S, \mathbb{H})$ is called an* admissible strategy *if*

$$(\vartheta \cdot S)_t = E^{\mathbb{H}}\left[(\vartheta \cdot S)_T \middle| \mathcal{H}_t\right] \quad \text{for all } t \in [0,T]. \tag{4.2}$$

The set of admissible strategies is denoted by $Adm^{\mathbb{H}}$.

Note that (4.2) requires that the right-hand side is well-defined and finite; this is satisfied for any $t \in [0,T]$ if and only if $E^{\mathbb{H}}[|(\vartheta \cdot S)_T| \,|\mathcal{H}_0] < \infty$. This property is part of the definition. However, our use of generalized conditional expectations as in (2.1) means that $(\vartheta \cdot S)_T$ need not be $Q^{\mathbb{H}}$-integrable. We will discuss Definition 4.3 in more detail after Lemma 4.4.

Intuitively, ϑ_t^i represents the number of shares of risky asset i held by an investor at time t. The *wealth process* of a strategy $\vartheta \in Adm^{I\!H}$ with initial capital x is then given by

$$V_t = x + (\vartheta \cdot S)_t = x + \int_0^t \vartheta \, dS\,, \quad 0 \le t \le T\,.$$

In particular, strategies are self-financing. A *utility function* is a strictly increasing, strictly concave and continuously differentiable function $U : (a, \infty) \to I\!R$, $a \in [-\infty, 0]$, which satisfies $\lim_{x\uparrow\infty} U'(x) = 0$ and $\lim_{x\downarrow a} U'(x) = +\infty$. We use the convention that $U(x) := -\infty$ for $x \le a$. Observe that this setting covers utility functions both on all of $I\!R$ and on $(0, \infty)$.
An investor with information flow $I\!H$ and initial capital $x > a$ who wants to maximize his expected utility from terminal wealth has to solve the optimization problem

$$u^{I\!H}(x) := \sup_{V \in \mathcal{V}^{I\!H}(x)} E^P[U(V_T)] \tag{4.3}$$

on the set

$$\mathcal{V}^{I\!H}(x) := \left\{ V \;\middle|\; V = x + \vartheta \cdot S,\ \vartheta \in Adm^{I\!H},\ U^-(V_T) \in L^1(P) \right\}.$$

$u^{I\!H}$ is called *indirect utility function*. We also consider the related optimization problem

$$\operatorname*{ess\,sup}_{V \in \mathcal{V}^{I\!H}(x)} E^P[U(V_T)|\mathcal{H}_0]. \tag{4.4}$$

If this supremum is attained by some element of $\mathcal{V}^{I\!H}(x)$ then this ω-wise optimum is also a solution to the optimization problem (4.3).

Our goal in the next result is to show that our definition of admissibility is quite natural in the present general context.

Lemma 4.4 *1. A measure $Q \sim P$ is a local martingale measure for S if and only if its density with respect to P on $\mathcal{H}_T$ is of the form $\frac{dQ}{dP}\Big|_{\mathcal{H}_0} \cdot Z_T^{I\!H}$.*

2. A process V satisfies $V_t = E^{I\!H}[V_T|\mathcal{H}_t]$ for all $t \in [0,T]$ if and only if V is a martingale with respect to some equivalent local martingale measure Q for S. In particular, $\vartheta \in L(S, I\!H)$ is an admissible strategy if and only if there exists an equivalent local martingale measure Q for S such that $\vartheta \cdot S$ is a $(Q, I\!H)$-martingale.

3. If the lower bound a for the domain of the utility function U is finite, we could replace $Adm^{I\!H}$ by $\{\vartheta \in L(S, I\!H)\,|\vartheta \cdot S \ge c$ for some $c \in I\!R\}$ without changing the supremum in (4.3) (or (4.4)) and the optimal solution, if this exists.

Proof:
1: The product YM of a (local) $(P, I\!H)$-martingale M and a finite $\mathcal{H}_0$-measurable random variable Y is a local $(P, I\!H)$-martingale. To see this, note that with $M \in \mathcal{M}(P, I\!H)$ $\tau^n := T 1_{\{Y \le n\}}$ is a sequence of localizing stopping times such that $1_{\{\tau^n > 0\}} Y M^{\tau^n}$ is a $(P, I\!H)$-martingale. This

will be used in the sequel. The "if"-part then follows from the observation that $dQ/dP|_{\mathcal{H}_0} Z_T^{I\!H}$ is a P-density and $dQ/dP|_{\mathcal{H}_0} Z^{I\!H} S$ is a local $(P, I\!H)$-martingale. For the "only if"-part let $Q \sim P$ with $S \in \mathcal{M}_{\rm loc}(Q, I\!H)$. Let Z denote the $I\!H$-density process of Q with respect to P. From $Z_0 \frac{Z}{Z_0} S \in \mathcal{M}_{\rm loc}(P, I\!H)$ follows $\frac{Z}{Z_0} S \in \mathcal{M}_{\rm loc}(P, I\!H)$. By Proposition 3.6, Assumption 4.1 ($I\!H$-C) then implies $\frac{Z}{Z_0} = Z^{I\!H}$, i.e. $\left.\frac{dQ}{dP}\right|_{\mathcal{H}_T} = Z_0 Z_T^{I\!H}$.

2: For the "only if"-part assume $V_t = E^{I\!H}[V_T|\mathcal{H}_t]$, $t \in [0,T]$, and recall (2.1). It follows that $E^{I\!H}[|V_T|\,|\mathcal{H}_0] < \infty$, $Y := 1 \wedge 1/E^{I\!H}[|V_T|\,|\mathcal{H}_0]$ is $Q^{I\!H}$-integrable and $dQ := Y/E^{I\!H}[Y]\, dQ^{I\!H}$ defines a measure Q equivalent to P. By part 1 we obtain $S \in \mathcal{M}_{\rm loc}(Q, I\!H)$. For the converse let $V \in \mathcal{M}(Q, I\!H)$ for some equivalent local martingale measure Q for S. Let Z denote the $I\!H$-density process of Q with respect to P. Then part 1 implies $Z/Z_0 = Z^{I\!H}$ and Bayes' formula leads to $E^{I\!H}[V_T|\mathcal{H}_t] = E^Q[V_T|\mathcal{H}_t] = V_t$, $t \in [0,T]$. By applying this result to $V := \vartheta \cdot S$ with $\vartheta \in L(S, I\!H)$ we obtain the second part of the claim.

3: We first show that for any $\psi \in L(S, I\!H)$ with $\psi \cdot S$ uniformly bounded from below, there exists $\vartheta \in Adm^{I\!H}$ such that $(\vartheta \cdot S)_T \geq (\psi \cdot S)_T$: By Corollaire 3 in Ansel and Stricker (1994) (which is also valid for non-trivial initial σ-algebras) $\psi \cdot S$ is a local $\left(Q^{I\!H}, I\!H\right)$-martingale. Hence $\psi \cdot S$ is a $\left(Q^{I\!H}, I\!H\right)$-supermartingale and $E^{I\!H}[(\psi \cdot S)_T \mid \mathcal{H}_0] \leq 0$. Assumption 4.1 implies the existence of some $\vartheta \in Adm^{I\!H}$ such that

$$(\vartheta \cdot S)_t = E^{I\!H}[(\psi \cdot S)_T \mid \mathcal{H}_t] - E^{I\!H}[(\psi \cdot S)_T \mid \mathcal{H}_0]\,, \quad t \in [0,T]. \tag{4.5}$$

It follows that $\vartheta \cdot S$ is uniformly bounded from below and satisfies $(\vartheta \cdot S)_T \geq (\psi \cdot S)_T$. In the case $(\vartheta \cdot S)_T = (\psi \cdot S)_T$ we obtain that $\psi \cdot S = \vartheta \cdot S$ and so $\psi \in Adm^{I\!H}$.

The claim now easily follows from the observation that by our conventions on U, every wealth process in $\mathcal{V}^{I\!H}(x)$ is uniformly bounded from below by a. □

We now discuss Definition 4.3 in more detail: If $\mathcal{H}_0$ is not trivial, there is no unique equivalent local martingale measure for S on $\mathcal{H}_T$ since there is complete freedom in the choice of such a measure on the initial σ-algebra $\mathcal{H}_0$. At first sight, our definition of admissibility seems to involve the particular measure $Q^{I\!H}$ via (4.2) in a crucial way. But part 2 of Lemma 4.4 shows that that we could equally well require (4.2) with any equivalent local martingale measure Q for S. In the usual setting of utility maximization (see e.g. Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987) or Kramkov and Schachermayer (1999)) the domain of the utility function is bounded from below and $\vartheta \in L(S, I\!H)$ is defined as admissible if $\vartheta \cdot S$ is uniformly bounded from below. Although our definition differs from this approach, part 3 of Lemma 4.4 shows that it is consistent with the usual setting. For these reasons our concept of admissibility is quite natural in the context of a general complete market and a utility function whose domain might be unbounded from below. For an alternative definition of admissibility and of the set of random variables over which the optimization problem is formulated we refer the interested reader to Schachermayer (1999).

To solve the optimization problem (4.4) we introduce the (continuous, strictly decreasing) inverse

I of the derivative U'. I maps $(0,\infty)$ onto (a,∞) and satisfies

$$\begin{aligned} \lim_{y\downarrow 0} I(y) &= +\infty\,, \\ \lim_{y\uparrow\infty} I(y) &= a\,. \end{aligned} \tag{4.6}$$

Theorem 4.5 *Suppose Assumption 4.1 ($\mathbb{H}$-C) is satisfied and there exists an $\mathcal{H}_0$-measurable random variable $\Lambda^{\mathbb{H}}(x):\Omega\to(0,\infty)$ with*

$$E^{\mathbb{H}}\left[I\left(\Lambda^{\mathbb{H}}(x)Z_T^{\mathbb{H}}\right)\Big|\mathcal{H}_0\right]=x \tag{4.7}$$

and such that

$$V_t^{\mathbb{H}}:=E^{\mathbb{H}}\left[I\left(\Lambda^{\mathbb{H}}(x)Z_T^{\mathbb{H}}\right)\Big|\mathcal{H}_t\right]\,,\quad t\in[0,T]\,,$$

satisfies $U^-(V_T^{\mathbb{H}})\in L^1(P)$. Then $V^{\mathbb{H}}$ is the solution to the optimization problem (4.4), i.e. $V^{\mathbb{H}}\in\mathcal{V}^{\mathbb{H}}(x)$ and $E^P\left[U\left(V_T^{\mathbb{H}}\right)\Big|\mathcal{H}_0\right]=\operatorname*{ess\,sup}_{V\in\mathcal{V}^{\mathbb{H}}(x)} E^P[U(V_T)|\mathcal{H}_0]$.

Proof:
By part 2 of Lemma 4.4 we obtain that $V^{\mathbb{H}}$ is an $\mathbb{H}$-martingale with respect to some equivalent local martingale measure Q for S. Assumption 4.1 ($\mathbb{H}$-C) implies that there is no other equivalent local martingale measure for S that coincides with Q on $\mathcal{H}_0$. Hence S has the strong predictable representation property with respect to $(Q,\mathbb{H})$ and this yields $V^{\mathbb{H}}\in\mathcal{V}^{\mathbb{H}}(x)$. Since U is concave we have

$$U(y)\geq U(x)+U'(y)(y-x)\,,\quad x,y\in(a,\infty)\,.$$

Since $V_T^{\mathbb{H}}=I(\Lambda^{\mathbb{H}}(x)Z_T^{\mathbb{H}})$, the above inequality implies

$$U(V_T^{\mathbb{H}})\geq U(V_T)+\Lambda^{\mathbb{H}}(x)Z_T^{\mathbb{H}}(V_T^{\mathbb{H}}-V_T)\,,\quad V\in\mathcal{V}^{\mathbb{H}}(x)\,. \tag{4.8}$$

Even if $\Lambda^{\mathbb{H}}(x)Z_T^{\mathbb{H}}\left(V_T^{\mathbb{H}}-V_T\right)$ is not integrable, we can take generalized conditional expectations to obtain

$$E^P\left[\Lambda^{\mathbb{H}}(x)Z_T^{\mathbb{H}}(V_T^{\mathbb{H}}-V_T)\Big|\mathcal{H}_0\right]=\Lambda^{\mathbb{H}}(x)E^{\mathbb{H}}\left[V_T^{\mathbb{H}}-V_T\Big|\mathcal{H}_0\right]=0\,,$$

since $\Lambda^{\mathbb{H}}(x)$ is $\mathcal{H}_0$-measurable by assumption. In combination with (4.8) this yields

$$E^P\left[U\left(V_T^{\mathbb{H}}\right)\,\Big|\,\mathcal{H}_0\right]\geq E^P[U(V_T)\,|\,\mathcal{H}_0]\,,\quad V\in\mathcal{V}^{\mathbb{H}}(x)\,. \tag{4.9}$$

Note that both conditional expectations in (4.9) are well-defined in the usual sense due to the definition of $\mathcal{V}^{\mathbb{H}}(x)$ and the assumption that $U^-(V_T^{\mathbb{H}})\in L^1(P)$. □

Remark 4.6 1. If $\sup_{V\in\mathcal{V}^{\mathbb{H}}(x)}E^P[U(V_T)]$ is finite then the strict concavity of U implies that the solution $V^{\mathbb{H}}$ is unique if it exists.

2. For a trivial initial σ-algebra $\mathcal{H}_0$ we recover the classical problem of a small investor maximizing his expected utility from terminal wealth in a complete and arbitrage free market; see Karatzas, Lehoczky and Shreve (1987) or Cox and Huang (1989) and the remarks following Lemma 4.4.

For specific utility functions, $\Lambda^{\mathbb{H}}$ and $V^{\mathbb{H}}$ can often be calculated explicitly in terms of $Z^{\mathbb{H}}$ and x. The following result provides such formulae for common utility functions. The relative entropy of P with respect to $Q^{\mathbb{H}}$ is denoted by $H\left(P|Q^{\mathbb{H}}\right)$.

Corollary 4.7 *Suppose Assumption 4.1 ($\mathbb{H}$-C) is satisfied. Then the optimal wealth process $V^{\mathbb{H}}(x)$ and the indirect utility function $u^{\mathbb{H}}$ for the utility functions U below are given as follows:*

1. *Logarithmic utility $U:(0,\infty)\to\mathbb{R}$, $x\mapsto\log x$: We have*

$$\begin{aligned} V_t^{\mathbb{H}}(x) &= \frac{x}{Z_t^{\mathbb{H}}}, \quad t\in[0,T], \\ u^{\mathbb{H}}(x) &= \log x + E^P\left[\log\frac{1}{Z_T^{\mathbb{H}}}\right] = \log x + H\left(P|Q^{\mathbb{H}}\right). \end{aligned}$$

   $u^{\mathbb{H}}$ is finite if and only if $H\left(P|Q^{\mathbb{H}}\right)$ is finite.

2. *Power utility $U:(0,\infty)\to\mathbb{R}$, $x\mapsto x^\gamma/\gamma$, $\gamma\in(0,1)$: If $E^P\left[\left(Z_T^{\mathbb{H}}\right)^{\frac{\gamma}{\gamma-1}}\middle|\mathcal{H}_0\right]<\infty$ then*

$$\begin{aligned} V_t^{\mathbb{H}}(x) &= \frac{x}{E^{\mathbb{H}}\left[\left(Z_T^{\mathbb{H}}\right)^{\frac{1}{\gamma-1}}\middle|\mathcal{H}_0\right]}E^{\mathbb{H}}\left[\left(Z_T^{\mathbb{H}}\right)^{\frac{1}{\gamma-1}}\middle|\mathcal{H}_t\right], \quad t\in[0,T], \\ u^{\mathbb{H}}(x) &= \frac{x^\gamma}{\gamma}E^P\left[E^P\left[\left(Z_T^{\mathbb{H}}\right)^{\frac{\gamma}{\gamma-1}}\middle|\mathcal{H}_0\right]^{1-\gamma}\right] \end{aligned}$$

   and $u^{\mathbb{H}}$ is finite if $E^P\left[\left(Z^{\mathbb{H}}\right)^{\frac{\gamma}{\gamma-1}}\middle|\mathcal{H}_0\right]^{1-\gamma}$ is P-integrable.

3. *Exponential utility $U:\mathbb{R}\to\mathbb{R}$, $x\mapsto -e^{-\alpha x}$, $\alpha>0$: If $H\left(Q^{\mathbb{H}}|P\right)$ is finite then $u^{\mathbb{H}}$ is finite and we have*

$$\begin{aligned} V_t^{\mathbb{H}}(x) &= x+\frac{1}{\alpha}E^P\left[Z_T^{\mathbb{H}}\log Z_T^{\mathbb{H}}\middle|\mathcal{H}_0\right]-\frac{1}{\alpha}E^{\mathbb{H}}\left[\log Z_T^{\mathbb{H}}\middle|\mathcal{H}_t\right], \quad t\in[0,T], \\ u^{\mathbb{H}}(x) &= -\frac{1}{e^{\alpha x}}E^P\left[Z_T^{\mathbb{H}}\exp\left(-E^P\left[Z_T^{\mathbb{H}}\log Z_T^{\mathbb{H}}\middle|\mathcal{H}_0\right]\right)\right]. \end{aligned}$$

Proof:
In order to apply Theorem 4.5 we first calculate $\Lambda^{\mathbb{H}}$ and verify the required assumptions. Note that a solution $\Lambda^{\mathbb{H}}$ to (4.7) is unique if it exists. Theorem 4.5 yields the formulae for the optimal wealth process $V^{\mathbb{H}}$ and the formulae for $u^{\mathbb{H}}$ then follow by straightforward calculations of $E^P[U(V_T^{\mathbb{H}})]$ which are left to the reader.
1: $I(y)=1/y$ and $\Lambda^{\mathbb{H}}(x)=1/x$ is the solution to (4.7). Obviously $x/Z^{\mathbb{H}}$ is a $(Q^{\mathbb{H}},\mathbb{H})$-martingale. $E^P\left[\log\left(1/Z_T^{\mathbb{H}}\right)\right]=E^{\mathbb{H}}\left[\left(1/Z_T^{\mathbb{H}}\right)\log\left(1/Z_T^{\mathbb{H}}\right)\right]$ is the relative entropy of P with

respect to $Q^{I\!H}$ on $\mathcal{H}_T$ and thus $E^P\left[\log\left(1/Z_T^{I\!H}\right)\right]$ is well-defined with values in $[0,\infty]$. By Theorem 4.5 we then obtain the formula for $V^{I\!H}$ and thus $u^{I\!H}$.
2: $I(y) = y^{\frac{1}{\gamma-1}}$, $E^{I\!H}\left[\left(Z_T^{I\!H}\right)^{\frac{1}{\gamma-1}}\middle|\mathcal{H}_0\right] = E^P\left[\left(Z_T^{I\!H}\right)^{\frac{\gamma}{\gamma-1}}\middle|\mathcal{H}_0\right]$ is finite by assumption and the solution to (4.7) is given by

$$\Lambda^{I\!H}(x) = \frac{x^{\gamma-1}}{E^{I\!H}\left[Z_T^{I\!H\,\frac{1}{\gamma-1}}\middle|\mathcal{H}_0\right]^{\gamma-1}}.$$

By Theorem 4.5 we obtain the formula for $V^{I\!H}$ and thus $u^{I\!H}$.
3: $I(y) = -\frac{1}{\alpha}\log\frac{y}{\alpha}$, $E^P\left[Z_T^{I\!H}\log Z_T^{I\!H}\right] < \infty$ by assumption and the solution to (4.7) is given by

$$\Lambda^{I\!H}(x) = \alpha\exp\left(-\alpha x - E^P\left[Z_T^{I\!H}\log Z_T^{I\!H}\middle|\mathcal{H}_0\right]\right).$$

By Theorem 4.5 we obtain the formula for $V^{I\!H}$ and thus $u^{I\!H}$. $u^{I\!H}$ is finite since $E^P\left[Z_T^{I\!H}\log Z_T^{I\!H}|\mathcal{H}_0\right]$ is bounded from below. □

The following lemma provides sufficient conditions for the existence of an $\mathcal{H}_0$-measurable random variable $\Lambda^{I\!H}(x)$ satisfying (4.7). For this we fix one version of the regular conditional distribution $Q^{I\!H}[Z_T^{I\!H} \in dz\,|\,\mathcal{H}_0]$ and define for each $\omega\in\Omega$ the function $\Psi_\omega : (0,\infty)\to(a,\infty]$ by

$$\Psi_\omega(\lambda) \;:=\; \int I(\lambda z)\,Q^{I\!H}\left[Z_T^{I\!H}\in dz\,|\,\mathcal{H}_0\right](\omega)$$

whenever this integral is finite and $\Psi_\omega(\lambda) := +\infty$ otherwise. Note that the integral is always well-defined in $(-\infty,\infty]$ if $a > -\infty$ since $I(\cdot)\geq a$ and that $\omega\mapsto\Psi_\omega(\lambda)$ is a version of $E^{I\!H}[I(\lambda Z_T^{I\!H})\,|\,\mathcal{H}_0]$ for all λ by (2.1).

Lemma 4.8 *There exists an $\mathcal{H}_0$-measurable $\Lambda^{I\!H}$ taking values in $(0,\infty)$ that satisfies (4.7) if one of the following conditions is valid:*

1. *For P-a.a. ω there are $\lambda_1,\lambda_2\in(0,\infty)$ such that $\Psi_\omega(\lambda_1)\leq x\leq\Psi_\omega(\lambda_2)<\infty$.*
2. *For P-a.a. ω the functions Ψ_ω are finite on $(0,\infty)$.*

Note that the second condition is in particular satisfied if $E^{I\!H}\left[\left|I(\lambda Z^{I\!H})\right|\right]<\infty$ for all $\lambda\in I\!R$. The latter one is a classical condition in the theory of utility maximization, see (5.6) in Karatzas, Lehoczky and Shreve (1987).

Proof:
1: Let $\mathrm{dom}\Psi_\omega$ denote the subset of $(0,\infty)$ where Ψ_ω is finite. As I is decreasing on $(0,\infty)$ it follows that Ψ_ω is decreasing on $\mathrm{dom}\Psi_\omega$ and $\mathrm{dom}\Psi_\omega$ is an interval. Furthermore, dominated convergence implies that Ψ_ω is continuous on $\mathrm{dom}\Psi_\omega$.
By assumption there is $N\subset\Omega$ with $P[N]=0$ such that for each $\omega\in\Omega\setminus N$ there are $\lambda_1,\lambda_2\in\mathrm{dom}\Psi_\omega$ such that $\Psi_\omega(\lambda_1)\leq x\leq\Psi_\omega(\lambda_2)$; hence continuity of Ψ_ω implies that there is $\lambda\in(0,\infty)$ such that $\Psi_\omega(\lambda)=x$. The set

$$B := \left\{(\omega,\lambda)\in\Omega\times(0,\infty)\;\middle|\;\Psi_\omega(\lambda)=x\right\}$$

is $\mathcal{H}_0 \otimes \mathcal{B}((0,\infty))$-measurable and if we denote by Γ the projection of B onto Ω, we have $\Omega \backslash N \subseteq \Gamma$. By the measurable selection theorem (see e.g. Dellacherie and Meyer (1980), III.44-45), B admits a measurable selector, i.e. there exists a $\mathcal{H}_0$-measurable mapping $\Lambda^{\mathbb{H}} : \Omega \to (0,\infty)$ such that $\left(\omega, \Lambda^{\mathbb{H}}(\omega)\right) \in B$ for all $\omega \in \Gamma$. Hence $E^{\mathbb{H}}\left[I\left(\Lambda^{\mathbb{H}} Z_T^{\mathbb{H}}\right) \mid \mathcal{H}_0\right](\omega) = \Psi_\omega\left(\Lambda^{\mathbb{H}}(\omega)\right) = x$ for P-a.a. ω.

2: We can choose a version of $Q^{\mathbb{H}}\left[Z_T^{\mathbb{H}} \in \cdot \,|\mathcal{H}_0\right]$ such that Ψ_ω is finite for all ω. Then $I(\lambda z)$ is decreasing in λ and $Q^{\mathbb{H}}\left[Z_T^{\mathbb{H}} \in dz \mid \mathcal{H}_0\right](\omega)$-integrable for all λ and ω. In combination with (4.6), monotone convergence and dominated convergence imply for all ω

$$\begin{aligned} \lim_{\lambda \downarrow 0} \Psi_\omega(\lambda) &= +\infty\,, \\ \lim_{\lambda \uparrow \infty} \Psi_\omega(\lambda) &= a\,. \end{aligned}$$

Since $a < x < \infty$, part 1 implies the claim. □

5 Utility indifference value of initial information in a complete market

We now consider an investor with information flow $\mathbb{F}$ trading in a complete financial market where the discounted prices of the risky assets are given by an $\mathbb{F}$-semimartingale $S = (S^1, \ldots, S^d)^{tr}$. The aim of Section 5.1 is to introduce and study the subjective monetary value of the additional initial information $\mathcal{G}$ for the investor. Section 5.2 provides explicit calculations of this value in a diffusion model where the additional information basically consists of a noisy signal about the terminal stock price.

5.1 Utility indifference value

We impose the following assumption throughout this section.

Assumption 5.1 *Suppose Assumption 2.1 (D) is satisfied and Assumption 4.1($\mathbb{F}$-C) is satisfied with respect to $\mathbb{F}$.*

It follows by Theorem 3.8 that Assumption 4.1($\mathbb{G}$-C) is also satisfied with respect to $\mathbb{G}$ and the martingale preserving measure $Q^{\mathbb{G}}$ corresponding to $Q^{\mathbb{F}}$. We denote by $Z_T^{\mathbb{F}}$ and $Z_T^{\mathbb{G}}$ the densities of $Q^{\mathbb{F}}$ and $Q^{\mathbb{G}}$ with respect to P. Recall from Theorem 3.4 that we have the relation $Z_T^{\mathbb{F}}/Z_T^{\mathbb{G}} = p^G$ since Assumption 2.3 (E) holds by Corollary 3.5. For both filtrations we are therefore within the complete market framework of the previous section with $\mathbb{H} \in \{\mathbb{F}, \mathbb{G}\}$ and can use the corresponding results and notations.

Definition 5.2 *The* utility indifference value *of the additional initial information $\mathcal{G} = \sigma(G)$ is defined as a solution $\pi = \pi(x)$ of the equation*

$$u^{\mathbb{F}}(x) = u^{\mathbb{G}}(x - \pi)\,. \tag{5.1}$$

Equation (5.1) means that the investor with the goal to maximize his expected utility from terminal wealth is indifferent between the alternatives:

1. Invest the initial capital x optimally by using the information flow $\mathbb{F}$.
2. Acquire the additional information $\mathcal{G}$ by paying π and then invest the remaining capital $x-\pi$ optimally with the help of the enlarged information flow $\mathbb{G}$.

If the indirect utility functions $u^{\mathbb{F}}$ and $u^{\mathbb{G}}$ are finite, continuous, strictly increasing and satisfy $\lim_{y\downarrow a} u^{\mathbb{G}}(y) < u^{\mathbb{F}}(x)$, then the utility indifference value exists and is unique: Corollary 3.3 and $Q^{\mathbb{G}} = Q^{\mathbb{F}}$ on $\mathcal{F}_T$ imply that $Adm^{\mathbb{F}} \subseteq Adm^{\mathbb{G}}$. Thus, $u^{\mathbb{F}}$ is dominated by $u^{\mathbb{G}}$. From $\lim_{y\downarrow a} u^{\mathbb{G}}(y) < u^{\mathbb{F}}(x) \leq u^{\mathbb{G}}(x)$ we then obtain the existence of a nonnegative solution π for (5.1) by the continuity of $u^{\mathbb{G}}$. The strict monotonicity of $u^{\mathbb{G}}$ implies the uniqueness of this solution. We note that these conditions are satisfied in all subsequent examples.
The following theorem provides explicit expressions for π for common utility functions.

Theorem 5.3 *Suppose Assumption 5.1 is satisfied. Then the utility indifference value π for the respective utility functions below is given as follows.*

1. *Logarithmic utility $U:(0,\infty)\to\mathbb{R}$, $x\mapsto \log x$: If $H\left(P|Q^{\mathbb{G}}\right) = E^P\left[\log\frac{1}{Z_T^{\mathbb{G}}}\right] < \infty$ then*

$$\pi = x\left(1-\exp\left(-E^P\left[\log\frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\right]\right)\right). \tag{5.2}$$

2. *Power utility $U:(0,\infty)\to\mathbb{R}$, $x\mapsto x^\gamma/\gamma$, $\gamma\in(0,1)$: If $E^P\left[\left(Z_T^{\mathbb{G}}\right)^{\frac{\gamma}{\gamma-1}}\right] < \infty$ then*

$$\pi = x\left(1-\frac{E^P\left[\left(Z_T^{\mathbb{F}}\right)^{\frac{\gamma}{\gamma-1}}\right]^{\frac{1-\gamma}{\gamma}}}{E^P\left[E^P\left[\left(Z_T^{\mathbb{G}}\right)^{\frac{\gamma}{\gamma-1}}\middle|\mathcal{G}_0\right]^{1-\gamma}\right]^{\frac{1}{\gamma}}}\right). \tag{5.3}$$

3. *Exponential utility $U:\mathbb{R}\to\mathbb{R}$, $x\mapsto -e^{-\alpha x}$, $\alpha>0$: If $H(Q^{\mathbb{G}}|P) = E^P\left[Z_T^{\mathbb{G}}\log Z_T^{\mathbb{G}}\right] < \infty$ then*

$$\pi = -\frac{1}{\alpha}\log E^P\left[Z_T^{\mathbb{G}}\exp\left(E^P\left[Z_T^{\mathbb{G}}\log\frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\middle|\mathcal{G}_0\right]\right)\right]. \tag{5.4}$$

Note that we can replace $Z_T^{\mathbb{F}}/Z_T^{\mathbb{G}}$ with p^G.

Proof:
By Theorem 3.8, Assumption 5.1 implies Assumption 4.1 ($\mathbb{G}$-C). For each part we show also that the integrability assumptions from Corollary 4.7 are satisfied in both $\mathbb{G}$ and $\mathbb{F}$. Then $u^{\mathbb{F}}$ and $u^{\mathbb{G}}$ are given explicitly by Corollary 4.7 and we just have to verify (5.1). Since $u^{\mathbb{G}}$ is strictly increasing, the solution π is unique.

1.: Jensen's inequality yields $E^P\left[\log \frac{1}{Z_T^{\mathbb{F}}}\right] = E^P\left[\log \frac{1}{E^P\left[Z_T^{\mathbb{G}}|\mathcal{F}_T\right]}\right] \leq E^P\left[\log \frac{1}{Z_T^{\mathbb{G}}}\right] < \infty$. Part 1 of Corollary 4.7 implies

$$u^{\mathbb{G}}(x-\pi) - u^{\mathbb{F}}(x) \quad = \quad \log(x-\pi) + E^P\left[\log \frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\right] - \log(x)\,, \tag{5.5}$$

and inserting π given by (5.2) gives (5.1).
2.: From $E^P\left[\left(Z_T^{\mathbb{G}}\right)^{\frac{\gamma}{\gamma-1}}\right] < \infty$ follows $E^P\left[\left(Z_T^{\mathbb{G}}\right)^{\frac{\gamma}{\gamma-1}}\Big|\mathcal{G}_0\right]^{1-\gamma} \in L^1(P)$ since $\gamma \in (0,1)$. Moreover, by Jensen's inequality we obtain

$$E^P\left[\left(Z_T^{\mathbb{F}}\right)^{\frac{\gamma}{\gamma-1}}\right] = E^P\left[E^P\left[Z_T^{\mathbb{G}}|\mathcal{F}_T\right]^{\frac{\gamma}{\gamma-1}}\right] \leq E^P\left[\left(Z_T^{\mathbb{G}}\right)^{\frac{\gamma}{\gamma-1}}\right] < \infty\,.$$

Thus part 2 of Corollary 4.7 shows that (5.1) is equivalent to

$$\frac{x^\gamma}{\gamma} E^P\left[E^P\left[\left(Z_T^{\mathbb{F}}\right)^{\frac{\gamma}{\gamma-1}}\right]^{1-\gamma}\right] = \frac{(x-\pi)^\gamma}{\gamma} E^P\left[E^P\left[\left(Z_T^{\mathbb{G}}\right)^{\frac{\gamma}{\gamma-1}}\Big|\mathcal{G}_0\right]^{1-\gamma}\right]$$

and solving for π leads to (5.3).
3.: Again Jensen's inequality shows $E^P\left[Z_T^{\mathbb{F}} \log Z_T^{\mathbb{F}}\right] \leq E^P\left[Z_T^{\mathbb{G}} \log Z_T^{\mathbb{G}}\right] < \infty$. The martingale preserving measure $Q^{\mathbb{G}}$ has properties as stated in Proposition 3.1. Using these we calculate

$$\begin{aligned}
-E^P\left[Z_T^{\mathbb{G}} \log Z_T^{\mathbb{G}}\Big|\mathcal{G}_0\right] &= E^{\mathbb{G}}\left[\log \frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}} - \log Z_T^{\mathbb{F}}\Big|\mathcal{G}_0\right] \\
&= E^{\mathbb{G}}\left[\log \frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\Big|\mathcal{G}_0\right] - E^{\mathbb{G}}\left[\log Z_T^{\mathbb{F}}\Big|\mathcal{G}_0\right] \\
&= E^P\left[Z_T^{\mathbb{G}} \log \frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\Big|\mathcal{G}_0\right] - E^P\left[Z_T^{\mathbb{F}} \log Z_T^{\mathbb{F}}\right]\,.
\end{aligned}$$

In combination with part 3 of Corollary 4.7 we get

$$\begin{aligned}
u^{\mathbb{G}}(y) &= -e^{-\alpha y} E^{\mathbb{G}}\left[\exp\left(-E^P\left[Z_T^{\mathbb{G}} \log Z_T^{\mathbb{G}}\Big|\mathcal{G}_0\right]\right)\right] \\
&= -\exp\Big(-\alpha y - E^P\left[Z_T^{\mathbb{F}} \log Z_T^{\mathbb{F}}\right]\Big) E^{\mathbb{G}}\left[\exp\left(E^P\left[Z_T^{\mathbb{G}} \log \frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\Big|\mathcal{G}_0\right]\right)\right]
\end{aligned} \tag{5.6}$$

for $y \in \mathbb{R}$. For $y = x - \pi$ with π given by (5.4) this leads to

$$u^{\mathbb{G}}(x-\pi) = -\exp\left(-\alpha x - E^P\left[Z_T^{\mathbb{F}} \log Z_T^{\mathbb{F}}\right]\right) = u^{\mathbb{F}}(x)$$

by part 3 of Corollary 4.7. □

Remark 5.4 We emphasize that our results are derived under the assumption of a small investor since asset prices are exogenously given and not affected by the investor's trading strategy. This assumption is quite usual in the literature on utility maximization but becomes questionable if the optimal strategy takes very large positions in the available securities. In our situation, this might happen in particular for a $\mathbb{G}$-investor who receives some highly relevant additional information. A prime example discussed in more detail in Section 5.2 is a signal about the terminal stock price distorted by only very small noise. However a systematic analysis of large investor effects and equilibrium questions is beyond the scope of this paper and left to future research.

5.2 Examples

We now calculate closed form expressions for the utility indifference value for logarithmic and exponential utility functions when the financial market is given by the standard complete diffusion model and the additional initial information consists as in Amendinger, Imkeller and Schweizer (1998), Section 4.3, of a noisy signal about the terminal value of the Brownian motion driving the asset prices. We have not yet found a closed form solution in the power utility case where an explicit computation of (5.3) appears more difficult.

Let the (discounted) prices of the risky assets be given by the stochastic differential equations

$$\frac{dS_t^i}{S_t^i} = \mu_t^i dt + \sum_{j=1}^{d} \sigma_t^{ij}\, dW_t^j, \quad S_0^i > 0 \quad \text{for } i = 1, \dots, d, \tag{5.7}$$

where $W = (W^j)_{j=1,\dots,d}$ is a d-dimensional Brownian motion and $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the P-augmentation of the filtration generated by W. The excess return vector $\mu = (\mu^i)_{i=1,\dots,d}$ and the volatility matrix $\sigma = (\sigma^{ij})_{i,j=1,\dots,d}$ are assumed predictable with $\int_0^T \left(|\mu_t| + |\sigma_t|^2\right) dt < \infty$ P-a.s. and σ_t has full rank P-a.s. for all $t \in [0,T]$. The *relative risk process* is given by $\lambda_t := \sigma_t^{-1}\mu_t$ and we suppose that $\int_0^T |\lambda_t|^2\, dt$ is finite and the stochastic exponential $Z^{\mathbb{F}} := \mathcal{E}(-\int \lambda\, dW)$ is a $(P, \mathbb{F})$-martingale. Then $dQ^{\mathbb{F}} := Z_T^{\mathbb{F}}\, dP$ is the unique equivalent local martingale measure for S on $\mathcal{F}_T$ so that Assumption 4.1($\mathbb{F}$-C) is satisfied.

Suppose the additional information is a noisy signal about the outcome of W_T, i.e., $\mathcal{G} = \sigma(G)$ with

$$G := (\delta_i W_T^i + (1-\delta_i)\varepsilon_i)_{i=1,\dots,d}\,,$$

where the ε_i are i.i.d. $\mathcal{N}(0,1)$-distributed and independent of $\mathcal{F}_T$ and the δ_i are constant numbers in $[0,1)$. If all δ_i are strictly positive, the additional information is also generated by

$$\widetilde{G} := \left(W_T^i + \frac{1-\delta_i}{\delta_i}\varepsilon_i\right)_{i=1,\dots,d}$$

which is an unbiased signal for W_T. The regular conditional distribution of G given $\mathcal{F}_t$ exists for all $t \in [0,T]$ and is multivariate normal with mean vector $\left(\delta_i W_t^i\right)_{i=1,\dots,d}$ and covariance matrix $\operatorname{diag}\left(\delta_i^2(T-t) + (1-\delta_i)^2\right)_{i=1,\dots,d}$. Hence the conditional distribution of G given $\mathcal{F}_T$ is a.s. equivalent to the distribution of G which is also normal. Assumption 2.3 (E) is therefore satisfied and a straightforward computation gives

$$p^G = \prod_{i=1}^{d} \sqrt{\frac{\delta_i^2 T + (1-\delta_i)^2}{(1-\delta_i)^2}}\, \exp\left(\frac{1}{2}\left(\frac{G_i^2}{\delta_i^2 T + (1-\delta_i)^2} - \frac{(G_i - \delta_i W_T^i)^2}{(1-\delta_i)^2}\right)\right). \tag{5.8}$$

As in previous sections we denote by $Q^{\mathbb{G}}$ the martingale preserving measure corresponding to $Q^{\mathbb{F}}$ and by $Z_T^{\mathbb{G}}$ its density with respect to P. We recall the relation $Z_T^{\mathbb{G}} = Z_T^{\mathbb{F}}/p^G$.

5.2.1 Logarithmic utility indifference value

Let the utility function of the investor be given by $U(x) = \log x$ and assume $E^P\left[\int_0^T |\lambda_t|^2\,dt\right] < \infty$ so that $H\left(P|Q^{\mathbb{F}}\right) = E^P\left[\log(1/Z_T^{\mathbb{F}})\right] = \frac{1}{2}E^P\left[\int_0^T |\lambda_t|^2\,dt\right] < \infty$. In combination with

$$E^P\left[\log p^G\right] = \frac{1}{2}\sum_{i=1}^{d} \log \frac{\delta_i^2 T + (1-\delta_i)^2}{(1-\delta_i)^2} < \infty$$

this leads to $E^P\left[\log \frac{1}{Z_T^{\mathbb{G}}}\right] = E^P\left[\log \frac{1}{Z_T^{\mathbb{F}}}\right] + E^P\left[\log p^G\right] < \infty$. By (5.2) we obtain that the logarithmic utility indifference value is given by

$$\pi = x\left(1 - \prod_{i=1}^{d} \sqrt{\frac{(1-\delta_i)^2}{\delta_i^2 T + (1-\delta_i)^2}}\right)$$

and we can analyze the behavior of this quantity as the parameters vary. If all δ_i converge to zero, then π tends to zero and in particular $\pi = 0$ if $\delta_i = 0$ for all i. Intuitively this shows that the information delivered by G becomes useless when increasing noise hides all information about W_T. Furthermore π is increasing in T, increasing in each δ_i and converges to x if $\delta_i \uparrow 1$ for one i with all other parameters fixed. For very small noise, the insider information intuitively almost offers an arbitrage opportunity; this is best seen in the case of constant coefficients μ and σ where S_T is a function of W_T. In fact, the value of the information for the ordinary investor then comes close to his total initial capital x and the investor can not pay more than x since the logarithmic utility function enforces an strictly positive remaining initial capital $x - \pi$ by requiring an a.s. strictly positive final wealth. Note that the limiting case $\delta_i = 1$ is not included in our framework since it would violate Assumption 2.3 (E).

5.2.2 Exponential utility indifference value

Now consider the case where the investor's utility function is given by $U(x) = -\exp(-\alpha x)$ with $\alpha > 0$ and assume $H(Q^{\mathbb{F}}|P) < \infty$. By Girsanov's theorem, $\widetilde{W} := W + \int \lambda_t\,dt$ is a $(Q^{\mathbb{F}}, \mathbb{F})$-Brownian motion. The relative entropy $H(Q^{\mathbb{F}}|P)$ is given by

$$E^P\left[Z_T^{\mathbb{F}} \log Z_T^{\mathbb{F}}\right] = E^{\mathbb{F}}\left[-\int_0^T \lambda_t\,d\widetilde{W}_t + \frac{1}{2}\int_0^T |\lambda_t|^2\,dt\right] = \frac{1}{2}E^{\mathbb{F}}\left[\int_0^T |\lambda_t|^2\,dt\right]$$

and finite by assumption. From $E^{\mathbb{F}}\left[\left(\int_0^T \lambda_s^i\,ds\right)^2\right] \le E^{\mathbb{F}}\left[\int_0^T |\lambda_s^i|^2\,ds\right] < \infty$ we conclude that W_T^i is in $L^2(Q^{\mathbb{F}})$ and hence in $L^2(Q^{\mathbb{G}})$ for all $i = 1, \ldots, d$. Since $Q^{\mathbb{G}} = P$ on $\sigma(G)$, the random variables G_i, $i = 1, \ldots, d$ are independent and normally distributed under $Q^{\mathbb{G}}$. By (5.8) we obtain $\log p^G \in L^1(Q^{\mathbb{G}})$ and

$$E^P\left[\frac{Z_T^{\mathbb{F}}}{p^G} \log \frac{Z_T^{\mathbb{F}}}{p^G}\right] = E^P\left[Z_T^{\mathbb{F}} \log Z_T^{\mathbb{F}}\right] + E^{\mathbb{G}}\left[\log p^G\right] < \infty\,.$$

Straightforward calculation yields

$$\exp\left(E^{\mathbb{G}}\left[\log p^G\middle|\mathcal{G}_0\right]\right) = \exp\left(E^{\mathbb{G}}\left[\log p^g\right]\Big|_{g=G}\right) \tag{5.9}$$
$$= \prod_{i=1}^{d}\exp\left(\frac{1}{2}\left(\log\frac{\delta_i^2T+(1-\delta_i)^2}{(1-\delta_i)^2}+\frac{G_i^2}{\delta_i^2T+(1-\delta_i)^2}-\frac{E^{\mathbb{G}}\left[(g_i-\delta_iW_T^i)^2\right]\big|_{g=G}}{(1-\delta_i)^2}\right)\right).$$

As $Q^{\mathbb{G}}=Q^{\mathbb{F}}$ on $\mathcal{F}_T$, we obtain

$$E^{\mathbb{G}}\left[(g_i-\delta_iW_T^i)^2\right]\big|_{g=G} = G_i^2-2\delta_iE^{\mathbb{F}}\left[W_T^i\right]G_i+\delta_i^2E^{\mathbb{F}}\left[\left(W_T^i\right)^2\right] \tag{5.10}$$

Further calculation yields

$$E^{\mathbb{G}}\left[\exp\left(\frac{-\delta_i^2T}{2\left(\delta_i^2T+(1-\delta_i)^2\right)(1-\delta_i)^2}G_i^2+\frac{\delta_iE^{\mathbb{F}}[W_T^i]}{(1-\delta_i)^2}G_i-\frac{\delta_i^2E^{\mathbb{F}}\left[(W_T^i)^2\right]}{2(1-\delta_i)^2}\right)\right]$$
$$= \sqrt{\frac{(1-\delta_i)^2}{\delta_i^2T+(1-\delta_i)^2}}\exp\left(\frac{-\delta_i^2}{2(1-\delta_i)^2}\mathrm{Var}_{Q^{\mathbb{F}}}[W_T^i]\right). \tag{5.11}$$

The d factors in the product from (5.9) are $Q^{\mathbb{G}}$-independent since $Q^{\mathbb{G}}=P$ on $\sigma(G)$. Hence $E^{\mathbb{G}}\left[\exp\left(E^{\mathbb{G}}[\log p^G|\mathcal{G}_0]\right)\right]$ is equal to the product of the $Q^{\mathbb{G}}$-expectations of the single d factors; computing $E^{\mathbb{G}}\left[(g_i-\delta_iW_T^i)^2\right]\big|_{g=G}$ by (5.10) and using (5.11) then leads to

$$E^{\mathbb{G}}\left[\exp\left(E^{\mathbb{G}}\left[\log p^G|\mathcal{G}_0\right]\right)\right] = \prod_{i=1}^{d}\exp\left(\frac{-\delta_i^2}{2(1-\delta_i)^2}\mathrm{Var}_{Q^{\mathbb{F}}}[W_T^i]\right).$$

By (5.4) we obtain that the exponential utility indifference value is given by

$$\pi = \frac{1}{2\alpha}\sum_{i=1}^{d}\frac{\delta_i^2}{(1-\delta_i)^2}\mathrm{Var}_{Q^{\mathbb{F}}}[W_T^i]$$
$$= \frac{1}{2\alpha}\sum_{i=1}^{d}\frac{\delta_i^2}{(1-\delta_i)^2}\left(T-2\mathrm{Cov}_{Q^{\mathbb{F}}}\left[\widetilde{W}_T^i,\int_0^T\lambda_s^i\,ds\right]+\mathrm{Var}_{Q^{\mathbb{F}}}\left[\int_0^T\lambda_s^i\,ds\right]\right).$$

If the relative risk process λ is deterministic this simplifies to

$$\pi = \frac{T}{2\alpha}\sum_{i=1}^{d}\frac{\delta_i^2}{(1-\delta_i)^2}.$$

π is decreasing in the risk-aversion coefficient α, increasing in δ_i and tends to zero if all δ_i converge to zero. π tends to infinity if $\delta_i\uparrow 1$ for one i with all other parameters being fixed. Again this is precisely what intuition suggests should happen.

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